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TRACE THEOREM ON THE HEISENBERG GROUP

by Hajer BAHOURI, Jean-Yves CHEMIN & Chao-Jiang XU

Abstract. — We prove in this work the trace and trace lifting theorem for Sobolev spaces on the Heisenberg groups for hypersurfaces with characteristics submanifolds.

Résumé. — Dans ce travail, nous démontrons des théorèmes de trace et de relèvement pour les espaces de Sobolev sur le groupe de Heisenberg pour des hypersurfaces dont l'ensemble caractéristique est une sous-variété.

1. Introduction

In this work, we continue the study of the problem of restriction of functions that belong to Sobolev spaces associated to left invariant vector fields for the Heisenberg group $\mathbb{H}^d$ initiated in [2]. As observed in [2], the case when $d=1$ is not very different from the case when $d\geqslant 2$, but the statement in this particular case are less pleasant. Thus, for the sake of simplicity, we shall assume that $d\geqslant 2$. Let us recall that the Heisenberg group is the space $\mathbb{R}^{2d+1}$ of the (non commutative) law of product

$$w\cdot w' = (x,y,s)\cdot (x',y',s') = (x+x', y+y', s+s'+(y|x')-(y'|x))\,.$$

The left invariant vector fields are

$$X_j = \partial_{x_j} + y_j\partial_s,\ Y_j = \partial_{y_j} - x_j\partial_s,\ j=1,\dots,d \text{ and } S=\partial_s = \frac{1}{2}[Y_j,X_j].$$

In all that follows, we shall denote by $\mathcal{Z}=(Z_\ell)_{1\leqslant \ell\leqslant 2d}$ defined by $Z_j=X_j$ and $Z_{j+d}=Y_j$ for j in $\{1,\dots,d\}$ and by $\widehat{\mathcal{Z}}$ the modulus of smooth vector fields generated by $\mathcal{Z}$. Moreover, for any C^1 function f, we shall state

$$\nabla_{\mathbb{H}} f \stackrel{\text{def}}{=} (Z_1\cdot f,\dots,Z_{2d}\cdot f).$$

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The key point is that $\mathcal{Z}$ satisfies Hörmander's condition at order 2, which means that the family $(Z_1, \dots, Z_{2d}, [Z_1, Z_{d+1}])$ spans the whole tangent space $T\mathbb{R}^{2d+1}$.

For $k \in \mathbb{N}$ and V an open subset of $\mathbb{H}^d$, we define the associated Sobolev space as following

$$H^k(\mathbb{H}^d, V) = \Big\{ f \in L^2(\mathbb{R}^{2d+1}) \mid \operatorname{Supp} f \subset V \\ \text{and } \forall \alpha \mid |\alpha| \leqslant k,\ Z^\alpha f \in L^2(\mathbb{R}^{2d+1}) \Big\},$$

where if $\alpha \in \{1, \dots, 2d\}^{k'}$, $|\alpha| \overset{\text{def}}{=} k'$ and $Z^\alpha \overset{\text{def}}{=} Z_{\alpha_1} \cdots Z_{\alpha_{k'}}$. As in the classical case, when s is any real number, we can define the function space $H^s(\mathbb{H}^d)$ through duality and complex interpolation, Littlewood-Paley theory on the Heisenberg group (see [4]), or Weyl-Hörmander calculus (see [7], [8] and [9]).

It turns out that these spaces have properties which look very much like the ones of usual Sobolev spaces, see [2] and their references.

The purpose of this paper is the study of the problems of trace and trace lifting on a smooth hypersurface of $\mathbb{H}^d$ in the frame of Sobolev spaces. Let us point out that the problem of existence of trace appears only when s is less than or equal to 1. Indeed, under the sub-ellipticity of system $\mathcal{Z}$, the space $H^s(\mathbb{H}^d)$ is included locally in $H^{\frac{s}{2}}(\mathbb{R}^{2d+1})$. So if s is strictly larger than 1, this implies that the trace on any smooth hypersurface exists and belongs locally to the usual Sobolev space $H^{\frac{s}{2}-\frac{1}{2}}$ of the hypersurface. Thus the case when $s = 1$ appears as the critical one. It is the case we study here.

1.1. Statement of the results

Two very different cases then appear: the one when the hypersurface is non characteristic, which means that any point w_0 of the hypersurface Σ is such that $\mathcal{Z}_{|w_0} \not\subset T_{w_0}\Sigma$, and the one when some point w_0 of the hypersurface Σ is characteristic, which means that $\mathcal{Z}_{|w_0} \subset T_{w_0}\Sigma$.

The non characteristic case is now well understood. In [2], we give a full account of trace and trace lifting results on smooth non characteristic hypersurfaces for $s > \frac{1}{2}$. This result generalize various previous results (see among others [6], [10] and [12]).

Let us recall this theorem in the case of H^1 (see [2] for the details). If w_0 is any non characteristic point of Σ, then there exists at last one of the vector fields $Z_1, \dots, Z_{2d}$ which is transverse to Σ at w_0. We define $\widehat{\mathcal{Z}}_\Sigma$ by

$\widehat{\mathcal{Z}}_{\Sigma|_w} = T_w\Sigma \cap \widehat{\mathcal{Z}}_{|w}$ (for w in Σ). It is easily checked that, if g is a local defining function of Σ, the family $\mathcal{R}_\Sigma$

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j\cdot g)Z_k - (Z_k\cdot g)Z_j$$

generates $\widehat{\mathcal{Z}}_\Sigma$ and that it satisfies the Hörmander condition at order 2 (see for instance Lemma 4.1 of [2]). We define, for an open subset W of Σ,

$$H^k(\mathcal{R}_\Sigma, W) = \left\{f \in L^2(W) \mid \forall (j,k),\ R_{j,k}u \in L^2(W)\right\}.$$

We have proved the following trace and trace lifting theorem in [2]:

Theorem 1.1. — *Let us suppose that Σ is non characteristic on an open subset V of $\mathbb{H}^d$, then the trace operator on Σ denoted by γ_{Σ} is an onto continuous map from $H^1(\mathbb{H}^d, V)$ onto*

$$\Big[H^1(\mathcal{R}_\Sigma, V\cap\Sigma), L^2(V\cap\Sigma)\Big]_{\frac{1}{2}} \stackrel{\text{def}}{=} H^{\frac{1}{2}}\big(\mathcal{R}_\Sigma, V\cap\Sigma\big).$$

Remark. — As the system $\mathcal{R}_\Sigma$ satisfies the Hörmander's condition at order 2, Theorem 1.1 implies in particular that γ_{Σ} maps $H^1(\mathbb{H}^d, V)$ into the classical Sobolev space $H^{\frac{1}{4}}(V\cap\Sigma)$.

We shall now consider the characteristic case. The set of characteristic points of Σ

$$\Sigma_c = \Big\{w \in \Sigma \mid \mathcal{Z}_{|w} \subset T_w\Sigma\Big\},$$

may have a complicated structure. Let us introduce the following definition.

Definition 1.1. — *A characteristic point w_0 of a hypersurface Σ is a regular point of order r if and only if*

i) *for any 1-form $\theta \in T^\star\mathbb{R}^{2d+1}$ that vanishes on $T\Sigma$ and such that $\theta(w_0) \neq 0$, the system $(\mathcal{L}_{Z_j}\theta_{|T_{w_0}\Sigma})_{1\leqslant j\leqslant 2d}$ is of rank r;*

ii) *near w_0, the characteristic set Σ_c is a submanifold of Σ of codimension r in Σ.*

Let us make some comments about this definition. A regular characteristic point of order $2d$ is exactly the familiar notion of non degenerate characteristic point. This notion of non degenerate characteristic point have been used in our preceding work [2] to study this problem of trace.

As we shall prove in forthcoming Proposition 2.2, if g is a local defining function of Σ, the condition i) means exactly that the matrix $(Z_i\cdot Z_j\cdot g)_{1\leqslant i,j\leqslant 2d}$ is of rank r at w_0. Let us notice that, because, if $i \in \{1,\dots,d\}$ and $j \neq i+d$,

$$(Z_i\cdot Z_{i+d}\cdot g)(w_0) - (Z_{i+d}\cdot Z_i\cdot g)(w_0) = -2\partial_s g(w_0) \neq 0$$

and

$$(Z_i\cdot Z_j\cdot g) = (Z_j\cdot Z_i\cdot g),$$

the rank of the matrix $(Z_i\cdot Z_j\cdot g)_{1\leqslant i,j\leqslant 2d}$ is at least d at w_0.

Let us give some examples. First let us consider the case when the hypersurface Σ is given by an equation of the type $s - P(x,y)$ where P is a homogeneous polynomial of degree 2 on $\mathbb{R}^{2d}$. Let us observe that this equation is homogeneous of order 2 with respect to the dilations of Heisenberg group $d_\lambda(x,y,s) \stackrel{\text{def}}{=} (\lambda x, \lambda y, \lambda^2 s)$. In this case $w_0 = (0,0,0)$ is always a regular characteristic point. Indeed the family $(Z_j\cdot g)_{1\leqslant j\leqslant 2d}$ is a family of linear forms on $\mathbb{R}^{2d}$. As $X_{j\,|w_0} = \partial_{x_j}$ and $Y_{j\,|w_0} = \partial_{y_j}$, the rank of the family is exactly the rank of the matrix $(Z_i\cdot Z_j\cdot g)_{1\leqslant i,j\leqslant 2d}$ at the point w_0. Thus Σ_c is obviously a linear submanifold of codimension r of Σ. This particular case is treated in [3].

