ANNALES

DE

L'INSTITUT FOURIER

Stefano FRANCAVIGLIA

Constructing equivariant maps for representations

Tome 59, nº 1 (2009), p. 393-428.

<http://aif.cedram.org/item?id=AIF_2009__59_1_393_0>

Ann. Inst. Fourier, Grenoble
59, 1 (2009) 393-428

CONSTRUCTING EQUIVARIANT MAPS FOR REPRESENTATIONS

by Stefano FRANCAVIGLIA (*)

Abstract. — We show that if Γ is a discrete subgroup of the group of the isometries of $\mathbb{H}^k$, and if ρ is a representation of Γ into the group of the isometries of $\mathbb{H}^n$, then any ρ-equivariant map $F : \mathbb{H}^k \to \mathbb{H}^n$ extends to the boundary in a weak sense in the setting of Borel measures. As a consequence of this fact, we obtain an extension of a result of Besson, Courtois and Gallot about the existence of volume non-increasing, equivariant maps. Then, we show that the weak extension we obtain is actually a measurable ρ-equivariant map in the classical sense. We use this fact to obtain measurable versions of Cannon-Thurston-type results for equivariant Peano curves. For example, we prove that if Γ is of divergence type and ρ is non-elementary, then there exists a measurable map $D : \partial\mathbb{H}^k \to \partial\mathbb{H}^n$ conjugating the actions of Γ and $\rho(\Gamma)$. Related applications are discussed.

Résumé. — On montre que pour chaque groupe discrète d'isométries G de l'espace hyperbolique de dimension k, chaque représentation R de G dans le groupe $\mathrm{Isom}(\mathbb{H}^n)$ et pour chaque application R-équivariante F de $\mathbb{H}^k$ en $\mathbb{H}^n$, il existe une extension de F dans le sens faible des mesures. On obtient donc, comme conséquence de ce fait, une extension d'un résultat de Besson, Courtois et Gallot sur l'existence d'une application équivariante qui n'augmente pas le volume. En plus, avec une hypothèse supplémentaire, on montre que notre extension faible est effectivement une vraie application mesurable du bord à l'infini de $\mathbb{H}^k$. On utilise alors ce résultat pour obtenir une version mesurable du résultat de Cannon et Thurston sur l'existence de courbes de Peano équivariantes. Enfin, on discute quelques applications.

1. Introduction

Let Γ be a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^k)$and let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation. Then, it is easy to construct a piecewise smooth map

Keywords: Hyperbolic spaces, discrete groups, isometries, representation, equivariant, barycenter, natural map, volume.
Math. classification: 57M50, 37A99.
(*) This work was supported by INdAM and the European Research Council (MEIF-CT-2005-010975 and MERG-CT-2007-046557.)

$D : \mathbb{H}^k \to \mathbb{H}^n$ which is ρ-equivariant, that is $D(\gamma x) = \rho(\gamma)D(x)$ for all $\gamma \in \Gamma$ and $x \in \mathbb{H}^k$, and the problem arises of whether such a map continuously extends to the boundaries of the hyperbolic spaces (this is a key step in the proofs of rigidity results for hyperbolic manifolds, see for example [29, 3, 12, 13, 16]). In some cases, for example if the representation is not discrete, such an extension is not possible. Moreover, in general it is hard even to construct a ρ-equivariant map between the boundaries with some regularity properties like continuity and measurability. If $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$ is a surface group and if $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ is an isomorphism such that $\mathbb{H}^3/\rho(\Gamma)$ is an hyperbolic 3-manifold, under certain assumptions, Cannon and Thurston [9] and Minsky [20], proved the existence of a continuous, ρ-equivariant, surjective map $\partial\mathbb{H}^2 \to \partial\mathbb{H}^3$, and Soma [25] proved that, outside zero-measure sets, such a map is a homeomorphism (see also the recent works [17, 18]).

The starting point of this paper is the existence, for any $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$, of a measurable, ρ-equivariant map from the limit set of Γ to the set of probability measures on $\partial\mathbb{H}^n$, that can be stated as follows (see Section 2 for precise definitions).

Theorem 1.1 (Existence of developing measures). — *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^k)$ be an infinite, non-elementary discrete group. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation. Then, a family of developing measures for ρ exists.*

The fact that we work in the world of measure with the weak-* topology introduces a lot of compactness, so that we will able to establish results of existence and convergence for equivariant maps. The main applications of our technique belong to the framework of *Barycentric maps* and the one of *Cannon Thurston maps*.

Barycentric maps. — First of all, we get a generalisation of the celebrated Theorem of Besson, Courtois and Gallot on existence of natural maps.

Theorem 1.2 (Existence of B-C-G-natural maps). — *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^k)$ be an infinite discrete group. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation whose image is non-elementary. Then, there exists a map $F : \mathbb{H}^k \to \mathbb{H}^n$, called* natural map*, such that:*

1. *F is smooth.*
2. *F is ρ-equivariant, i.e., $F(\gamma x) = \rho(\gamma)F(x)$ for all $x \in \mathbb{H}^k$ and $\gamma \in \Gamma$.*
3. *For all $p \geqslant 3$, $\mathrm{Jac}_p F(x) \leqslant \left(\dfrac{\delta(\Gamma)}{p-1}\right)^p$.*

(4) *If* $||d_xF(u_1) \wedge \cdots \wedge d_xF(u_p)|| = \left(\frac{\delta(\Gamma)}{p-1}\right)^p$ *for an orthonormal* p-*frame* $u_1, \ldots, u_p$ *at* $x \in \mathbb{H}^k$, *then the restriction of* d_xF *to the subspace generated by* $u_1, \ldots, u_p$ *is a homothety.*

Theorem 1.2 was proved by Besson Courtois and Gallot in the special case in which ρ is discrete and faithful and both Γ and $\rho(\Gamma)$ are convex co-compact ([6].)

In our setting, Theorem 1.2 will follow directly from Theorem 1.1 and a modification of the construction of Besson, Courtois and Gallot (see [4, 5, 6].)

In [6], for $\varepsilon > 0$ the authors construct a smooth ρ-equivariant map $F_\varepsilon : \mathbb{H}^k \to \mathbb{H}^n$ such that for all $p \geqslant 3$, $\mathrm{Jac}_p F_\varepsilon(x) \leqslant \left(\frac{\delta(\Gamma)(1+\varepsilon)}{p-1}\right)^p$. We call such maps ε-natural maps. We will see that the natural map we construct is actually the limit of a sequence of such ε-natural maps.

Theorem 1.3. — *In the hypotheses of Theorem 1.2, there exists a family* $\{F_\varepsilon\}$ *of* ε*-natural maps (constructed as in* [6]*) and a sequence* $\varepsilon_i \to 0$ *such that* F_{ε_i} *converges to the natural map* F.

The proofs of Theorems 1.1 and 1.3 both start with a ρ-equivariant map $D : \mathbb{H}^k \to \mathbb{H}^n$. While the natural map F does not depend on D, the maps F_ε's can be constructed in such a way to keep memory of D. More precisely, the construction of the ε-natural maps depends on the choice of a probability measure on a fundamental domain for the action of Γ, and they depends on the restriction of D to the support of such measure. This is useful to study non-compact manifolds. For example, suppose that $f : M \to N$ is a proper map between complete non-compact hyperbolic manifolds. If D is the lift of f to the universal covers, then the natural maps F_ε descend to maps $f_\varepsilon : M \to N$, and one can show that, if one used a suitable measure to construct such maps, then they are proper.

Such results can be used to prove rigidity results for representations. The following theorem (whose proof will be sketched in Section 6 and completely described in [14]) is an example of applications of Theorems 1.2 and 1.3.

Theorem 1.4 (Rigidity of representations). — *Let* M *be a complete hyperbolic* k*-manifold of finite volume. Let* $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n)$ *be an irreducible representation. Then* $\mathrm{vol}(\rho) \leqslant \mathrm{vol}(M)$ *and equality holds if and only if* ρ *is a discrete and faithful representation into the group of isometries of a* k*-dimensional hyperbolic subspace of* $\mathbb{H}^n$.

Measurable Cannon-Thurston maps. — A Cannon-Thurston map for ρ is a continuous, ρ-equivariant map from the limit set of γ to $\partial\mathbb{H}^n$. It is quite difficult to show the existence of such maps (and in general there are obstructions to continuity, and one needs to impose geometric constraints.) In fact, the existence problem has been solved for $k = 2$ and $n = 3$ with geometric hypotheses (see for instance [9, 19, 20, 17, 18]).

Theorem 1.1 is a weak existence result, and its proof use a kind of weak extension result; namely, the family of developing measures weakly extends the orbit of a point. The main result here, is the proof of a stronger measurable extension result, which will be the base for the study of measurable versions of the Cannon-Thurston map (the measures we consider on $\partial\mathbb{H}^k$ are the Patterson-Sullivan measures, see Section 2.) The strong extension theorem can be stated in his general form as follows.

Theorem 1.5 (Existence and uniqueness of measurable extensions). — *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^k)$ be a discrete group and let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation whose image is non-elementary. Suppose that there exists a family $\{\lambda_z\}_{z\in\partial\mathbb{H}^k}$ of developing measures for ρ such that for almost all z, the measure λ_z is not the sum of two Dirac deltas with equal weights. Then, the natural map F constructed using $\{\lambda_z\}$ extends to the conical limit set. More precisely, there exists a measurable, ρ-equivariant map $\overline{F} : \overline{\mathbb{H}}^k \to \overline{\mathbb{H}}^n$ which agrees with F on $\mathbb{H}^k$ and such that, for almost all ω in the conical limit set, if $\xi \in \mathbb{H}^k$ and $\{\gamma_n\xi\}$ is a sequence conically converging to ω, then $F(\gamma_n x) \to \overline{F}(\omega)$ for all $x \in \mathbb{H}^k$.*

Moreover, the map $\overline{F}$ is unique. More precisely, if $\overline{F}_1$ and $\overline{F}_2$ are two measurable, ρ-equivariant maps from $\Lambda(\Gamma)$ to the set of probability measures on $\partial\mathbb{H}^n$, then $\overline{F}_1$ and $\overline{F}_2$ are in fact ordinary functions, that is to say, they map almost every point of $\Lambda(\Gamma)$ to a Dirac delta concentrated on a point of $\partial\mathbb{H}^n$. Moreover, they agree almost everywhere on $\partial\mathbb{H}^k$.

