



ANNALES

DE

L'INSTITUT FOURIER

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Tome 59, n° 3 (2009), p. 1105-1134.

http://aif.cedram.org/item?id=AIF_2009__59_3_1105_0

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PROOF OF THE KNOP CONJECTURE

by Ivan V. LOSEV

ABSTRACT. — In this paper we prove the Knop conjecture asserting that two smooth affine spherical varieties with the same weight monoid are equivariantly isomorphic. We also state and prove a uniqueness property for (not necessarily smooth) affine spherical varieties.

RÉSUMÉ. — Dans cet article nous prouvons la conjecture de Knop qui affirme que deux variétés affines sphériques lisses avec le même monoïde des poids sont isomorphes de manière équivariante. On énonce et prouve également une propriété d'unicité pour des variétés affines sphériques non nécessairement lisses.

1. Introduction

Throughout the paper the base field \mathbb{K} is algebraically closed and of characteristic zero.

Let G be a connected reductive group, X an irreducible G -variety. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$.

The algebra of regular functions $\mathbb{K}[X]$ has a natural structure of a G -module. It is known that $\mathbb{K}[X]$ is the sum of its finite dimensional G -submodules. By the *weight monoid* $\mathfrak{X}_{G,X}^+$ of X we mean the set of all highest weights of the G -module $\mathbb{K}[X]$. Since $\mathbb{K}[X]$ is an integral domain, the product of two highest vectors is non-zero, whence again a highest vector. It follows that $\mathfrak{X}_{G,X}^+$ is indeed a submonoid of the character lattice $\mathfrak{X}(T)$ of T . By results of Popov [21], $\mathfrak{X}_{G,X}^+$ is finitely generated whenever X is affine.

Recall X is said to be *spherical* iff X is normal and B has an open orbit in X . If X is affine, then X is spherical iff $\mathbb{K}[X]$ is a multiplicity free

Keywords: Spherical varieties, weight monoids, systems of spherical roots, multiplicity free Hamiltonian actions.

Math. classification: 14R20, 53D20.

G -module, that is, the multiplicity of every irreducible module in $\mathbb{K}[X]$ is at most 1. In this case $\mathbb{K}[X] = \bigoplus_{\lambda \in \mathfrak{X}_{G,X}^+} V(\lambda)$ and the monoid $\mathfrak{X}_{G,X}^+$ is saturated, that is, $\mathfrak{X}_{G,X}^+$ is the intersection of a lattice with a finitely generated cone in $\mathfrak{X}(T)_{\otimes \mathbb{Z}} \mathbb{Q}$.

Suppose X is an affine spherical variety. Let $\lambda, \mu, \nu \in \mathfrak{X}_{G,X}^+$ be such that $V(\nu) \subset V(\lambda)V(\mu)$ (the product is taken in $\mathbb{K}[X]$). An element of the form $\lambda + \mu - \nu$ is said to be a *tail* of X . By the *tail cone* of X we mean the closure of the cone in $\mathfrak{t}(\mathbb{R})^*$ generated by all tails. Here $\mathfrak{t}(\mathbb{R})^*$ (real form of the dual space to Cartan subalgebra) stands for $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The tail cone has a distinguished system of generators called the system of spherical roots of X and denoted by $\Psi_{G,X}$, see Section 3 for details.

DEFINITION 1.1. — *Let X_1, X_2 be affine spherical varieties. We say that X_1, X_2 are \mathfrak{X}^+ -equivalent (resp., $\mathfrak{X}^+\Psi$ -equivalent) if $\mathfrak{X}_{G,X_1}^+ = \mathfrak{X}_{G,X_2}^+$ (resp., $\mathfrak{X}_{G,X_1}^+ = \mathfrak{X}_{G,X_2}^+, \Psi_{G,X_1} = \Psi_{G,X_2}$).*

The main objective of the paper is to prove the following assertions.

THEOREM 1.2. — *Any two $\mathfrak{X}^+\Psi$ -equivalent affine spherical varieties are equivariantly isomorphic.*

THEOREM 1.3. — *Any two smooth \mathfrak{X}^+ -equivalent affine spherical varieties are equivariantly isomorphic.*

It is known that there are only finitely many systems of spherical roots for given G . See [23] for details.

Theorem 1.3 was conjectured by F. Knop. Note that it fails if the smoothness assumption is omitted. In fact, for $G = \text{SO}_3$ the spherical varieties

$$\begin{aligned} X_0 &= \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 0\}, \\ X_1 &= \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\} \end{aligned}$$

are \mathfrak{X}^+ -equivalent but not isomorphic. In this example we have $\Psi_{G,X_0} = \emptyset$, while Ψ_{G,X_1} consists of one element.

If G is a torus, i.e. $G = T$, then spherical is the same as toric. If X is an affine toric T -variety, then there is an isomorphism of T -algebras $\mathbb{K}[X] \cong \mathbb{K}[\mathfrak{X}_{T,X}^+]$. Therefore both theorems hold. For G of type A (that is, when all simple ideals of \mathfrak{g} are of type A) Theorem 1.3 was proved by Camus [4]. His approach uses Luna’s classification of spherical varieties, see [18]. We do not use his result in this paper.

We remark that for any finitely generated saturated monoid $\mathfrak{X}^+ \subset \mathfrak{X}(T)$ consisting of dominant weights there is an affine spherical variety X with $\mathfrak{X}_{G,X}^+ = \mathfrak{X}^+, \Psi_{G,X} = \emptyset$, see [21] for details. However, if G is not a torus,

then a smooth affine spherical variety usually has at least one spherical root.

Apart of being interesting in its own right, Theorem 1.3 is important for the theory of multiplicity-free Hamiltonian actions of compact groups on compact symplectic smooth manifolds. Let us recall all necessary definitions.

Let K be a connected compact Lie group and M be a real smooth manifold equipped with a symplectic form ω . An action $K : M$ is called Hamiltonian if it preserves ω and is equipped with a *moment map* μ , that is, a K -equivariant map $\mu : M \rightarrow \mathfrak{k}^*$ satisfying

$$\omega(\xi_*x, v) = \langle d_x\mu(v), \xi \rangle, \quad \forall x \in M, \xi \in \mathfrak{k}, v \in T_xM.$$

Here ξ_*x denotes the vector field at x corresponding to ξ . By a Hamiltonian K -manifold we mean a symplectic manifold equipped with a Hamiltonian action of K .

Choosing a K -invariant scalar product on \mathfrak{k} , one identifies \mathfrak{k}^* with \mathfrak{k} and considers μ as a map $M \rightarrow \mathfrak{k}$. Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$ and a positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$. Define the map $\psi : M \rightarrow \mathfrak{t}_+$ by $\psi(x) = K\mu(x) \cap \mathfrak{t}_+$.

Recall that a compact Hamiltonian K -manifold M is called *multiplicity-free* if it satisfies the following equivalent conditions:

- (1) A general K -orbit in M is a coisotropic submanifold.
- (2) Any fiber of ψ is a single K -orbit.
- (3) The algebra $C^\infty(M)^K$ is commutative with respect to the Poisson bracket induced from $C^\infty(M)$.

The proof of equivalence is similar, for example, to the proof of Proposition A.1 in [24].

An important and interesting problem is to classify (in reasonable terms) all compact multiplicity-free Hamiltonian K -manifolds. To solve this problem we need to introduce certain simple invariants associated with such a manifold. Firstly, by the Kirwan theorem [9], $\psi(M)$ is a convex polytope called the *moment polytope* of M . This is the first invariant we need. The second one is the stabilizer K_x for $x \in \mu^{-1}(\eta)$, where η is a general element of $\psi(M)$. This stabilizer does not depend on choices of η and x so is determined uniquely. It is called the *principal isotropy subgroup* of M . Note that for an arbitrary action one can define a principal isotropy subgroup only up to conjugacy but in our situation we have a "distinguished" general point x , that lies in $\mu^{-1}(C)$.

CONJECTURE 1.4 (Delzant). — *Any two multiplicity-free compact Hamiltonian K -manifolds M_1, M_2 with the same moment polytope and*

principal isotropy subgroup are isomorphic (that is, there is a K -equivariant symplectomorphism $\varphi: M_1 \rightarrow M_2$ commuting with the moment maps).

Currently, the conjecture is proved only in some special cases: see [5], [6], [24]. F. Knop reduced the Delzant conjecture to Theorem 1.3 (unpublished). So this paper completes the proof of the Delzant conjecture (modulo Knop's reduction).

Theorem 1.2 also has some applications to symplectic geometry. Using this theorem, one proves a certain uniqueness result for invariant Kähler structures on a given multiplicity free compact Hamiltonian K -manifold, Theorem 8.3.

Let us describe briefly the content of the paper. Section 2 contains conventions used in the paper and the list of notation. In Section 3 we recall the definitions and some properties of important combinatorial invariants of spherical varieties. These invariants are the Cartan space $\mathfrak{a}_{G,X}$, the weight lattice $\mathfrak{X}_{G,X}$, the valuation cone $\mathcal{V}_{G,X}$, the system of spherical roots $\Psi_{G,X}$, and the set of B -divisors $\mathcal{D}_{G,X}$ equipped with certain two maps.

Section 4 contains some auxiliary results concerning affine spherical varieties and weight monoids. Further, we state there two auxiliary statements – Theorems 4.8, 4.9. Theorem 4.8 asserts, roughly speaking, that affine spherical varieties X_1, X_2 have the same set of B -divisors provided they are $\mathfrak{X}^+\Psi$ -equivalent or smooth and \mathfrak{X}^+ -equivalent. Theorem 4.9 states that $\Psi_{G,X_1} = \Psi_{G,X_2}$ provided X_1, X_2 are smooth and \mathfrak{X}^+ -equivalent. In the end of Section 4 we deduce Theorems 1.2, 1.3 from Theorems 4.8, 4.9 and results of [16].

Section 5 is devoted to reduction procedures, which are based on the local structure theorem and play a crucial role in the proofs of Theorems 4.8, 4.9. These proofs are presented in Sections 6, 7. At the end of Section 6 we also give an algorithm recovering $\mathcal{D}_{G,X}$ from $\mathfrak{X}_{G,X}^+$ and $\Psi_{G,X}$.

Finally in Section 8 we prove a uniqueness result for invariant Kähler structures on a given compact multiplicity free Hamiltonian manifold.

Acknowledgements. — This paper was partially written during author's stay in Rutgers University, New Brunswick, in the beginning of 2007. I would like to thank this institution and especially Professor F. Knop for hospitality. I am also grateful to B. Van Steirteghem for some useful remarks on an earlier version of this text.

2. Notation, terminology and conventions

If an algebraic group is denoted by a capital Latin letter, then we denote its Lie algebra by the corresponding small German letter.

Recall that we fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. This allows us to define the root system $\Delta(\mathfrak{g})$, the Weyl group $W(\mathfrak{g})$, and the system of simple roots $\Pi(\mathfrak{g})$ of \mathfrak{g} . For simple roots and fundamental weights of \mathfrak{g} we use the notation of [19]. Put $U = \text{Rad}_u(B)$.

Note that the character groups $\mathfrak{X}(T), \mathfrak{X}(B)$ are naturally identified. We recall that the character group $\mathfrak{X}(H)$ of an algebraic group H , by definition, consists of all algebraic group homomorphisms $H \rightarrow \mathbb{K}^\times$, where \mathbb{K}^\times is the one-dimensional torus. If H is connected, then $\mathfrak{X}(H)$ is a lattice, so we call it the *character lattice*.

Since G is reductive, there is a G -invariant symmetric form (\cdot, \cdot) , whose restriction to $\mathfrak{t}(\mathbb{Q})$ is positive definite. Note that (\cdot, \cdot) is nondegenerate on the Lie algebra of any reductive subgroup of G . We fix such a form and identify \mathfrak{g} with \mathfrak{g}^* , \mathfrak{t} with \mathfrak{t}^* .

If X_1, X_2 are G -varieties, then we write $X_1 \cong^G X_2$ when X_1, X_2 are G -equivariantly isomorphic. Note that if V_1, V_2 are G -modules, then $V_1 \cong^G V_2$ iff V_1, V_2 are isomorphic as G -modules.

Let Q be a parabolic subgroup of G containing either B or B^- . There is a unique Levi subgroup of Q containing T , we call it the *standard* Levi subgroup of Q . For $\Sigma \subset \Pi(\mathfrak{g})$ we denote by P_Σ the parabolic subgroup of G whose Lie algebra is generated by \mathfrak{b} and the root subspaces corresponding to $-\alpha$ with $\alpha \in \Sigma$.

$A_\lambda^{(B)}$	$= \{a \in A \mid b \cdot a = \lambda(b)a, \quad \forall b \in B\}.$
$A^{(B)}$	$= \cup_{\lambda \in \mathfrak{X}(B)} A_\lambda^{(B)}.$
$\mathfrak{a}_{G,X}$	the Cartan space of a spherical G -variety X , see Section 3 for the definition.
$\mathcal{D}_{G,X}$	the set of B -divisors of a spherical G -variety, see Section 3 for the definition.
$\mathcal{D}_{G,X}(\alpha)$	$= \{D \in \mathcal{D}_{G,X} \mid P_\alpha \not\subset G_D\}.$
$\mathcal{D}_{G,X}^G$	$= \{D \in \mathcal{D}_{G,X} \mid G_D = G\}.$
f_λ	a nonzero element of $\mathbb{K}(X)_\lambda^{(B)}$.
(f)	the zero divisor of a rational function f .
(G, G)	the derived subgroup of a group G
$[\mathfrak{g}, \mathfrak{g}]$	the derived subalgebra of a Lie algebra \mathfrak{g} .
$G *_H V$	the homogeneous bundle over G/H with a fiber V .
G_y	the stabilizer of y under an action of G .

\mathbb{K}^\times	the one-dimensional torus.
$N_G(H)$	$= \{g \in G \mid gHg^{-1} = H\}$.
$\text{Pic}(X)$	the Picard group of a variety X .
$\text{Rad}_u(G)$	the unipotent radical of an algebraic group G .
$\text{rank}_G(X)$	$= \text{rank } \mathfrak{X}_{G,X}$.
$\text{Span}_A(M)$	$= \{a_1m_1 + \dots + a_km_k, a_i \in A, m_i \in M\}$.
$\text{Supp}(\gamma)$	the support of $\gamma \in \text{Span}_{\mathbb{Q}}(\Pi(\mathfrak{g}))$, that is, the set $\{\alpha \in \Pi(\mathfrak{g}) \mid n_\alpha \neq 0\}$, where $\gamma = \sum_{\alpha \in \Pi(\mathfrak{g})} n_\alpha \alpha$.
$V(\mu)$	the irreducible module with highest weight μ .
X^G	$= \{x \in X \mid gx = x, \forall g \in G\}$.
$\mathfrak{X}(G)$	the character group of an algebraic group G .
$\mathfrak{X}_{G,X}$	the weight lattice of a spherical G -variety X , see Section 3 for definition.
$\mathfrak{X}_{G,X}^+$	the weight monoid of a spherical G -variety X .
$\#X$	the cardinality of a set X .
$W(\mathfrak{g})$	the Weyl group of a reductive Lie algebra \mathfrak{g} .
$Z_G(\mathfrak{h})$	$= \{g \in G \mid \text{Ad}(g) _{\mathfrak{h}} = \text{id}\}$
$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$	$= \{\xi \in \mathfrak{g} \mid [\xi, \mathfrak{h}] = 0\}$.
α^\vee	the dual root corresponding to a root α .
$\langle \alpha, v \rangle$	the pairing of elements α, v of dual vector spaces.
$\Delta(\mathfrak{g})$	the root system of a reductive Lie algebra \mathfrak{g} .
$\Pi(\mathfrak{g})$	the system of simple roots of \mathfrak{g} associated with B .
$\Pi(\mathfrak{g})_X^a, \dots, \Pi(\mathfrak{g})_X^d$	the set of all simple roots of type a), ..., d) for a spherical G -variety X , see Section 3 for the definition.
$\Psi_{G,X}$	the system of spherical roots of a spherical G -variety X , see Section 3 for the definition.
φ_D	the vector in $\mathfrak{a}_{G,X}^*$ associated with $D \in \mathcal{D}_{G,X}$, see Section 3 for the definition.