Now let us exhibit an example of non regular characteristic point. In the case when $d=2$, let us define, for a in $\mathbb{R}$,

$$\Sigma_a = \Big\{(x_1,y_1,x_2,y_2,s)\in\mathbb{R}^5 \mid s = x_1y_1 + a(x_1^3+y_1^3)\Big\}.$$

If $a=0$, as observed above, the origin is a regular characteristic point. A very easy computation shows that the rank of the matrix $(Z_i\cdot Z_j\cdot g)_{1\leqslant i,j\leqslant 4}$ is three. But the characteristic set $\Sigma_{a,c}$ is the set of points of Σ_a such that

$$3ax_1^2 = -2x_1 + 3ay_1^2 = y_2 = x_2 = 0.$$

If $a\neq 0$, the characteristic set $\Sigma_{a,c}$ reduces to the origin.

Let us introduce some rings of functions adapted to our situation.

Definition 1.2. — *Let W be any open subset of Σ and F a closed subset of W. Let us denote by $C^\infty_F(W)$ the set of smooth functions a on $W\smallsetminus F$ such that for any multi-index α, a constant C_α exists such that*

$$\forall\alpha\in\mathbb{N}^d\ \ |\partial^\alpha a(z)| \leqslant C_\alpha d(z,F)^{-|\alpha|},$$

where d denotes the distance on Σ induced by the euclian distance on $\mathbb{R}^{2d+1}$.

Now let us define the vector fields on Σ which will describe the regularity on Σ.

Definition 1.3. — *Let w_0 a characteristic point of a hypersurface Σ. Let W be a neighbourhood of w_0. We denote by $\mathcal{Z}_\Sigma$ the $C^\infty_{\Sigma_c}(W)$ modulus spanned by the set vector fields of $\mathcal{Z}\cap T\Sigma_{|W}$ that vanish on Σ_c.*

As we shall see in Proposition 3.1, the modulus $\mathcal{Z}_\Sigma$ is of finite type (of course as a $C^\infty_{\Sigma_c}(W)$ modulus) if w_0 is a regular characteristic point and W

is chosen sufficiently small. If g is a local defining function of Σ, a generating system is given by

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j\cdot g)Z_k - (Z_k\cdot g)Z_j \quad \text{for} \quad 1 \leqslant j < k \leqslant 2d. \tag{1.1}$$

Now we are ready to introduce the space of traces.

Definition 1.4. — *Let w_0 a regular characteristic point of a hypersurface Σ. Let W be a sufficiently small neighbourhood of w_0. We denote by $H^1(\mathcal{Z}_\Sigma, W)$ the space of functions v of $L^2(W)$ such that*

$$\|v\|^2_{H^1(\mathcal{Z}_\Sigma,W)} \stackrel{\text{def}}{=} \|v\|^2_{L^2(W)} + \sum_{1\leqslant j,k\leqslant 2d} \|R_{j,k}v\|^2_{L^2(W)} < \infty,$$

where the family $(R_{j,k})_{1\leqslant j,k\leqslant 2d}$ is given by (1.1). If $s \in [0,1]$, we define $H^s(\mathcal{Z}_\Sigma, V)$ by complex interpolation.

Our theorem is the following.

Theorem 1.2. — *Let w_0 a regular characteristic point of a hypersurface Σ. Let V be a sufficiently small neighbourhood of w_0. Then the restriction map γ_Σ is an onto continuous map from $H^1(\mathbb{H}^d, V)$ onto $H^{\frac12}(\mathcal{Z}_\Sigma, V\cap\Sigma)$.*

Let us remark that, if w_0 is a non degenerate characteristic point (*i.e.* a regular characteristic point of order $2d$) this theorem is Theorem 1.8 of [2].

1.2. Structure of the proof

In our paper [2], we use a blow up of the point w_0 (which is Σ_c in the case when the characteristic point w_0 is of order $2d$). Here we shall blow up the submanifold Σ_c. In order to do it, let us introduce a function $\varphi \in \mathcal{D}(\mathbb{R}_+ \smallsetminus \{0\})$ such that

$$\forall t \in [-1,1] \smallsetminus \{0\},\ \sum_{p=0}^{\infty} \varphi(2^p t) = 1. \tag{1.2}$$

Let us define the function ρ_c by $\rho_c \stackrel{\text{def}}{=} \left(g^2 + |\nabla_{\mathbb{H}} g|^4\right)^{\frac14}$. Now writing that for any function u in $L^2(\rho_c \leqslant 1)$,

$$u = \sum_{p=0}^{\infty} \varphi_p u \quad \text{with} \quad \varphi_p(w) \stackrel{\text{def}}{=} \varphi(2^p \rho_c(w)), \tag{1.3}$$

we apply Theorem 1.1 of trace and trace lifting to each piece $\varphi_p u$ which is supported in a domain where Σ is non characteristic because $\rho_c \sim 2^{-p}$ in this domain. This decomposition leads immediately to the problem of estimating the norm $H^1(\mathbb{H}^d)$ of each piece $\varphi_p u$. Leibnitz formula and the chain rule tell us that

$$\nabla_{\mathbb{H}}(\varphi_p u) = \varphi_p \nabla_{\mathbb{H}} u + 2^p \varphi'(2^p \rho_c) u \nabla_{\mathbb{H}} \rho_c.$$

Let us observe that, as

$$Z_j \rho_c^4 = 2g Z_j \cdot g + 4|\nabla_{\mathbb{H}} g|^2 \sum_{k=1}^{2d} (Z_k \cdot g) Z_j \cdot (Z_k \cdot g),$$

we have, for any real number s, $|\nabla_{\mathbb{H}} \rho_c^s| \leqslant C_s \rho_c^{s-1}$. As the support of $\varphi'(2^p \rho_c)$ included in $\rho_c \sim 2^{-p}$, the supports of $\varphi'(2^p \rho_c)$ and $\varphi'(2^{p'} \rho_c)$ are disjoint if $|p - p'| \geqslant N_0$ for some N_0. Thus, we get that

$$\sum_{p=0}^{\infty} 2^{2p} \|\varphi'(2^p \rho_c) u \nabla_{\mathbb{H}} \rho_c\|_{L^2}^2 \leqslant C \Big\| \frac{u}{\rho_c} \Big\|_{L^2}^2.$$

This leads to the proof of the following Hardy type inequality.

Theorem 1.3. — *If w_0 is a regular characteristic point of Σ, a neighbourhood V of w_0 exists such that, for any u in the space $H^1(\mathbb{H}^d, V)$ of $H^1(\mathbb{H}^d)$ functions supported in V,*

$$\int_{\mathbb{H}^d} \frac{u^2}{\rho_c^2} dw \leqslant C \|\nabla_{\mathbb{H}} u\|_{L^2}^2 \text{ with } \rho_c = \left(g^2 + |\nabla_{\mathbb{H}} g|^4\right)^{\frac{1}{4}}.$$

This theorem implies that, for any u in $H^1(\mathbb{H}^d, V)$,

$$\sum_{p=0}^{\infty} \|\nabla_{\mathbb{H}}(\varphi_p u)\|_{L^2}^2 \leqslant C \|\nabla_{\mathbb{H}} u\|_{L^2}^2. \tag{1.4}$$

The proof of this theorem, which is the core of this work, is the purpose of the second section.

In the third section, we first straighten the submanifolds Σ and Σ_c, and after dilation, we apply Theorem 1.1. This gives a rather unpleasant description of the trace space. Then, we prove an interpolation result which allows to conclude the proof of Theorem 1.2.

2. A Hardy type inequality

2.1. The classical Hardy inequality

As a warm up, let us recall briefly the usual proof of the classical Hardy inequality [1]

(1) For a different approach based on Fourier analysis, see [1].

$$(2.1)\qquad \int_{\mathbb{H}^d} \frac{u^2}{\rho^2} dw \leqslant C\|\nabla_{\mathbb{H}} u\|_{L^2}^2 \quad \text{with} \quad \rho(w) = \big(s^2 + (|x|^2+|y|^2)^2\big)^{\frac14}\,.$$

As $\mathcal{D}(\mathbb{H}^d \smallsetminus \{0\})$ is dense in $H^1(\mathbb{H}^d)$, we restrict ourselves to functions u in $\mathcal{D}(\mathbb{H}^d \smallsetminus \{0\})$. Then the proof mainly consists in an integration by parts with respect to the radial vector field $R_{\mathbb{H}}$ adapted to the structure of $\mathbb{H}^d$, namely

$$R_{\mathbb{H}} \stackrel{\text{def}}{=} 2s\partial_s + \sum_{j=1}^d \big(x_j\partial_{x_j} + y_j\partial_{y_j}\big) = s[Y_1, X_1] + \sum_{j=1}^d (x_jX_j + y_jY_j)\,,$$

once noticed that $R_{\mathbb{H}}\cdot \rho^{-2} = -2\rho^{-2}$ and $\operatorname{div} R_{\mathbb{H}} = 2d+2$. More precisely, this gives

$$\begin{aligned} -d\int \frac{u^2}{\rho^2}dw = \int \sum_{j=1}^d \frac{u}{\rho}\Big(\frac{x_j}{\rho}X_j + \frac{y_j}{\rho}Y_j\Big)u\,dw \\ - \int \Big(Y_1\frac{s}{\rho^2}\Big)u(X_1u)dw + \int \Big(X_1\frac{s}{\rho^2}\Big)u(Y_1u)dw. \end{aligned}$$

As we have $\Big|Z_j\Big(\dfrac{s}{\rho^2}\Big)\Big| \leqslant C\rho^{-1}$, Cauchy-Schwarz inequality gives (2.1).