Theorem 1.5 in particular applies to the case of fundamental groups of hyperbolic manifolds. Indeed, we will prove that in this case the developing measures are almost never the sum of two Dirac deltas with equal weights. In particular, this gives the following theorem.

Theorem 1.6 (Existence and uniqueness of Cannon-Thurston maps). *Let $\mathbb{H}^k/\Gamma$ be a complete hyperbolic manifold of finite volume and let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation whose image is not elementary. Then there exists a measurable ρ-equivariant map $\overline{F} : \partial\mathbb{H}^k \to \partial\mathbb{H}^n$ Such a map is the extension of a natural map, and two such maps are equal almost everywhere.*

In particular when a classical Cannon-Thurston map exists, it coincides with the extension provided by Theorem 1.6. An immediate consequence of these facts is the following result.

THEOREM 1.7 (Inverse of Cannon-Thurston maps). — *Let* $\Gamma < \mathrm{Isom}(\mathbb{H}^k)$ *and* $\Gamma' < \mathrm{Isom}(\mathbb{H}^n)$ *be non-elementary discrete groups such that they diverge respectively at* $\delta(\Gamma)$ *and* $\delta(\Gamma')$. *Let* $\rho : \Gamma \to \Gamma'$ *be an isomorphism.*

Then, there exist measurable maps $\overline{F} : \partial\mathbb{H}^k \to \partial\mathbb{H}^n$ *and* $\overline{G} : \partial\mathbb{H}^n \to \partial\mathbb{H}^k$ *which are respectively* ρ *and* ρ^{-1} *equivariant. Moreover, almost everywhere*

$$\overline{F} \circ \overline{G} = \mathrm{Id}_{\mathbb{H}^k} \qquad \overline{G} \circ \overline{F} = \mathrm{Id}_{\mathbb{H}^n}.$$

As noticed above, working with measures helps when one has to deal with convergence problems. For example, an application of the above techniques provides an answer to the following question.

Suppose that $\rho_i \to \rho$ is a converging sequence of non-elementary representations. Do the corresponding Cannon-Thurston maps converge?

Miyachi proved [21] that if Γ is a surface-group without parabolics, and if the injectivity radius is bounded away from zero along the whole sequence, then the answer is "yes" (in fact, one has uniform convergence,) and it is conjectured that the same holds in the case of cusped surfaces with a uniform bound on the injectivity radius outside the cusps (see for example [24, Conjecture 5.2].) In general, as Example 9.1 shows, one can construct sequences having no uniformly converging sub-sequence (the condition on injectivity radii is violated.) We prove here that in the general case (no geometric hypotheses, no bounds on dimensions) the answer is "yes" for the point-wise convergence almost everywhere.

THEOREM 1.8 (Convergence of Cannon-Thurston maps). — *Let* $\Gamma < \mathrm{Isom}(\mathbb{H}^k)$ *be a discrete group that diverges at its critical exponent* $\delta(\Gamma)$. *Let* $\rho_i : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ *be a sequence of representations with non-elementary images. Suppose that* ρ_i *converges to a representation* ρ *whose image is non-elementary. Let* f_i *and* f *be the corresponding measurable Cannon-Thurston maps for* ρ_i *and* ρ *respectively. Then,* f_i *converges to* f *almost everywhere with respect to the Patterson-Sullivan measures on the limit set of* Γ.

ACKNOWLEDGEMENTS. I need to thank Jeff Brock, Ben Klaff, Carlo Mantegazza, Joan Porti, Juan Souto and Xavi Tolsa for their fundamental help.

2. Definitions, notation and preliminary facts

First of all, we recall some definitions.

Definition 2.1. — *A subgroup* $\Gamma < \mathrm{Isom}(\mathbb{H}^k)$ *is said* non-elementary *if any* Γ*-invariant, non-empty set* $A \subset \partial\mathbb{H}^k$ *contains at least three points. Otherwise,* Γ *is said* elementary.

Definition 2.2. — *Let* $F : \mathbb{H}^k \to \mathbb{H}^n$ *be a smooth map. The* p*-Jacobian* $\mathrm{Jac}_p F$ *of* F *is defined by*

$$\mathrm{Jac}_p F(x) = \sup ||d_x F(u_1)\wedge, \dots, \wedge d_x F(u_p)||_{\mathbb{H}^n},$$

where $\{u_i\}_{i=1}^p$ *varies on the set of orthonormal* p*-frames at* $x \in \mathbb{H}^k$*.*

Definition 2.3. — *Let* (X, g) *be a complete Riemannian manifold and let* Γ *be a group of isometries of* X*. We denote by* $\delta(\Gamma)$ *the critical exponent of the Poincaré series of* Γ*, that is:*

$$\delta(\Gamma) = \inf\{s > 0 : \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} < +\infty\}$$

where $d(\cdot,\cdot)$ *denotes the distance induced by* g *on* X *and* x *is a point of* X*.*

It is readily checked that $\delta(\Gamma)$ does not depend on x. We notice that, when $X = \mathbb{H}^k$, the critical exponent is the Hausdorff dimension of the conical limit set of Γ ([7]). Moreover, $\delta(\Gamma)$ can be computed by

$$\delta(\Gamma) = \lim_{R\to\infty} \frac{1}{R} \log(\#\{\gamma \in \Gamma :\ d(\gamma O, O) < R\}).$$

We refer to [22, 28, 30] for details.

Definition 2.4. — *Let* (X, g) *be a complete Riemannian manifold and let* Γ *be a group of isometries of* X*. We say that* Γ diverges *at* $\delta(\Gamma)$ *if*

$$\lim_{s\to\delta(\Gamma)^+} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} = +\infty.$$

The following lemma is an immediate consequence of [22, Theorem 1.6.1], [22, Theorem 1.6.3] and [22, Corollary 3.4.5].

Lemma 2.5. — *Let* Γ *be a infinite, non-elementary discrete subgroup of* $\mathrm{Isom}(\mathbb{H}^k)$*, then*

$$0 < \delta(\Gamma) \leqslant k - 1.$$

Moreover, if $\mathbb{H}^k/\Gamma$ *has finite volume, then* $\delta(\Gamma) = k - 1$ *and* Γ *diverges at* $\delta(\Gamma)$*.*

Notation. — For the rest of the paper we fix the following notation. Γ will be an infinite discrete subgroup of $\mathrm{Isom}(\mathbb{H}^k)$, and

$$\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$$

will be a representation. We fix base-points in $\mathbb{H}^k$ and $\mathbb{H}^n$, both denoted by O. We denote by B_K and B_N the Busemann functions, respectively of $\mathbb{H}^k$ and $\mathbb{H}^n$, normalised at O. Namely, for $x \in \mathbb{H}^k$ and $\theta \in \partial\mathbb{H}^k$ (resp. $\mathbb{H}^n$ and $\partial\mathbb{H}^n$) we set

$$B_K(x,\theta) = \lim_{t\to\infty} \big(d_{\mathbb{H}^k}\big(x,\gamma_\theta(t)\big) - t\big),$$

where γ_θ is the geodesic ray from O to θ, parametrised by arc length. We denote by π_K (respectively π_N) the projection of $\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n$ to $\overline{\mathbb{H}}^k$ (resp. $\overline{\mathbb{H}}^n$):

$$\pi_K : \overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n \to \overline{\mathbb{H}}^k.$$

Finally, we fix a continuous piecewise smooth ρ-equivariant map

$$D : \mathbb{H}^k \to \mathbb{H}^n.$$

We notice that such a map can be easily constructed by triangulating a fundamental domain for Γ and then arguing by induction on the i-skeleta.

Patterson-Sullivan measures. — A fundamental tool for our purpose is the family of Patterson-Sullivan measures. We recall the main results we need, which we summarise in Theorem 2.6, referring to [6], [22, Chap. 3, 4] and [30, 31] for proofs and details.

Theorem 2.6. — *Let Γ be an infinite, non-elementary discrete subgroup of* $\mathrm{Isom}(\mathbb{H}^k)$ *with critical exponent $\delta(\Gamma)$. For all $x \in \mathbb{H}^k$ there exists a positive Borel measure μ_x of finite, non-zero mass such that, for all $x, y \in \mathbb{H}^k$ and $\gamma \in \Gamma$:*

(1) *The measures μ_x and μ_y are in the same density class of measures and are concentrated on $\partial\mathbb{H}^k$.*
(2) *The measure μ_x satisfies*

$$d\mu_x(\theta) = e^{-\delta(\Gamma)B_K(x,\theta)}d\mu_O(\theta)$$

where $\theta \in \partial\mathbb{H}^k$.
(3) *The measures μ_x are Γ-equivariant, that is*

$$\mu_{\gamma x} = \gamma_*\mu_x.$$

Proof. — We only sketch the proof, recalling the construction of the Patterson-Sullivan measures because we will explicitly use it in the following.