3. Combinatorial invariants of spherical varieties

In this section G is a connected reductive algebraic group and X is a spherical G -variety.

By the *weight lattice* of X we mean the set

$$\mathfrak{X}_{G,X} := \left\{ \mu \in \mathfrak{X}(T) \mid \mathbb{K}(X)_\mu^{(B)} \neq \{0\} \right\}.$$

This is a sublattice in $\mathfrak{X}(T)$. Note that $\dim \mathbb{K}(X)_\mu^{(B)} = 1$ for any $\mu \in \mathfrak{X}_{G,X}$.

Define the Cartan space of X by $\mathfrak{a}_{G,X} := \mathfrak{X}_{G,X} \otimes_{\mathbb{Z}} \mathbb{Q}$. This is a subspace in $\mathfrak{t}(\mathbb{Q})^*$. Note that $\mathfrak{a}_{G,X}$ is equipped with a positive definite symmetric bilinear form (the restriction of the form on $\mathfrak{t}(\mathbb{Q})^*$).

Let v be a \mathbb{Q} -valued discrete valuation of $\mathbb{K}(X)$. One defines the element $\varphi_v \in \mathfrak{a}_{G,X}^*$ by

$$\langle \varphi_v, \mu \rangle = v(f_\mu), \quad \forall \mu \in \mathfrak{X}_{G,X}.$$

It is known, see [10], that the restriction of the map $v \mapsto \varphi_v$ to the set of all G -invariant \mathbb{Q} -valued discrete valuations is injective. Its image is a finitely generated cone in $\mathfrak{a}_{G,X}^*$. We denote this image by $\mathcal{V}_{G,X}$ and call it the valuation cone.

Let X be, in addition, affine and \mathcal{T} denote the tail cone of X . It is known, see, for example [10], Lemma 5.1, that $-\mathcal{V}_{G,X}$ is the dual cone of \mathcal{T} . In turn \mathcal{T} is the dual cone of $-\mathcal{V}_{G,X}$.

Moreover, the valuation cone is a Weyl chamber for a (uniquely determined) group $W_{G,X}$ generated by reflections, see [11], Theorem 7.4. This group is called the Weyl group of X .

Denote by $\Psi_{G,X}$ the set of primitive elements $\alpha \in \mathfrak{X}_{G,X}$ such that $\ker \alpha \subset \mathfrak{a}_{G,X}^*$ is a wall of $\mathcal{V}_{G,X}$ and α is nonpositive on $\mathcal{V}_{G,X}$. It is clear from construction that $\Psi_{G,X}$ is a system of simple roots in a certain root system with Weyl group $W_{G,X}$. An element of $\Psi_{G,X}$ is called a spherical root of X . So if X is affine, then its tail cone is generated by $\Psi_{G,X}$.

By a B -divisor we mean a prime B -stable divisor. Let $\mathcal{D}_{G,X}$ denote the set of all B -divisors on X . We write φ_D instead of φ_{ord_D} . Further, set $G_D := \{g \in G \mid gD = D\}$. Clearly, G_D is a parabolic subgroup of G containing B . For a subset $\mathcal{D} \subset \mathcal{D}_{G,X}$ we put $G_{\mathcal{D}} := \bigcap_{D \in \mathcal{D}} G_D$.

Let $\alpha \in \Pi(\mathfrak{g})$. Set

$$\begin{aligned} \mathcal{D}_{G,X}(\alpha) &:= \{D \in \mathcal{D}_{G,X} \mid P_\alpha \not\subset G_D\}, \\ \mathcal{D}_{G,X}^\alpha &:= \{D \in \mathcal{D}_{G,X} \mid G_D = G\}. \end{aligned}$$

Clearly, $\mathcal{D}_{G,X} = \mathcal{D}_{G,X}^\alpha \sqcup \bigcup_{\alpha \in \Pi(\mathfrak{g})} \mathcal{D}_{G,X}(\alpha)$.

Now we are going to recall Luna's results ([17], see also [18], Section 2) concerning the structure of the sets $\mathcal{D}_{G,X}(\alpha)$ and the vectors $\varphi_D, D \in \mathcal{D}_{G,X}(\alpha)$.

PROPOSITION 3.1 ([17], Proposition 3.4, [18], Lemma 6.4.2). — For $\alpha \in \Pi(\mathfrak{g})$ exactly one of the following possibilities takes place:

- (a) $\mathcal{D}_{G,X}(\alpha) = \emptyset$.
- (b) $\alpha \in \Psi_{G,X}$. Here $\mathcal{D}_{G,X}(\alpha) = \{D^+, D^-\}$, $\varphi_{D^+} + \varphi_{D^-} = \alpha^\vee|_{\mathfrak{a}_{G,X}}$, $\langle \varphi_{D^\pm}, \alpha \rangle = 1$.
- (c) $2\alpha \in \Psi_{G,X}$. In this case $\mathcal{D}_{G,X}(\alpha) = \{D\}$ and $\varphi_D = \frac{1}{2}\alpha^\vee|_{\mathfrak{a}_{G,X}}$.

(d) $\mathbb{Q}\alpha \cap \Psi_{G,X} = \emptyset$, $\mathcal{D}_{G,X}(\alpha) = \{D\}$ and $\varphi_D = \alpha^\vee|_{\mathfrak{a}_{G,X}}$.

We say that a root $\alpha \in \Pi(\mathfrak{g})$ is of type a) (or b),c),d)) if the corresponding possibility takes place for α . The set of all simple roots of type a),... , d) is denoted by $\Pi(\mathfrak{g})_X^a, \dots, \Pi(\mathfrak{g})_X^d$.

PROPOSITION 3.2 ([18], Proposition 3.2). — *Let $\alpha, \beta \in \Pi(\mathfrak{g})$. If $\mathcal{D}_{G,X}(\alpha) \cap \mathcal{D}_{G,X}(\beta) \neq \emptyset$, then exactly one of the following possibilities takes place:*

- (1) $\alpha, \beta \in \Pi(\mathfrak{g})_X^b$ and $\#\mathcal{D}_{G,X}(\alpha) \cap \mathcal{D}_{G,X}(\beta) = 1$.
- (2) $\alpha, \beta \in \Pi(\mathfrak{g})_X^d$, $\langle \alpha^\vee, \beta \rangle = 0$, $\alpha^\vee - \beta^\vee|_{\mathfrak{a}_{G,X}} = 0$ and $\alpha + \beta = \gamma$ or 2γ for some $\gamma \in \Psi_{G,X}$.

Conversely, if $\alpha, \beta \in \Pi(\mathfrak{g})$ are as in (2), then $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta)$.

The following lemma is proved analogously to Lemma 4.1.12 from [16].

LEMMA 3.3. — *Let X be a spherical G -variety, and $\alpha \in \Psi_{G,X}$ have one of the following forms:*

- (1) $\alpha = \alpha_1$, where $\alpha_1 \in \Pi(\mathfrak{g})$.
- (2) $\alpha = 2\alpha_1$, where $\alpha_1 \in \Pi(\mathfrak{g})$.
- (3) $\alpha = k(\alpha_1 + \alpha_2)$, where α_1, α_2 are orthogonal elements of $\Pi(\mathfrak{g})$ and $k = 1$ or $\frac{1}{2}$.

If $D \in \mathcal{D}_{G,X} \setminus \mathcal{D}_{G,X}(\alpha_1)$, then $\langle \varphi_D, \alpha \rangle \leq 0$.

COROLLARY 3.4. — *Let $\alpha \in \Pi(\mathfrak{g})$, $D \in \mathcal{D}_{G,X}(\alpha)$.*

- (1) *If $\alpha \in \Pi(\mathfrak{g})_X^b \cup \Pi(\mathfrak{g})_X^c$, then $G_D = P_{\Pi(\mathfrak{g}) \setminus A}$, where $A := \{\beta \in \Pi(\mathfrak{g}) \mid \langle \beta, \varphi_D \rangle = 1\}$. In particular, if $\alpha \in \Pi(\mathfrak{g})_X^c$, then $A = \{\alpha\}$.*
- (2) *If $\alpha \in \Pi(\mathfrak{g})_X^d$, then either $G_D = P_{\Pi(\mathfrak{g}) \setminus \{\alpha\}}$ or $G_D = P_{\Pi(\mathfrak{g}) \setminus \{\alpha, \beta\}}$, where β is the unique simple root such that $\langle \alpha^\vee, \beta \rangle = 0$, $\alpha^\vee - \beta^\vee|_{\mathfrak{a}_{G,X}} = 0$ and $\alpha + \beta$ is a positive multiple of a spherical root.*

Proof. — The first assertion follows from Proposition 3.2 and Lemma 3.3. In assertion 2 one only needs to prove that β is unique. Assume the contrary, let β, γ be such that $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta) = \mathcal{D}_{G,X}(\gamma)$. Note that $k_1(\alpha + \beta)$, $k_2(\alpha + \gamma)$, $k_3(\beta + \gamma) \in \Psi_{G,X}$ for some $k_1, k_2, k_3 > 0$. But $\langle \alpha^\vee - \beta^\vee, \beta + \gamma \rangle \neq 0$, which contradicts Proposition 3.2. □

4. Some remarks on affine spherical varieties and weight monoids

Throughout this section X is an affine spherical variety.

The definition of the weight monoid $\mathfrak{X}_{G,X}^+$ given in the introduction can be rewritten as $\mathfrak{X}_{G,X}^+ = \left\{ \lambda \in \mathfrak{X}(T) \mid \mathbb{K}[X]_\lambda^{(B)} \neq \{0\} \right\}$. The following lemma is easy. A proof can be found, for instance, in [16], Lemma 3.6.2.

LEMMA 4.1. — $\mathfrak{X}_{G,X} = \text{Span}_{\mathbb{Z}}(\mathfrak{X}_{G,X}^+)$.

LEMMA 4.2. — *If X_1, X_2 are \mathfrak{X}^+ -equivalent affine spherical G -varieties, then $\dim X_1 = \dim X_2$.*

Proof. — Since $\mathbb{K}[X_1]^U \cong \mathbb{K}[\mathfrak{X}_{G,X_1}^+] \cong \mathbb{K}[X_2]^U$, the claim follows easily from results of [21]. □

The following proposition follows from the Luna slice theorem, see [13], Corollary 2.2, for details.

PROPOSITION 4.3. — *If X is smooth, then $X \cong^G G *_H V$, where H is a reductive subgroup of G and V is an H -module.*

Remark 4.4. — Note that G acts on the set of all pairs $(H, V): g \cdot (H, V) = (H', V')$, where $H' = gHg^{-1}$ and there is a linear isomorphism $\iota: V \rightarrow V'$ such that $(ghg^{-1})\iota(v) = \iota(hv)$ for all $h \in H$. One easily shows, see, for instance [16], Lemma 3.6.6, that $G *_H V \cong^G G *_H' V'$ iff $(H, V) \sim_G (H', V')$. So if V_0 is a G -module and $V_0 \times (G *_H V) \cong V_0 \times (G *_H' V')$, then $G *_H V \cong G *_H' V'$.

The following lemma follows directly from highest weight theory.

LEMMA 4.5. — *A simple normal subgroup $G_1 \subset G$ acts trivially on X iff $\langle \mathfrak{X}_{G,X}^+, T \cap G_1 \rangle = 1$. An element $t \in Z(G)$ acts trivially on X iff $\langle \mathfrak{X}_{G,X}^+, t \rangle = 1$.*

DEFINITION 4.6. — *A G -variety X is said to be decomposable if there exist nontrivial connected normal subgroups $G_1, G_2 \subset G$ and G_i -varieties $X_i, i = 1, 2$, such that G is decomposed into a locally direct product of G_1, G_2 (that is, $G = G_1G_2$ and $G_1 \cap G_2$ is finite) and $X \cong^{G_1 \times G_2} X_1 \times X_2$. Under these assumptions we say that the pair G_1, G_2 decomposes X .*

LEMMA 4.7. — *Let $G = G_1G_2$ be a decomposition into a locally direct product. Then the following conditions are equivalent.*

- (1) *The pair (G_1, G_2) decomposes X .*
- (2) $\mathfrak{X}_{G_1 \times G_2, X}^+ = \Gamma_1 + \Gamma_2$, where $\Gamma_i \subset \mathfrak{X}(T \cap G_i), i = 1, 2$.

In particular, if affine spherical varieties X_1, X_2 are \mathfrak{X}^+ -equivalent, and the pair (G_1, G_2) decomposes X_1 , then it decomposes X_2 .

Proof. — Essentially, this lemma was proved in the proof of Lemma 3.6.4 in [16]. In order to make the present paper more self-contained we present an argument below.

Clearly, (1) \Rightarrow (2). Let us check the opposite implication. Set $X^i := X//G_i$. From highest weight theory one easily deduces that $\mathfrak{X}_{G_i, X^i}^+ = \mathfrak{X}_{G, X}^+ \cap \mathfrak{X}(T \cap G^i)$, $i = 1, 2$. Thus $\mathfrak{X}_{G, X}^+ = \mathfrak{X}_{G_1, X^1}^+ + \mathfrak{X}_{G_2, X^2}^+$. In other words, $\mathbb{K}[X]^U = \mathbb{K}[X^1]^{U \cap G_1} \otimes \mathbb{K}[X^2]^{U \cap G_2}$. It follows from highest weight theory that $\mathbb{K}[X] = \mathbb{K}[X^1] \otimes \mathbb{K}[X^2]$. \square

To prove Theorems 1.2, 1.3 we need the following two theorems.

THEOREM 4.8. — *Let X_1, X_2 be affine spherical varieties. If X_1, X_2 are $\mathfrak{X}^+\Psi$ -equivalent or X_1, X_2 are \mathfrak{X}^+ -equivalent and smooth, then there is a bijection $\iota: \mathcal{D}_{G, X_1} \rightarrow \mathcal{D}_{G, X_2}$ such that $\varphi_{\iota(D)} = \varphi_D, G_{\iota(D)} = G_D$.*

THEOREM 4.9. — *Let X_1, X_2 be smooth \mathfrak{X}^+ -equivalent affine spherical varieties. Then $\Psi_{G, X_1} = \Psi_{G, X_2}$.*

Proof of Theorems 1.2, 1.3 modulo Theorems 4.8, 4.9. — Let X_1, X_2 be either $\mathfrak{X}^+\Psi$ -equivalent or smooth and \mathfrak{X}^+ -equivalent affine spherical varieties. Let $X_i^0, i = 1, 2$, denote the open G -orbit of X_i . Clearly, $\mathfrak{X}_{G, X_i^0} = \mathfrak{X}_{G, X_i}, \mathcal{V}_{G, X_i^0} = \mathcal{V}_{G, X_i}, \mathcal{D}_{G, X_i^0} = \mathcal{D}_{G, X_i} \setminus \mathcal{D}_{G, X_i}^G, i = 1, 2$. Thanks to Lemma 4.1, Theorems 4.8, 4.9, one can apply Theorem 1 from [16] to X_1^0, X_2^0 and obtain that $X_1^0 \cong^G X_2^0$. It follows from [16], Proposition 3.6.5, that $X_1 \cong^G X_2$. \square

In the proof of Theorems 4.8, 4.9 we may (and will) assume that Theorems 1.2, 1.3 are already proved for all groups G' and spherical ($\mathfrak{X}^+\Psi$ -equivalent or smooth and \mathfrak{X}^+ -equivalent) G' -varieties X'_1, X'_2 provided one of the following assumptions holds:

- (A1) $\dim G' < \dim G$.
- (A2) $G = G'$ and $\min_{i=1,2}(\#\mathcal{D}_{G, X_i}^G) > \min_{i=1,2}(\#\mathcal{D}_{G, X'_i}^G)$.