2.2. Reduction to the construction of substitute of ρ and $R_{\mathbb{H}}$

The proof of Theorem 1.3 is based on the following proposition that we admit for the time being.

Proposition 2.1. — *There is a couple of vector fields $(Z_0, \overline{Z}_0)$ in $(T\Sigma \cap \widehat{\mathcal{Z}}) \times \mathcal{Z}$ such that*

$$D(\overline{Z}_0\cdot g)(w_0) \neq 0 \text{ and } [Z_0, \overline{Z}_0] - 2\partial_s \in \widehat{\mathcal{Z}}.$$

Moreover, a smooth function β exists such that, if

$$R_1 \stackrel{\text{def}}{=} 2g\partial_s + \beta(\overline{Z}_0\cdot g)Z_0 \text{ and } \rho_0 \stackrel{\text{def}}{=} (g^2 + (\overline{Z}_0\cdot g)^4)^{\frac14},$$

then, we have

$$R_1\rho_0^4 = 4\rho_0^4 \text{ and } (\operatorname{div} R_1)(w_0) = 3.$$

With this proposition, we shall prove that a neighbourhood V of w_0 exists such that, for any u in $H^1(\mathbb{H}^d, V)$,

$$(2.2)\qquad \int \frac{u^2}{\rho_0^2} dw \leqslant C\|\nabla_{\mathbb{H}} u\|_{L^2}^2.$$

It is obvious that this inequality implies Theorem 1.3. Surprisingly, we are not able to prove directly the inequality of Theorem 1.3.

In order to prove the inequality (2.2), let us first reduce to the case of smooth functions compactly supported outside $\rho_0^{-1}(0)$. Indeed, Proposition 2.1 implies that, near w_0, the set $\rho_0^{-1}(0)$ is a submanifold of $\mathbb{H}^d$ of codimension 2. The following lemma will allow us to assume all along the proof that u belongs to $\mathcal{D}(V \setminus \rho_0^{-1}(0))$.

Lemma 2.1. — *Let V be a bounded domain of $\mathbb{H}^d$ and Γ a submanifold of codimension $\geqslant 2$. Then $\mathcal{D}(V \setminus \Gamma)$ is dense in the space $H_0^1(\mathbb{H}^d, V)$ of functions of $H_0^1(\mathbb{H}^d)$ supported in V equipped with the norm*

$$\left(\|u\|_{L^2}^2 + \|\nabla_{\mathbb{H}} u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Proof of Lemma 2.1. — As $H_0^1(\mathbb{H}^d, V)$ is a Hilbert space, it is enough to prove that the orthogonal of $\mathcal{D}(V \setminus \Gamma)$ is $\{0\}$. Let u be in this space. For any v in $\mathcal{D}(V \setminus \Gamma)$, we have

$$(u|v)_{L^2} + (\nabla_{\mathbb{H}} u|\nabla_{\mathbb{H}} v)_{L^2} = 0.$$

By integration by parts, this implies that

$$\forall v \in \mathcal{D}(V \setminus \Gamma),\ \langle u - \Delta_{\mathbb{H}} u, v\rangle = 0.$$

Thus the support of $u - \Delta_{\mathbb{H}} u$ is included in Γ. As $Z_j u$ belongs to L^2, then $Z_j^2 u$ belongs to $H^{-1}(\mathbb{R}^{2d+1})$ (the classical Sobolev space). And except 0, no distribution of $H^{-1}(\mathbb{R}^{2d+1})$ can be supported in a submanifold of codimension greater than 1. Thus $u - \Delta_{\mathbb{H}} u = 0$. Taking the L^2 scalar product with u implies that $u \equiv 0$. □

Thanks to Proposition 2.1, we have

$$\rho_0^{-2} = -\frac{1}{2} R_1 \cdot \rho_0^{-2}. \tag{2.3}$$

Thus by integration by parts, we have, using Proposition 2.1,

$$\int \frac{u^2}{\rho_0^2} dw = \frac{3}{2} \int \frac{u^2}{\rho_0^2} dw + \int \theta \frac{u^2}{\rho_0^2} dw + I$$
$$\text{with } I \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} (R_1 \cdot u) dw \text{ and } \theta \stackrel{\text{def}}{=} \operatorname{div} R_1 - 3.$$

As the function θ vanishes at the point w_0, we can assume that V is sufficiently small such that $\|\theta\|_{L^\infty(V)} \leqslant \frac{1}{4}$. This gives

$$\int \frac{u^2}{\rho_0^2} dw \leqslant 4|I|. \tag{2.4}$$

In order to estimate I, which contains terms of the type $g\partial_s u$, we have to compute the vector field R_1 in term of elements of $\mathcal{Z}$. Proposition 2.1 claims that there are two families $(\beta_k)_{1\leqslant k\leqslant 2d}$ and $(\gamma_k)_{1\leqslant k\leqslant 2d}$ of smooth functions such that

$$R_1 = 2g[Z_0, \overline{Z}_0] + \sum_{k=1}^{2d} \big(\beta_k g + \gamma_k(\overline{Z}_0\cdot g)\big)\, Z_k. \tag{2.5}$$

We deduce that

$$I = J_1 + J_2 \quad \text{with} \quad J_1 \stackrel{\text{def}}{=} \sum_{k=1}^{2d} \int \frac{u}{\rho_0} \frac{\beta_k g + \gamma_k(\overline{Z}_0\cdot g)}{\rho_0} (Z_k\cdot u) dw$$
$$\text{and} \quad J_2 \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g[Z_0, \overline{Z}_0]\cdot u dw.$$

As V is supposed bounded, we have that the functions

$$\frac{\beta_k g + \gamma_k(\overline{Z}_0\cdot g)}{\rho_0}$$

are bounded on V. Cauchy-Schwarz inequality yields

$$|J_1| \leqslant C\Big\|\frac{u}{\rho_0}\Big\|_{L^2} \|\nabla_{\mathbb{H}} u\|_{L^2}. \tag{2.6}$$

The estimate about J_2 is a little bit more difficult to obtain. Let us write that $J_2 = K_1 - K_2$ with

$$K_1 \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g Z_0\cdot (\overline{Z}_0\cdot u) dw \quad \text{and} \quad K_2 \stackrel{\text{def}}{=} \int \frac{u}{\rho_0^2} g \overline{Z}_0\cdot (Z_0\cdot u) dw.$$

By integration by parts, we have $K_1 = -K_{11} - K_{12}$ with

$$K_{11} \stackrel{\text{def}}{=} \int \frac{g}{\rho_0^2} (Z_0\cdot u)(\overline{Z}_0\cdot u) dw \quad \text{and}$$
$$K_{12} \stackrel{\text{def}}{=} \int f \frac{u}{\rho_0} (\overline{Z}_0\cdot u) dw \text{ with } f \stackrel{\text{def}}{=} (\operatorname{div} Z_0)\frac{g}{\rho_0} + \rho_0\Big(Z_0\cdot \frac{g}{\rho_0^2}\Big)\cdot$$

By definition of ρ_0, it is obvious that

$$|K_{11}| \leqslant C\|\nabla_{\mathbb{H}} u\|_{L^2}^2. \tag{2.7}$$

As we can assume that V is included in $\rho_0^{-1}([0,1])$, we have that $\rho_0^{-1} g|\operatorname{div} Z_0| \leqslant C$ on V. Moreover using that $Z_0\cdot g = 0$, we get

$$\Big|Z_0\cdot \frac{g}{\rho_0^2}\Big| = \frac{2g}{\rho_0^6}\big|Z_0\cdot (\overline{Z}_0\cdot g)\big|\,|\overline{Z}_0\cdot g|^3 \leqslant C\frac{g}{\rho_0^3} \leqslant \frac{C}{\rho_0}.$$

This ensures that f is bounded on V and thus by Cauchy-Schwarz inequality,

$$\|K_{12}\| \leqslant C\Big\|\frac{u}{\rho_0}\Big\|_{L^2} \|\nabla_{\mathbb{H}} u\|_{L^2}.$$

Together with (2.7), this proves that

$$|K_1| \leqslant C\Big(\Big\|\frac{u}{\rho_0}\Big\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2}\Big)\|\nabla_{\mathbb{H}} u\|_{L^2}. \tag{2.8}$$

In order to estimate K_2, let us write that, by integration by parts,

$$K_2 = -\int \frac{g}{\rho_0^2}(\overline{Z}_0\cdot u)(Z_0\cdot u)dw - \int \rho_0\Big(\overline{Z}_0\cdot \frac{g}{\rho_0^2}\Big)\frac{u}{\rho_0}(Z_0\cdot u)dw.$$

Using that

$$\overline{Z}_0\cdot \rho_0^4 = 2g(\overline{Z}_0\cdot g) + 4\left(\overline{Z}_0\cdot(\overline{Z}_0\cdot g)\right)(\overline{Z}_0\cdot g)^3,$$

we immediately get that the function $\rho_0\Big(\overline{Z}_0\cdot \dfrac{g}{\rho_0^2}\Big)$ is bounded on V and we deduce that

$$|K_2| \leqslant C\Big(\Big\|\frac{u}{\rho_0}\Big\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2}\Big)\|\nabla_{\mathbb{H}} u\|_{L^2}.$$

Together with (2.4), (2.6) and (2.8), we infer that

$$\Big\|\frac{u}{\rho_0}\Big\|_{L^2}^2 \leqslant C\Big(\Big\|\frac{u}{\rho_0}\Big\|_{L^2} + \|\nabla_{\mathbb{H}} u\|_{L^2}\Big)\|\nabla_{\mathbb{H}} u\|_{L^2}$$

which concludes the proof of the inequality 2.2 (and thus Theorem 1.3) provided of course we prove Proposition 2.1

2.3. Construction of substitute of ρ and $R_{\mathbb{H}}$

Let us start with some remarks about the relations between Σ_c and the vector fields Z_j in the case when w_0 is a regular characteristic point.

