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## TWO GENERALIZATIONS OF CHEEGER-GROMOLL SPLITTING THEOREM VIA BAKRY-EMERY RICCI CURVATURE

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ABSTRACT. — In this paper, we prove two generalized versions of the Cheeger-Gromoll splitting theorem via the non-negativity of the Bakry-Émery Ricci curvature on complete Riemannian manifolds.

RÉSUMÉ. — Dans cet article, nous obtenons deux généralisations du théorème de scindage de Cheeger-Gromoll sur les variétés riemanniennes complètes à courbure de Ricci non-négative au sens de Bakry-Émery.

### 1. Introduction

Cheeger and Gromoll's splitting theorem [6] played an important role in the study of manifolds with nonnegative or almost nonnegative Ricci curvature. In this paper, we consider the manifolds with nonnegative Bakry-Émery Ricci curvature and prove two generalized versions of the splitting theorem on such manifolds.

Following Bakry-Émery [1], see also [17, 3, 15, 12], given a Riemannian manifold  $(M, g)$  and a  $C^2$ -smooth function  $\phi$ ,  $M$  is said to have nonnegative  $\infty$ -dimensional Bakry-Émery Ricci curvature associated to  $\phi$  if  $Ric + Hess(\phi) \geq 0$ , where  $Ric$  denotes the Ricci curvature of  $g$  and  $Hess$  denotes the Hessian with respect to  $g$ . As pointed out in Lott [15], in general, the splitting theorem does not hold for manifolds with nonnegative

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$\infty$ -dimensional Bakry-Émery Ricci curvature. A trivial counterexample is given by the hyperbolic  $n$ -space form  $\mathbb{H}^n$ , where  $Ric + \frac{1}{\delta} Hess(\rho^2) \geq 0$  for some small constant  $\delta > 0$  and the distance function  $\rho$ . Obviously there are many lines in this space but it doesn't split off a line. See [19].

If the manifold  $M$  is compact and its  $\infty$ -dimensional Bakry-Émery Ricci curvature is positive, then  $\pi_1(M)$  is finite. This was first proved by X.-M. Li [13], see also [8, 19, 20, 21]. Also, from Lott's work [15, Theorem 1], a compact manifold with nonnegative  $\infty$ -dimensional Bakry-Émery Ricci curvature has  $b_1$  parallel vector fields where  $b_1$  is the first Betti number of  $M$ , which are orthogonal to the gradient field of  $\phi$ . This indicates that the universal Riemannian covering space of  $(M, g)$  should split off  $b_1$  lines. We confirm this in this paper, as a corollary of the following theorem.

**THEOREM 1.1.** — *Let  $(M, g)$  be a complete connected Riemannian manifold with  $Ric + Hess(\phi) \geq 0$  for some  $\phi \in C^2(M)$  which is bounded above uniformly on  $M$ . Then it splits isometrically as  $N \times \mathbb{R}^l$ , where  $N$  is a complete Riemannian manifold without lines and  $\mathbb{R}^l$  is the  $l$ -Euclidean space. Furthermore, the function  $\phi$  is constant on each  $\mathbb{R}^l$  in this splitting.*

Then the corollary reads as

**COROLLARY 1.2.** — *Let  $(M, g)$  be a closed connected Riemannian manifold with  $Ric + Hess(\phi) \geq 0$  for some smooth function  $\phi$  on  $M$ . Then we have an isometric decomposition for its universal Riemannian covering space:  $\widetilde{M} \cong N \times \mathbb{R}^l$ , where  $N$  is a closed manifold,  $\mathbb{R}^l$  is the  $l$ -Euclidean space and  $l \geq b_1$ , the first Betti number of  $M$ . Furthermore, the lifted function of  $\phi$ , say  $\tilde{\phi}$ , is constant on each  $\mathbb{R}^l$ -factor.*

If  $b_1$  equals the dimension of  $M$ , then  $(M, g)$  is the flat torus.