5. Reductions

Our reductions are based on the local structure theorem. First variants of this theorem were proved independently in [3], [7]. We give here a version due to Knop.

THEOREM 5.1 ([11], Theorem 2.3 and Lemma 2.1). — *Let X be an irreducible normal G -variety, D an effective B -stable Cartier divisor on X ,*

L_D the standard Levi subgroup of G_D . Then there exists a closed L_D -subvariety $\Sigma \subset X^0 := X \setminus D$ such that the morphism $\text{Rad}_u(G_D) \times \Sigma \rightarrow X^0, [g, s] \mapsto gs$, is a G_D -equivariant isomorphism. Here $\text{Rad}_u(G_D)$ acts by left translations on itself and trivially on Σ , while L_D acts by conjugations on $\text{Rad}_u(G_D)$ and initially on Σ .

A subvariety Σ satisfying the claim of the previous theorem is said to be a section of X associated with D .

Remark 5.2. — Being the quotient for the action $\text{Rad}_u(G_D): X^0$, the L_D -variety Σ depends only on the support of D . If $D = \sum_{i=1}^k a_i D_i, a_i \in \mathbb{N}$, we denote the L_D -variety Σ by $X(D_1, \dots, D_k)$ or $X(\{D_1, \dots, D_k\})$. Note also that Σ is smooth (spherical) provided X is.

Till the end of this subsection X is an affine spherical variety.

Here is the first version of our reduction procedure.

PROPOSITION 5.3. — Choose a subset $\mathcal{D} \subset \mathcal{D}_{G,X}$ and let M be the standard Levi subgroup of $G_{\mathcal{D}}$. Suppose there is a Cartier divisor of the form $\sum_{D \in \mathcal{D}} a_D D, a_D > 0$, (which is the case, for instance, when X is smooth). Then

- (1) $X(\mathcal{D})$ is an affine M -variety.
- (2) The map $\iota: \mathcal{D}_{M,X(\mathcal{D})} \rightarrow \mathcal{D}_{G,X}, D \mapsto \text{Rad}_u(G_D) \times D$, is an injection with image $\mathcal{D}_{G,X} \setminus \mathcal{D}$. Furthermore, $\varphi_D = \varphi_{\iota(D)}, G_{\iota(D)} \cap M = M_D$ for any $D \in \mathcal{D}_{M,X(\mathcal{D})}$.
- (3) $\mathfrak{X}_{M,X(\mathcal{D})} = \mathfrak{X}_{G,X}, \Psi_{M,X(\mathcal{D})} = \Psi_{G,X} \cap \text{Span}_{\mathbb{Q}}(\Delta(\mathfrak{m}))$.
- (4) $\mathfrak{X}_{M,X(\mathcal{D})}^+ = \{\chi \in \mathfrak{X}_{G,X} \mid \langle \varphi_D, \chi \rangle \geq 0, \forall D \in \mathcal{D}_{G,X} \setminus \mathcal{D}\}$.

Proof. — To prove assertion 1 note that, being a complement to a Cartier divisor in an affine variety, X^0 is affine. Being isomorphic to a closed subvariety of X^0 , the variety Σ is affine. Assertions 2, 3 follow from [16], Lemma 3.5.5. Assertion 4 follows from assertion 2 and [16], Lemma 3.6.2. □

In the second version of our reduction procedure we do not need to know $\varphi_D, D \notin \mathcal{D}, G_{\mathcal{D}}$.

PROPOSITION 5.4. — In the above notation suppose $(f_{\mu}) = \sum_{D \in \mathcal{D}} a_D D$ for some positive a_D . Then $G_{\mathcal{D}} = BG_{\mu}, \mathfrak{X}_{G_{\mu},X(\mathcal{D})}^+ = \mathfrak{X}_{G,X}^+ + \mathbb{Z}\mu$ and the image of $\mathcal{D}_{G_{\mu},X(\mathcal{D})}$ in $\mathcal{D}_{G,X}$ (see assertion 2 of Proposition 5.3) coincides with $\{D \in \mathcal{D}_{G,X} \mid \langle \mu, \varphi_D \rangle = 0\}$.

Proof. — An element $g \in \mathbb{K}(X)$ lies in $\mathbb{K}[X \setminus (f)]$ iff $g = g_1/f^n$ for some $n \in \mathbb{N}, g_1 \in K[X]$. Clearly, $g \in \mathbb{K}[X^0]^{(B)}$ iff $g_1 \in \mathbb{K}[X]^{(B)}$. The

equality for $\mathfrak{X}_{G_\mu, X(\mathcal{D})}^+$ stems now from the natural isomorphism $\mathbb{K}[X^0]^U \cong \mathbb{K}[X(\mathcal{D})]^{U \cap G_\mu}$. The description of $\mathcal{D}_{G_\mu, X(\mathcal{D})}$ is clear. \square

Under the assumptions of the previous proposition, we put $X(\mu) := X(\mathcal{D})$. Note that $X(\mu) \neq X$ iff f_μ is not invertible in $\mathbb{K}[X]$ iff $\mu \in \mathfrak{X}_{G, X}^+ \setminus -\mathfrak{X}_{G, X}^+$.

COROLLARY 5.5. — *Let $\mu \in \mathfrak{X}_{G, X}^+$ be such that $\mathfrak{X}_{G, X}^+ + \mathbb{Z}\mu = \mathfrak{X}_{G, X}$ (such μ always exists). Then $\Pi(\mathfrak{g})_X^a = \{\alpha \in \Pi(\mathfrak{g}) \mid \langle \alpha^\vee, \mu \rangle = 0\}$.*

Proof. — Any $f \in \mathbb{K}[X(\mu)]^{(B \cap G_\mu)}$ is invertible. Thus $\mathcal{D}_{G(\mu), X(\mu)} = \emptyset$. From Proposition 5.4 it follows that $G_{\mathcal{D}_{G, X}} = BG_\mu$. But $G_{\mathcal{D}_{G, X}} = P_{\Pi(\mathfrak{g})_X^a}$. \square

The following statement stems directly from Propositions 5.3, 5.4.

COROLLARY 5.6. — *Let X_1, X_2 be affine spherical G -varieties.*

- (1) *If X_1, X_2 are \mathfrak{X}^+ -equivalent (resp., $\mathfrak{X}^+\Psi$ -equivalent), then for any $\mu \in \mathfrak{X}_{G, X}^+ \setminus -\mathfrak{X}_{G, X}^+$ the G_μ -varieties $X_1(\mu), X_2(\mu)$ are \mathfrak{X}^+ -equivalent (resp., $\mathfrak{X}^+\Psi$ -equivalent).*
- (2) *Suppose X_1, X_2 are smooth and \mathfrak{X}^+ -equivalent, $D_1 \in \mathcal{D}_{G, X_1}, D_2 \in \mathcal{D}_{G, X_2}$ are such that $G_{D_1} = G_{D_2}$ and $\{\varphi_D, D \in \mathcal{D}_{G, X_1} \setminus \{D_1\}\} = \{\varphi_D, D \in \mathcal{D}_{G, X_2} \setminus \{D_2\}\}$. Then $X_1(D_1), X_2(D_2)$ are \mathfrak{X}^+ -equivalent (as M -varieties, where M is the standard Levi subgroup in G_D).*

Remark 5.7. — Let X, Y be smooth spherical varieties and $\varphi: X \rightarrow Y$ be a smooth surjective morphism with irreducible fibers. Then φ induces an embedding $\varphi^*: \mathcal{D}_{G, Y} \hookrightarrow \mathcal{D}_{G, X}, D \mapsto \varphi^{-1}(D)$. Note that $G_D = G_{\varphi^*(D)}$.

In general, it is difficult to understand the structure of $X(\mathcal{D})$ as a homogeneous vector bundle. However, there is a special case when the description is easy.

LEMMA 5.8. — *Let Q^- be a parabolic subgroup of G containing B^-, M the standard Levi subgroup of G . Suppose $X = G *_H V$, where $H \subset M$. Denote by π the natural homomorphism $X = G *_H V \rightarrow G/Q^-, [g, v] \mapsto gQ^-,$ and set $\mathcal{D} := \pi^*(\mathcal{D}_{G, G/Q^-})$. Then $G_{\mathcal{D}} = BM$ and $X(\mathcal{D}) \cong^M \text{Rad}_u(\mathfrak{q}^-) \times M *_H V$.*

Proof. — Note that $X(\mathcal{D}) \cong^M \pi^{-1}(eQ^-) \cong^M Q^- *_H V \cong^M Q^- *_M (M *_H V) \cong^M \text{Rad}_u(Q^-) \times M *_H V$. The exponential mapping defines an M -equivariant isomorphism $\text{Rad}_u(\mathfrak{q}^-) \rightarrow \text{Rad}_u(Q^-)$. \square

6. Correspondence between B -divisors

The goal of this section is to prove Theorem 4.8. We assume that G is not a torus. Throughout the section X is an affine spherical variety and the

action $G: X$ is assumed to be locally effective, that is, its kernel is finite. We write $\mathfrak{a}, \mathfrak{X}^+, \mathfrak{X}, \Psi, \mathcal{D}, \Pi^a, \dots, \Pi^d$ instead of $\mathfrak{a}_{G,X}, \mathfrak{X}_{G,X}^+, \mathfrak{X}_{G,X}, \Psi_{G,X}, \mathcal{D}_{G,X}, \Pi(\mathfrak{g})_X^a, \dots, \Pi(\mathfrak{g})_X^d$.

Let us briefly describe the scheme of proof. The key idea in the proof is to use Proposition 5.4. This proposition allows to recover (partially) elements of \mathcal{D}_{G,X_i} satisfying $\langle \varphi_D, \mu \rangle = 0$ for some $\mu \in \mathfrak{X}_{G,X_i}^+ \setminus -\mathfrak{X}_{G,X_i}^+$. This motivates us to define a certain subset of *hidden* elements of \mathcal{D}_{G,X_i} (Definition 6.2), those that can not be recovered by using Proposition 5.4. Then we study some properties of hidden divisors, Lemma 6.3, Propositions 6.4, 6.6. The former proposition deals with the case when all divisors in $\mathcal{D}_{G,X_i}(\alpha)$ are hidden for some i . Proposition 6.6 describes the set of hidden divisors for smooth X_i . In the proof of Theorem 4.8 we first construct a certain bijection between the sets of nonhidden divisors. Its existence is deduced essentially from Proposition 5.4. Then we show (and this is the most complicated and technical part of the proof) that this bijection can be extended in the required way to the whole sets of divisors.

LEMMA 6.1. — *Suppose X is a spherical G -variety and the action $G: X$ is locally effective. If $\Pi^a \cup \bigcup_{\alpha \in \Psi} \text{Supp}(\alpha) = \Pi(\mathfrak{g})$, then $\bigcup_{\alpha \in \Psi} \text{Supp}(\alpha) = \Pi(\mathfrak{g})$.*

Proof. — Assume the contrary. It follows from Corollary 5.5 that $\Pi(\mathfrak{g}_1) \not\subset \Pi^a$ for any simple ideal $\mathfrak{g}_1 \subset \mathfrak{g}$. Therefore there is $\beta \in \Pi^a$ such that $\beta \notin \text{Supp}(\alpha)$ for any $\alpha \in \Psi$ but β is adjacent to $\text{Supp}(\alpha)$ for some $\alpha \in \Psi$. It follows that $\langle \beta^\vee, \alpha \rangle \neq 0$. Contradiction with Lemma 3.5.8 from [16]. \square

The following notion plays a central role in the proof.

DEFINITION 6.2. — *An element $D \in \mathcal{D}$ is said to be hidden if any noninvertible function from $\mathbb{K}[X]^{(B)}$ is zero on D , equivalently, $\langle \varphi_D, \mu \rangle > 0$ for any $\mu \in \mathfrak{X}^+ \setminus -\mathfrak{X}^+$.*

The set of all hidden elements of \mathcal{D} is denoted by $\underline{\mathcal{D}}^\emptyset (= \underline{\mathcal{D}}_{G,X}^\emptyset)$. Set $\overline{\mathcal{D}}^\emptyset := \mathcal{D} \setminus \underline{\mathcal{D}}^\emptyset$. Let us establish some properties of $\underline{\mathcal{D}}^\emptyset$ and $\overline{\mathcal{D}}^\emptyset$.

LEMMA 6.3. — $\mathcal{D}^G \subset \overline{\mathcal{D}}^\emptyset$.

Proof. — Let $D \in \mathcal{D}^G \cap \underline{\mathcal{D}}^\emptyset$. Since G is not a torus and (G, G) acts nontrivially on X , it follows from [16], Lemma 3.5.7, that $\mathcal{D} \neq \{D\}$. Let $\lambda \in \mathfrak{X}$ be such that $\langle \lambda, \varphi_{D'} \rangle \geq 0$ for any $D' \in \mathcal{D} \setminus \{D\}$. Let us check that $\text{ord}_D(f_\lambda) \geq 0$. Assume the contrary. Choose a function $g \in \mathbb{K}[X]^{(B)}$ that is zero on some $D' \in \mathcal{D} \setminus \{D\}$. One can find positive m, n such that $f^n g^m \in \mathbb{K}[X]^{(B)}$ and $\text{ord}_D(f^n g^m) = 0$. Thence $f^n g^m$ is invertible, which

contradicts $\text{ord}_{D'}(f) \geq 0, \text{ord}_{D'}(g) > 0$. Therefore $\mathbb{K}[X]^{(B)} = \mathbb{K}[X \setminus D]^{(B)}$. From highest weight theory it follows that $\mathbb{K}[X] = \mathbb{K}[X \setminus D]$, which is nonsense. \square

PROPOSITION 6.4. — *Let $\alpha, \alpha_1, \alpha_2$ be as in Lemma 3.3. If $\mathcal{D}(\alpha_1) \subset \underline{\mathcal{D}}^\emptyset$, then $\mathcal{D} = \mathcal{D}(\alpha_1)$ and $\varphi_{D_1} = \varphi_{D_2}$ for any $D_1, D_2 \in \mathcal{D}(\alpha_1)$ (the last condition is essential only if $\alpha = \alpha_1$).*

Proof. — Consider the case $\alpha = \alpha_1$. Let D_1, D_2 be different elements of $\mathcal{D}(\alpha_1)$. Note that $\text{ord}_{D_i}(f_\alpha) = 1, i = 1, 2$. By Lemma 3.3, $\text{ord}_D(f_\alpha) \leq 0$ for $D \in \mathcal{D} \setminus \mathcal{D}(\alpha_1)$. Choose $\lambda \in \mathfrak{X}^+ \setminus -\mathfrak{X}^+$. Set $m := \min_{i=1,2} \langle \varphi_{D_i}, \lambda \rangle$. Since $\mathcal{D}(\alpha_1) \subset \underline{\mathcal{D}}^\emptyset$, we have $m > 0$. Put $f := f_\lambda / f_\alpha^m$. Clearly, $\text{ord}_D(f) \geq 0$ for any $D \in \mathcal{D}$ and $\text{ord}_{D_i}(f) = 0$ for some i . It follows that f is an invertible element of $\mathbb{K}[X]^{(B)}$. Therefore $\langle \varphi_D, \alpha \rangle = \langle \varphi_D, \lambda \rangle = 0$ for $D \notin \mathcal{D}(\alpha_1)$ and $\langle \varphi_{D_1}, \lambda \rangle = \langle \varphi_{D_2}, \lambda \rangle$. This implies the claim.