Proposition 2.2. — *The condition i) of Definition 1.1 is equivalent to the fact that, for any defining function g of Σ, the rank of the matrix $(Z_i\cdot Z_j\cdot g(w_0))_{1\leqslant i,j\leqslant 2d}$ is r.*

Proof of Proposition 2.2. — Let g be a local defining function of Σ. Of course, Dg vanishes on $T\Sigma$. As $Z_j(w_0)$ belongs to $T_{w_0}\Sigma$, we have $\mathcal{L}_{Z_j}(Dg)(w_0) = D(Z_j\cdot g)(w_0)$. By definition of $\mathcal{Z}$, we infer that

$$D(Z_j\cdot g)(w_0) = \sum_{i=1}^{2d}(Z_i\cdot Z_j\cdot g)(w_0)dz_i.$$

Thus the rank of matrix $(Z_i\cdot Z_j\cdot g)_{1\leqslant i,j\leqslant 2d}$ is the rank of $\mathcal{L}_{Z_j}(Dg)(w_0)$.

Conversely, let θ be a 1-form that vanishes on $T\Sigma$ and such that $\theta(w_0)\neq 0$ and g a local defining function of Σ. A function a that does not vanish at w_0 exists such that $\theta = aDg$. Thanks to Leibnitz formula, $\mathcal{L}_{Z_j}(\theta)(w_0)_{|T_{w_0}\Sigma} =$

$a(w_0)D(Z_j\cdot g)(w_0)_{|T_{w_0}\Sigma}$. The fact that the function a does not vanish at point w_0 implies the proposition. $\square$

In all that follows, g will denote a defining function of Σ of the form $g(x,y,s) = s+f(x,y)$ (this is allowed by the implicit function theorem) near w_0, assumed to be the origin of $\mathbb{H}^d$ which is assumed to be a characteristic regular point of order $r < 2d$.

As the matrix $(Z_i\cdot Z_j\cdot g)_{1\leqslant i,j\leqslant 2d}$ is of rank r in w_0, and as $Z_{i|w_0} = \partial_{z_i}$, a family $(j_\ell)_{1\leqslant \ell\leqslant r}$ exists in $\{1,\dots,2d\}^r$ such that the linear forms $(D(Z_{j_\ell}\cdot g))_{1\leqslant \ell\leqslant r}$ are linearly independent near w_0. Moreover, the function $Z_i\cdot g$ are independent of s and $Dg(w_0) = (ds,0,0)$. Thus the family of functions

$$(g,\ (Z_{j_1}\cdot g),\dots,(Z_{j_r}\cdot g)) \tag{2.9}$$

is a family of $r+1$ independent functions. They vanish on the submanifold Σ_c which is by hypothesis a submanifold of $\mathbb{H}^d$ of codimension $r+1$. This implies that, near w_0,

$$\Sigma_c = \big\{w \mid g(w) = (Z_{j_1}\cdot g)(w) = \cdots = (Z_{j_r}\cdot g)(w) = 0\big\}. \tag{2.10}$$

We shall keep these notations all along this text.

The definition of substitute to ρ and $R_{\mathbb{H}}$ relies on the following two lemmas.

Lemma 2.2. — *A couple of vector fields $(\widetilde{Z}_0,\overline{Z}_0)$ exists in $(\mathcal{Z}\setminus\{Z_{j_1},\dots,Z_{j_r}\})\times(\pm\mathcal{Z})$ such that*

$$[\widetilde{Z}_0,\overline{Z}_0] = 2\partial_s \text{ and } D(\overline{Z}_0\cdot g)(w_0)\neq 0.$$

Proof of Lemma 2.2. — Let us consider $\widetilde{Z}_0\in\mathcal{Z}\setminus\{Z_{j_1},\dots,Z_{j_r}\}$ and $\underline{Z}_0$ in $\pm\mathcal{Z}$ such that $[\widetilde{Z}_0,\underline{Z}_0] = 2\partial_s$. If $\pm\underline{Z}_0$ belongs to $\{Z_{j_1},\dots,Z_{j_r}\}$, then (2.9) implies that $D(\underline{Z}_0\cdot g)(w_0)$ is different from 0 and then $\overline{Z}_0 = \underline{Z}_0$ fits. If $\pm\underline{Z}_0$ is not in $\{Z_{j_1},\dots,Z_{j_r}\}$, as

$$\big(\widetilde{Z}_0\cdot(\underline{Z}_0\cdot g)\big)(w_0) - \big(\underline{Z}_0\cdot(\widetilde{Z}_0\cdot g)\big)(w_0) = 2,$$

either $D(\widetilde{Z}_0\cdot g)(w_0)$ or $D(\underline{Z}_0\cdot g)(w_0)$ is different from 0. Thus if

$$D(\underline{Z}_0\cdot g)(w_0) = 0,$$

we get the lemma interchanging the role of $\underline{Z}_0$ and $\widetilde{Z}_0$.

For the reader's convenience, we recall the proof given in [3] in the case when $g = s - P(x,y)$ where P is a homogeneous polynomial of degree 2. In this case, the functions $Z_j\cdot g$ are linear forms on $\mathbb{R}^{2d}$. Thus, by hypothesis, there is a family of real numbers $(\alpha_\ell)_{1\leqslant\ell\leqslant r}$ such that

$$\widetilde{Z}_0\cdot g=\sum_{\ell=1}^{r}\alpha_\ell Z_{j_\ell}\cdot g.$$

Now let us define

$$Z_0\stackrel{\text{def}}{=}\widetilde{Z}_0-\sum_{\ell=1}^{r}\alpha_\ell Z_{j_\ell},\quad R_1\stackrel{\text{def}}{=}2g\partial_s+\frac{1}{2}(\overline{Z}_0\cdot g)Z_0 \quad\text{and}\quad \rho_0\stackrel{\text{def}}{=}\left(g^2+(\overline{Z}_0\cdot g)^4\right)^{\frac14}. \tag{2.11}$$

As $\overline{Z}_0$ and the Z_{j_ℓ} commute, we have

$$[Z_0,\overline{Z}_0]=2\partial_s,\ Z_0\cdot g=0 \text{ and } Z_0\cdot\overline{Z}_0\cdot g=2.$$

This implies that

$$R_1\cdot g=2,\ R_1\cdot\overline{Z}_0\cdot g=1 \text{ and } \operatorname{div}R_1=2\partial_s g+\frac{1}{2}Z_0\cdot\overline{Z}_0\cdot g=3.$$

We immediately infer that $R_1\cdot\rho_0^4=4\rho_0^4$. Proposition 2.1 is proved in that case.

Let us treat now the general case. Noticing that the functions $Z_j\cdot g$ do not depend on s, the set of the zeros of the functions $Z_{j_\ell\cdot g}$ is a submanifold of $\mathbb{R}^{2d}$, the standard division theorem implies the existence (on a sufficiently small neighbourhood V of w_0) of a family $(\alpha_\ell)_{1\leqslant\ell\leqslant r}$ of functions of $C^\infty(V)$ such that

$$\widetilde{Z}_0\cdot g=\sum_{\ell=1}^{r}\alpha_\ell(Z_{j_\ell}\cdot g).$$

As in (2.11), let us define

$$Z_0\stackrel{\text{def}}{=}\widetilde{Z}_0-\sum_{\ell=1}^{r}\alpha_\ell Z_{j_\ell},\quad R_1\stackrel{\text{def}}{=}2g\partial_s+\beta(\overline{Z}_0\cdot g)Z_0$$

and

$$\rho_0\stackrel{\text{def}}{=}\left(g^2+(\overline{Z}_0\cdot g)^4\right)^{\frac14}$$

where β is a function determined later on. By definition of the function ρ_0, we have

$$R_1\cdot\rho_0^4=2g(R_1\cdot g)+4(\overline{Z}_0\cdot g)^3\left(R_1\cdot(\overline{Z}_0\cdot g)\right).$$

By definition of Z_0, it is tangent to Σ. Using that $\partial_s g\equiv1$, this implies that $R_1\cdot g=2g$. Let us compute $R_1\cdot(\overline{Z}_0\cdot g)$. As $\partial_s(\overline{Z}_0\cdot g)=0$, we have

$$R_1\cdot(\overline{Z}_0\cdot g)=\beta(\overline{Z}_0\cdot g)\left(Z_0\cdot(\overline{Z}_0\cdot g)\right).$$