For all $s > \delta(\Gamma)$ let

$$c(s) = \sum_{\gamma\in\Gamma} e^{-sd(O,\gamma O)}$$

where $d(\cdot,\cdot)$ denotes the hyperbolic distance of $\mathbb{H}^k$. For simplicity, here we stick to the case that Γ diverges at $\delta(\Gamma)$ (this happens for example if Γ is geometrically finite, see [22, p. 87]). For any $x \in \mathbb{H}^k$ and $s > \delta(\Gamma)$, define

$$\mu_x^s = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma O)} \delta_{\gamma O}$$

where $\delta_{\gamma O}$ is the Dirac measure concentrated on γO. For $s > \delta(\Gamma)$, μ_x^s is a well-defined positive Borel measure on $\mathbb{H}^k \subset \overline{\mathbb{H}}^k$. It can be shown that for $s \to \delta(\Gamma)^+$, the measures μ_x^s weakly converge to a positive Borel measure μ_x on $\overline{\mathbb{H}}^k$. More precisely, for all $x \in \mathbb{H}^k$ and $\varphi \in C(\overline{\mathbb{H}}^k)$

$$\int_{\overline{\mathbb{H}}^k} \varphi \, d\mu_x^s \to \int_{\overline{\mathbb{H}}^k} \varphi \, d\mu_x.$$

Moreover, the fact that $\lim_{s \to \delta(\Gamma)^+} c(s) = +\infty$ easily implies that μ_x is concentrated on the boundary (in fact μ_x is concentrated on the limit set of Γ). $\square$

Barycentre of a measure. — We recall now the definition and the main properties of the barycentre of a measure. We refer the reader to [11] and [4] for complete proofs and details.

Let β be a positive Borel measure on $\partial\mathbb{H}^n$ of finite mass. Define the function $\mathcal{B}_\beta : \mathbb{H}^n \to \mathbb{R}$ by

$$\mathcal{B}_\beta : y \mapsto \int_{\partial\mathbb{H}^n} B_N(y,\theta)\, d\beta(\theta).$$

Since we are working in the hyperbolic space, the Busemann functions are convex. Thus, its β-average $\mathcal{B}_\beta$ is strictly convex, provided that β is not the sum of two deltas. Moreover, one can show that $\mathcal{B}_\beta(y) \to \infty$ as y approaches $\partial\mathbb{H}^n$. It follows that $\mathcal{B}$ has a unique minimum in $\mathbb{H}^n$.

Definition 2.7. — *For any positive Borel measure β on $\partial\mathbb{H}^n$ of finite mass which is not concentrated on two points, we define the* barycentre $\mathrm{bar}(\beta)$ *of β as the unique minimum point of the function $y \mapsto \int_{\partial\mathbb{H}^n} B_N(y,\theta)\, d\beta(\theta)$.*

We refer to [11], [4] and [5] for a proof of the following lemma.

Lemma 2.8. — *The barycentre of a measure β satisfies the following properties:*

(1) *The barycentre is characterised by the equation*

$$\int_{\partial\mathbb{H}^n} dB_{N(\mathrm{bar}(\beta),\theta)}(\cdot)\, d\beta(\theta) = 0.$$

(2) *The barycentre is* $\mathrm{Isom}(\mathbb{H}^n)$*-equivariant, that is, for any* $g \in \mathrm{Isom}(\mathbb{H}^n)$

$$\mathrm{bar}(g_*\beta) = g(\mathrm{bar}(\beta)).$$

(3) *The barycentre is continuous respect the weak convergence of measures. That is, if* $\beta_i \rightharpoonup \beta$*, then* $\mathrm{bar}(\beta_i) \to \mathrm{bar}(\beta)$.

The first property follows from the definition after differentiating the function $\mathcal{B}_\beta$. The equivariance follows from the properties of the Busemann functions. The continuity can be easily proved using that, if $\beta_i \rightharpoonup \beta$, then $\mathcal{B}_{\beta_i}$ and $d\mathcal{B}_{\beta_i}$ point-wise converge to $\mathcal{B}_\beta$ and $d\mathcal{B}_\beta$ respectively.

Remark 2.9. — If $\beta = a\delta_{\theta_1} + b\delta_{\theta_2}$, with $0 < a < b$, then it can be checked that the minimum of $\mathcal{B}_\beta$ is the point $\theta_2 \in \partial\mathbb{H}^n$. Thus, one can define the barycentre of a measure β whenever β is not the sum of two deltas with the same weights. Note that, since the barycentre of a measure concentrated on two points belongs to $\partial\mathbb{H}^n$, equation 1 of Lemma 2.8 makes no sense for such measures.

Developing measures. — We now introduce the notion of *family of developing measures* for ρ, which extends the one of ρ-equivariant map. We recall that $\{\mu_x\}$ is the family of Patterson-Sullivan measures.

Definition 2.10. — *A family of* developing measures *for* ρ *is a set* $\{\lambda_z\}_{z \in \partial\mathbb{H}^k}$ *of positive Borel measures on* $\partial\mathbb{H}^n$*, of finite mass, and such that:*

(1) *The measures* λ_z*'s are* ρ*-equivariant, that is, for* μ_O*-almost all* z *and all* $\gamma \in \Gamma$

$$\lambda_{\gamma z} = \rho(\gamma)_* \lambda_z$$

(2) *For any* $\varphi \in C(\partial\mathbb{H}^n)$*, the function*

$$z \mapsto \int_{\partial\mathbb{H}^n} \varphi(\theta)\, d\lambda_z(\theta)$$

is μ_O*-integrable (whence, by points 1 and 2 of Theorem 2.6, it is* μ_x*-integrable for all* x*).*

(3) *The function* $z \mapsto ||\lambda_z||$ *belongs to* $L^\infty(\partial\mathbb{H}^k, \mu_O)$.

(4) *For* μ_O*-almost* $z \in \partial\mathbb{H}^k$, $||\lambda_z|| > 0$.

As an example, consider a μ_O-measurable ρ-equivariant map $\overline{D} : \partial\mathbb{H}^k \to \partial\mathbb{H}^n$. Then the family $\{\lambda_z = \delta_{\overline{D}(z)}\}$, where $\delta_{\overline{D}(z)}$ is the Dirac measure, is a family of developing measures for ρ. In this sense the notion of developing measures extends the one of equivariant map.

Convolutions of measures. — Let X, Y be topological spaces and let μ be a Borel measure on X. Let $\{\alpha_x\}_{x\in X}$ be a family of Borel measures on Y such that for each $\varphi \in C_0(Y)$ the function $x \mapsto \int_Y \varphi\, d\alpha_x$ is μ-integrable. The convolution $\mu * \{\alpha_x\}$ is the Borel measure on Y defined by

$$\int_Y \varphi(y)\, d(\mu * \{\alpha_x\}) = \int_X \left(\int_Y \varphi(y)\, d\alpha_x(y) \right) d\mu(x)$$

for any $\varphi \in C_0(Y)$. Similarly, we define the product $\mu \times \{\alpha_x\}$ on $X \times Y$ by

$$\int_Y \varphi(x,y)\, d(\mu \times \{\alpha_x\}) = \int_X \left(\int_Y \varphi(x,y)\, d\alpha_x(y) \right) d\mu(x).$$

The measure $\mu * \{\alpha_x\}$ is the μ-average of the α_x's. Moreover, if $\pi : X \times Y \to Y$ is the projection, then $\mu * \{\alpha_x\} = \pi_*(\mu \times \{\alpha_x\})$.

We say that a sequence of measures $\{\mu_i\}$ weakly converges to μ if, for any continuous function f with compact support, $\int f\, d\mu_i \to \int f\, d\mu$. The proof of following lemmas are left to the reader.

Lemma 2.11. — *Suppose that $\{\mu_i\}$ is a sequence of measures on X, weakly converging to μ. If for each $\varphi \in C_0(Y)$ the function $x \mapsto \int_Y \varphi\, d\alpha_x$ belongs to $C_0(X)$, then the sequence $\mu_i * \{\alpha_x\}$ weakly converges to $\mu * \{\alpha_x\}$.*

Lemma 2.12. — *Let Z be a topological space and let $\{\nu_y\}_{y\in Y}$ be a family of Borel measures on Z such that for all $\psi \in C_0(z)$ the function $y \mapsto \int_Z \psi\, d\nu_y$ is α_x-integrable for μ-almost all x and $x \mapsto \int_Y \int_Z \psi\, d\nu_y d\alpha_x$ is μ-measurable. Then*

$$\mu * \{\alpha_x * \{\nu_y\}\} = (\mu * \{\alpha_x\}) * \{\nu_y\}.$$

By Lemma 2.12 we can omit the parentheses and write $\mu * \{\alpha_x\} * \{\nu_y\}$.

Some remarks on the hypotheses of Theorems 1.1, 1.2 and 3.1. — First of all, we notice that the hypotheses of such theorems can be relaxed by replacing the spaces $\mathbb{H}^k$ and $\mathbb{H}^n$ with Riemannian manifolds with suitable bounds on the curvatures. We refer the reader to [6] for further details on that direction.

Moreover, even if the hypothesis that the image of ρ is non-elementary is crucially used in Corollary 3.5, it is not strictly necessary. More precisely, suppose that $\rho(\Gamma)$ is elementary. Then there exists a $\rho(\Gamma)$-invariant set $A \subset \partial\mathbb{H}^n$ with either one or two points.

In the latter case, there exists a whole geodesic c in $\mathbb{H}^n$ which is $\rho(\Gamma)$-invariant and it is easy to see that a natural map whose image is contained in c exists. Therefore, Theorem 1.2 is true in this case.

In the former case, we can suppose that, in the half-space model $\mathbb{H}^{n-1} \times \mathbb{R}^+$ of $\mathbb{H}^n$, the point ∞ is fixed by $\rho(\Gamma)$. Then, one can easily construct a ρ-equivariant map D whose image is contained in the horosphere $\{(z,1) : z \in \mathbb{R}^{n-1}\}$. Thus, if $\mathrm{Jac}_p D(x)$ is bounded, then Theorem 1.2 is proved by raising D to a sufficiently high horosphere. It follows that, if $\mathbb{H}^k/\Gamma$ is compact, or simply if $\mathbb{H}^k/\Gamma$ retracts to a compact set (and this is the case if for example Γ is geometrically finite), then Theorem 1.2 is true even if $\rho(\Gamma)$ has a fixed point.