The proof in the remaining two cases is analogous (except for $\alpha = \alpha_1 + \alpha_2$ or $2\alpha_1$ one has to consider $f = f_\lambda^2 / f_\alpha^m$ instead of f_λ / f_α^m). \square

LEMMA 6.5. — *In the notation of Proposition 6.4, $\Psi = \{\alpha\}, \Pi(\mathfrak{g}) = \text{Supp}(\alpha)$.*

Proof. — Consider the case $\alpha = k(\alpha_1 + \alpha_2), k = 1$ or $\frac{1}{2}$ (the other two cases are analogous, even easier). Let $\beta \in \Pi(\mathfrak{g}) \setminus \text{Supp}(\alpha)$ be such that $\mathcal{D}(\beta) \neq \emptyset$. Corollary 3.4 implies $\mathcal{D}(\beta) \neq \mathcal{D}(\alpha_1)$, contradiction with Proposition 6.4. So $\Pi(\mathfrak{g}) \setminus \text{Supp}(\alpha) \subset \Pi^\alpha$. By Lemma 6.1, $\Pi(\mathfrak{g}) = \text{Supp}(\alpha)$. So $[\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$. Since $\langle \alpha_i, \alpha_1 + \alpha_2 \rangle > 0$ for $i = 1, 2$, we see that Ψ does not contain multiples of α_1, α_2 . Therefore $\Psi = \{\alpha\}$. \square

PROPOSITION 6.6. — *Suppose $X \cong^G G *_H V, \underline{\mathcal{D}}^\emptyset, \overline{\mathcal{D}}^\emptyset \neq \emptyset$. Then $\underline{\mathcal{D}}^\emptyset$ consists of one element, say D , and there exist a simple ideal $\mathfrak{g}_1 \subset \mathfrak{g}, \mathfrak{g}_1 \cong \mathfrak{sl}_n$, and $i \in \{1, n - 1\}$ such that*

- (1) *The simple root α_i of \mathfrak{g}_1 is the unique simple root of \mathfrak{g} positive on $\mathfrak{X}^+ \setminus -\mathfrak{X}^+$.*
- (2) *$G_D = P_{\Pi(\mathfrak{g}) \setminus \{\alpha_i\}}$.*
- (3) *H is G -conjugate to a subgroup in G_D .*

Proof. — We may assume that $G = Z(G)^\circ \times \overline{G_1} \times \cdots \times G_k$, where G_i is a simple simply connected group. Let $H_i, i = \overline{1, k}$, denote the projection of H to G_i and ρ_i denote the natural projection $G *_H V \rightarrow G_i / H_i$.

Step 1. Since both $\overline{\mathcal{D}}^\emptyset, \underline{\mathcal{D}}^\emptyset$ are nonempty, we see that no multiple of any element in $\overline{\mathcal{D}}^\emptyset$ is principal (otherwise, there is a noninvertible B -semiinvariant function nonvanishing on a hidden divisor). So $\text{Pic}(X)$ is

infinite. Note that $\text{Pic}(X) \cong \text{Pic}(G/H) \cong \mathfrak{X}(H)/p(\mathfrak{X}(G))$ (the last isomorphism is due to Popov [20]), where p is the restriction of characters. It follows that $\mathfrak{h}^H \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$. In particular, for some $i \in \{1, \dots, k\}$ the group H_i is not semisimple whence $H_i \neq G_i$. To be definite, suppose $H_1 \neq G_1$.

Step 2. Let Y be an affine spherical G -variety of positive dimension, $\varphi: X \rightarrow Y$ a smooth G -morphism and $D \in \underline{\mathcal{D}}^\emptyset$. If $Y \not\cong G/G_0$ with $(G, G) \subset G_0$, then $D \in \varphi^*(\underline{\mathcal{D}}_{G,Y}^\emptyset)$. Indeed, if $\overline{\varphi(D)} = Y$, then $\text{ord}_D(\varphi^*(f)) = 0$ for any $f \in \mathbb{K}[Y]^{(B)}$. By the assumption on Y , there is a noninvertible element in $\mathbb{K}[Y]^{(B)}$. Thus $\overline{\varphi(D)}$ is a divisor. To see that $D \in \varphi^*(\underline{\mathcal{D}}_{G,Y}^\emptyset)$ note that $\text{ord}_{\overline{\varphi(D)}}(f) = \text{ord}_D(\varphi^*(f))$. We also remark that $G_D = G_{\overline{\varphi(D)}}$.

Applying this observation to $\rho_i: X \rightarrow G_i/H_i$ we get that either $\underline{\mathcal{D}}^\emptyset = \rho_i^*(\underline{\mathcal{D}}_{G,G_i/H_i}^\emptyset)$ or $H_i = G_i$. It follows that $G_i = H_i$ for any $i > 1$ and $\underline{\mathcal{D}}_{G_1,G_1/H_1}^\emptyset \neq \emptyset$.

Step 3. Let us check that $\mathfrak{g}_1 \cong \mathfrak{sl}_n$ and H_1 is conjugate to a subgroup in $Z_{G_1}(\pi_1)$, where π_1 is the fundamental weight of \mathfrak{sl}_n corresponding to the simple root α_1 .

By step 1, $\mathfrak{h}_1^{H_1} \neq \{0\}$. Put $L_1 = Z_{G_1}(\mathfrak{h}_1^{H_1})$. Let us check now that $\text{rank}_{G_1}(G_1/L_1) = 1$. Assume the contrary. Let G_1/L_1 be symmetric. By results of Vust [22], $\varphi_D, D \in \mathcal{D}_{G,G_1/L_1}$, is the half of a simple coroot of the symmetric space G_1/L_1 . In particular, for any $D \in \mathcal{D}_{G,G_1/L_1}$ there exists a noninvertible (=nonconstant) function $f \in \mathbb{K}[G_1/L_1]^{(B)}$ such that $\text{ord}_D(f) = 0$, so $D \in \overline{\mathcal{D}}_{G,G_1/L_1}^\emptyset$.

So G_1/L_1 is not symmetric. We deduce from the classification of [14] that the pair $(\mathfrak{g}_1, \mathfrak{l}_1)$ is either $(\mathfrak{so}_{2n+1}, \mathfrak{gl}_n)$ or $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2} \times \mathbb{K})$. However, in both these cases $\mathfrak{l}_1^{N_{G_1}(\mathfrak{l}_1)} = 0$. Applying step 2 to the projection $G_1/H_1 \rightarrow G_1/N_{G_1}(\mathfrak{l}_1)$, we get $\underline{\mathcal{D}}_{G,G_1/H_1}^\emptyset = \emptyset$. This contradicts step 2.

So $\text{rank}_{G_1}(G_1/L_1) = 1$. It follows from the classification in [14] that $\mathfrak{g}_1 = \mathfrak{sl}_n, \mathfrak{l}_1 \sim_G \mathfrak{z}_{\mathfrak{g}_1}(\pi_1)$. We note that $\mathfrak{X}_{G,G_1/L_1}^+ = \mathbb{Z}_{\geq 0}(\pi_1 + \pi_{n-1})$, $\#\mathcal{D}_{G,G_1/L_1} = 2$, the stabilizers of elements of $\#\mathcal{D}_{G,G_1/L_1}$ are $P_{\Pi(\mathfrak{g}) \setminus \{\alpha_i\}}$, $i = 1, n - 1$, and the classes of two elements of $\rho_1^*(\mathcal{D}_{G,G_1/L_1})$ in $\text{Pic}(X)$ are opposite. By step 2, $\#\underline{\mathcal{D}}^\emptyset \leq 2$.

Note that, by the above, any Levi subgroup of G_1 containing H_1 is conjugate to L_1 . Therefore $\dim \mathfrak{h}_1^{H_1} = 1$. Since $H_i = G_i$ for any $i > 1$, it follows that $\text{rank Pic}(X) = 1$.

Step 4. Let us check that $\#\underline{\mathcal{D}}^\emptyset = 1$. For $D \in \mathcal{D}$ we denote by $[D]$ the class of D in $F := \text{Pic}(X)/\text{Tor}(\text{Pic}(X))$. The group F is generated by $[D]$ for any $D \in \mathcal{D}_{G,G_1/L_1}$. The equality $\#\underline{\mathcal{D}}^\emptyset = 1$ will follow if we check that

for any $D \in \mathcal{D}^\emptyset, D' \in \mathcal{D}, D \neq D'$ there is a positive integer n such that $[D'] = -n[D]$. Assume the contrary. Then there is $f \in \mathbb{K}(X)^{(B)}$ such that $(f) = D' - nD, n \geq 0$. If $n = 0$, then f is a noninvertible element of $\mathbb{K}[X]^{(B)}$ and f is nonzero on D . So let $n > 0$. There is a function $g \in \mathbb{K}[X]^{(B)}$ such that $\text{ord}_D(g) = kn, k \in \mathbb{N}$. It remains to note that $f^k g \in \mathbb{K}[X]^{(B)}, f^k g$ is noninvertible and $\text{ord}_D(f^k g) = 0$, contradiction.

Without loss of generality, assume that $\mathcal{D}^\emptyset \subset \mathcal{D}(\alpha_1)$.

Step 5. If $\mu \in \mathfrak{X}^+$ is such that $\langle \alpha_1^\vee, \mu \rangle = 0$, then $\langle \varphi_D, \mu \rangle = 0$. Indeed, since $\langle \alpha_1^\vee, \mu \rangle = 0$, we see that $f_\mu \in \mathbb{K}[X]^{(P_{\alpha_1})}$. To prove the claim note that $\overline{P_{\alpha_1} D} = X$ whence f_μ is nonzero on D .

Step 6. From step 5 it follows that $\langle \alpha_1^\vee, \mathfrak{X}^+ \setminus -\mathfrak{X}^+ \rangle > 0$. To prove the proposition it remains to check that there is no other simple root with this property. Assume the contrary, let $\alpha \in \Pi(\mathfrak{g}) \setminus \{\alpha_1\}$ be such that $\langle \alpha^\vee, \mathfrak{X}^+ \setminus -\mathfrak{X}^+ \rangle > 0$. Since $\pi_1 + \pi_{n-1} \in \mathfrak{X}^+$, we see that $\alpha = \alpha_{n-1}$ whence $n \geq 2$. Let D_1 denote the unique element in $\rho_1^*(\mathcal{D}_{G, G_1/L_1}(\alpha_{n-1}))$.

Suppose that there is $D_2 \in \mathcal{D} \setminus \mathcal{D}(\alpha_{n-1}) \setminus \{D\}$. Then, by step 4, there exists $\mu \in \mathfrak{X}^+$ such that $(f_\mu) = nD + D_2, n > 0$. In particular, $f_\mu \in \mathbb{K}[X]^{(P_{\alpha_{n-1}})}$ whence $\langle \alpha_{n-1}^\vee, \mu \rangle = 0$.

It remains to consider the case when $\mathcal{D} = \{D\} \cup \mathcal{D}(\alpha_{n-1})$. Suppose $\mathcal{D}(\alpha_{n-1}) = \{D_1\}$. Recall that $F \cong \mathbb{Z}, [D] + [D_1] = 0$. It follows that (f) is proportional to $D + D_1$ for any $f \in \mathbb{K}(X)^{(B)}$. Therefore $D_1 \in \mathcal{D}^\emptyset$, contradiction.

Suppose $\#\mathcal{D}(\alpha_{n-1}) = 2$. Let D_2 be the unique element in $\mathcal{D}(\alpha_{n-1}) \setminus \{D_1\}$. By Proposition 3.1, $\alpha_{n-1} \in \Psi$ and $\langle \varphi_{D_i}, \alpha_{n-1} \rangle = 1, i = 1, 2$. By Lemma 3.3, $\langle \varphi_D, \alpha_{n-1} \rangle \leq 0$. It follows that $[D_1] + [D_2] = m[D]$ for some nonnegative m . Contradiction with step 4. □

Proof of Theorem 4.8. — Below in the proof we write $\mathfrak{X}^+, \mathfrak{X}, \mathfrak{a}, \Pi^a$ instead of $\mathfrak{X}_{G, X_i}^+, \mathfrak{X}_{G, X_i}, \mathfrak{a}_{G, X_i}, \Pi(\mathfrak{g})_{X_i}^a$ (recall that $\Pi(\mathfrak{g})_{X_1}^a = \Pi(\mathfrak{g})_{X_2}^a$ in virtue of Corollary 5.5), $\mathcal{D}_i, \mathcal{D}_i^\emptyset, \overline{\mathcal{D}}_i^\emptyset, \Psi_i, \Pi_i^b, \dots, \Pi_i^d$ instead of $\mathcal{D}_{G, X_i}, \text{etc}$.

Assume the contrary: there is no bijection $\iota: \mathcal{D}_1 \rightarrow \mathcal{D}_2$ with the desired properties. Thanks to the assumptions made in the end of Section 4 and Corollary 5.6, the theorem holds for $X_1(\mu), X_2(\mu)$ for any $\mu \in \mathfrak{X}^+ \setminus -\mathfrak{X}^+$, see Corollary 5.6.

It follows from Lemma 4.5 that both actions $G: X_1, G: X_2$ are locally effective. We may also assume that $Z(G)^\circ$ acts on X_1, X_2 effectively. Thanks to Lemma 4.7 both X_1, X_2 are indecomposable.

Step 1. Here we construct a bijection $\iota: \overline{\mathcal{D}}_1^\emptyset \rightarrow \overline{\mathcal{D}}_2^\emptyset$ such that $\varphi_D = \varphi_{\iota(D)}$. By our assumptions, for any $\mu \in \mathfrak{X}^+ \setminus -\mathfrak{X}^+$ there is a bijection

$\iota_\mu: \mathcal{D}_{G_\mu, X_1(\mu)} \rightarrow \mathcal{D}_{G_\mu, X_2(\mu)}$ such that $\varphi_D = \varphi_{\iota_\mu(D)}$, $G_D \cap G_\mu = G_{\iota_\mu(D)} \cap G_\mu$. By Proposition 5.4, there is a natural embedding $\mathcal{D}_{G_\mu, X_i(\mu)} \hookrightarrow \mathcal{D}_i$ with image $\{D \in \mathcal{D}_i \mid \langle \mu, \varphi_D \rangle = 0\}$. In the sequel we identify $\mathcal{D}_{G_\mu, X_i(\mu)}$ with this image. Since $\mathcal{D}_{G_\mu, X_i(\mu)} = \{D \in \mathcal{D}_i \mid f_\mu|_D \neq 0\}$, we have

$$(6.1) \quad \mathcal{D}_{G_\lambda, X_i(\lambda)} \cap \mathcal{D}_{G_\mu, X_i(\mu)} = \mathcal{D}_{G_{\lambda+\mu}, X_i(\lambda+\mu)}, \forall \lambda, \mu \in \mathfrak{X}^+ \setminus -\mathfrak{X}^+.$$

By definition,

$$(6.2) \quad \overline{\mathcal{D}}_i^\emptyset = \bigcup_{\mu \in \mathfrak{X}^+ \setminus -\mathfrak{X}^+} \mathcal{D}_{G_\mu, X_i(\mu)}.$$

We remark that ι_λ, ι_μ , in general, do not coincide on $\mathcal{D}_{G_{\lambda+\mu}, X_1(\lambda+\mu)}$. Indeed, any G -equivariant automorphism ψ of X_1 induces a bijection $\mathcal{D}_1 \rightarrow \mathcal{D}_1, D \mapsto \psi(D)$. This bijection can be nontrivial (take SL_2 for G and G/T for X_1).