Let us notice that Z_0 does not belong to the family $(Z_{j_\ell})_{1\leqslant\ell\leqslant r}$. Thus $\overline{Z}_0$ commutes with the vector fields Z_{j_ℓ}. By definition of Z_0, we infer

$$\begin{aligned}[Z_0, \overline{Z}_0] &= [Z_0, \overline{Z}_0] - \sum_{\ell=1}^{r} [\alpha_\ell Z_{j_\ell}, \overline{Z}_0] \\ &= 2\partial_s + \sum_{\ell=1}^{r} (\overline{Z}_0\cdot \alpha_\ell) Z_{j_\ell}.\end{aligned} \tag{2.12}$$

By definition of Z_0, we have $Z_0\cdot g = 0$. Thus we get

$$\begin{aligned}Z_0\cdot (\overline{Z}_0\cdot g) &= \overline{Z}_0\cdot (Z_0\cdot g) + 2\partial_s g + \sum_{\ell=1}^{r} (\overline{Z}_0\cdot \alpha_\ell)(Z_{j_\ell}\cdot g) \\ &= 2 + \widetilde{\theta} \quad \text{with} \quad \widetilde{\theta} \stackrel{\text{def}}{=} \sum_{\ell=1}^{r} (\overline{Z}_0\cdot \alpha_\ell)(Z_{j_\ell}\cdot g).\end{aligned} \tag{2.13}$$

It turns out that $R_1\cdot \rho_1^4 = 4g^2 + 4(\overline{Z}_0\cdot g)^4\beta(2+\widetilde{\theta})$. Choosing $\beta \stackrel{\text{def}}{=} (2+\widetilde{\theta})^{-1}$ ensures that $R_1\cdot \rho_0^4 = \rho_0^4$. Now, let us compute $\operatorname{div} R_1$. We have

$$\operatorname{div} R_1 = 2\partial_s g + \beta Z_0\cdot (\overline{Z}_0\cdot g) + (\overline{Z}_0\cdot g)(\operatorname{div} Z_0 + Z_0\cdot \beta).$$

Using that $\partial_s g \equiv 1$ and (2.13), we get

$$\begin{aligned}\operatorname{div} R_1 &= 2 + \beta(2+\widetilde{\theta}) + (\overline{Z}_0\cdot g)(\operatorname{div} Z_0 + Z_0\cdot \beta) \\ &= 3 + (\overline{Z}_0\cdot g)(\operatorname{div} Z_0 + Z_0\cdot \beta).\end{aligned}$$

This proves the lemma with $\theta \stackrel{\text{def}}{=} (\overline{Z}_0\cdot g)(\operatorname{div} Z_0 + Z_0\cdot \beta)$. □

3. The proof of the trace and trace lifting theorem

3.1. Some preliminary properties

PROPOSITION 3.1. — *A neighbourhood W of w_0 exists such that the $C_{\Sigma_c}(W)$ modulus $\mathcal{Z}_\Sigma$ spanned by the vector fields of $\mathcal{Z}\cap T\Sigma_{|W}$ which vanish on the characteristic submanifold Σ_c is of finite type and generated by*

$$R_{j,k} \stackrel{\text{def}}{=} (Z_j\cdot g)Z_k - (Z_k\cdot g)Z_j.$$

Proof of Proposition 3.1. — It is enough to prove that any element L of $\mathcal{Z}\cap T\Sigma$ which vanish on Σ_c is a combinaison (with coefficients in $C^\infty_{\Sigma_c}(W)$) of the $R_{j,k}$. By definition

$$L = \sum_{j=1}^{2d} \alpha_j Z_j \text{ with } \alpha_{j|\Sigma_c} = 0 \text{ and } \sum_{j=1}^{2d} \alpha_j (Z_j\cdot g) = 0.$$

Let us introduce a partition of unity $(\widetilde{\psi}_j)_{1\leqslant j\leqslant 2d}$ of the sphere $\mathbb{S}^{2d-1}$ such that the support of $\widetilde{\psi}_j$ is included in the set of ζ of $\mathbb{S}^{2d-1}$ such that $|\zeta_j|\geqslant (4d)^{-1}$. Let us state

$$\psi_j \stackrel{\text{def}}{=} \widetilde{\psi}_j\Big(\frac{\nabla_{\mathbb{H}} g}{|\nabla_{\mathbb{H}} g|}\Big).$$

It is an exercise left to the reader to check that ψ_j belongs to $C^\infty_{\Sigma_c}(W)$. On $\Sigma\setminus\Sigma_c$, we have, for any j in $\{1,\ldots,2d\}$,

$$\psi_j(L\cdot g)=\sum_{k=1}^{2d}\psi_j\alpha_k(Z_k\cdot g)=0.$$

By definition of ψ_j, $(Z_j\cdot g)$ does not vanish on the support of ψ_j. Thus we have

$$\alpha_j\psi_j=-\frac{1}{(Z_j\cdot g)}\sum_{k\neq j}\psi_j\alpha_k(Z_k\cdot g).$$

From this, we deduce that

$$\begin{aligned}\psi_j L&=\sum_{k\neq j}\psi_j\alpha_k\Big(Z_k-\frac{(Z_k\cdot g)}{(Z_j\cdot g)}Z_j\Big)\\&=\sum_{k\neq j}\frac{\psi_j\alpha_k}{(Z_j\cdot g)}\big((Z_j\cdot g)Z_k-(Z_k\cdot g)Z_j\big).\end{aligned}$$

Now the facts that $\alpha_k\in C^\infty_{\Sigma_c}$ and that $(Z_j\cdot g)$ does not vanish on the support of ψ_j ensure that

$$\alpha_{j,k}\stackrel{\text{def}}{=}\frac{\psi_j\alpha_k}{(Z_j\cdot g)}\in C^\infty_{\Sigma_c}.$$

So we have

$$L=\sum_{1\leqslant j<k\leqslant 2d}\alpha_{j,k}\big((Z_j\cdot g)Z_k-(Z_k\cdot g)Z_j\big)$$

and the proposition is proved. □

The blow up procedure requires to straighten the submanifolds Σ and Σ_c.

Lemma 3.1. — *A neighbourhood V of w_0 and a diffeomorphism χ from V onto $\chi(V)$ exist which satisfy the following properties.*

- *It straighten the submanifolds Σ and Σ_c, namely*

$$\chi(\Sigma\cap V)=\{w\mid s=0\}\cap\chi(V)$$

and

$$\chi(\Sigma_c\cap V)=\{w\mid s=z_1=\cdots z_r=0\}\cap\chi(V).$$

- *The transported vector fields are of the form*

$$\chi^\star(\partial_s) = \partial_s$$

and

$$Z_j^D \stackrel{\text{def}}{=} \chi^\star(Z_j) = \frac{\partial}{\partial e_j} + \Big(\sum_{\ell=1}^{r} \alpha_j^\ell(z) z_\ell\Big)\partial_s + h_j(z,\partial_z)$$

where $(e_j)_{1\leqslant j\leqslant 2d}$ is a basis of $\mathbb{R}^{2d}$, the (α_k^ℓ) are smooth bounded functions on V such that, for $j\in\{1,\dots,r\}$, $\alpha_j^\ell\equiv\delta_j^\ell$ and $(h_j)_{1\leqslant j\leqslant 2d}$ is a family of smooth vector fields which vanish at $z=0$.

Proof of Lemma 3.1. — Let $(L_k)_{r+1\leqslant k\leqslant 2d}$ be a family of linear forms on $\mathbb{R}^{2d}$ such that

$$\big(D(Z_{j_\ell}\cdot g)(w_0)\big)_{1\leqslant \ell\leqslant r}\,,\,(L_k)_{r+1\leqslant k\leqslant 2d}$$

is a basis of the dual space of $\mathbb{R}^{2d}$. It is easily checked that the map defined by

$$\chi(x,y,s) = \begin{pmatrix} g(x,y,s) & = s + f(x,y) & & \\ z_k & = (Z_{j_\ell}\cdot g)(x,y) & \text{if} & k\leqslant r \\ z_k & = \langle L_k,(x,y)\rangle & \text{if} & k > r \end{pmatrix}$$

is a local diffeomorphism which satisfies the property of the lemma. □

From now on, we shall work only in the straighten situation and to avoid excessive heaviness of notations, we shall still denote Z_j^D by Z_j.

3.2. The blow up procedure

Let us write that, for any function u, we can write (at least in L^2) that

$$u = \sum_{p=0}^{\infty} \varphi_p u \text{ with } \varphi_p(z,s) \stackrel{\text{def}}{=} \varphi\Big(2^p(s^2+|z'|^4)^{\frac14}\Big)$$

where φ is the function introduced in (1.2) and if $z=(z_j)_{1\leqslant j\leqslant 2d}$, $z' \stackrel{\text{def}}{=} (z_1,\dots,z_r,0,\dots,0)$. Let us define the space of trace in this straightened situation.