3. Construction of B-C-G natural maps

In this section we prove the following Theorem, which, together with Theorem 1.1 (proved in Section 4,) directly implies Theorem 1.2. For this section we keep the notation fixed in Section 2.

THEOREM 3.1. — *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^k)$ be an infinite discrete group. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation whose image is non-elementary. If there exists a family $\{\lambda_z\}$ of developing measures for ρ, then a natural map exists.*

Proof. — Fix a family

$$\{\lambda_z\}_{z\in\partial\mathbb{H}^k}$$

of developing measures for ρ. The idea of the proof is to use the developing measures to push-forward the Patterson-Sullivan measures μ_x's to measures β_x's on $\partial\mathbb{H}^n$, and define the natural map by

$$x \mapsto \mu_x \mapsto \beta_x \mapsto \mathrm{bar}(\beta_x).$$

Then, the properties of the natural map will follow as in [6].

The push-forward the measures μ_x's is defined as follows. For each $x \in \mathbb{H}^k$, define β_x as the positive Borel measure on $\partial\mathbb{H}^n$ given by $\beta_x = \mu_x * \{\lambda_z\}$. Namely, for all $\varphi \in C(\partial\mathbb{H}^n)$

$$\begin{aligned}\int_{\partial\mathbb{H}^n} \varphi(y)\, d\beta_x(y) &= \int_{\partial\mathbb{H}^k} \left(\int_{\partial\mathbb{H}^n} \varphi(y)\, d\lambda_z(y)\right) d\mu_x(z)\\ &= \int_{\partial\mathbb{H}^k} \left(\int_{\partial\mathbb{H}^n} \varphi(y)\, d\lambda_z(y)\right) e^{-\delta(\Gamma)B_K(x,z)}\, d\mu_O(z).\end{aligned}$$

Note that the measure β_x is well-defined and has finite mass because of conditions (2) and (3) of Definition 2.10. Moreover, since $\beta_x = \pi_{N*}(\mu_x \times \{\lambda_z\})$, if the family $\{\lambda_z\}$ is of the form $\{\delta_{\overline{D}(z)}\}$ for a μ_O-measurable function $\overline{D} : \partial\mathbb{H}^k \to \partial\mathbb{H}^n$, then $\beta_x = \overline{D}_* \mu_x$. □

Lemma 3.2. — *The family of measures* $\{\beta_x\}_{x\in\mathbb{H}^k}$ *is* ρ*-equivariant, that is to say, for all* $x \in \mathbb{H}^k$ *and* $\gamma \in \Gamma$

$$\beta_{\gamma x} = \rho(\gamma)_*\beta_x.$$

Proof. — For any $\varphi \in C(\partial\mathbb{H}^n)$

$$\int_{\partial\mathbb{H}^n} \varphi(y)\, d\beta_{\gamma x}(y) = \int_{\partial\mathbb{H}^k} \left(\int_{\partial\mathbb{H}^n} \varphi(y)\, d\lambda_z(y)\right) d\mu_{\gamma x}(z)$$
$$= \int_{\partial\mathbb{H}^k} \left(\int_{\partial\mathbb{H}^n} \varphi(y)\, d\lambda_{\gamma z}(y)\right) d\mu_x(z)$$
$$= \int_{\partial\mathbb{H}^k} \left(\int_{\partial\mathbb{H}^n} \varphi(\rho(\gamma)y)\, d\lambda_z(y)\right) d\mu_x(z) = \int_{\partial\mathbb{H}^n} \varphi(y)\, d(\rho(\gamma)_*\beta_x)(y).$$

□

Lemma 3.3. — *For all* $x, y \in \mathbb{H}^k$*, the measures* β_x *and* β_y *are in the same density class of measures.*

Proof. — We have to show that for all positive functions $\varphi \in C(\partial\mathbb{H}^n)$ we have

$$\int_{\partial\mathbb{H}^n} \varphi\, d\beta_x = 0 \iff \int_{\partial\mathbb{H}^n} \varphi\, d\beta_y = 0.$$

This follows from the fact that μ_x and μ_y are in the same density class. Indeed, if Φ denotes the function $z \mapsto \int_{\partial\mathbb{H}^n} \varphi\, d\lambda_z$, since the developing measures are positive, Φ is positive, and

$$\int_{\partial\mathbb{H}^n} \varphi\, d\beta_x = \int_{\partial\mathbb{H}^n} \Phi\, d\mu_x = 0 \iff 0 = \int_{\partial\mathbb{H}^n} \Phi\, d\mu_y = \int_{\partial\mathbb{H}^n} \varphi\, d\beta_y.$$

□

Lemma 3.4. — *For all* $x \in \mathbb{H}^k$*,* $||\beta_x|| > 0$*.*

Proof. — It follows from condition (4) of Definition 2.10 and from the fact that $||\mu_x|| > 0$. □

Corollary 3.5. — *For all* $x \in \mathbb{H}^k$*, the measure* β_x *is not concentrated on two points.*

Proof. — By Lemma 3.4, β_x is not the zero-measure. Suppose that β_x has an atom of positive weight at $y_0 \in \partial\mathbb{H}^n$. By Lemmas 3.2 and 3.3, β_x has an atom of positive weight at each point of the $\rho(\Gamma)$-orbit of y_0, which contains most than two points because $\rho(\Gamma)$ is non-elementary. □

It follows that for all $x \in \mathbb{H}^k$, the barycentre of the measure β_x is well-defined and belongs to $\mathbb{H}^n$. We define the natural map $F : \mathbb{H}^k \to \mathbb{H}^n$ by

$$F(x) = \mathrm{bar}(\beta_x).$$

By condition (1) of Lemma 2.8, the natural map is characterised by the implicit equation

$$G(x, F(x)) = 0 \tag{3.1}$$

where

$$\begin{aligned} G(x,\xi) &= \int_{\partial\mathbb{H}^n} dB_{N(\xi,\theta)}(\cdot)\, d\beta_x(\theta) \\ &= \int_{\partial\mathbb{H}^k}\int_{\partial\mathbb{H}^n} dB_{N(\xi,\theta)}(\cdot)\, d\lambda_z(\theta) e^{-\delta(\Gamma)B_K(x,z)}\, d\mu_O(z). \end{aligned}$$

The function G is smooth because the Busemann functions B_K and B_N are smooth. Then, by the implicit function theorem, we get that F is smooth. Moreover, by Lemma 3.2 and claim (2) of Lemma 2.8, it follows that F is ρ-equivariant.

By differentiating equation (3.1) we get that for all $u \in T_x\mathbb{H}^k$ and $v \in T_{F(x)}\mathbb{H}^n$

$$\begin{aligned} &\int_{\partial\mathbb{H}^k}\int_{\partial\mathbb{H}^n} DdB_N|_{(F(x),\theta)}(dF_x(u), v)\, d\lambda_z\, d\mu_x \\ &= \delta(\Gamma)\int_{\partial\mathbb{H}^k}\int_{\partial\mathbb{H}^n} dB_{N(F(x),\theta)}(v) dB_{K(x,z)}(u)\, d\lambda_z\, d\mu_x. \end{aligned} \tag{3.2}$$

Equation (3.2) is the analogous of equation (2.3) of [6]. The proof of the properties of the natural map now goes exactly as in [6, p. 152-154], and the proof of Theorem 3.1 is complete. □

An immediate corollary of Theorem 1.2 is the following fact.

Corollary 3.6. — *Let X, Y be two compact hyperbolic manifold of (possibly different) dimension at least tree. Then in each homotopy class of maps $f : Y \to X$ there exists a smooth map $F : Y \to X$ such that $|\mathrm{Jac}\,(F)| \leqslant 1$. Moreover, if $|\mathrm{Jac}\, F(y)| = 1$, then d_yF is an isometry.*

Proof. — By Theorem 1.2, any $f : Y \to X$ is homotopic to the natural map corresponding to the representation f_*. Such a map has the requested properties because, by Lemma 2.5, we have $\delta(\pi_1(Y)) \leqslant \dim(Y)$. □

As Theorem 1.2, Corollary 3.6 was proved by Besson, Courtois and Gallot in the special case in which ρ is discrete and faithful and both Γ and $\rho(\Gamma)$ are convex co-compact ([6].)

4. Weak extension of equivariant maps: existence of developing measures

In this section we prove Theorem 1.1. In particular, to obtain a family of developing measures, we show that any equivariant map weakly extend to the boundary in the setting of Borel measures. We keep here the notation fixed in Section 2, included the one of Theorem 2.6 for the Patterson-Sullivan measures.

The rough idea is the following. Consider the graphic $G(D)$ of D as a subspace of $\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n$. Then, for each measure μ on $\mathbb{H}^k$ we can consider the graphic measure η on $\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n$, that is, the only measure which is concentrated on $G(D)$ and whose push-forward $\pi_{K*}\eta$ is μ. The sequence $\{\eta_x^s\}$ of the graphic measures corresponding to the measures $\{\mu_x^s\}$, weakly converges to a measure η_x concentrated on $\partial\mathbb{H}^k \times \overline{\mathbb{H}}^n$. Then, by disintegrating the measure η_x we obtain a family $\{\alpha_z\}_{z\in\partial\mathbb{H}^k}$ of measures on $\overline{\mathbb{H}}^n$. By making the convolution with the family of visual measures of $\partial\mathbb{H}^n$, we get a family of developing measures.