Another generalized version of splitting theorem can be described as follows. According to [2, 3], we say that the symmetric diffusion operator  $L = \Delta - \nabla\phi \cdot \nabla$  satisfies the curvature-dimension condition  $CD(0, m)$  if

$$L|\nabla u|^2 \geq \frac{2|Lu|^2}{m} + 2 \langle \nabla u, \nabla Lu \rangle, \quad \forall u \in C_0^\infty(M).$$

Following the notation used in [12], which is slightly different from [2, 3, 15], we define the  $m$ -dimensional Bakry-Émery Ricci curvature of  $L = \Delta - \nabla\phi \cdot \nabla$  on an  $n$ -dimensional Riemannian manifold as follows

$$Ric_{m,n}(L) := Ric + Hess(\phi) - \frac{\nabla\phi \otimes \nabla\phi}{m - n},$$

where  $m = \dim_{BE}(L) > n$  is called the Bakry-Émery dimension of  $L$ , which is a constant and is not necessarily an integer. By [2, 3, 12], we know that  $CD(0, m)$  holds if and only if  $Ric_{m,n}(L) \geq 0$ . We now state the following

**THEOREM 1.3.** — *Let  $(M, g)$  be a complete connected Riemannian  $n$ -manifold and  $\phi \in C^2(M)$  be a function satisfying that  $\text{Ric}_{m,n}(L) \geq 0$  for some constant  $m = \dim_{BE}(L) > n$  which is not necessarily an integer. Then  $M$  splits isometrically as  $N \times \mathbb{R}^l$ , where  $N$  is a complete Riemannian manifold  $N$  without line, and  $\mathbb{R}^l$  is the  $l$ -dimensional Euclidean space. Furthermore, the function  $\phi$  is constant on each  $\mathbb{R}^l$ -factor, and  $N$  has non-negative  $(m - l)$ -dimensional Bakry-Émery Ricci curvature.*

The Bakry-Émery Ricci curvature has been widely used in the literature. Recently, it was used by G. Perelman [16] to modify R. Hamilton's Ricci flow equation. Our paper provides us with two extensions of the Cheeger-Gromoll splitting theorem on complete Riemannian manifolds via the Bakry-Émery Ricci curvature. We would like to mention that a very relevant and independent paper by Wei and Wylie [19] has been posted recently in the Arxiv. One of their results (see Theorem 1.4 in [19]) says that if  $M$  is an  $n$ -dimensional complete Riemannian manifold with  $\text{Ric} + \text{Hess}(f) \geq 0$  for some bounded function  $f$  and contains a line, then  $M$  splits into  $M \cong N^{n-1} \times \mathbb{R}$  and  $f$  is constant along the line. This result is originally due to A. Lichnerowicz [14].

The paper is organized as follows: In Section 2, we show that the Busemann function associated to the line has parallel gradient field. Then we prove Theorems 1.1, 1.3 and Corollary 1.2 in Section 3. In Section 4, we give some remarks on the Bakry-Émery Ricci curvature and the Cheeger-Gromoll splitting theorem.

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## 2. Estimation of the Laplacian on Busemann function

So far, there have been at least three different proofs of the Cheeger-Gromoll splitting theorem. All these proofs amount to showing that the Busemann function is harmonic. The original proof of Cheeger and Gromoll [6] uses the Jacobi fields theory and the elliptic regularity. The second one by Eschenburg-Heintze [7] uses only the Laplacian comparison theorem on distance function and the Hopf-Calabi maximum principle. The third one, given by Schoen-Yau [18], uses the Laplacian comparison theorem on distance and the sub-mean value inequality rather than the maximum principle. For an elegant description of the proof of [7], see Besse [4]. We will follow the lines given by [4, 6, 7, 18] to prove the  $L$ -harmonicity of the Busemann function on  $M$ .