Let $\mu_1, \dots, \mu_k \in \mathfrak{X}^+$ be the minimal set of generators of \mathfrak{X}^+ modulo $X^+ \cap -\mathfrak{X}^+$, i.e. $\{\mu_1, \dots, \mu_k\} = X^+ \setminus \mathfrak{X}^+ + \mathfrak{X}^+$. For a subset $I \subset \{1, 2, \dots, k\}$ put $\mu_I := \sum_{j \in I} \mu_j, M_I := G_{\mu_I}, \mathcal{D}_i^I := \mathcal{D}_{G_{\mu_I}, X_i(\mu_I)}, \overline{\mathcal{D}}_i^I := \cup_{J \supseteq I} \mathcal{D}_i^J, \underline{\mathcal{D}}_i^I := \mathcal{D}_i^I \setminus \overline{\mathcal{D}}_i^I, i = 1, 2$. For $I \neq \emptyset$ set $\iota_I := \iota_{\mu_I}$. It follows from (6.1) that $\mathcal{D}_j^I = \cap_{i \in I} \mathcal{D}_j^{\{i\}}$ whence $\mathcal{D}_j^I \cap \mathcal{D}_j^J = \mathcal{D}_j^{I \cup J}, j = 1, 2$. Note that $\iota_I(\mathcal{D}_1^J) = \mathcal{D}_2^J$ for any $J \supset I$ because $\mathcal{D}_j^J = \{D \in \mathcal{D}_j^I \mid \langle \varphi_D, \sum_{i \in J} \mu_i \rangle = 0\}$ and $\varphi_{\iota_I(D)} = \varphi_D$. Thus the map

$$\iota: \overline{\mathcal{D}}_1^\emptyset \rightarrow \overline{\mathcal{D}}_2^\emptyset, \iota|_{\underline{\mathcal{D}}_1^I} = \iota_I,$$

is a well-defined bijection such that $\varphi_{\iota(D)} = \varphi_D$. Clearly, ι^{-1} coincides with ι_I^{-1} on $\underline{\mathcal{D}}_2^I$.

Step 2. Choose $\alpha_1 \in \Pi(\mathfrak{g}) \setminus \Pi^a$. Suppose $\mathcal{D}_i(\alpha_1) \subset \underline{\mathcal{D}}_i^\emptyset$. Let us check that $\alpha_1 \in \Pi_i^d$ and $G_D = P_{\Pi(\mathfrak{g}) \setminus \{\alpha_1\}}$. To be definite, put $i = 1$. Assume the contrary. Then one of the following cases takes place.

Case 1. $\alpha_1 \in \Pi_1^b$. Let D_1^+, D_1^- denote different elements of $\mathcal{D}_1(\alpha_1)$. By Proposition 6.4, $\mathcal{D}_1 = \mathcal{D}_1(\alpha_1)$ and $\varphi_{D_1^+} = \varphi_{D_1^-}$. By Lemma 6.5, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}_2$. If $\Psi_1 = \Psi_2$, then $\alpha_1 \in \Psi_2$. Applying Proposition 6.4 again, we see that $\varphi_{D_2^+} = \varphi_{D_2^-}$ for $D_2^+, D_2^- \in \mathcal{D}_2$. Set $\iota(D_1^\pm) = D_2^\pm$.

Now suppose X_1, X_2 are smooth. Let $H \subset G, V$ be such that $X_1 \cong^G G *_H V$. Let π denote the natural projection $X_1 \twoheadrightarrow G/H$. By Proposition 4.2.4 from [16], the pull-back map $\pi^*: \mathcal{D}_{G, G/H} \rightarrow \mathcal{D}_1$ is a bijection. Take $f_\lambda \in \mathbb{K}[X_1]_\lambda^{(B)}$. The divisor of f_λ is a pull-back of a divisor on G/H whence f_λ is constant on fibers of the vector bundle $\pi: X_1 \rightarrow G/H$. This means that $f_\lambda \in \pi^* \mathbb{K}[G/H]$. Therefore $V = \{0\}$ and $X_1 = G/H$. By Theorem 2 from [16], $N_G(H)$ is not connected, whence $H = T_1 \times T_0$, where T_1

is a maximal torus of SL_2 , and $T_0 \subset Z(G)^\circ$. By our assumptions X_1 is indecomposable. Thus $\mathfrak{g} = \mathfrak{sl}_2$. Now it is easy to check that $X_1 \cong^G X_2$.

Case 2. $\alpha_1 \in \Pi_1^c$. By Proposition 6.4, it is enough to check that $\alpha_1 \in \Pi_2^c$. Similarly to the previous case, one can check that $G = SL_2, X_1 = G/N_G(T)$. Then it is easy to see that $X_1 \cong^G X_2$.

Case 3. $\alpha_1 \in \Pi_1^d$ and there is $\alpha_2 \in \Pi(\mathfrak{g}) \setminus \{\alpha_1\}$ such that $\mathcal{D}_1(\alpha_1) = \mathcal{D}_1(\alpha_2)$. Again we only need to consider the case when X_1, X_2 are smooth. Analogously to case 1, $[\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2, X_1 = G/H$. From $\mathcal{D}_1(\alpha_1) = \mathcal{D}_1(\alpha_2)$ one can deduce that \mathfrak{h} contains the diagonally embedded subalgebra $\mathfrak{sl}_2 \subset [\mathfrak{g}, \mathfrak{g}]$. Again, since X_1 is indecomposable, we see that $\mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{sl}_2$. Then we easily check that $X_1 \cong^G X_2$.

Step 3. Let $D \in \overline{\mathcal{D}}_1^\emptyset \cap \mathcal{D}_1(\alpha)$. In this step we show that

- (1) If $\alpha \in \Pi_1^d$, then $\iota(D) \in \mathcal{D}_2(\alpha)$ and $\alpha \in \Pi_2^d$.
- (2) If $\alpha \in \Pi_1^c$, then $\iota(D) \in \mathcal{D}_2(\alpha)$ and $\alpha \in \Pi_2^c$.
- (3) If $\alpha \in \Pi_1^b$, then either $\iota(D) \in \mathcal{D}_2(\alpha)$ and $\alpha \in \Pi_2^b$ or $\alpha \in \Pi_2^d$ and $\mathcal{D}_2(\alpha) \cap \overline{\mathcal{D}}_2^\emptyset = \emptyset$.

Let $I \subset \{1, \dots, k\}$ be such that $D \in \underline{\mathcal{D}}_1^I$.

Case 1. Suppose $\alpha \in \Pi_1^d$. Since $G_D \cap M_I = G_{\iota(D)} \cap M_I$, the inclusion $\iota(D) \in \mathcal{D}_2(\alpha)$ will follow if we check that $\alpha \in \Delta(\mathfrak{m}_I)$. Assume the contrary. Then f_{μ_I} is not P_α -semiinvariant. It follows that (f_{μ_I}) contains a P_α -unstable prime divisor. But D is the only such divisor. Since $D \in \mathcal{D}_I$, we have $\text{ord}_D(f_{\mu_I}) = 0$. Contradiction. The inclusion $\alpha \in \Pi_2^d$ is easily deduced from Proposition 3.1 and the equality $\varphi_{\iota(D)} = \varphi_D = \alpha^\vee|_{\mathfrak{a}}$.

Case 2. Suppose $\alpha \in \Pi_1^c$. Then $\varphi_{\iota(D)} = \varphi_D = \alpha^\vee/2|_{\mathfrak{a}}$. If $\alpha \in \Pi_2^d$, then $\mathcal{D}_2(\alpha) \subset \overline{\mathcal{D}}_2^\emptyset$ because $\varphi_{D'} = 2\varphi_{\iota(D)}$ for $D' \in \mathcal{D}_2(\alpha)$. This contradicts the previous case (recall that X_1, X_2 have equal rights). Therefore $\alpha \in \Pi_2^b \cup \Pi_2^c$. By Corollary 3.4, $\iota(D) \in \mathcal{D}_2(\alpha)$. If $\alpha \in \Pi_2^b$, then there is $D' \in \mathcal{D}_2(\alpha) \setminus \{\iota(D)\}$ such that $\varphi_{D'} = \varphi_D$. Therefore $D' \in \underline{\mathcal{D}}_2^I$ and $\iota^{-1}(D') \in \underline{\mathcal{D}}_1^I$. Again, by Corollary 3.4, $\iota^{-1}(D') \in \mathcal{D}_1(\alpha)$. Contradiction with $\alpha \in \Pi_1^c$.

Case 3. Suppose $\alpha \in \Pi_1^b$. If $\mathcal{D}_2(\alpha) \cap \overline{\mathcal{D}}_2^\emptyset \neq \emptyset$, then, thanks to cases 1,2, $\alpha \in \Pi_2^b$. If $\mathcal{D}_2(\alpha) \subset \underline{\mathcal{D}}_2^\emptyset$, then, by step 2, $\alpha \in \Pi_2^d$, q.e.d.

From case 2 it follows that $\Pi_1^c = \Pi_2^c$.

Step 4. Suppose $\Pi_1^b = \Pi_2^b, \Pi_1^d = \Pi_2^d$. Let us construct a bijection $\iota: \mathcal{D}_1 \rightarrow \mathcal{D}_2$ extending the bijection $\overline{\mathcal{D}}_1^\emptyset \rightarrow \overline{\mathcal{D}}_2^\emptyset$ constructed above and such that $G_{\iota(D)} = G_D, \varphi_{\iota(D)} = \varphi_D$ for any $D \in \mathcal{D}_1$. By step 3, for any $\alpha \in \Pi(\mathfrak{g}), D \in \overline{\mathcal{D}}_1^\emptyset \cap \mathcal{D}_1(\alpha)$ we have $\iota(D) \in \mathcal{D}_2(\alpha)$. It follows that $G_D = G_{\iota(D)}$.

Now choose $D \in \underline{\mathcal{D}}_1^\emptyset$. Thanks to steps 2, 3, exactly one of the following possibilities holds:

- (1) $\alpha \in \Pi_1^b$ and $\overline{\mathcal{D}}_i^\emptyset \cap \mathcal{D}_i(\alpha) \neq \emptyset$ for any $\alpha \in \Pi(\mathfrak{g})$ such that $D \in \mathcal{D}_1(\alpha)$ and any $i = 1, 2$.
- (2) There is a unique root $\alpha \in \Pi(\mathfrak{g})$ such that $D \in \mathcal{D}_1(\alpha)$. This root is in $\Pi_1^d = \Pi_2^d$. Let D' be the unique element of $\mathcal{D}_2(\alpha)$. Then $G_D = G_{D'} = P_{\Pi(\mathfrak{g}) \setminus \{\alpha\}}, D' \in \underline{\mathcal{D}}_2^\emptyset$.

In case 2 put $\iota(D) := D'$. In case 1 set $A := \{\alpha \in \Pi(\mathfrak{g}) \mid D \in \mathcal{D}_1(\alpha)\}$. For $\alpha \in A$ let D_α denote the unique element in $\mathcal{D}_1(\alpha) \cap \overline{\mathcal{D}}_1^\emptyset$. Clearly, $\varphi_D \neq \varphi_{D_\alpha}$ for any $\alpha \in A$. Further, $\langle \alpha, \varphi_{D_\beta} \rangle = \langle \alpha, \beta^\vee - \varphi_D \rangle < 0$ whenever α, β are different elements of A . By step 3, $\iota(D_\alpha) \in \mathcal{D}_2(\alpha)$. For $\alpha \in A$ let D'_α denote the unique element in $\mathcal{D}_2(\alpha) \setminus \{\iota(D_\alpha)\}$. Note that $\langle \varphi_{D'_\alpha}, \beta \rangle = \langle \alpha^\vee - \varphi_{D_\alpha}, \beta \rangle = 1$ for any $\alpha, \beta \in A$. Applying Proposition 3.2 and Corollary 3.4, we see that $D'_\alpha = D'_\beta$ for any $\alpha, \beta \in A$. Put $\iota(D) := D'_\alpha, \alpha \in A$.

Since $\langle \varphi_D, \gamma \rangle \leq 0$ for any $\gamma \in \Pi_1^b \setminus A$, we see that $\iota: \mathcal{D}_1 \rightarrow \mathcal{D}_2$ is injective. As already checked, $G_D \subset G_{\iota(D)}$ for any $D \in \underline{\mathcal{D}}_1^\emptyset$. Finally, by the symmetry between X_1, X_2 , we obtain that $G_D = G_{\iota(D)}$ and $\iota: \mathcal{D}_1 \rightarrow \mathcal{D}_2$ is surjective.

Step 5. We may assume that X_1, X_2 are smooth and there is $\alpha \in \Pi_1^d \cap \Pi_2^b$. In this case, according to case 3 of step 3, $\mathcal{D}_1(\alpha) \subset \underline{\mathcal{D}}_1^\emptyset$.

Since $\Pi_2^b \neq \emptyset$, it follows from step 2 that $\overline{\mathcal{D}}_1^\emptyset, \overline{\mathcal{D}}_2^\emptyset \neq \emptyset$. Therefore X_1 satisfies conditions (1)-(3) of Proposition 6.6. Let \mathfrak{g}_1 be as in Proposition 6.6. Note that α is the unique simple root of \mathfrak{g} such that $\langle \alpha^\vee, \mathfrak{X}^+ \setminus -\mathfrak{X}^+ \rangle > 0$. Suppose $\underline{\mathcal{D}}_2^\emptyset \neq \emptyset$. It follows that X_2 satisfies conditions (1)-(3) for the same simple ideal \mathfrak{g}_1 . So we have $G_{D_1} = G_{D_2}$ for $D_i \in \underline{\mathcal{D}}_i^\emptyset, i = 1, 2$. It follows from Corollary 5.6 and the assumptions made in the end of Section 4 that $X_1(D_1) \cong^M X_2(D_2)$, where M is the standard Levi subgroup of G_{D_i} . Suppose that $X_1 \cong^G G *_{H_1} V_1, X_2 \cong^G G *_{H_2} V_2$. We may assume that $H_i \subset M, i = 1, 2$. Lemma 5.8 implies $X_i(D_i) \cong^M \text{Rad}_u(\mathfrak{q}^-) \times M *_{H_i} V_i, i = 1, 2$, where $\mathfrak{q}^- := \mathfrak{b}^- + \mathfrak{m}$. By Remark 4.4, $X_1 \cong^G X_2$.

So it remains to consider the situation when $\underline{\mathcal{D}}_2^\emptyset = \emptyset$. If $\beta \in \Pi_1^d \setminus \Pi_2^d, \beta \neq \alpha$, then, by step 3, $\mathcal{D}_1(\beta) \in \underline{\mathcal{D}}_1^\emptyset$, which is impossible. Analogously, $\Pi_2^d \subset \Pi_1^d$. So $\Pi_1^b = \Pi_2^b \setminus \{\alpha\}$ and $\Pi_1^d = \Pi_2^d \sqcup \{\alpha\}$. Note also that $\mathcal{D}_1(\beta) \subset \overline{\mathcal{D}}_1^\emptyset$ for any $\beta \in \Pi(\mathfrak{g}), \beta \neq \alpha$.