For each $x \in \mathbb{H}^k$ and $s > \delta(\Gamma)$ we define a positive Borel measure of finite mass on $\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n$ as follows:

$$\eta_x^s = \frac{1}{c(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)} \delta_{(\gamma O, D(\gamma O))} = \frac{1}{c(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)} \delta_{(\gamma O, \rho(\gamma) D(O))}$$

where $\delta_{(x,y)}$ denotes the Dirac measure concentrated on $(x,y) \in \overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n$. The measures $\{\eta_x^s\}$ are the graphic measures associated to $\{\mu_x^s\}$ and D, and they are concentrated on $\mathbb{H}^k \times \mathbb{H}^n$. Note that $\eta_x^s = \mu_x^s \times \{\delta_{D(z)}\}$ (see Section 2.)

The group $\mathrm{Isom}(\mathbb{H}^k) \times \mathrm{Isom}(\mathbb{H}^n)$ acts on $\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n$ by

$$(g_1, g_2)(x, y) = (g_1 x, g_2 y).$$

Lemma 4.1. — *For all $s > \delta(\Gamma)$ the family $\{\eta_x^s\}$ is ρ-equivariant, that is, for all $x \in \mathbb{H}^k$ and $\psi \in \Gamma$*

$$\eta_{\psi x}^s = (\psi, \rho(\psi))_* \eta_x^s.$$

Proof.

$$\begin{aligned}\eta^s_{\psi x} &= \frac{1}{c(s)}\sum_{\gamma\in\Gamma} e^{-sd(\psi x,\gamma O)}\delta_{(\gamma O, D(\gamma O))} = \frac{1}{c(s)}\sum_{\gamma\in\Gamma} e^{-sd(x,\psi^{-1}\gamma O)}\delta_{(\gamma O, D(\gamma O))}\\ &= \frac{1}{c(s)}\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)}\delta_{(\psi\gamma O, D(\psi\gamma O))}\\ &= \frac{1}{c(s)}\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)}\delta_{(\psi\gamma O, \rho(\psi)(D(\gamma O)))} = (\psi,\rho(\psi))_*\eta^s_x.\end{aligned}$$

□

Now, focus on the point O and consider the family $\{\eta^s_O\}_{s>\delta(\Gamma)}$. Since $||\eta^s_O|| = 1$ for all s, there exists a sequence $s_i \to \delta(\Gamma)^+$ such that $\eta^{s_i}_O$ weakly converges to a positive Borel measure η_O of finite mass on $\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n$

$$\eta^{s_i}_O \rightharpoonup \eta_O.$$

The sequence $\{s_i\}$ depends on the point O, but we will show later that the same sequence works for any point x (see Theorem 4.3 below.) Moreover, by definition, for all $s > \delta(\Gamma)$ and $x \in \mathbb{H}^k$, the measure μ^s_x is the π_K-push-forward of η^s_x. Then, weak continuity of the push-forward implies

$$\mu_O = (\pi_K)_*\eta_O.$$

In particular, since the support of the Patterson-Sullivan measures is the limit set $\Lambda(\Gamma)$ of Γ, the measure η_O is concentrated on $\Lambda(\Gamma)\times\overline{\mathbb{H}}^n$. A priori, η_O is non supported on a graphic; nonetheless, as it is a positive Borel measure, we can use the theorem of disintegration of measures ([1, Theorem 2.28], see also [10],) which asserts that for any positive Borel measure η on $\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n$, there exists a family of positive Borel measures $\{\alpha^\eta_z\}_{z\in\overline{\mathbb{H}}^k}$ such that, if $\mu = (\pi_K)_*\eta$, then $\eta = \mu\times\{\alpha^\eta_z\}$. Thus, for all $\varphi \in C(\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n)$

$$\iint_{\overline{\mathbb{H}}^k\times\overline{\mathbb{H}}^n} \varphi\, d\eta = \int_{\overline{\mathbb{H}}^k}\left(\int_{\overline{\mathbb{H}}^n}\varphi(x,y)\, d\alpha^\eta_z(y)\right) d\mu(z),$$

and for μ-almost z, the measure α^η_z is a probability measure. We say that the family $\{\alpha^\eta_z\}$ *disintegrates* the measure η (compare disintegration with property (2) of Definition 2.10).

Moreover, the measures α^η_z's are characterised by the following property. For all $z \in \overline{\mathbb{H}}^k$, let $\{U_j(z) \subset \overline{\mathbb{H}}^k\}_{j\in\mathbb{N}}$ be a sequence of nested balls centred at z such that $\cap_j U_j(z) = z$. For $j \in \mathbb{N}$, let $\psi^{z,\eta}_j : \overline{\mathbb{H}}^k \to \mathbb{R}$ be the following function (defined for μ-almost all z)

$$\psi_j^{z,\eta} = \frac{\chi_{U_j(z)}}{\mu(U_j(z))}$$

where χ_A denotes the characteristic function of the set A (note that $\psi_j^{z,\eta} \to \delta_z$ for μ-almost all z). Then, for μ-almost all z and for all $\varphi \in C(\overline{\mathbb{H}}^n)$

$$\int_{\overline{\mathbb{H}}^n} \varphi(y)\, d\alpha_z^\eta(y) = \lim_{j\to\infty} \iint_{\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n} \psi_j^{z,\eta}(x)\varphi(y)\, d\eta(x,y).$$

From now on, the set $\{\alpha_z\}_{z\in\overline{\mathbb{H}}^k}$ will denote the family of measures that disintegrates η_O, where we set $\alpha_z = 0$ for $z \in \mathbb{H}^k$. It seems worth mentioning that the choice of α_z for z in the interior of $\mathbb{H}^k$ is not relevant for our purposes. Indeed, since any μ_x is concentrated on the limit set of Γ, which is contained in the boundary at infinity of $\mathbb{H}^k$, we have $\mu_x(\mathbb{H}^k) = 0$. In particular, the value of α_z for z in the interior of $\mathbb{H}^k$ does not affect the product measure $\mu_x \times \{\alpha_z\}$.

Definition 4.2. — *For each $x \in \mathbb{H}^k$ we set $\eta_x = \mu_x \times \{\alpha_z\}$.*

Now, we have two problems. First, while it is true that any $\{\eta_x^s\}$ has a converging sub-sequence, it is not clear a priori that the same sub-sequence works for all x. Second, by definition, the family $\{\alpha_z\}$ disintegrates the limit measure η_O, and there is no reasons for such a family to disintegrate the limits of η_x^s for all x. In other words, it is not clear a priori that η_x^s converges to η_x. The following theorem settles both questions. This is the core of the proof of Theorem 1.1, as it implies that the family $\{\alpha_z\}$ is equivariant (see Lemma 4.4 below.)

Theorem 4.3. — *For all $x \in \mathbb{H}^k$, the sequence $\{\eta_x^{s_i}\}$ weakly converges to η_x. Where $\{s_i\}$ is the sequence of indices such that $\eta_O^{s_i} \rightharpoonup \eta_O$.*

Proof. — We fix $x \in \mathbb{H}^k$. Since the measures $\eta_x^{s_i}$ are bounded in norm, up to pass to sub-sequences, we can suppose that $\{\eta_x^{s_i}\}$ weakly converges to a measure $\widetilde{\eta}_x$. We prove now that any such possible limit $\widetilde{\eta}_x$ coincides with η_x, and this will prove the thesis. Let $\{\widetilde{\alpha}_z\}$ be the family that disintegrates $\widetilde{\eta}_x$. By weak continuity of push forward

$$\pi_{K*}(\widetilde{\eta}_x) = \mu_x = \pi_{K*}(\eta_x).$$

Therefore, it is sufficient to show that $\widetilde{\alpha}_z = \alpha_z$ for μ_O-almost all z (recall that since μ_x and μ_O are in the same density-class, the notions of μ_O-negligible set and μ_x-negligible set coincide.) For any positive function $\varphi \in$

$C(\overline{\mathbb{H}}^n)$ and $z \in \partial\mathbb{H}^k$

$$\begin{aligned}
&\int_{\overline{\mathbb{H}}^n} \varphi(y)\, d\widetilde{\alpha}_z(y) = \lim_{j\to\infty} \iint_{\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n} \psi_j^{z,\widetilde{\eta}_x}(\xi)\varphi(y)\, d\widetilde{\eta}_x(\xi, y)\\
&= \lim_{j\to\infty} \lim_{s_i\to\delta(\Gamma)^+} \iint_{\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n} \psi_j^{z,\widetilde{\eta}_x}(\xi)\varphi(y)\, d\eta_x^{s_i}(\xi, y)\\
&= \lim_{j\to\infty} \lim_{s_i\to\delta(\Gamma)^+} \int_{\overline{\mathbb{H}}^k} \psi_j^{z,\widetilde{\eta}_x}(\xi)\varphi(D(\xi))\, d\mu_x^{s_i}(\xi)\\
&= \lim_{j\to\infty} \lim_{s_i\to\delta(\Gamma)^+} \int_{U_j(z)} \varphi(D(\xi)) \frac{1}{\mu_x(U_j(z))}\, d\mu_x^{s_i}(\xi)\\
&= \lim_{j\to\infty} \lim_{s_i\to\delta(\Gamma)^+} \int_{U_j(z)} \varphi(D(\xi)) \frac{\mu_O(U_j(z))e^{-\delta(\Gamma)B_K(x,z)}}{\mu_x(U_j(z))\mu_O(U_j(z))e^{-\delta(\Gamma)B_K(x,z)}}\, d\mu_x^{s_i}(\xi)
\end{aligned}$$

whence, using the definition of $\mu_x^{s_i}$, and setting

$$A_j(z) = \frac{\mu_O(U_j(z))e^{-\delta(\Gamma)B_K(x,z)}}{\mu_x(U_j(z))},$$

we get

$$\begin{aligned}
&\int_{\overline{\mathbb{H}}^n} \varphi(y)\, d\widetilde{\alpha}_z(y)\\
&= \lim_{j\to\infty} A_j(z) \lim_{s_i\to\delta(\Gamma)^+} \int_{U_j(z)} \frac{\varphi(D(\xi))}{\mu_O(U_j(z))e^{-\delta(\Gamma)B_K(x,z)}}\, d\mu_x^{s_i}(\xi) \qquad (4.1)\\
&= \lim_{j\to\infty} A_j(z) \lim_{s_i\to\delta(\Gamma)^+} \frac{1}{c(s_i)} \sum_{\substack{g\,\in\,\Gamma\\ gO\,\in\,U_j}} \frac{\varphi(D(gO))e^{-s_i d(x,gO)}}{\mu_O(U_j(z))e^{-\delta(\Gamma)B_K(x,z)}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{\varphi(D(gO))e^{-s_i d(x,gO)}}{\mu_O(U_j(z))e^{-\delta(\Gamma)B_K(x,z)}}\\
&= \frac{\varphi(D(gO))}{\mu_O(U_j(z))} \cdot \frac{e^{-s_i(d(x,gO)-d(O,gO))}}{e^{-\delta(\Gamma)B_K(x,z)}} \cdot e^{-s_i d(O,gO)}\\
&= \frac{\varphi(D(gO))}{\mu_O(U_j(z))} \cdot \frac{e^{-\delta(\Gamma)(d(x,gO)-d(O,gO))}}{e^{-\delta(\Gamma)B_K(x,z)}} \cdot \frac{e^{-s_i(d(x,gO)-d(O,gO))}}{e^{-\delta(\Gamma)(d(x,gO)-d(O,gO))}} \cdot e^{-s_i d(O,gO)}.
\end{aligned}$$