Assume that  $(M, g)$  is a complete Riemannian manifold and  $\phi \in C^2(M)$  satisfies  $Ric + Hess(\phi) \geq 0$  over  $M$ . Fix  $p \in M$  as a base point and denote  $\rho(x) = \text{dist}(p, x)$  the distance function. Given any  $q \in M$ , let  $\gamma : [0, \rho] \rightarrow M$  be a minimal normal geodesic from  $p$  to  $q$  and  $\{E_i(t)\}_{i=1}^{n-1}$  be parallel orthonormal vector fields along  $\gamma$  which are orthogonal to  $\dot{\gamma}$ . Constructing vector fields  $\{X_i(t) = \frac{t}{\rho} E_i(t)\}_{i=1}^{n-1}$  along  $\gamma$  and by the second variation formula, we have the estimate

$$\begin{aligned} \Delta\rho(q) &\leq \int_0^\rho \sum_{i=1}^{n-1} (|\nabla_{\dot{\gamma}} X_i|^2 - \langle X_i, R_{X_i, \dot{\gamma}} \dot{\gamma} \rangle) dt \\ &= \int_0^\rho \left( \frac{n-1}{\rho^2} - \frac{t^2}{\rho^2} Ric(\dot{\gamma}, \dot{\gamma}) \right) dt \\ &\leq \frac{n-1}{\rho} + \int_0^\rho \frac{t^2}{\rho^2} Hess(\phi)(\dot{\gamma}, \dot{\gamma}) dt \\ &= \frac{n-1}{\rho} + \frac{1}{\rho^2} \int_0^\rho t^2 \frac{d^2}{dt^2} (\phi \circ \gamma) \\ &= \frac{n-1}{\rho} + \langle \nabla\phi, \dot{\gamma} \rangle (q) - \frac{2}{\rho^2} \int_0^\rho t \frac{d}{dt} (\phi \circ \gamma) dt \\ &= \frac{n-1}{\rho} + \langle \nabla\phi, \dot{\gamma} \rangle (q) - \frac{2}{\rho} \phi(q) + \frac{2}{\rho^2} \int_0^\rho \phi \circ \gamma dt. \end{aligned}$$

Thus

$$(2.1) \quad L\rho(q) \leq \frac{n-1}{\rho} - \frac{2}{\rho} \phi(q) + \frac{2}{\rho^2} \int_0^\rho \phi \circ \gamma dt, \quad \forall q \in M \setminus \text{cut}(p),$$

where  $\gamma$  is any minimal normal geodesic connecting  $p$  and  $q$ .

LEMMA 2.1. — *Let  $(M, g)$  be a complete Riemannian manifold and  $\gamma$  be a ray. If  $Ric + Hess(\phi) \geq 0$  for some smooth function  $\phi$  which is bounded*

from above uniformly on  $M$ , then the associated Buesman function of  $\gamma$ , say  $b^\gamma$ , satisfies that  $Lb^\gamma \geq 0$  in the barrier sense.

*Remark 2.2.* — We say that a continuous function  $f$  on  $M$  satisfies  $Lf \geq 0$  in the barrier sense, if for any given  $q \in M$  and  $\epsilon > 0$ , there is a  $C^2$  function  $f_{q,\epsilon}$  in a neighborhood of  $q$ , such that  $f_{q,\epsilon} \leq f$ ,  $f_{q,\epsilon}(q) = f(q)$ , and  $Lf_{q,\epsilon} \geq -\epsilon$ . Such  $f_{q,\epsilon}$  is called a support function of  $f$ . We say that  $Lf \leq 0$  in the barrier sense if  $L(-f) \geq 0$  in the barrier sense.

*Proof of Lemma 2.1.* — We use the same argument as in [4, 7], see also Lemma 4.7 of [21]. Denote  $p = \gamma(0)$ . The Busemann function along the ray  $\gamma$  is defined by  $b^\gamma(q) := \lim_{t \rightarrow \infty} (t - d(q, \gamma(t)))$ . By [4, 7, 21, 18],  $b^\gamma$  is 1-Lipshitz. Following [4, 7, 21], for any fixed  $q \in M$ , we define the support functions around  $q$  as follows.