Definition 3.1. — *Let us denote by $[A,B]_\theta$ the complex interpolation between A and B and by $H^1(\mathcal{R},W)$ the space of functions of $H^1(\mathcal{R})$ supported in W. The space $T^{\frac12}$ is the space of functions $v\in L^2(|z'|\leqslant 1)$ such* *that*

$$\|v\|^2_{T^{\frac12}} \overset{def}{=} \sum_{p=0}^{\infty} \|\varphi_p^{\Sigma} v\|^2_{H^{\frac12}(\mathcal{R}_p)} < \infty$$

$$\text{with} \quad H^s(\mathcal{R}_p) \overset{def}{=} \left[L^2(2^{-p}\mathcal{C}_\Sigma), H^1(\mathcal{R}, 2^{-p}\mathcal{C}_\Sigma)\right]_s$$

where $\mathcal{C}_\Sigma \overset{def}{=} \{c \leqslant |z'| \leqslant C\}$ *and* $\varphi_p^{\Sigma}(z) \overset{def}{=} \varphi_p(z,0) = \varphi(2^p|z'|)$.

We shall prove the following theorem.

Theorem 3.1. — *The restriction map to the hypersurface* $(s=0)$ *can be extended in a continuous onto map from* $H^1(\mathcal{Z};\{\rho_c \leqslant 1\})$ *onto* $T^{\frac12}$*.*

Proof of Theorem 3.1. — Once noticed that the Hardy inequality given by Theorem 1.3 becomes

$$\int \frac{u^2(z,s)}{(s^2+|z'|^4)^{\frac12}} dzds \leqslant C \sum_{j=1}^{2d} \|Z_j u\|^2_{L^2}, \tag{3.1}$$

we get, by computations very similar to the ones done at the beginning of Subsection 1.2, an analogous of (1.4), namely

$$\sum_{p=0}^{\infty}\sum_{j=1}^{2d} \|Z_j(\varphi_p u)\|^2_{L^2} \leqslant C \sum_{j=1}^{2d} \|Z_j u\|^2_{L^2}. \tag{3.2}$$

Let us notice that outside $\Sigma_c = \{(z,s) \mid s=0,\ z'=0\}$, thus in particular in the support of φ_p, the hypersurface Σ is non characteristic for $\mathcal{Z}$. Thus locally we can apply Theorem 1.1 to each piece $\varphi_p u$. The key point is the control of the constant when p tends to ∞. In order to do so, it is convenient to use the quasi-homogeneous dilations $\delta_p(z,s) \overset{\text{def}}{=} (2^p z, 2^{2p} s)$. Let us define

$$u_p(z,s) \overset{\text{def}}{=} \varphi_0(z,s) u(2^p z, 2^{2p} s)$$

and

$$Z_{j,p} \overset{\text{def}}{=} \frac{\partial}{\partial e_j} + \sum_{\ell=1}^{r} \alpha_j^{\ell}(2^{-p}z) z_\ell \partial_s + h_j(2^{-p}z, \partial_z).$$

It is obvious that a one to one map σ of $\{1,\dots,2d\}$ exists such that

$$[Z_{j,p}, Z_{k,p}] = 2\delta_{k,\sigma(j)}\partial_s. \tag{3.3}$$

Moreover, as $\|u_p\|^2_{L^2} = 2^{2p(d+1)}\|\varphi_p u\|_{L^2}$, we have, thanks to Hardy's inequality (3.1),

$$\sum_{p=0}^{\infty} 2^{-2pd}\|u_p\|^2_{L^2} \leqslant C \sum_{j=1}^{2d} \|Z_j u\|^2_{L^2}.$$

Applying (3.2), we infer

$$\sum_{p=0}^{\infty} 2^{-2pd}\Big(\|u_p\|^2_{L^2} + \sum_{j=1}^{2d} \|Z_{j,p}u_p\|^2_{L^2}\Big) \leqslant C\sum_{j=1}^{2d} \|Z_j u\|^2_{L^2}. \tag{3.4}$$

On the support of φ_0, the hypersurface $(s=0)$ is non characteristic with respect to the family $(Z_{j,p})_{1\leqslant j\leqslant 2d}$ because, for j between 1 and r,

$$Z_{j,p} = \frac{\partial}{\partial e_j} + h_j(2^{-p}z, \partial_z) + z_j\partial_s.$$

Let us notice that the transverse component of $Z_{j,p}$ does not depend on p. Thus we can apply Theorem 4.6 of [2] together with a result of interpolation between Sobolev spaces (see Remark 4.2 page 89 in [7]) to each u_p. Using that $\|u_p\|_{H^1(\mathcal{Z}_p)} = 2^{2pd}\|\varphi_p u\|_{H^1(\mathcal{Z})}$, this gives in particular that a constant C exists (independent of p) such that

$$\|\gamma(u_p)\|_{[L^2(\mathbb{R}^{2d}), H^1(\widetilde{\mathcal{R}}_p, \mathbb{R}^{2d})]_{\frac{1}{2}}} \leqslant C2^{2pd}\|\varphi_p u\|_{H^1(\mathcal{Z})} \tag{3.5}$$

with $\widetilde{\mathcal{R}}_p$ is the union of

$$\mathcal{R}_p = \big(\widetilde{\varphi}^\Sigma_0(Z_{j,p}\cdot s)Z_{k,p} - \varphi^\Sigma_0(Z_{k,p}\cdot s)Z_{j,p}\big)_{1\leqslant j,k\leqslant 2d} \text{ and } \big((1-\widetilde{\varphi}^\Sigma_0)\partial_j\big)_{1\leqslant j\leqslant 2d}$$

where $\widetilde{\varphi}^\Sigma_0$ is a smooth function supported in $\mathcal{C}_\Sigma$ such that $\widetilde{\varphi}^\Sigma_0 \equiv 1$ near the support of φ^Σ_0.

At this point, let us recall the definition of complex interpolation. For details of this theory, we refer in particular to [5] and [11].

Definition 3.2. — *Let $(\mathcal{H}_j, \|\cdot\|_j)_{j\in\{0,1\}}$ be two Hilbert spaces such that $\mathcal{H}_1$ is densely included in $\mathcal{H}_0$. Let $\mathcal{F}(\mathcal{H}_0,\mathcal{H}_1)$ be the space of holomorphic functions f from the strip $0 < \Re e\,\zeta < 1$ into $\mathcal{H}_0$ such that $f(j+it)$ is continuous and vanishes at infinity in $\mathcal{H}_j$. Then, for $\theta \in]0,1[$, the space $[\mathcal{H}_0,\mathcal{H}_1]_\theta$ is*

$$[\mathcal{H}_0,\mathcal{H}_1]_\theta \stackrel{\text{def}}{=} \Big\{v \in L^2 \mid \exists f \in \mathcal{F}(\mathcal{H}_0,\mathcal{H}_1) \mid f(\theta) = v\Big\}$$

equipped with the norm

$$\|v\|_{[\mathcal{H}_0,\mathcal{H}_1]_\theta} \stackrel{\text{def}}{=} \inf_{f\in\mathcal{F}(\mathcal{H}_0,\mathcal{H}_1)} \max_{j\in\{0,1\}} \sup_{t\in\mathbb{R}} \|f(j+it)\|_{\mathcal{H}_j}.$$

As the support of $\gamma(u_p)$ is included in the support of φ^Σ_0, let us consider a smooth function φ^Σ_1 supported in the set where φ^Σ_0 has value 1 and such that φ^Σ_1 has value 1 near the support of φ^Σ_0. If f is a function in $\mathcal{F}(L^2(\mathbb{R}^{2d}), H^1(\widetilde{\mathcal{R}}_p, \mathbb{R}^{2d}))$ such that $f(1/2) = v$, then the function

$\zeta \mapsto \varphi_1 f(\zeta)$ belongs to $\mathcal{F}(L^2(\mathcal{C}_\Sigma), H^1(\widetilde{\mathcal{R}}_p, \mathcal{C}_\Sigma))$ and $\varphi_1 f(1/2) = v$. As we obviously have that $H^1(\widetilde{\mathcal{R}}_p, \mathcal{C}_\Sigma) = H^1(\mathcal{R}_p, \mathcal{C}_\Sigma)$, inequality (3.5) becomes

$$\|\gamma(u_p)\|_{[L^2(\mathcal{C}_\Sigma), H^1(\mathcal{R}_p, \mathcal{C}_\Sigma)]_{\frac{1}{2}}} \leqslant C 2^{2pd} \|\varphi_p u\|_{H^1(\mathcal{Z})}. \tag{3.6}$$

Moreover, dilations on $\mathbb{R}^{2d}$ of ratio 2^{-p} maps $L^2(\mathcal{C}_\Sigma)$ (resp. $H^1(\mathcal{R}_p, \mathcal{C}_\Sigma)$) into $L^2(2^{-p}\mathcal{C}_\Sigma)$ (resp. $H^1(\mathcal{R}, 2^{-p}\mathcal{C}_\Sigma)$) with norm equal to 2^{-pd}. Thus by the functorial property of complex interpolation, inequality (3.6) becomes

$$\|\gamma(\varphi_p u)\|_{H^{\frac{1}{2}}(\mathcal{R}_p)} \leqslant C \|\varphi_p u\|_{H^1(\mathcal{Z})}. \tag{3.7}$$

Inequality (3.2) implies that γ can be extended to a continuous linear map from $H^1(\mathcal{Z})$ into $T^{\frac{1}{2}}$.