From the definition of the Busemann function B_K, it follows that for all $z \in \partial\mathbb{H}^k$ and $p \in \mathbb{H}^k$

$$\lim_{\xi\to z} (d(p,\xi) - d(O,\xi)) = B_K(p,z). \qquad (4.2)$$

Therefore, there exist two sequences $\{E_j^+\}$ and $\{E_j^-\}$, converging to 1 as $j\to\infty$, and such that for all $g\in\Gamma$ with $gO\in U_j(z)$

$$E_j^- \leqslant \frac{e^{-\delta(\Gamma)(d(x,gO)-d(O,gO))}}{e^{-\delta(\Gamma)B_K(x,z)}} \leqslant E_j^+.$$

Moreover, equation (4.2) implies that the term $(d(x,gO)-d(O,gO))$ is bounded, so for each j

$$\lim_{s_i\to\delta(\Gamma)^+} \frac{e^{-s_i(d(x,gO)-d(O,gO))}}{e^{-\delta(\Gamma)(d(x,gO)-d(O,gO))}} = 1$$

uniformly on U_j. Whence, since φ is positive,

$$\begin{aligned}
&\lim_{j\to\infty} A_j(z) \lim_{s_i\to\delta(\Gamma)^+} \frac{1}{c(s_i)} \sum_{\substack{g\,\in\,\Gamma\\ gO\,\in\, U_j}} \frac{\varphi(D(gO))e^{-s_i d(x,gO)}}{\mu_O(U_j(z))e^{-\delta(\Gamma)B_K(x,z)}}\\
&\leqslant \lim_{j\to\infty} A_j(z)E_j^+ \lim_{s_i\to\delta(\Gamma)^+} \frac{1}{c(s_i)} \sum_{\substack{g\,\in\,\Gamma\\ gO\,\in\, U_j}} \frac{\varphi(D(gO))}{\mu_O(U_j(z))}\cdot e^{-s_i d(O,gO)}\\
&= \lim_{j\to\infty} A_j(z)E_j^+ \lim_{s_i\to\delta(\Gamma)^+} \int_{U_j(z)} \frac{\varphi(D(x))}{\mu_O(U_j(z))}\, d\mu_O^{s_i}(x)
\end{aligned} \tag{4.3}$$

and similarly for E_j^-. Since for μ_O-almost all z we have $\lim_j A_j(z)=1$, and since $E_j^\pm\to 1$, we get that μ_O-almost everywhere

$$\begin{aligned}
&\lim_{j\to\infty} A_j(z)E_j^\pm \lim_{s_i\to\delta(\Gamma)^+} \int_{U_j(z)} \frac{\varphi(D(\xi))}{\mu_O(U_j(z))}\, d\mu_O^{s_i}(\xi)\\
&= \lim_{j\to\infty} \lim_{s_i\to\delta(\Gamma)^+} \int_{U_j(z)} \varphi(D(\xi)) \frac{1}{\mu_O(U_j(z))}\, d\mu_O^{s_i}(\xi)\\
&= \lim_{j\to\infty} \lim_{s_i\to\delta(\Gamma)^+} \int_{\overline{\mathbb{H}}^k} \psi_j^{z,\eta_O}(\xi)\varphi(D(\xi))\, d\mu_O^{s_i}(\xi)\\
&= \lim_{j\to\infty} \lim_{s_i\to\delta(\Gamma)^+} \iint_{\overline{\mathbb{H}}^k\times\overline{\mathbb{H}}^n} \psi_j^{z,\eta_O}(\xi)\varphi(y)\, d\eta_O^{s_i}(\xi,y)\\
&= \lim_{j\to\infty} \iint_{\overline{\mathbb{H}}^k\times\overline{\mathbb{H}}^n} \psi_j^{z,\eta_O}(\xi)\varphi(y)\, d\eta_O(\xi,y)\\
&= \int_{\overline{\mathbb{H}}^n} \varphi(y)\, d\alpha_z(y).
\end{aligned} \tag{4.4}$$

Finally, from equations (4.1) – (4.4), μ_O-almost everywhere we get

$$\int_{\overline{\mathbb{H}}^n} \varphi(y)\, d\alpha_z(y) \leqslant \int_{\overline{\mathbb{H}}^n} \varphi(y)\, d\widetilde{\alpha}_z(y) \leqslant \int_{\overline{\mathbb{H}}^n} \varphi(y)\, d\alpha_z(y)$$

and the claim follows. □

In particular, Theorem 4.3 implies that the measures η_x's are ρ-equivariant. Indeed, since the measures $\{\eta_x^s\}$ are ρ-equivariant, and since the push-forward is continuous for the weak convergence, for all $x \in \mathbb{H}^k$ and $\gamma \in \Gamma$

$$\eta_{\gamma x}^{s_i} = (\gamma, \rho(\gamma))_* \eta_x^{s_i} \rightharpoonup (\gamma, \rho(\gamma))_* \eta_x.$$

Moreover, since $\pi_{K*}(\eta_x) = \mu_x$, each measure η_x is concentrated in $\partial\mathbb{H}^k \times \overline{\mathbb{H}}^n$.

Lemma 4.4. — *The family $\{\alpha_z\}$ is ρ-equivariant, that is, for all $\gamma \in \Gamma$ and μ_O-almost all $z \in \partial\mathbb{H}^k$*

$$\alpha_{\gamma z} = \rho(\gamma)_* \alpha_z.$$

Proof. — From point (3) of Theorems 2.6 and the the ρ-equivariance of the η_x's, it follows that for all $\varphi \in C(\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n)$

$$\begin{aligned}
&\iint_{\partial\mathbb{H}^k \times \overline{\mathbb{H}}^n} \varphi(\gamma z, y)\, d\alpha_{\gamma z}(y)\, d\mu_O(z) \\
&= \iint_{\partial\mathbb{H}^k \times \overline{\mathbb{H}}^n} \varphi(z, y)\, d\alpha_z(y)\, d\mu_{\gamma O}(z) \\
&= \iint_{\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n} \varphi(z, y)\, d\eta_{\gamma O} = \iint_{\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n} \varphi(\gamma z, \rho(\gamma) y)\, d\eta_O \\
&= \iint_{\partial\mathbb{H}^k \times \overline{\mathbb{H}}^n} \varphi(\gamma z, \rho(\gamma) y)\, d\alpha_z(y)\, d\mu_O(z) \\
&= \iint_{\overline{\mathbb{H}}^k \times \overline{\mathbb{H}}^n} \varphi(\gamma z, y)\, d(\rho(\gamma)_* \alpha_z)(y)\, d\mu_O(z).
\end{aligned}$$

Whence, the measures $\alpha_{\gamma z}$ and $\rho(\gamma)_* \alpha_z$ equal μ_O-almost everywhere. $\square$

Now, for each $y \in \overline{\mathbb{H}}^n$ let ν_y be the visual measure on $\partial\mathbb{H}^n$ centred at y. More precisely, choose the disc model of $\mathbb{H}^n$ whose centre is O and let ν_O be the standard probability measure on $S^{n-1} \simeq \partial\mathbb{H}^n$. Then, for all $g \in \mathrm{Isom}(\mathbb{H}^n)$ define

$$\nu_{gO} = g_* \nu_O.$$

This definition is not ambiguous because ν_O is $\mathrm{Stab}(O)$-invariant, where $\mathrm{Stab}(O) = \{g \in \mathrm{Isom}(\mathbb{H}^n) : g(O) = O\}$. For $y \in \partial\mathbb{H}^n$ simply define $\nu_y = \delta_y$.

For all $z \in \partial\mathbb{H}^k$ define λ_z a measure on $\partial\mathbb{H}^n$ by $\lambda_z = \alpha_z * \{\nu_y\}$. That is, for all $\varphi \in C(\partial\mathbb{H}^n)$

$$\int_{\partial\mathbb{H}^n} \varphi\, d\lambda_z = \int_{\overline{\mathbb{H}}^n} \left(\int_{\partial\mathbb{H}^n} \varphi(\theta)\, d\nu_y(\theta) \right) d\alpha_z(y).$$

Note that such an integral is well-defined because, for any φ, the function $y \mapsto \int_{\partial\mathbb{H}^n} \varphi \, d\nu_y$ is continuous in y.

Remark 4.5. — Since the measure ν_y depends continuously on y, by Lemma 2.11, the convolution with the family of visual measures is weakly continuous.

We show now that $\{\lambda_z\}_{z\in\partial\mathbb{H}^k}$ is a family of developing measures for ρ.