Let  $\delta_{t_k}$  be a minimal geodesic connecting  $q$  and  $\gamma(t_k)$ . By [4, 7, 21], there exists a subsequence of  $t_k$  such that the initial vector  $\dot{\delta}_{t_k}(0)$  converges to some  $X \in T_qM$ . Let  $\delta$  be the ray emanating from  $q$  and generated by  $X$ . Then  $q$  does not belong to the cut-locus of  $\delta(r)$  for any  $r > 0$ . So  $b_r^\gamma(x) = r - d(x, \delta(r)) + b^\gamma(q)$  is  $C^\infty$  around  $q$  and satisfies that  $b_r^\gamma \leq b^\gamma$  with  $b_r^\gamma(q) = b^\gamma(q)$ . On the other hand, by the estimate (2.1), we have

$$\begin{aligned}
 -Lb_r^\gamma(x) = Ld(\delta(r), x) &\leq \frac{n-1}{d(\delta(r), x)} - \frac{2\phi(x)}{d(\delta(r), x)} \\
 &\quad + \frac{2}{d(\delta(r), x)^2} \int_0^{d(\delta(r), x)} \phi \circ \sigma dt
 \end{aligned}$$

where  $\sigma$  is a minimal geodesic connecting  $\delta(r)$  and  $x$ . Thus for any given  $\epsilon > 0$ , when  $r$  is large enough,  $Lb_r^\gamma \geq -\epsilon$  for  $x$  in a small neighborhood of  $q$ . This shows that  $b_r^\gamma$  is the desired support function for  $b^\gamma$ . □

*Remark 2.3.* — If  $b^\gamma$  is smooth at  $q$ , then  $\nabla b^+(q) = \dot{\delta}(0)$ , where  $\delta$  is the ray emanating from  $q$  constructed in the proof of Lemma 2.1. See [6, 21].

**LEMMA 2.4** (The Calabi-Hopf maximum principal). — *Let  $(M, g)$  be a connected complete Riemannian manifold,  $\phi \in C^2(M)$ , and  $L = \Delta - \nabla\phi \cdot \nabla$ . Let  $f$  be a continuous function on  $M$  such that  $Lf \geq 0$  in the barrier sense. Then  $f$  attains a maximum if and only if it is constant.*

*Proof.* — The proof is similar to the one in [5], see also [4]. □

**LEMMA 2.5.** — *Let  $(M, g)$  and  $\phi$  be as in Lemma 2.1. Suppose  $M$  contains a line  $\gamma$ , then the Busemann functions  $b^\pm$  associated to rays  $\gamma^\pm(t) = \gamma(\pm t), t \geq 0$ , are both smooth and satisfy that  $Lb^\pm = 0$ .*

*Proof.* — By Lemma 2.1,  $L(b^+ + b^-) \geq 0$  in the barrier sense. On the other hand,  $b^+ + b^- = 0$  on the line  $\gamma$  and the triangle inequality implies that  $b^+ + b^- \leq 0$  over  $M$ . So  $b^+ + b^- \equiv 0$  over  $M$  by Lemma 2.4. Now  $Lb^+ \geq 0$  and  $L(-b^+) = Lb^- \geq 0$  show that  $Lb^+ = 0$  in the barrier sense, then from the elliptic regularity theorem  $b^+$  is smooth and  $Lb^+ = 0$  in the canonical way, cf. section 6.3-6.4 of [9]. Similarly  $b^-$  is smooth satisfying that  $Lb^- = 0$ . □

Next we consider the case where  $Ric_{m,n}(L) \geq 0$ . We have the following

LEMMA 2.6. — *Let  $M$  be a complete Riemannian manifold,  $\phi \in C^2(M)$ . Suppose that there exists a constant  $m > n$  such that  $Ric_{m,n}(L) \geq 0$ . Then the Busemann functions  $b^\pm$  associated to rays  $\gamma^\pm(t) = \gamma(\pm t), t \geq 0$ , are both smooth and satisfy that  $Lb^\pm = 0$ .*

*Proof.* — Let  $b_r^\gamma(x) = t - d(x, \delta(r)) + b^\gamma(q)$  be the support function defined in the proof of Lemma 2.1. By [12, Remark 3.2](pp. 1317-1318), we have the Laplacian comparison theorem

$$Ld(\cdot, x)|_y \leq \frac{m-1}{d(y, x)}, \quad \forall x \in M, y \in M \setminus cut(x).$$