In order to prove that γ is onto, let us consider $v \in T^{\frac{1}{2}}$. By definition of $T^{\frac{1}{2}}$, and after dilation, we infer that

$$\|\varphi_0^\Sigma v(2^{-p}\cdot)\|_{[L^2(\mathcal{C}_\Sigma), H^1(\mathcal{R}_p, \mathcal{C}_\Sigma)]_{\frac{1}{2}}} \leqslant C 2^{pd} c_p \|v\|_{T^{\frac{1}{2}}} \text{ with } \sum_p c_p^2 = 1. \tag{3.8}$$

As $L^2(\mathcal{C}_\Sigma)$ (resp. $H^1(\mathcal{R}_p, \mathcal{C}_\Sigma)$) is a subspace of $L^2(\mathbb{R}^{2d})$ (resp. $H^1(\widetilde{\mathcal{R}}_p; \mathbb{R}^{2d})$), using Theorem 4.6 of [2] together with Remarque 4.2 of [7] and (3.8), we claim the existence of a function $\widetilde{u}_p$ in the space $H^1(\widetilde{Z}_p)$ such that a constant C (independent of p) exists which satisfies, for any p,

$$\|\widetilde{u}_p\|_{H^1(\widetilde{\mathcal{Z}}_p)} \leqslant C 2^{pd} c_p \|v\|_{T^{\frac{1}{2}}} \text{ with } \sum_p c_p^2 = 1 \tag{3.9}$$

where $\widetilde{\mathcal{Z}}_p$ is the union of the families

$$(\widetilde{\varphi}_0 Z_{j,p})_{1 \leqslant j,k \leqslant 2d}, \ ((1-\widetilde{\varphi}_0)\partial_j)_{1 \leqslant j \leqslant 2d} \text{ and } (1-\widetilde{\varphi}_0)\partial_s.$$

Let us consider a smooth function φ_1 supported in the domain where $\widetilde{\varphi}$ has value 1 and such that $\varphi_1 \equiv 1$ near the support of φ_0. Defining $u_p \stackrel{\text{def}}{=} \varphi_1 \widetilde{u}_p$, we have, by definition of $\mathcal{R}_p$ and by (3.9)

$$u_p \in H^1(\widetilde{\mathcal{Z}}_p, \mathcal{C}_\Sigma) = H^1(\mathcal{Z}_p, \mathcal{C}_\Sigma) \text{ and } \|u_p\|_{H^1(\mathcal{R}_p)} \leqslant C \|\widetilde{u}_p\|_{H^1(\widetilde{\mathcal{R}}_p, \mathcal{C}_\Sigma)}.$$

After dilation, this gives

$$\sum_p \|u_p(2^p\cdot)\|^2_{H^1(\mathcal{Z})} \leqslant C \|v\|^2_{T^{\frac{1}{2}}}.$$

As an integer N_0 exists such that

$$|p-p'| \geqslant N_0 \Longrightarrow u_p(2^p\cdot) \perp u_{p'}(2^{p'}\cdot) \quad \text{in} \quad H^1(\mathcal{Z}),$$

the series $\left(u_p(2^p\cdot)\right)_p$ converge in $H^1(\mathcal{Z})$ to a function u whose trace is obviously v. This concludes the proof of Theorem 3.1. $\square$

Remark. — The trace lifting theorem provides functions in $H^1(\mathbb{H}^d)$ supported in a set of the form $s^2 \leqslant C(|x|^2+|y|^2)^2$. It obviously prevents, using this method, to prove trace lifting theorem for very regular (for instance continuous) functions.

3.3. The space of trace as an interpolation space

The description given by Theorem 3.1 is not totally satisfactory. We want to describe this space of trace as an interpolation space to get Theorem 1.2. In order to do so, let us define, for $s \in [0,1]$, the space

$$T^s \stackrel{\text{def}}{=} \Big\{ v \in L^2 \mid \|v\|^2_{T^s} \stackrel{\text{def}}{=} \sum_p \|\varphi_p^\Sigma v\|^2_{H^s(\mathcal{R}_p)} < \infty \Big\}.$$

Let us start with the proof of the following lemma.

Lemma 3.2. — *The space T^1 is equal to $H^1(\mathcal{R})$ and the norm are equivalents.*

Proof of Lemma 3.2. — By definition of the norm on $H^1(\mathcal{R}_p)$, we have

$$\|(\varphi_p^\Sigma v)(2^{-p}\cdot)\|^2_{H^1(\mathcal{R}_p)} = \|(\varphi_p^\Sigma v)(2^{-p}\cdot)\|^2_{L^2} + \sum_{j,k} \|\mathcal{R}_{j,k,p}\left((\varphi_p^\Sigma v)(2^{-p}\cdot)\right)\|^2_{L^2}.$$

By definition of $R_{j,k,p}$, we have

$$2^{-2pd}\|\mathcal{R}_{j,k,p}\left((\varphi_p^\Sigma v)(2^{-p}\cdot)\right)\|^2_{L^2} = \|\mathcal{R}_{j,k}(\varphi_p^\Sigma v)\|^2_{L^2}.$$

By Leibnitz formula and by definition of φ_p^Σ, we have

$$\begin{aligned} R_{j,k}(\varphi_p^\Sigma v)(z) &= \varphi_p^\Sigma R_{j,k}\cdot v(z) + (R_{j,k}\cdot \varphi_p^\Sigma)v(z) \\ &= \varphi_p^\Sigma(z)(R_{j,k}\cdot v)(z) + 2^p(R_{j,k}\cdot |z'|)\varphi'(2^p|z'|)v(z). \end{aligned}$$

As the vector fields $R_{j,k}$ vanishes at 0, we have

$$\sup_{p,j,k} \|R_{j,k}\varphi_p^\Sigma\|_{L^\infty} < \infty.$$

This gives that

$$\left|R_{j,k}(\varphi_p^\Sigma v)(z) - \varphi_p^\Sigma R_{j,k} v(z)\right| \leqslant C\varphi'(2^p|z'|)|v(z)|.$$

As, for some positive integer N_0, the support of the two functions $\varphi(2^p|z'|)$ and $\varphi(2^{p'}|z'|)$ are disjoint when $|p-p'| \geqslant N_0$, this gives the lemma. □

Now Theorem 1.2 will be an easy consequence of the following abstract interpolation lemma.

LEMMA 3.3. — *Let us consider* $(\mathcal{H}_j, \|\cdot\|_j)_{j\in\{0,1\}}$ *two Hilbert spaces such that* $\mathcal{H}_1$ *is densely included in* $\mathcal{H}_0$ *and a family* $(\mathcal{H}_{j,p})_{(j,p)\in\{0,1\}\times\mathbb{N}}$ *such that, for any* p, $\mathcal{H}_{j,p}$ *is a closed subset of* $\mathcal{H}_j$.

Let us assume that a family of $(\Lambda_p)_{p\in\mathbb{N}}$ *of (unbounded) selfadjoints operators on* $\mathcal{H}_{0,p}$ *exists such that* $\mathcal{H}_{1,p}$ *equals to the domain of* Λ_p *and*

$$\forall u \in \mathcal{H}_{1,p},\ \|u\|_{\mathcal{H}_1} \sim \|\Lambda_p u\|_{\mathcal{H}_0}. \tag{3.10}$$

Let us assume in addition that a family of operators $(A_p)_{p\in\mathbb{N}}$ *exists such that, for any* (j,p) *in* $\{0,1\}\times\mathbb{N}$, *the operator* A_p *is continuous from* $\mathcal{H}_j$ *into* $\mathcal{H}_{j,p}$ *and*

$$\forall v \in \mathcal{H}_j,\ \lim_{p\to\infty} \Big\| v - \sum_{p=0}^{N} v_p \Big\| = 0 \text{ and } \|v\|^2_{\mathcal{H}_j} \sim \sum_p \|A_p v\|^2_{\mathcal{H}_j}. \tag{3.11}$$

Then,

$$[\mathcal{H}_0,\mathcal{H}_1]_s = \Big\{ v \in \mathcal{H}_0 \mid \|v\|^2_{T^s} \stackrel{def}{=} \sum_{p=0}^{\infty} \|A_p v\|^2_{\mathcal{H}_{s,p}} \Big\}$$
$$\text{with } \mathcal{H}_{s,p} \stackrel{\text{def}}{=} [\mathcal{H}_{0,p},\mathcal{H}_{1,p}]_s.$$

Proof of Lemma 3.3. — It is enough to prove that the two norms are equivalent on the dense space of v such that

$$v = \sum_{p=0}^{N} v_p \text{ with } v_p \in \mathcal{H}_{1,p}.$$

Let us first estimate $\|v\|_{[\mathcal{H}_0,\mathcal{H}_1]_s}$. By definition of the norm on $\mathcal{H}_{s,p}$, a function f_p exists in $\mathcal{F}(\mathcal{H}_{0,p},\mathcal{H}_{1,p})$ such that

$$f_p(s) = A_p v \text{ and } \max_{j\in\{0,1\}} \sup_{t\in\mathbb{R}} \|f_p(j+it)\|_{\mathcal{H}_j} \leqslant 2\|A_p v\|_{\mathcal{H}_{s,p}}.$$

Now let us define

$$F_N(\zeta) \stackrel{\text{def}}{=} e^{\zeta^2 - s^2} \sum_{p=0}^{N} f_p(\zeta).$$