LEMMA 4.6. — *The family $\{\lambda_z\}$ is ρ-equivariant, that is, for μ_O-almost all $z \in \partial\mathbb{H}^k$ and all $\gamma \in \Gamma$, we have $\lambda_{\gamma z} = \rho(\gamma)_*\lambda_z$.*

Proof. — By Lemma 4.4, for μ_O-almost all $z \in \partial\mathbb{H}^k$ and all $\varphi \in C(\partial\mathbb{H}^n)$

$$\begin{aligned}\int_{\partial\mathbb{H}^n} \varphi \, d\lambda_{\gamma z} &= \int_{\overline{\mathbb{H}}^n}\int_{\partial\mathbb{H}^n} \varphi(\theta)\, d\nu_y(\theta)\, d\alpha_{\gamma z}(y) = \int_{\overline{\mathbb{H}}^n}\int_{\partial\mathbb{H}^n} \varphi(\theta)\, d\nu_{\rho(\gamma)y}(\theta)\, d\alpha_z(y)\\ &= \int_{\overline{\mathbb{H}}^n}\int_{\partial\mathbb{H}^n} \varphi(\rho(\gamma)\theta)\, d\nu_y(\theta)\, d\alpha_z(y)\\ &= \int_{\partial\mathbb{H}^n} \varphi\circ\rho(\gamma)\, d\lambda_z = \int_{\partial\mathbb{H}^n} \varphi \, d(\rho(\gamma)_*\lambda_z).\end{aligned}$$

□

LEMMA 4.7. — *For any $\varphi \in C(\partial\mathbb{H}^n)$, the function $z \mapsto \int_{\partial\mathbb{H}^n} \varphi(\theta)\, d\lambda_z(\theta)$ is μ_O-integrable.*

Proof. — This follows directly from the definition of λ_z, because the family $\{\alpha_z\}$ disintegrates η_O, and $\mu_O = \pi_{K*}\eta_O$. □

LEMMA 4.8. — *For μ_O-almost all $z \in \partial\mathbb{H}^k$, we have $||\lambda_z|| = 1$.*

Proof. — For all $x \in \mathbb{H}^k$ and $y \in \partial\mathbb{H}^n$, the measures η_x and ν_y are positive. Then the measures α_z's are positive, and this implies that the measures λ_z's are positive. Thus

$$||\lambda_z|| = \int_{\partial\mathbb{H}^n} 1\, d\lambda_z = \int_{\overline{\mathbb{H}}^n}\int_{\partial\mathbb{H}^n} 1\, d\nu_y(\theta)\, d\alpha_z(y) = \int_{\overline{\mathbb{H}}^n} 1\, d\alpha_z = ||\alpha_z|| = 1$$

μ_O-almost everywhere because $\{\alpha_z\}$ disintegrates η_O.

Therefore, the family $\{\lambda_z\}$ satisfies properties $(1)-(4)$ of Definition 2.10. So it is a family of developing measure for ρ, and the proof of Theorem 1.1 is complete. □

5. Sequence of ε-natural maps

In this section we prove Theorem 1.3, showing that the natural map constructed in Sections 3 and 4 is the limit of a sequence of ε-natural maps.

We keep the notation of Sections 2-4. Through this section we suppose that Γ diverges at $\delta(\Gamma)$; by Lemma 2.5, this is the case if $\mathbb{H}^k/\Gamma$ has finite volume.

Let $A \subset \mathbb{H}^k$ be the Dirichlet domain of O. The set A is a fundamental domain for Γ containing O. Let σ be any Borel probability measure on A.

Definition 5.1. — *For each $s > \delta(\Gamma)$ and $x \in \mathbb{H}^k$ we define m_x^s a positive Borel measure on $\mathbb{H}^k$ by*

$$m_x^s = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma O)} \gamma_* \sigma.$$

Note that if $\sigma = \delta_O$, then $m_x^s = \mu_x^s$.

Lemma 5.2. — *Let $\{\gamma_n\}$ be any sequence of elements of Γ. If $\gamma_n(O)$ converges to a point $\theta \in \partial\mathbb{H}^k$, then for any sequence $\{x_n\} \in A$, the sequence $\{\gamma_n(x_n)\}$ converges to θ.*

Proof. — Suppose the contrary. Since $\overline{\mathbb{H}}^k$ is compact, up to passing to a sub-sequence, we can suppose that $\{\gamma_n(x_n)\}$ converges to a point $\zeta \neq \theta$ in $\overline{\mathbb{H}}^k$. Then, the geodesics joining $\gamma_n(x_n)$ to $\gamma_n(O)$ accumulate near the geodesic between ζ and θ. This can not happen because Γ is discrete. □

Theorem 5.3. — *For each $x \in \mathbb{H}^k$, if μ_x denotes the Patterson-Sullivan measure constructed as in Theorem 2.6, then*

$$m_x^s \rightharpoonup \mu_x$$

in $\overline{\mathbb{H}}^k$, when $s \to \delta(\Gamma)^+$.

Proof. — We have to show that for each $\varphi \in C(\overline{\mathbb{H}}^k)$, $\int \varphi \, dm_x^s \to \int \varphi \, d\mu_x$. Let $\varphi \in C(\overline{\mathbb{H}}^k)$. Since $\mu_x^s \to \mu_x$, we will have finished by proving that

$$\lim_{s \to \delta(\Gamma)^+} \left| \int_{\overline{\mathbb{H}}^k} \varphi \, dm_x^s - \int_{\overline{\mathbb{H}}^k} \varphi \, \mu_x^s \right| = 0.$$

Let $C > 0$ be a small constant and let $A_1 \subset A$ be a compact set such that $O \in A_1$ and $\sigma(A \setminus A_1) < C$. Since the supports of the measures m_x^s and μ_x^s are contained in $\mathbb{H}^k$, and since $O \in A$, we have

$$(5.1) \quad \int_{\overline{\mathbb{H}}^k} \varphi \, d(m_x^s - \mu_x^s) = \sum_{\gamma \in \Gamma} \int_{\gamma(A_1)} \varphi \, d(m_x^s - \mu_x^s) + \sum_{\gamma \in \Gamma} \int_{\gamma(A \setminus A_1)} \varphi \, dm_x^s.$$

Looking at the second summand,

$$\begin{aligned}
&\Big|\sum_{\gamma\in\Gamma}\int_{\gamma(A\setminus A_1)}\varphi\, dm_x^s\Big|\\
&\leqslant \max(|\varphi|)\frac{1}{c(s)}\sum_{\gamma\in\Gamma}\int_{\gamma(A\setminus A_1)} e^{-sd(x,\gamma O)}\, d\gamma_*\sigma(\xi) \qquad (5.2)\\
&= \max(|\varphi|)\frac{1}{c(s)}\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)}\sigma(A\setminus A_1) \leqslant C\cdot\max(\varphi)\cdot||\mu_x^s||.
\end{aligned}$$

We estimate now the first summand. Since $c(s)\to\infty$, for any finite subset Γ_1 of Γ

$$\lim_{s\to\delta(\Gamma)^+}\Big|\sum_{\gamma\in\Gamma_1}\int_{\gamma(A_1)}\varphi\, dm_x^s\Big| = \lim_{s\to\delta(\Gamma)^+}\Big|\sum_{\gamma\in\Gamma_1}\int_{\gamma(A_1)}\varphi\, d\mu_x^s\Big| = 0. \tag{5.3}$$

Moreover, since φ is continuous, it is uniformly continuous on $\overline{\mathbb{H}}^k$ for any metric that induces the usual topology on $\overline{\mathbb{H}}^k$ (recall that the hyperbolic metric of $\mathbb{H}^k$ is not a metric on $\overline{\mathbb{H}}^k$). Therefore, Lemma 5.2 implies that, except for a finite number of elements of Γ, we have

$$|\varphi(\gamma O)-\varphi(\gamma\xi)| < C \tag{5.4}$$

independently on γ and on $\xi\in A_1$. Let Γ_1 be a finite subset of Γ such that (5.4) holds for $\gamma\in\Gamma\setminus\Gamma_1$. Then,

$$\begin{aligned}
&\Big|\sum_{\gamma\in\Gamma\setminus\Gamma_1}\int_{\gamma(A_1)}\varphi\, d(m_x^s-\mu_x^s)\Big|\\
&= \Big|\frac{1}{c(s)}\sum_{\gamma\in\Gamma\setminus\Gamma_1} e^{-sd(x,\gamma O)}\Big(\int_{\gamma(A_1)}\varphi(\xi)\, d\gamma_*\sigma(\xi)\\
&\qquad\qquad - \varphi(\gamma O)\big(\sigma(A_1)+\sigma(A\setminus A_1)\big)\Big)\Big|\\
&\leqslant C\int_{\overline{\mathbb{H}}^k}|\varphi|\, d\mu_x^s + \frac{1}{c(s)}\sum_{\gamma\in\Gamma\setminus\Gamma_1} e^{-sd(x,\gamma O)}\int_{A_1}|\varphi(\gamma\xi)-\varphi(\gamma O)|\, d\sigma(\xi)\\
&\leqslant C\int_{\overline{\mathbb{H}}^k}|\varphi|\, d\mu_x^s + \frac{C}{c(s)}\sum_{\gamma\in\Gamma\setminus\Gamma_1} e^{-sd(x,\gamma O)}\cdot\sigma(A_1) \leqslant C\int_{\overline{\mathbb{H}}^k}(1+|\varphi|)\, d\mu_x^s.
\end{aligned}$$

Whence the claim follows, combining with (5.1), (5.2) and (5.3), since C can be chosen arbitrarily small. □

Now we proceed as in Sections 3 and 4. Namely, we fix a ρ-equivariant map D, we define measures $n_x^s = m_x^s\times\{\delta_{D(z)}\}$, and we chose a sequence $s_i\to\delta(\Gamma)^+$ such that $n_O^{s_i}$ converges to a measure n_O. We disintegrate n_O as $n_O=\mu_O\times\{a_z\}$, and we define $n_x=\mu_x\times\{a_z\}$. As in Theorem 4.3 one can show that $n_x^{s_i} \rightharpoonup n_x$. We define then $b_x^s = (D_* m_x^s) * \{\nu_y\} = \pi_{N*}(n_x^s) * \{\nu_f\}$, and $b_x = \pi_{N*}(n_x) * \{\nu_f\}$.