Let D_1 denote the unique element of $\mathcal{D}_1(\alpha), D_2^+, D_2^-$ denote different elements of $\mathcal{D}_2(\alpha)$ and $D_1^\pm := \iota^{-1}(D_2^\pm)$. Then $\varphi_{D_1} = \varphi_{D_1^+} + \varphi_{D_1^-}$. Put $\mathcal{D}'_1 := \mathcal{D}_1 \setminus \{D_1, D_1^+, D_1^-\}, \mathcal{D}'_2 := \mathcal{D}_2 \setminus \{D_2^+, D_2^-\}$. Let us check that $G_{\mathcal{D}'_1} = G_{\mathcal{D}'_2}$. Indeed, choose $\beta \in \Pi(\mathfrak{g})$. Suppose there is $D' \in \mathcal{D}_2(\beta) \cap \mathcal{D}'_2$. By step 3, $\iota^{-1}(D') \in \mathcal{D}_1(\beta)$. Further, $\iota^{-1}(D') \neq D, D_1^\pm$. Therefore $G_{\mathcal{D}'_1} \subset G_{\mathcal{D}'_2}$. The opposite inclusion is proved in the same way.

Let M denote the standard Levi subgroup of $G_{\mathcal{D}'_i}$. Since $\varphi_{D_1} = \varphi_{D_1^+} + \varphi_{D_1^-}$, we have

$$\mathfrak{X}_{M, X_i(\mathcal{D}'_i)}^+ = \left\{ \lambda \in \mathfrak{X}_{G, X} \mid \langle \lambda, \varphi_{D_j^\pm} \rangle \geq 0 \right\}, \quad j = 1, 2.$$

But $\#\mathcal{D}_{M, X_1(\mathcal{D}'_1)} \neq \#\mathcal{D}_{M, X_2(\mathcal{D}'_2)}$. The assumptions made in the end of Section 4 yield $\mathcal{D}'_i = \emptyset$.

Since $\Pi_2^b \neq \emptyset$ and $\#\mathcal{D}_2 = 2$, we see that $\Pi(\mathfrak{g}) = \Pi^a \cup \Pi_2^b$. By Lemma 6.1, $[\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{sl}_2$. However, in this case Proposition 6.6 implies $\#\mathcal{D}_1(\alpha) = 2$. Contradiction with $\alpha \in \Pi_1^d$. \square

Let us present an algorithm recovering the set $\mathcal{D}_{G, X}$ with the maps $D \mapsto \varphi_D, D \mapsto G_D$ from $\mathfrak{X}_{G, X}^+, \Psi_{G, X}$.

Algorithm 6.7. — Put $\mathfrak{X}^+ := \mathfrak{X}_{G, X}^+, \mathfrak{X} := \mathfrak{X}_{G, X}, \mathfrak{a} := \mathfrak{a}_{G, X}, \Psi := \Psi_{G, X}, \Pi^a := \Pi(\mathfrak{g})_{\mathfrak{X}}^a, \dots, \Pi^d := \Pi(\mathfrak{g})_{\mathfrak{X}}^d$. We only need to determine the set $\mathcal{D}'(G, \mathfrak{X}^+, \Psi) := \mathcal{D}_{G, X}^G \cup_{\alpha \in \Pi^b} \mathcal{D}_{G, X}(\alpha)$ and the maps $D \mapsto \varphi_D$ for $D \in \mathcal{D}'(G, \mathfrak{X}^+, \Psi)$. Then for $D \in \mathcal{D}'$ we recover G_D from φ_D by using Corollary 3.4.

Step 1. Compute $\{\mu_1, \dots, \mu_k\} := \mathfrak{X}^+ \setminus (\mathfrak{X}^+ \cup \mathfrak{X}^+)$. For $I \subset \{1, \dots, k\}$ put $M^I := Z_G(\sum_{i \in I} \mu_i), \Pi^I = \{\alpha \in \Pi(\mathfrak{g}) \mid \langle \alpha^\vee, \sum_{i \in I} \mu_i \rangle = 0\}, \mathfrak{X}^{+I} := \mathfrak{X}^+ + \sum_{i \in I} \mathbb{Z}\mu_i, \Psi^I := \{\alpha \in \Psi \mid \text{Supp}(\alpha) \subset \Pi^I\}$. Put $\mathcal{D}^I := \mathcal{D}'(M^I, \mathfrak{X}^{+I}, \Psi^I), \overline{\mathcal{D}}^I := \cup_{J \supseteq I} \mathcal{D}^J, \underline{\mathcal{D}}^I := \mathcal{D}^I \setminus \overline{\mathcal{D}}^I$. Note that $\mathcal{D}'(G, \mathfrak{X}^+, \Psi) = \sqcup_{I \subset \{1, 2, \dots, k\}} \underline{\mathcal{D}}^I$ and $\underline{\mathcal{D}}^{\{1, \dots, k\}} = \emptyset$.

Step 2I. Here we compute the set $\underline{\mathcal{D}}^I$ together with the map $D \mapsto \varphi_D$. Suppose we have already computed the set $\overline{\mathcal{D}}^I$ together with the map $D \mapsto \varphi_D$. Note that $\underline{\mathcal{D}}^I \subset \underline{\mathcal{D}}_{M^I, X}^\emptyset(\sum_{i \in I} \mu_i)$.

Case 1. $\Pi^I = \Pi^a$.

Case 1a: $\mathfrak{X}^{+I} = -\mathfrak{X}^{+I}$. In this case $\underline{\mathcal{D}}^I = \emptyset$.

Case 1b: $\text{rank}(\mathfrak{X}^{+I} \cap -\mathfrak{X}^{+I}) \leq \text{rank } \mathfrak{X} - 2$. Using the argument of the proof of Lemma 6.3, one can show that $\underline{\mathcal{D}}^I = \emptyset$.

Case 1c: $\text{rank}(\mathfrak{X}^{+I} \cap -\mathfrak{X}^{+I}) = \text{rank } \mathfrak{X} - 1$. Let i be such that the image of μ_i generates $\mathfrak{X}^{+I}/(\mathfrak{X}^{+I} \cap -\mathfrak{X}^{+I})$. Then $\underline{\mathcal{D}}^I = \{D\}$ and φ_D is given by $\langle \varphi_D, \mu_j \rangle = \delta_{ij}, \langle \varphi_D, \mathfrak{X}^{+I} \cap -\mathfrak{X}^{+I} \rangle = 0$.

Case 2. $\Pi^I = \{\alpha\} \sqcup \Pi^a, \alpha \in \Pi^b, \langle \alpha, \varphi_D \rangle \leq 0$ for any $D \in \overline{\mathcal{D}}^I$. Then $\underline{\mathcal{D}}^I = \{D_1, D_2\}$ and $\varphi_{D_i} = \frac{1}{2}\alpha^\vee|_{\mathfrak{a}}$.

Case 3. Otherwise, let Σ denote the set of all $\alpha \in \Pi^I \cap \Pi^b \setminus \cup_{J \supseteq I} \Pi^J$ such that there is only one divisor $D \in \overline{\mathcal{D}}^I$ with $\langle \alpha, \varphi_D \rangle = 1$ and put $\varphi_\alpha = \alpha^\vee - \varphi_D$. Let $\varphi_1, \dots, \varphi_l$ be all different values of φ_α . Then $\underline{\mathcal{D}}^I := \{D_1, \dots, D_l\}$ with $\varphi_{D_i} = \varphi_i, i = \overline{1, l}$.

Note that in the algorithm one does not need to know the whole set $\Psi_{G,X}$ but only the subset consisting of all roots α of one of the three forms indicated in Lemma 3.3.

7. Equality of the systems of spherical roots

The goal of this section is to prove Theorem 4.9. The proof will be given at the end of the section. It is rather close in spirit to that of Theorem 4.8. It is based on Proposition 5.3 and Theorem 4.8 and uses the notion of a hidden spherical root, see Definition 7.1. We shall see that this proof is much easier than that of Theorem 4.8.

Let X be a spherical G -variety. We suppose that G is not a torus and the action $G: X$ is locally effective. \mathfrak{a} , \mathfrak{X} , \mathfrak{X}^+ , etc. have the same meaning as in the previous section.

DEFINITION 7.1. — *An element $\gamma \in \Psi$ is called hidden if the following two conditions are satisfied.*

- (1) *For any $D \in \mathcal{D}$ there is $\alpha \in \text{Supp}(\gamma)$ such that $D \in \mathcal{D}(\alpha)$.*
- (2) *γ is not of any of the types (1)–(3) indicated in Lemma 3.3.*

By Ψ^0 we denote the subset of Ψ consisting of all nonhidden roots.

The terminology is justified by step 1 of the proof of Theorem 4.9 below.

PROPOSITION 7.2. — *Suppose $\Psi \neq \Psi^0$. If $\#\Psi = 1$, then $[\mathfrak{g}, \mathfrak{g}]$ is simple and $\text{Supp}(\gamma) = \Pi(\mathfrak{g})$ for $\gamma \in \Psi$. If $\#\Psi > 1$, then $(\Pi(\mathfrak{g}), \Psi, \Pi^a)$ is one of the triples listed below:*

- (1) $\Pi(\mathfrak{g}) = C_n, n \geq 2, \Psi = \{k\alpha_1, \alpha_1 + \alpha_n + 2 \sum_{i=2}^{n-1} \alpha_i\}$ for $k=1$ or 2 ,
 $\Pi^a := \{\alpha_3, \dots, \alpha_n\}$.
- (2) $\Pi(\mathfrak{g}) = G_2, \Psi = \{\alpha_2, \alpha_1 + \alpha_2\}, \Pi^a = \emptyset$.
- (3) $\Pi(\mathfrak{g}) = C_n \times A_1, n \geq 2, \Psi = \{\alpha_1 + \alpha'_1, \alpha_1 + \alpha_n + 2 \sum_{i=2}^{n-1} \alpha_i\}$
(here $\alpha_1, \dots, \alpha_n$, resp. α'_1 , are simple roots in C_n , resp. A_1),
 $\Pi^a = \{\alpha_3, \dots, \alpha_n\}$.
- (4) $\Pi(\mathfrak{g}) = B_4, \Psi = \{\alpha_2 + 2\alpha_3 + 3\alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$,
 $\Pi^a = \{\alpha_2, \alpha_3\}$.

In all cases the second root is hidden.

Proof. — Choose $\gamma \in \Psi \setminus \Psi^0$. Inspecting Table 1 from [23], we see that $\text{Supp}(\gamma)$ is connected. Recall that $\langle \alpha^\vee, \mathfrak{a} \rangle = 0$ for any $\alpha \in \Pi^a$ ([16], Lemma 3.5.8). In particular, if $\beta \in \Psi$ and $\alpha \in \Pi^a$, then either $\alpha \in \text{Supp}(\beta)$ or α is not adjacent to $\text{Supp}(\beta)$.

Step 1. Let $\alpha \in \Pi^c$. We claim that $\alpha \in \text{Supp } \gamma$. Indeed, by Corollary 3.4, $G_D = P_{\Pi(\mathfrak{g}) \setminus \{\alpha\}}$ for $D \in \mathcal{D}(\alpha)$. Further, any simple root adjacent to α is not of type a) and $\langle \alpha^\vee, \gamma \rangle \leq 0$.

Step 2. Let $\alpha \in \Pi^b$. Let D_1, D_2 be different elements of $\mathcal{D}(\alpha)$. If β is adjacent to α , then $\beta \notin \Pi^a$. Let $\alpha \notin \text{Supp}(\gamma)$. Note that $\langle \alpha^\vee, \gamma \rangle \leq 0$. Then, thanks to Corollary 3.4, there are $\alpha^1, \alpha^2 \in \Pi^b \cap \text{Supp}(\gamma)$ such that $\alpha^1 \neq \alpha^2$, $D_i \in \mathcal{D}(\alpha) \cap \mathcal{D}(\alpha^i)$, $i = 1, 2$.

Step 3. Let $\alpha \in \Pi^d$. If $\alpha \notin \text{Supp}(\gamma)$, then, by Corollary 3.4, there is (uniquely determined) $\alpha' \in \Pi^d \cap \text{Supp}(\gamma)$ such that $\mathcal{D}(\alpha) = \mathcal{D}(\alpha')$, $\langle \alpha^\vee - \alpha'^\vee, \mathfrak{a} \rangle = 0$ and $k(\alpha + \alpha') \in \Psi$ for $k = 1$ or $\frac{1}{2}$. Again, $\beta \notin \Pi^a$ whenever β is adjacent to α or α' . Since $\langle \alpha^\vee, \gamma \rangle \leq 0$, we have $\langle \alpha'^\vee, \gamma \rangle \leq 0$.

Step 4. If $\#\Psi = 1$, then, by the previous steps, $\text{Supp}(\gamma) \cup \Pi^a = \Pi(\mathfrak{g})$. Applying Lemma 6.1, we see that $\text{Supp}(\gamma) = \Pi(\mathfrak{g})$. Till the end of the proof we suppose that $\#\Psi > 1$. Exactly one of following possibilities takes place:

- (A) $\Pi^a \cup \Pi^d = \Pi(\mathfrak{g}) = \text{Supp}(\gamma)$.
- (B) There are two adjacent roots $\alpha^1, \alpha^2 \in \text{Supp}(\gamma) \setminus \Pi^a$ such that $\langle \alpha^{1\vee}, \gamma \rangle \leq 0$ and either $\alpha^1 \in \Pi^b \cup \Pi^c$ or $\alpha^1 \in \Pi^d$ and there is $\beta \in \Pi^d$ such that $\mathcal{D}(\beta) = \mathcal{D}(\alpha^1)$.

Step 5. Let us consider possibility (B). Inspecting Table 1 in [23], we see that only the following three cases are possible:

- (BC) $\text{Supp}(\gamma) = C_n, n \geq 2, \gamma = \alpha_1 + \alpha_n + 2 \sum_{i=2}^{n-1} \alpha_i, \alpha^1 = \alpha_1, \alpha^2 = \alpha_2$.
- (BG) $\text{Supp}(\gamma) = G_2, \gamma = \alpha_1 + \alpha_2, \alpha^1 = \alpha_2, \alpha^2 = \alpha_1$.

If $\Pi^b \not\subset \text{Supp}(\gamma)$, then, according to step 2, $\#\Pi^b \cap \text{Supp}(\gamma) \geq 2$. Note also that $\langle \beta, \gamma \rangle \leq 0$ for any $\beta \in \Pi^b$. Now we check case by case that $\Pi^b \subset \text{Supp}(\gamma)$.

Consider case (BG). Since $\langle \alpha^{1\vee}, \gamma \rangle < 0, \langle \beta^\vee, \gamma \rangle = 0$ for any $\beta \in \Pi(\mathfrak{g}) \setminus \text{Supp}(\gamma)$, we get $\alpha^1 \in \Pi^b$. Thus $\Pi(\mathfrak{g}) = \text{Supp}(\gamma) \cup \Pi^a$. By Lemma 6.1, $\Pi(\mathfrak{g}) = G_2$. So we get possibility 2 of the proposition.

Consider case (BC). Note that $\text{Supp}(\gamma) \setminus \{\alpha_1, \alpha_2\} \subset \Pi^a, \langle \alpha_1^\vee, \gamma \rangle = 0$. Suppose $\alpha_1 \in \Pi^b \cup \Pi^c$. Since $\langle \alpha_2, \gamma \rangle > 0$, we get from step 3 that $\Pi^d \subset \text{Supp}(\gamma)$. So $\Pi(\mathfrak{g}) \setminus \text{Supp}(\gamma) \subset \Pi^a$. By Lemma 6.1, $\Pi(\mathfrak{g}) = \text{Supp}(\gamma)$ and we get possibility 1 of the proposition.