As the sum is finite, it is obvious that F_N belongs to $\mathcal{F}(\mathcal{H}_0,\mathcal{H}_1)$. Because of (3.11), we have, for $j\in\{0,1\}$,

$$\begin{aligned} \|F_N(j+it)\|^2_{\mathcal{H}_j} &\leqslant Ce^{-t^2} \sum_{p=0}^{N} \|f_p(j+it)\|^2_{\mathcal{H}_j} \\ &\leqslant Ce^{-t^2} \sum_{p=0}^{N} \|A_p v\|^2_{\mathcal{H}_{s,p}} \leqslant C\|v\|^2_{T^s}. \end{aligned}$$

Thus by definition of the complex interpolation norm, we deduce that

$$\|v\|_{[\mathcal{H}_0,\mathcal{H}_1]_s} \leqslant C\|v\|_{T^s}.$$

Now let us estimate $\|v\|_{T^s}$. In order to do so, let us consider F in $\mathcal{F}(\mathcal{H}_0,\mathcal{H}_1)$ such that

$$F(s) = v \text{ and } \max_{j\in\{0,1\}} \sup_t \|F(j+it)\|_{\mathcal{H}_j} \leqslant 2\|v\|_{[\mathcal{H}_0,\mathcal{H}_1]_s}.$$

For a greater than 1, let us introduce

$$\mathcal{N}_a(\zeta) \stackrel{\text{def}}{=} e^{\zeta^2-s^2} \sum_{p=0}^{N} \int_1^a \lambda^{2\zeta} d\mu_p(A_pF(\zeta), A_pF(\zeta))$$

where μ_p is the spectral measure of Λ_p. Then, by using (3.10) and (3.11),

$$\begin{aligned}
|\mathcal{N}_a(j+it)| &\leqslant Ce^{-t^2}\left|\int_1^a \lambda^{2it}\lambda^{2j} d\mu_p\Big(A_pF(j+it), A_pF(j+it)\Big)\right| \\
&\leqslant Ce^{-t^2}\sum_{p=0}^{N}\int_1^a \lambda^{2j} d\mu_p\Big(A_pF(j+it), A_pF(j+it)\Big) \\
&\leqslant Ce^{-t^2}\sum_{p=0}^{N}\|A_pF(j+it)\|^2_{\mathcal{H}_j} \\
&\leqslant Ce^{-t^2}\|F_j(j+it)\|^2_{\mathcal{H}_j} \\
&\leqslant C\|v\|^2_{[\mathcal{H}_0,\mathcal{H}_1]_s}.
\end{aligned}$$

Then using the Phragmen-Lindelöf principle, we get that

$$\mathcal{N}_a(s) \leqslant \sup_t |\mathcal{N}_a(it)|^{1-s}|\mathcal{N}_a(1+it)|^s \leqslant C\|v\|^2_{[\mathcal{H}_0,\mathcal{H}_1]_s}.$$

Thus a constant C exists such that, for any a,

$$\sum_{p=0}^{N}\int_1^a \lambda^{2s} d\mu_p(A_pv, A_pv) \leqslant C\|v\|^2_{[\mathcal{H}_0,\mathcal{H}_1]_s}. \tag{3.12}$$

By definition of $\mathcal{H}_{s,p}$ and using that

$$\|w\|^2_{\mathcal{H}_{s,p}} = \int_1^\infty \lambda^{2s} d\mu_p(w,w),$$

we infer by passing to the limit when a tends to infinity in (3.12) that

$$\sum_{p=0}^{N}\|A_pv\|^2_{\mathcal{H}_{s,p}} \leqslant C\|v\|^2_{[\mathcal{H}_0,\mathcal{H}_1]_s}.$$

This conclude the proof of Lemma 3.3. □

3.4. Conclusion of the proof of Theorem 1.2

Theorem 1.2 follows, observing that the hypothesis of Lemma 3.3 are satisfied with $\mathcal{H}_0 = L^2$, $\mathcal{H}_1 = H^1(\mathcal{R})$, $\mathcal{H}_{j,p}$ is the set of v in $\mathcal{H}_j$ supported in $2^{-p}\mathcal{C}$ and Λ_p is the square root of Dirichlet realization on $2^{-p}\mathcal{C}$ of the operator

$$\mathrm{Id} + \Delta_\Sigma \text{ with } \Delta_\Sigma \stackrel{\text{def}}{=} \sum_{j,k} \mathcal{R}^\star_{j,k}\mathcal{R}_{j,k}.$$

To be able to apply Lemma 3.3, and then to conclude the proof of Theorem 1.2, it is enough to prove the following proposition.

Proposition 3.2. — *A neighbourhood V of w_0 exists such that the operator Δ_Σ is selfadjoint on $L^2(V)$ with domain*

$$\Big\{v \in L^2(V) \mid \forall (j,k,j',k') \in \{1,\ldots,2d\}^4,\ R_{j,k}v \in L^2(V) \text{ and } R_{j,k}R_{j',k'}v \in L^2(V)\Big\}.$$

Proof of Proposition 3.2. — Up to an omitted regularization process, it is enough to prove that, for any $v \in \mathcal{D}(V)$,

$$\sum_{j,k} \|R_{j,k}v\|^2_{L^2} + \sum_{j,k,j',k'} \|R_{j,k}R_{j',k'}v\|^2_{L^2} \leqslant C\Big(\|v\|^2_{L^2} + \|\Delta_\Sigma v\|^2_{L^2}\Big). \tag{3.13}$$

Let us start with the observation that

$$\begin{aligned}\sum_{j,k} \|R_{j,k}v\|^2_{L^2} &\leqslant C\sum_{j,k} (R_{j,k}v|R_{j,k}v)_{L^2}\\ &\leqslant C\sum_{j,k} (R^\star_{j,k}R_{j,k}v|v)_{L^2}\\ &\leqslant C(\Delta_\Sigma v|v)_{L^2}\\ &\leqslant C\|\Delta_\Sigma v\|_{L^2}\|v\|_{L^2}.\end{aligned} \tag{3.14}$$

In order to estimate $\|R_{j,k}R_{j',k'}v\|_{L^2}$, we are going to proceed as in the proof of Lemma 3.2. Let us write that

$$\begin{aligned}&R_{j,k}R_{j',k'}(\varphi_p v) - \varphi_p R_{j,k}R_{j',k'}v\\ &\qquad = (R_{j,k}\varphi_p)(R_{j',k'}v) + (R_{j',k'}\varphi_p)(R_{j,k}v) + \varphi_p(R_{j,k}R_{j',k'}v).\end{aligned} \tag{3.15}$$

As the coefficients of the vector fields $R_{j,k}$ vanish on Σ_c, we have

$$\sup_{p,j,k,j',k'} \|R_{j,k}\varphi_p\|_{L^\infty} + \|R_{j,k}R_{j',k'}\varphi_p\|_{L^\infty} < \infty.$$

Thus, using (3.14), we have

$$\|\varphi_p R_{j,k} R_{j',k'} v - R_{j,k} R_{j',k'}(\varphi_p v)\|_{L^2} \leqslant C c_p \|\Delta_\Sigma v\|_{L^2}^{\frac12} \|v\|_{L^2}^{\frac12} \quad \text{with} \sum_{p=0}^{\infty} c_p^2 = 1. \tag{3.16}$$

We have

$$R_{j,k} R_{j',k'}(\varphi_p v) = R_{j,k,p} R_{j',k',p}\big(\varphi_0 v(2^p\cdot\,)\big).$$

Lemma 4.1 of [2] tells us that the systems $(\mathcal{R}_{j,k,p})_{j,k}$ satisfy the Hörmander condition at order 2 uniformly with respect to p on $\mathcal{C}$. Thus, the classical maximal estimate tells us that

$$\|R_{j,k,p} R_{j',k',p} w\|_{L^2} \leqslant C\bigg(\Big\|\sum_{j,k} R^\star_{j,k,p} R_{j,k,p} w\Big\|_{L^2} + \|w\|_{L^2}^2\bigg).$$

Applied with $w = \varphi_0 v(2^p\cdot\,)$, this gives

$$\begin{aligned} \|R_{j,k} R_{j',k'}(\varphi_p v)\|_{L^2} & \\ &\leqslant C 2^{pd}\bigg(\Big\|\sum_{j,k} R^\star_{j,k,p} R_{j,k,p} \varphi_0 v(2^p\cdot\,)\Big\|_{L^2} + \|\varphi_0 v(2^p\cdot\,)\|_{L^2}^2\bigg) \\ &\leqslant C\left(\|\Delta_\Sigma(\varphi_p v)\|_{L^2} + \|\varphi_p v\|_{L^2}\right). \end{aligned} \tag{3.17}$$

Then (3.16) implies that

$$\|\Delta_\Sigma(\varphi_p v) - \varphi_p \Delta_\Sigma v\|_{L^2} \leqslant C c_p \|\Delta_\Sigma v\|_{L^2}^{\frac12} \|v\|_{L^2}^{\frac12} \text{ with } \sum_{p=0}^{\infty} c_p^2 = 1.$$

Thus, by using (3.15) and (3.17) we infer that

$$\|\varphi_p R_{j,k} R_{j',k'} v\|_{L^2} \leqslant C c_p(\|\Delta_\Sigma v\|_{L^2} + \|v\|_{L^2}) \text{ with } \sum_{p=0}^{\infty} c_p^2 = 1.$$

This proves (3.13) and thus Proposition 3.2. □

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