This yields that

$$Lb_r^\gamma(x) = -Ld(x, \delta(r)) \geq -\frac{m-1}{d(x, \delta(r))}.$$

Hence,  $Lb^+ \geq 0$  holds in the barrier sense. Similarly,  $Lb^- \geq 0$  holds in the barrier sense. By the same argument as used in the proof of Lemma 2.5, we can conclude the result. Below we follow [18] to give an alternative proof of Lemma 2.6. Indeed, for all  $\psi \in C_0^\infty(M)$  with  $\psi \geq 0$ , and for all  $t > 0$ ,

$$\begin{aligned} \int_M Lb_t^\gamma \psi d\mu &= - \int_M Ld(x, \gamma(t)) \psi d\mu \\ &\geq - \int_M \frac{m-1}{d(x, \gamma(t))} \psi d\mu. \end{aligned}$$

Taking  $t \rightarrow \infty$ , we have  $Lb^+ \geq 0$  in the sense of distribution. Similarly,  $Lb^- \geq 0$ . Hence  $L(b^+ + b^-) \geq 0$  holds in the sense of distribution. By the strong maximum principle, since the  $L$ -subharmonic function  $b^+ + b^-$  has an interior maximum on the geodesic ray  $\gamma$ , it must be identically constant. Thus,  $b^+ + b^- = 0, Lb^\pm = 0$  and  $b^\pm$  are smooth. □

LEMMA 2.7. — *Under the conditions as in Lemma 2.5 or Lemma 2.6,  $\nabla b^+$  and  $\nabla b^-$  are unit parallel vector fields.*

*Proof.* — By Remark 2.3,  $\nabla b^\pm$  are normal vector fields. To show they are parallel, we will use a generalized version of the Bochner-Weitzenböck formula. By Bakry-Emery [1], for any smooth function  $\psi$ , we have

$$(2.2) \quad L|\nabla\psi|^2 = 2|\nabla^2\psi|^2 + 2 \langle \nabla L\psi, \nabla\psi \rangle + 2(Ric + Hess(\phi))(\nabla\psi, \nabla\psi).$$

Using Lemma 2.5 or Lemma 2.6 and applying (2.2) to  $\psi = b^\pm$ , we see that  $0 = L|\nabla b^\pm|^2 \geq 2|\nabla^2 b^\pm|^2$  over  $M$ , since  $Ric + Hess(\phi) \geq 0$  in both cases. Now the result follows. □

### 3. Proof of Theorems 1.1 and 1.3

We now are in a position to give a

*Proof of Theorems 1.1 and 1.3.* — By Lemma 2.7,  $X = \nabla b^+$  is a parallel unit vector field. Let  $\phi(t) = e^{tX}$  be the one-parameter transformation group of isometries generated by  $X$ . The level surface  $N = \{x | b^+(x) = 0\}$  is a totally geodesic submanifold of  $M$ , and the induced metric  $h_N$  from  $g$  is complete. Define the map  $F : N \times \mathbb{R} \rightarrow M$  by

$$F(p, t) = \phi(t)(p).$$

We have  $\frac{d}{dt} b^+(\phi(t)p) = |\nabla b^+|^2(\phi(t)p) \equiv 1$ . This implies  $F(N, t) \subset \{x \in M | b^+(x) = t\}$ . We claim that  $F$  is bijective. In fact, for any  $x \in M$ , letting  $q \in N$  be the nearest point to  $x$  and  $\gamma$  be the shortest normal geodesic from  $q$  to  $x$ , then  $\dot{\gamma}(q) = X(q)$  and  $\gamma(t) = \phi(t)q$  by the uniqueness of the geodesic, as  $\phi(t)q$  is obviously a normal geodesic. So  $x \in \gamma \subset \text{Im}(F)$ . This proves that  $F$  is surjective. By the semi-group property  $F(\cdot, t) \circ F(\cdot, s) = F(\cdot, t + s)$ ,  $F$  is injective. The claim follows.