Now suppose $\alpha_1 \in \Pi^d$. Let β be a (unique, see Corollary 3.4) element of $\Pi^d \setminus \text{Supp}(\gamma)$ such that $\mathcal{D}(\beta) = \mathcal{D}(\alpha_1)$. Since $\langle \alpha_1^\vee, \gamma \rangle = 0$, we see that β is not adjacent to $\text{Supp}(\gamma)$. For $k = 1$ or $\frac{1}{2}$ we get $\gamma_1 := k(\alpha_1 + \beta) \in \Psi$. Since $\alpha_2 \in \Pi^d$, we have $k = 1$. By step 3, $\Pi^d \setminus \text{Supp}(\gamma) = \{\beta\}$. It follows that $\text{Supp}(\gamma) \cup \text{Supp}(\gamma_1) \cup \Pi^a = \Pi(\mathfrak{g})$ whence $\Pi(\mathfrak{g}) = \text{Supp}(\gamma) \cup \{\beta\}$. We get possibility 2 of the proposition.

Step 6. It remains to consider case (A). Choose $\gamma_1 \in \Psi \setminus \{\gamma\}$. Recall that $\Psi \subset \Pi^{\alpha^\perp}$. Therefore $\#\Pi^d \geq 2$. Inspecting Table 1 from [23], we see that one of the following possibilities takes place:

- (AA) $\Pi(\mathfrak{g}) = A_n, n \geq 2, \gamma = \alpha_1 + \dots + \alpha_n, \Pi^d = \{\alpha_1, \alpha_n\}$.
- (AB) $\Pi(\mathfrak{g}) = B_n, n \geq 2, \gamma = \alpha_1 + \dots + \alpha_n, \Pi^d = \{\alpha_1, \alpha_n\}$.
- (AC) $\Pi(\mathfrak{g}) = C_n, n > 2, \gamma = \alpha_1 + \alpha_n + 2 \sum_{i=2}^{n-1} \alpha_i, \Pi^d = \{\alpha_1, \alpha_2\}$.
- (AG) $\Pi(\mathfrak{g}) = G_2, \gamma = \alpha_1 + \alpha_2$.

Inspecting Table 1 from [23] again and taking into account that $(\gamma, \gamma_1) \leq 0$ and $\langle \gamma_1, \alpha^\vee \rangle = 0$ for any $\alpha \in \Pi^\alpha$, we get possibility 4 of the proposition. □

Proof of Theorem 4.9. — It follows from Theorem 4.8 that the type of a root $\alpha \in \Pi(\mathfrak{g})$ is the same for X_1 and X_2 . We put $\Pi^\alpha := \Pi(\mathfrak{g})_{X_1}^\alpha, \dots, \Pi^d := \Pi(\mathfrak{g})_{X_i}^d, \mathfrak{a} := \mathfrak{a}_{G, X_i}, \mathfrak{X}^+ := \mathfrak{X}_{G, X_i}^+, \Psi_i := \Psi_{G, X_i}, i = 1, 2$. We also identify $\mathcal{D}_{G, X_1}, \mathcal{D}_{G, X_2}$ and write \mathcal{D} instead of \mathcal{D}_{G, X_i} .

Suppose $\Psi_1 \neq \Psi_2$. Again, we impose the assumptions in the end of Section 4. In particular, it follows from Corollary 5.6 and Theorem 4.8 that $\Psi_{M, X_1(D)} = \Psi_{M, X_2(D)}$ for any $D \in \mathcal{D}$, where M denotes the standard Levi subgroup of G_D . So we may assume that $\mathcal{D}^G = \emptyset$. As in the proof of Theorem 4.8 we may also assume that both actions $G: X_1, G: X_2$ are locally effective and both G -varieties X_1, X_2 are indecomposable.

Step 1. Let us show that $\Psi_1^0 = \Psi_2^0$. It is enough to check that $\Psi_1^0 \subset \Psi_2^0$.

Let $\alpha \in \Psi_1^0$. If α has one of the forms indicated in Lemma 3.3, then the inclusion $\alpha \in \Psi_2$ follows from Theorem 4.8, Propositions 3.1, 3.2.

Now let $D \in \mathcal{D}$ and M be the standard Levi subgroup of G_D . Suppose $\text{Supp } \alpha \subset \Pi(\mathfrak{m})$. By Proposition 5.3, $\alpha \in \Psi_{M, X_1(D)} = \Psi_{M, X_2(D)} \subset \Psi_2$.

Step 2. So we may assume that $\Psi_1 \neq \Psi_1^0$ whence Ψ_1 is one of the systems described in Proposition 7.2. Let us check that $\Psi_2 = \Psi_1^0$. Otherwise $\Psi_2^0 \neq \Psi_2$ and Ψ_2 is also one of the systems from Proposition 7.2. If $\#\Psi_1^0 \neq \emptyset$, then we get $\Psi_1 = \Psi_2$ because all systems $\Pi(\mathfrak{g})$ in the list of Proposition 7.2 are distinct. So $\#\Psi_i = 1$ and $\text{Supp}(\gamma_i) = \Pi(\mathfrak{g})$ for the unique element $\gamma_i \in \Psi_i$. Using Table 1 from [23] and the equality $\Pi(\mathfrak{g})_{X_1}^\alpha = \Pi(\mathfrak{g})_{X_2}^\alpha$, we get $\gamma_1 = \gamma_2$.

Step 3. Let us check that $X_2 \cong^G G *_H V$, where $(G, G) \subset H$. Assume the contrary, let $(G, G) \not\subset H$ or, equivalently, $\tilde{H} := N_G(H)^\circ \neq G$. Let H_0 denote the stabilizer of a point from the open H -orbit in V . Then $H_0 \subset \tilde{H}$. It is clear that $\Psi_{G, G/H_0} = \Psi_2$. Applying Proposition 3.4.3 from [16] to the pair $H_0 \subset \tilde{H}$, we see that $\Psi_{G, G/\tilde{H}} = \emptyset$ or $\Psi_{G, G/\tilde{H}} = \Psi_{G, G/H_0}$. However, $\mathfrak{a}_{G, G/\tilde{H}}$ is generated by $\Psi_{G, G/\tilde{H}}$ (it follows, for example, from [16], Lemma

3.1.4) and $\mathfrak{a}_{G, G/\tilde{H}}$ contains a dominant weight. On the contrary, $\text{Span}_{\mathbb{Q}}(\Psi_1^0)$ does not contain a dominant weight.

Step 4. Suppose $\#\Psi_1 = 1$. Then $\Psi_2 = \emptyset$. It is known, see, for example, [10], Corollary 6.2, that H_v contains a maximal unipotent subgroup of G for any $v \in V$. One easily deduces from this that V as a (G, G) -module is the tautological SL_n - or Sp_{2n} -module. It follows that $\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}] \subset \mathbb{Q}\pi_1$. But, according to Table 1 from [23], $\Psi_1 \not\subset \mathbb{Q}\pi_1$, contradiction.

Suppose $\#\Psi_1 = 2$. Note that $(\Pi(\mathfrak{h}), \Pi(\mathfrak{h})_V^a) = (\Pi(\mathfrak{g}), \Pi(\mathfrak{g})_V^a)$ and so the l.h.s. is one of the four pairs listed in Proposition 7.2. Since $\mathcal{D}^G = \emptyset$, we get $\mathcal{D}_{H^\circ, V}^{H^\circ} = \emptyset$. The classification of spherical modules, see, for example, [15], shows that there are no pairs (H°, V) , where V is a spherical H° -module with $\Pi(\mathfrak{h}), \Pi(\mathfrak{h})_V^a$ listed in Proposition 7.2 and $\mathcal{D}_{H^\circ, V}^{H^\circ} = \emptyset$. The set $\Pi(\mathfrak{h})_V^a$ is determined from Leahy’s tables as follows: this is the set of all simple roots annihilated by all highest weights in $\mathbb{K}[V]$. □

8. Invariant Kähler structures

In this section K is a compact connected Lie group, G is the complexification of K , and M is a multiplicity free compact Hamiltonian K -manifold (see Introduction) with symplectic form ω and moment map μ . Put $\tilde{\omega} = \omega + \mu$. This is an equivariantly closed form on M called the *equivariant* symplectic form. We say that $\tilde{\omega}$ is an *equivariant Kähler* form if ω is Kähler. We denote by $[\tilde{\omega}]$ the class of $\tilde{\omega}$ in the second equivariant cohomology group $H_K^2(M, \mathbb{R})$.

As above, we fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We may assume that $T_K := T \cap K$ is a maximal torus in K . The choice of B and T defines the Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{k}^*$. Define the invariant moment map $\psi: M \rightarrow \mathfrak{t}_+$ as in Introduction.

DEFINITION 8.1. — *We say that a complex structure I on M is compatible (with K, ω) if I is K -invariant, and ω is a Kähler form with respect to I .*

PROPOSITION 8.2 ([25], Proposition 5.2). — *Let I be a compatible complex structure on M . Then the K -action on M extends to a unique action $G: M$ by holomorphic automorphisms. Moreover, M is a spherical (algebraic) projective G -variety.*

This proposition allows one to define the valuation cone of (M, I) , which we denote by $\mathcal{V}(M, I)$. The objective of this section is to prove the following uniqueness result.

THEOREM 8.3. — *Let I_1, I_2 be two compatible complex structures on M . Suppose $\mathcal{V}(M, I_1) = \mathcal{V}(M, I_2)$. Then there is a K -equivariant diffeomorphism $\varphi: M \rightarrow M$ preserving $[\tilde{\omega}]$ and such that $\varphi^*(I_2) = I_1$.*

The restriction $\mathcal{V}(M, I_1) = \mathcal{V}(M, I_2)$ is essential, see [25], Remark 4.4.

To prove the theorem we need to recall some more or less standard facts.

Let X be a smooth projective G -variety. Denote by $\text{Pic}_G(X)$ the equivariant Picard group of X . Choose $\mathcal{L} \in \text{Pic}_G(X)$. Suppose that \mathcal{L} is very ample (as a usual line bundle). To \mathcal{L} one can assign an equivariant Kähler form $\tilde{\omega}_{\mathcal{L}}$ as follows. Let V denote the G -module $H^0(X, \mathcal{L})^*$ and $\iota: X \hookrightarrow \mathbb{P}(V)$ be the embedding induced by \mathcal{L} . Choose a K -invariant hermitian form (\cdot, \cdot) on V . Let ω_{FS} denote the corresponding Fubini-Study form on $\mathbb{P}(V)$ and μ_{FS} be the corresponding moment map:

$$\langle \mu_{FS}(x), \xi \rangle = \frac{(\xi v, v)}{2\pi i(v, v)},$$

where v denotes a nonzero vector on the line x . Put $\omega_{\mathcal{L}} := \iota^*(\omega_{FS}), \mu_{\mathcal{L}} := \mu_{FS} \circ \iota, \tilde{\omega}_{\mathcal{L}} := \omega_{\mathcal{L}} + \mu_{\mathcal{L}}$. By $\psi_{\mathcal{L}}$ we denote the invariant moment map associated with $\mu_{\mathcal{L}}$. Note that $\tilde{\omega}_{\mathcal{L}}$ does not depend (up to a K -equivariant diffeomorphism) from the choice of (\cdot, \cdot)

We have a unique homomorphism $\text{Pic}_G(X) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow H_K^2(X, \mathbb{R})$ mapping a very ample G -bundle \mathcal{L} to the class $[\tilde{\omega}_{\mathcal{L}}]$.

LEMMA 8.4. — *Suppose X is spherical. Then the homomorphism $\text{Pic}_G(X) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow H_K^2(X, \mathbb{R})$ is an isomorphism.*

Proof. — For $\chi \in \mathfrak{X}(G)$ let \mathbb{C}_{χ} denote the trivial bundle on X on which G acts by χ . Note that $\tilde{\omega}_{\mathcal{L} \otimes \mathbb{C}_{\chi}} = \tilde{\omega}_{\mathcal{L}} + i\chi$. So we have a commutative diagram, where the top sequence is exact

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{X}(G) \otimes_{\mathbb{Z}} \mathbb{R} & \longrightarrow & \text{Pic}_G(X) \otimes_{\mathbb{Z}} \mathbb{R} & \longrightarrow & \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathfrak{k}/[\mathfrak{k}, \mathfrak{k}])^* & \longrightarrow & H_K^2(X, \mathbb{R}) & \longrightarrow & H^2(X, \mathbb{R}) \longrightarrow 0 \end{array}$$

As was noted in the proof of Proposition 5.2 in [25], there is an action of \mathbb{C}^{\times} on X with finitely many fixed points. The Bialynicki-Birula decomposition of X induced by this action (see [1]) consists of affine spaces. It follows that $H^1(X, \mathbb{R}) = \{0\}$ and the right vertical arrow is an isomorphism. Since $H^1(X, \mathbb{R}) = \{0\}$, we see that the the bottom sequence is exact. Note that the left vertical arrow is an isomorphism. Since the top sequence is exact, we see that the middle vertical arrow is an isomorphism. □

LEMMA 8.5. — *Let X be a smooth projective G -variety. Then the following assertions hold.*

- (1) *The subset $H_K^2(X, \mathbb{R})^+ \subset H_K^2(X, \mathbb{R})$ consisting of all classes of equivariant Kähler forms is open.*
- (2) *The moment polytope for an equivariant Kähler form $\tilde{\omega}$ depends only on $[\tilde{\omega}]$ and this dependence is continuous.*

Proof. — By definition, ω is Kähler iff $\omega_x(iv, v) > 0$ for all $x \in X$, $v \in T_x X$ whence assertion 1. To prove the first claim of assertion 2 one uses Moser’s trick exactly as in the proof of Proposition 5.2 in [25]. The second claim stems from the formula $\langle \mu(x), [\xi, \eta] \rangle = \omega_x(\xi_*, \eta_*)$. □

PROPOSITION 8.6. — *Let X be a smooth projective spherical G -variety and \mathcal{L} a very ample G -bundle. We consider X as a Hamiltonian K -manifold with respect to the equivariant form $\tilde{\omega}_{\mathcal{L}}$. Then the following assertions hold:*

- (1) *$2\pi i \operatorname{im} \psi_{\mathcal{L}}$ is a rational polytope and $(2\pi i \operatorname{im} \psi_{\mathcal{L}}) \cap \mathfrak{t}^*(\mathbb{Q}) = \bigcup_{d \in \mathbb{N}} \left\{ \lambda/d \mid H^0(X, \mathcal{L}^{\otimes d})_{\lambda}^{(B)} \neq \{0\} \right\}$.*
- (2) *Let σ be a rational B -semiinvariant section of \mathcal{L} of weight μ . Then $2\pi i \operatorname{im} \psi_{\mathcal{L}} = \mu + \{ \lambda \in \mathfrak{a}_{G,X}(\mathbb{R}) \mid \langle \lambda, \varphi_D \rangle \geq -\operatorname{ord}_D(\sigma), \forall D \in \mathcal{D}_{G,X} \}$.*
- (3) *Let K_0 denote the principal isotropy subgroup for the Hamiltonian action $K: X$ (see Introduction). The group $\mathfrak{X}(T_K/(T_K \cap K_0))$ coincides with $\mathfrak{X}_{G,X}$.*

Proof. — The first assertion is due to Brion [2]. Assertion 2 easily follows from the first one. The third assertion seems to be known but we failed to find its proof in the literature. So we give a proof here.

Below we put $\omega := \omega_{\mathcal{L}}, \mu := \mu_{\mathcal{L}}$.