Next we prove that  $F$  is an isometry. To do so, notice that  $F(\cdot, t)$  maps  $N$  isometrically onto  $\{x \in M | b^+(x) = t\}$  via  $\phi(t)$ . So it suffices to show that for any vector  $v \in TN$ , we have

$$\langle dF(\cdot, t)(v), dF(\frac{\partial}{\partial t}) \rangle = \langle d\phi(t)(v), X \rangle \equiv 0.$$

This is obviously true since  $d\phi(t)(TN) \perp X$ . So  $F$  is an isometry.

Now identifying  $(M, g)$  with  $(N \times \mathbb{R}, h_N \otimes dt^2)$  and applying (2.2) to  $\psi = b^+$ , we get

$$0 = L|\nabla b^+|^2 = 2(Ric + Hess(\phi))(\nabla b^+, \nabla b^+) = 2 \frac{\partial^2}{\partial t^2} \phi.$$

So  $\phi$  is linear on each line of  $M$ . Since  $\phi$  is bounded from above, it must be constant on each line. This proves Theorem 1.1.

Finally, if  $Ric_{m,n}(L) \geq 0$ , then (2.2) yields

$$0 = \frac{\partial^2}{\partial t^2} \phi \geq \frac{1}{m-n} \left| \frac{\partial}{\partial t} \phi \right|^2.$$

So  $\phi$  is constant along each line of  $M$  and Theorem 1.3 follows.  $\square$

*Proof of Corollary 1.2.* — By Theorem 1.1 and using the same argument as Cheeger and Gromoll in [6] we conclude.  $\square$

*Remark 3.1.* — All the arguments in the proof of Theorem 1.1 depend only on the fact that the limit

$$\overline{\lim}_{\rho \rightarrow \infty} \frac{1}{\rho^2} \int_0^\rho \phi \circ \sigma dt \leq 0$$

on any ray  $\sigma$ , see Estimate (2.1). If so, then Theorem 1.1 remains true. In particular, if  $\frac{\phi(q)}{d(p,q)} = o(1)$  as  $\frac{1}{d(p,q)} \rightarrow 0$ , where  $p$  is a fixed base point, then Theorem 1.1 and all corollaries considered above still hold.

Finally we state an alternative result about the splitting theorem, where the boundedness of the potential function  $\phi$  is removed.

**COROLLARY 3.2.** — *Let  $(M, g)$  be an open complete connected Riemannian manifold and  $X$  be a unit parallel vector field. If  $Ric + Hess(\phi) \geq cg$  for some  $\phi \in C^2(M)$  and a constant  $c > 0$ , then  $(M, g)$  splits off a line. In particular, any open shrinking Ricci soliton with a parallel vector field splits off a line.*

Recall that a Riemannian manifold  $(M, g)$  is a shrinking Ricci soliton if there exists a smooth function  $f$  such that  $Ric + Hess(f) = cg$  for some positive constant  $c$ .

*Proof.* — By the result of [19, 20], such a manifold  $M$  has finite fundamental group  $\pi_1(M)$ . Denote by  $(\widetilde{M}, \widetilde{g})$  the universal Riemannian covering of  $(M, g)$  and let  $\widetilde{X}$  be the lifting of  $X$ . Let  $\phi(t) = e^{t\widetilde{X}}$  and  $N$  be a maximal integral submanifold of  $\widetilde{X}^\perp$ , the distribution orthogonal to  $\widetilde{X}$ . Define the map  $F$  as in the proof of Theorem 1.1 and Theorem 1.3. Then it can be shown that  $F$  is an isometry by the simply connectedness of  $\widetilde{M}$ . So we can identify  $(\widetilde{M}, \widetilde{g})$  with  $(N \times \mathbb{R}, h_N \otimes dt^2)$ , where  $h_N$  is the restriction of  $\widetilde{g}$  on  $N$ . Then the vector field  $\widetilde{X}$  equals  $\frac{\partial}{\partial t}$  and is invariant under the action by  $\pi_1(M)$ . We claim that  $\pi_1(M)$  acts trivially on the  $\mathbb{R}$ -factor.

Suppose not, then there is  $\alpha \in \pi_1(M)$  and  $(z_0, t_0) \in N \times \mathbb{R}$  such that  $\alpha(z_0, t_0) = (z_1, t_1)$  with  $t_0 \neq t_1$ . Then  $\alpha$  must map the line  $\{z_0\} \times \mathbb{R}$  isometrically onto the line  $\{z_1\} \times \mathbb{R}$  in the same direction and maps the slice  $N \times \{t_0\}$  onto the slice  $N \times \{t_1\}$ , because it preserves  $\frac{\partial}{\partial t}$ . Denote

by  $p : N \times \mathbb{R} \rightarrow \mathbb{R}$  the projection to the  $\mathbb{R}$ -factor and  $i : \mathbb{R} \rightarrow N \times \mathbb{R}$  the injection  $i(t) = (z_0, t)$ . Then  $\bar{\alpha} = p \circ \alpha \circ i$  is a translation on  $\mathbb{R}$  with variation  $t_1 - t_0 \neq 0$ . Now  $\{\bar{\alpha}^k = p \circ \alpha^k \circ i\}_{k=1}^\infty$  forms a subgroup of the isometry group of  $\mathbb{R}$ , which is generated by a translation. This shows that  $\alpha$  is a free element of  $\pi_1(M)$ , which contradicts the finiteness of  $\pi_1(M)$ . Hence  $\pi_1(M)$  acts trivially on the  $\mathbb{R}$ -factor and consequently the base manifold  $(M, g)$  splits off a line.  $\square$

It is natural to ask the following questions.

*Question 1.* — Construct a compact Riemannian manifold with negative Ricci curvature somewhere and with positive Bakry-Émery-Ricci curvature everywhere.

*Question 2.* — Let  $(M, g)$  be an open complete Riemannian manifold with  $Ric + Hess(\phi) \geq cg$  for some function  $\phi \in C^2(M)$  and some constant  $c \in \mathbb{R}$ . If  $(M, g)$  contains a line, does it really split off a line? In particular, is it true on a shrinking Ricci soliton?

#### 4. The Gromov precompactness theorem

By the Bishop-Gromov volume comparison theorem for the weighted volume measure and using the standard argument as used in Gromov's original proof [10, 11] for his famous theorem, we can extend the Gromov precompactness theorem to compact Riemannian manifolds with weighted measures via the finite dimensional Bakry-Émery Ricci curvature. More precisely, we have the following

**THEOREM 4.1.** — *Let  $\mathcal{M}(m, n, d, K)$  be the set of  $n$ -dimensional compact Riemannian manifolds  $(M, g)$  equipped with  $C^2$ -weighted volume measures  $d\mu = e^{-\phi} dv$  such that:  $n \leq \dim_{BE}(L) \leq m$ ,  $\text{diam}(M) \leq d$ , and  $Ric_{\dim_{BE}(L), n}(L) \geq K$ , where  $\dim_{BE}(L)$  is the Bakry-Émery dimension of the diffusion operator  $L = \Delta - \nabla\phi \cdot \nabla$ . Then  $\mathcal{M}(m, n, d, K)$  is precompact in the sense of the measured Gromov-Hausdorff convergence.*

To our knowledge, at least in the case  $\dim_{BE}(L) = m$  and  $\phi \in C^\infty(M)$ , Theorem 4.1 has been already pointed out by Lott [15, Remark 3, p. 881]. Indeed, if  $L = \Delta - \nabla\phi \cdot \nabla$  is a symmetric diffusion operator with  $Ric_{m, n}(L) \geq K$  for some  $m \geq n$  and  $K \in \mathbb{R}$ , then it is obviously true that  $Ric_{m', n}(L) \geq K$  for all  $m' \geq m$ . So, if  $\dim_{BE}(L) \leq m$  and if  $Ric_{\dim_{BE}(L), n}(L) \geq K$ , then obviously we have  $Ric_{m, n}(L) \geq K$ . Therefore, Theorem 4.1 can be recaptured from the above mentioned result due to Lott [15], which holds obviously when  $\phi \in C^2(M)$ .

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