Step 1. Put $V = H^0(X, \mathcal{L})^*$ and let (\cdot, \cdot) be the hermitian form on V used to define $\tilde{\omega}_{\mathcal{L}}$. Put $\tilde{K} := K \times Z$, where Z is a one-dimensional compact torus, and let Z act on V by scalar multiplications. Denote by \tilde{X} the affine cone over X . The map $\Phi: V \rightarrow \tilde{\mathfrak{k}}^*$ given by $\langle \Phi(v), \xi \rangle = \frac{1}{2i}(\xi v, v)$ is a moment map for the action $\tilde{K}: V$. Note that $\mu^{-1}(\mathfrak{t}_+)$ is just the set of lines in $\Phi^{-1}(\mathfrak{t}_+ \times \mathfrak{z}) \cap \tilde{X}$. Let \tilde{K}_0 denote the stabilizer of a point $v \in \Phi^{-1}(\mathfrak{t}_+ \times \mathfrak{z}) \cap \tilde{X}$ in general position. It is easy to see that the restriction of the projection $\tilde{K} \rightarrow K$ to \tilde{K}_0 is injective and its image coincides with K_0 . So it is enough to prove that

$$(8.1) \quad \mathfrak{X}(\tilde{T}_K/(\tilde{T}_K \cap \tilde{K}_0)) = \mathfrak{X}_{\tilde{G}, \tilde{X}},$$

where $\tilde{T}_K := T_K \times Z, \tilde{G} := G \times \mathbb{C}^\times$.

Step 2. Let \mathfrak{t}_+^0 denote the interior of the smallest face of \mathfrak{t}_+ containing $i \operatorname{im} \psi_{\mathcal{L}}$. Put $\tilde{Y} := \Phi^{-1}(\mathfrak{t}_+^0 \times \mathfrak{z})$. Replacing K with a covering we may assume that K is the direct product of a torus and a simply connected semisimple group. It follows now from results of [8] (see especially Theorem 4.9, Theorem 6.11, and the proof of Theorem 6.17) that there is a \tilde{T}_K -equivariant embedding $\tilde{Y} \rightarrow \tilde{X} // U := \operatorname{Spec}(\mathbb{C}[\tilde{X}]^U)$ with dense image. To check (8.1) note that $\tilde{\mathfrak{X}}_{G, \tilde{X}} = \tilde{\mathfrak{X}}(\tilde{T}/\tilde{T}_0)$, where $\tilde{T} := T \times \mathbb{C}^\times$ and \tilde{T}_0 is the kernel for the action $\tilde{T}: \tilde{X} // U$ (i.e. the kernel of the corresponding homomorphism $\tilde{T} \rightarrow \operatorname{Aut}(\tilde{X} // U)$). \square

Let us generalize assertion 2 of Proposition 8.6 to the case of an arbitrary equivariant Kähler form $\tilde{\omega}$. Let ψ denote the invariant moment map corresponding to $\tilde{\omega}$.

Consider the embedding $\mathfrak{X}_{G, X} \rightarrow \mathfrak{X}(T) \times \mathbb{Z}^{\mathcal{D}_{G, X}}$ given by $\lambda \mapsto (\lambda, \sum_{D \in \mathcal{D}_{G, X}} \langle \varphi_D, \lambda \rangle D)$. Define the homomorphism $\chi: \operatorname{Pic}_G(X) \rightarrow \mathfrak{X}(T) \times \mathbb{Z}^{\mathcal{D}_{G, X}} / \mathfrak{X}_{G, X}$ by $\chi(\mathcal{L}) = (\lambda, \sum_{D \in \mathcal{D}_{G, X}} \operatorname{ord}_D(\sigma) D)$, where σ is a B -semi-invariant rational section of \mathcal{L} and λ is the weight of σ . Extend χ to a linear map $\chi: H_K^2(X, \mathbb{R}) \rightarrow \mathfrak{t}(\mathbb{R}) \oplus \mathbb{R}^{\mathcal{D}_{G, X}} / \mathfrak{a}_{G, X}(\mathbb{R})$. For $v \in H_K^2(X, \mathbb{R})$ put

$$P(\chi, v) = \chi_0(v) + \{ \lambda \in \mathfrak{a}_{G, X}(\mathbb{R}) \mid \langle \lambda, \varphi_D \rangle \geq -\chi_D(v), \forall D \in \mathcal{D}_{G, X} \},$$

where $(\chi_0, \sum_{D \in \mathcal{D}_{G, X}} \chi_D D)$ is a lifting of χ . It follows from assertion 2 of Proposition 8.6 and Lemma 8.5 that $2\pi i \operatorname{im} \psi = P(\chi, [\tilde{\omega}])$.

Proof of Theorem 8.3. — Let X_j denote the manifold M with the complex structure $I_j, j = 1, 2$. By assertion 3 of Proposition 8.6, $\mathfrak{X}_{G, X_1} = \mathfrak{X}_{G, X_2}$. In the sequel we write \mathfrak{X} instead of \mathfrak{X}_{G, X_j} and \mathfrak{a} instead of \mathfrak{a}_{G, X_j} . Put $\mathcal{D}_j := \mathcal{D}_{G, X_j}$. Let χ^j denote the map $H_K^2(M, \mathbb{R}) \rightarrow (\mathfrak{t}(\mathbb{R}) \oplus \mathbb{R}^{\mathcal{D}_j}) / \mathfrak{a}(\mathbb{R})$ defined above, and $\bar{\chi}^j = \chi_0^j + \sum_{D \in \mathcal{D}_{G, X}} \chi_D^i D: H_K^2(M, \mathbb{R}) \rightarrow \mathfrak{t}(\mathbb{R}) \oplus \mathbb{R}^{\mathcal{D}_j}$ be a rational lifting of χ^j .

Step 1. Let us reduce the proof to the case when $[\tilde{\omega}]$ is rational. By the above,

$$2\pi i \operatorname{im} \psi = P(\chi^i, [\tilde{\omega}]) := \chi_0^i([\tilde{\omega}]) + \{ \lambda \in \mathfrak{a}(\mathbb{R}) \mid \langle \varphi_D, \lambda \rangle \geq -\chi_D^i([\tilde{\omega}]), \forall D \in \mathcal{D}_i \}, \quad i = 1, 2.$$

Note that the projections of $\chi_0^1([\tilde{\omega}]), \chi_0^2([\tilde{\omega}])$ to $\mathfrak{t}(\mathbb{R}) / \mathfrak{a}(\mathbb{R})$ coincide. Thus, possibly after modifying $\bar{\chi}^1$, we may assume that $\chi_0^1(v) = \chi_0^2(v)$ for any $v \in V'$, where V' is a rational subspace $V' \subset H_K^2(M, \mathbb{R})$ such that $[\tilde{\omega}] \in V'(\mathbb{R})$.

The following lemma implies that there is a sequence $\tilde{\omega}_k \in \operatorname{Pic}_G(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $P^1(\chi^1, [\tilde{\omega}_k]) = P^2(\chi^2, [\tilde{\omega}_k])$ and $[\tilde{\omega}_k] \rightarrow [\tilde{\omega}]$.

LEMMA 8.7. — Let V, \mathfrak{a} be finite dimensional vector spaces over \mathbb{Q} , $\varphi_i^1, \varphi_j^2 \in \mathfrak{a}^*, \chi_i^1, \chi_j^2 \in V^*$, where $i = \overline{1, k_1}, j = \overline{1, k_2}$. Suppose there is $v \in V(\mathbb{R})$ such that the polytopes $P^l(v) \subset \mathfrak{a}(\mathbb{R}), l = 1, 2$, given by

$$P^l(v) := \{ \lambda \in \mathfrak{a}(\mathbb{R}) \mid \langle \varphi_i^l, \lambda \rangle \geq -\chi_i^l(v), i = \overline{1, k_i} \}.$$

coincide and have dimension $\dim \mathfrak{a}(\mathbb{R})$. Then for any neighborhood O of v in $V(\mathbb{R})$ there is $v' \in O \cap V$ such that $P^1(v') = P^2(v')$.

Proof of Lemma 8.7. — Let m be the number of facets of $P^l(v)$. We may assume that the facets are defined by the equations $\langle \varphi_i^l, \lambda \rangle = \chi_i^l(v), i = \overline{1, m}$. After rescaling $\varphi_i^1, \chi_i^1, i \leq l$, we get $\varphi_i^1 = \varphi_i^2, \chi_i^1 = \chi_i^2$ for any $i \leq m$. Let $P(v'), v' \in V(\mathbb{R})$, denote the polytope given by the inequalities $\langle \varphi_i^l, \lambda \rangle \geq -\chi_i^l(v'), i = \overline{1, m}$. If the hyperplane $\{ \lambda \mid \langle \varphi_j^l, \lambda \rangle = \chi_j^l(v') \}$ does not intersect $P(v')$ for $v' = v$, then the same holds for any v' from a certain neighborhood of v . Now suppose the hyperplane $\{ \lambda \mid \langle \varphi_j^l, \lambda \rangle = \chi_j^l(v) \}$ meets $P^l(v)$ at a face Γ . Let I denote the subset of $\{1, \dots, l\}$ consisting of all i such that $\langle \varphi_i^l, \cdot \rangle - \chi_i^l(v)$ is a facet of $P^l(v)$ containing Γ . Then there are positive rational numbers $a_i, i \in I$, such that $\varphi_j^l = \sum_{i \in I} a_i \varphi_i^l$. It follows that $\chi_j^l(v) = \sum_{i \in I} a_i \chi_i^l(v)$. This equation defines a subspace $V_j^l \subset V$. Let V_0 denotes the intersection of all subspaces V_j^l . Since $v \in V_0$, we see that V_0 is nonzero. It follows that $P^l(v') = P(v'), l = 1, 2$, for any v' from a certain neighborhood of v in V_0 . □

Suppose we have constructed diffeomorphisms $\varphi_k: M \rightarrow M$ such that $\varphi_k^*(I_2) = I_1, \varphi_k([\tilde{\omega}_k]) = [\tilde{\omega}_k]$. Note that $\psi_{kl} := \varphi_k^{-1} \circ \varphi_l$ is a G -equivariant (polynomial) automorphism of X_1 . By the definition of ψ_{kl} ,

$$(8.2) \quad \psi_{kl}^*([\tilde{\omega}_l]) = \varphi_k^{*-1}([\tilde{\omega}_l]).$$

By [12], the group $\text{Aut}^G(X_1)$ is algebraic. Clearly, $\text{Aut}^G(X_1)^\circ$ acts trivially on $H_K^2(M, \mathbb{R})$. So replacing the sequence φ_k with a subsequence, we may assume that the isomorphism ψ_{kl}^* is the identity on $H_K^2(M, \mathbb{R})$ for all k, l . Since $[\tilde{\omega}_k] \rightarrow [\tilde{\omega}]$, it follows from (8.2) that $\varphi_k^*([\tilde{\omega}]) = [\tilde{\omega}]$.

Step 2. Multiplying $\tilde{\omega}$ by a sufficiently large integer m , we may assume that $\tilde{\omega} = \tilde{\omega}_{\mathcal{L}^i}$ for a very ample line G -bundle \mathcal{L}^i . Let \tilde{X}_i denote the affine cone over X_i corresponding to \mathcal{L}^i . Set $\tilde{G} := G \times \mathbb{K}^\times$. The variety \tilde{X}_i has a natural structure of a (spherical) \tilde{G} -variety. By assertion 1 of Proposition 8.6, $\mathfrak{X}_{\tilde{G}, \tilde{X}_i}^+$ is generated by integral points in the moment polytope. From [11] it follows that $\mathcal{V}_{\tilde{G}, \tilde{X}_1} = \mathcal{V}_{\tilde{G}, \tilde{X}_2}$. Applying Theorem 1.2, we see that $\tilde{X}_1 \cong_{\tilde{G}} \tilde{X}_2$. Therefore there is a G -equivariant isomorphism $\varphi: X_1 \rightarrow X_2$ such that $\varphi^*(\mathcal{L}^2) = \mathcal{L}^1$. □

BIBLIOGRAPHY

- [1] A. BIALYNICKI-BIRULA, “Some properties of the decompositions of algebraic varieties determined by an action of a torus”, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **24** (1976), p. 667-674.
- [2] M. BRION, *Sur l'image de l'application moment*, Lect. Notes Math., vol. 1296, Springer Verlag, 1987, 177-192 pages.
- [3] M. BRION, D. LUNA & T. VUST, “Espaces homogènes sphériques”, *Invent. Math.* **84** (1986), p. 617-632.
- [4] R. CAMUS, “Variétés sphériques affines lisses”, PhD Thesis, Université J. Fourier, 2001, <http://www-fourier.ujf-grenoble.fr/THESE/html/a110>.
- [5] T. DELZANT, “Hamiltoniens périodiques et images convexes de l'application moment”, *Bul. Soc. Math. France* **116** (1988), p. 315-339.
- [6] ———, “Classification des actions hamiltoniennes complètement intégrables de rang deux”, *Ann. Global. Anal. Geom.* **8** (1990), p. 87-112.
- [7] F. GROSSHANS, *Constructing invariant polynomials via Tschirnhaus transformations*, Lect. Notes Math., vol. 1278, Springer Verlag, 1987, 95-102 pages.
- [8] V. GUILLEMIN, L. JEFFREY & R. SJAMAAR, “Symplectic implosion”, *Transform. Groups* **7** (2002), p. 155-184.
- [9] F. KIRWAN, “Convexity properties of the moment mapping III”, *Invent. Math.* **77** (1984), p. 547-552.
- [10] F. KNOP, “The Luna-Vust theory of spherical embeddings”, in *Proceedings of the Hyderabad conference on algebraic groups* (Madras), Manoj Prakashan, 1991, p. 225-249.
- [11] ———, “The asymptotic behaviour of invariant collective motion”, *Invent. Math.* **114** (1994), p. 309-328.
- [12] ———, “Automorphisms, root systems and compactifications”, *J. Amer. Math. Soc.* **9** (1996), p. 153-174.
- [13] F. KNOP & B. V. STEIRTEGHEM, “Classification of smooth affine spherical varieties”, *Transform. Groups* **11** (2006), p. 495-516.
- [14] M. KRAMER, “Spharische Untergruppen in kompakten zusammenhängenden Liegruppen”, *Compos. Math.* **38** (1979), p. 129-153.
- [15] A. LEAHY, “A classification of multiplicity free representations”, *J. Lie Theory* **8** (1998), p. 376-391.
- [16] I. LOSEV, “Uniqueness properties for spherical homogeneous spaces”, to appear in *Duke Math. J.*
- [17] D. LUNA, *Grosses cellules pour les variétés sphériques*, Austr. Math. Soc. Lect. Ser., vol. 9, Cambridge University Press, 1997, 267-280 pages.
- [18] ———, “Variétés sphériques de type A”, *IHES Publ. Math.* **94** (2001), p. 161-226.
- [19] A. ONISHCHIK & E. VINBERG, *Lie groups and algebraic groups*, Springer Verlag, 1990, xx+328 pages.
- [20] V. POPOV, “Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles”, *Math. USSR-Izv* **7** (1973), p. 301-327.
- [21] ———, “Contractions of the actions of reductive algebraic groups”, *Math. USSR Sbornik* **58** (1987), p. 311-355.
- [22] T. VUST, “Plongement d'espaces symétriques algébriques: une classification”, *Ann. Sc. Norm. Super. Pisa, Ser. IV* **17** (1990), p. 165-194.
- [23] B. WASSERMAN, “Wonderful varieties of rank 2”, *Transform. Groups* **1** (1996), p. 375-403.

- [24] C. WOODWARD, “The classification of transversal multiplicity-free Hamiltonian actions”, *Ann. Global. Anal. Geom.* **14** (1996), p. 3-42.
- [25] ———, “Spherical varieties and existence of invariant Kahler structure”, *Duke Math. J* **93** (1998), p. 345-377.

Manuscrit reçu le 8 janvier 2007,
accepté le 7 novembre 2008.

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