Finally, for each $s > \delta(\Gamma)$ we set $s = (1+\varepsilon)\delta(\Gamma)$ and we define maps $F(x) = \mathrm{bar}(b_x)$ and $F_\varepsilon(x) = \mathrm{bar}(b_x^s)$. The map F is a natural map, in the sense that it has the properties (1) – (4) of Theorem 1.2. The maps F_ε the ε-natural maps constructed in [6], so they are smooth, ρ-equivariant, and for all $p \geqslant 3$ and $\varepsilon > 0$, $\mathrm{Jac}_p F_\varepsilon(x) \leqslant \left(\dfrac{(1+\varepsilon)\delta(\Gamma)}{p-1}\right)^p$.

Proposition 5.4. — *The maps F_{ε_i} punctually converge to the map F.*

Proof. — From the weak continuity of the push-forward and from Lemma 2.11, we get

$$b_x^{s_i} \rightharpoonup b_x.$$

Then, the claim follows from Point 3 of Lemma 2.8. □

Remark 5.5. — The weak convergence of b_x^s to b_x is enough to prove stronger convergences. For example, it can be shown that the derivatives of F_{ε_i} converges the ones of F, whence one gets that the convergence of the ε-natural maps is locally uniform (see [14] for details).

6. Rigidity of representations

In this section we give a proof of Theorem 1.4, referring the reader to [14] for a fully detailed discussion on the matter.

Proof of Theorem 1.4. — Let M be a complete hyperbolic k-manifold of finite volume and let $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation. We consider $\mathbb{H}^k$ as the universal cover of M, and we identify $\pi_1(M) < \mathrm{Isom}(\mathbb{H}^k)$ with the group of deck transformations of $\mathbb{H}^k \to M$.

We denote by a *pseudo-developing map* any piecewise smooth, ρ-equivariant map $D : \mathbb{H}^k \to \mathbb{H}^n$. The volume of a pseudo developing map D is defined by integrating the pull-back of the volume form of $\mathbb{H}^n$ on a fundamental domain for M. Equivalently, if ω_N denotes the volume form of $\mathbb{H}^n$, by equivariance, the form $D^*\omega_N$ descends to M and we set

$$\mathrm{vol}(D) = \int_M D^*\omega_N.$$

Now, let us suppose that M is compact. In this case we define the volume of ρ as the infimum of the volumes of all the pseudo-developing maps for ρ:

$$\mathrm{vol}(\rho) = \inf_D \mathrm{vol}(D).$$

Note that the volume of any elementary representation vanishes, so that we can suppose ρ to be non-elementary.

The existence of a natural map immediately gives the inequality; indeed Theorem 1.2 point 3, together with the fact that $\delta(\pi_1(M)) = k-1$, tells us that the volume of a natural map is less than $\mathrm{vol}(M)$. Moreover, if ρ has maximum volume, then from Theorem 1.2 point 4 it follows that in each point the differential of a natural map is an isometry. Since $\mathrm{vol}(\rho)$ is maximal, we deduce that the image of a natural map is a locally minimal sub-manifold of $\mathbb{H}^n$. The claim now follows because a locally isometric and locally minimal immersion from a Riemannian manifold to another is totally geodesic.

In the non-compact case, some problem arises. From now on we suppose that M is non-compact. If we keep the above definition of volume of ρ we get that the volume of any representation vanishes. This is because any non-compact manifold collapses to a spine, which is a codimension-one object. To avoid such a pathology we need to require that a pseudo-developing map has a nice behaviour on the cusps. For G a group of isometries, we denote by $\mathrm{Fix}(G)$ the set of fixed points of G (including the points at infinity). If C is a cusp of M, it is readily checked that $\pi_1(C)$ is Abelian and that $\mathrm{Fix}(\pi_1(C))$ consists of a unique point of $\partial\mathbb{H}^k$.

We say that a pseudo-developing map D *properly ends* if for any cusp C of M, if $\xi = \mathrm{Fix}(\pi_1(C))$ and $\alpha(t)$ is a geodesic ray ending at ξ, then all limit points of $D(\alpha(t))$ lie either in $\mathrm{Fix}(\rho(\pi_1(C))) \subset \overline{\mathbb{H}}^n$ or in a finite union of $\rho(\pi_1(C))$-invariant geodesics. It easy to see that properly ending pseudo-developing maps always exist.

Now we can define the volume of ρ by taking the infimum of the volumes of properly ending pseudo-developing maps.

$$\mathrm{vol}(\rho) = \inf_{D \text{ properly ending}} \mathrm{vol}(D).$$

We would like to use a natural map as in the compact case, but the problem now is that a natural map given by Theorem 1.2 in general does not end properly.

Nevertheless, we can use the ε-natural maps. Indeed, it can be shown (see [14]) that any ε-natural map constructed as in Section 5 properly ends, and this gives immediately the inequality. Moreover, by Proposition 5.4 and Remark 5.5 we have that the volumes of the ε-natural maps converge to the volume of a natural map. Thus, if the volume of ρ is maximal, then the volume of a natural map is maximal and we conclude as in the compact case. □

7. Measurable extension of natural maps

This section is devoted to proving Theorem 1.5. We keep here the notation of previous sections. In particular we recall that $\{\mu_x\}_{x\in\mathbb{H}^k}$ is the family of Patterson-Sullivan measures, and that $\{\lambda_z\}_{z\in\partial\mathbb{H}^k}$ is a family of developing measures.

Definition 7.1. — *Let $\{\gamma_i O\}$ be a sequence in the Γ-orbit of O. We say that $\gamma_i O$* conically *converges to $\omega \in \partial\mathbb{H}^k$ if $\gamma_i O \to \omega$ and there exists a geodesic σ, ending at ω, such that the distance of $\gamma_i O$ from σ is bounded. The* conical limit set *of Γ, denoted by Λ_c, is the set of the limits of conically converging sequences in the Γ-orbit of O.*

Clearly, the conical limit set is a sub-set of the limit set of Γ. In order to prove that the natural maps extend to the boundary we need the following result.

Theorem 7.2. — *For each $f \in L^1(\partial\mathbb{H}^k, \mu_O)$ there exists a set Z with $\mu_O(Z) = 0$ such that for all $\omega \in \Lambda_c \setminus Z$, and for any sequence $\{\gamma_i\} \subset \Gamma$ such that $\gamma_i O$ conically converges to ω*

$$\int_{\partial\mathbb{H}^k} f(\theta)\, d\mu_{\gamma_i O}(\theta) \to f(\omega).$$

Before proving Theorem 7.2 we show how it implies Theorem 1.5.

Proof of Theorem 1.5. — First, we prove the existence part. By definition (see Section 2) we have

$$\begin{aligned}\mathcal{B}_{\beta_x}(y) &= \int_{\partial\mathbb{H}^n} B_N(y,\theta)\, d\beta_x(\theta) = \int_{\partial\mathbb{H}^k}\int_{\partial\mathbb{H}^n} B_N(y,\theta)\, d\lambda_z(\theta)\, d\mu_x(z)\\ &= \int_{\partial\mathbb{H}^k} \mathcal{B}_{\lambda_z}(y)\, d\mu_x(z)\end{aligned}$$

and a similar formula holds for the derivatives of $\mathcal{B}_{\beta_x}$. Therefore, for each y there exists a μ_O-negligible set $Z \subset \partial\mathbb{H}^k$ such that for all $\omega \in \partial\mathbb{H}^k \setminus Z$ we have

$$\lim_{x\to\omega} \mathcal{B}_{\beta_x}(y) = B_{\lambda_\omega}(y)$$

Where "$\lim_{x\to\omega}$" means "for any sequence $\{x_i\}$ in the Γ-orbit of O, conically converging to ω...". The same statement holds for the derivatives of $\mathcal{B}_{\beta_x}$. Now, let Y be a countable dense subset of $\mathbb{H}^n$. Then (since countable unions of negligible sets are negligible), there exists a μ_O-negligible set $W \subset \partial\mathbb{H}^k$ such that the above limit holds for all $\omega \in \partial\mathbb{H}^k \setminus W$, all $y \in Y$ and all the derivatives of $\mathcal{B}_{\beta_x}$. It follows that the barycentre of β_x, that is the unique point of minimum of $\mathcal{B}_{\beta_x}$, converges to the barycentre of λ_ω, which is well-defined because λ_ω is not the sum of two Dirac deltas with equal weights. Therefore, if F denotes the natural map constructed using the family $\{\lambda_z\}$, setting $\overline{F}(\omega) = \mathrm{bar}(\lambda_\omega)$, we have that for μ_O-almost all ω, for any sequence $\{\gamma_i O\}$ conically converging to ω

$$\lim_{\gamma_i O \to \omega} F(\gamma_i O) = \overline{F}(\omega) = \mathrm{bar}(\lambda_\omega). \tag{7.1}$$

The map $\overline{F}$ is measurable because it can be viewed as a limit of continuous functions. Finally, it is readily checked that $F(\gamma_i O)$ and $F(\gamma_i x)$ have the same limit, and this completes the proof of the existence part. It remains now to prove the last part of Theorem 1.5.

Given the maps $\overline{F}_1$ and $\overline{F}_2$, we construct the corresponding natural maps F_1, F_2. For $i = 1, 2$ and for almost all ω in the conical limit set of Γ, if $\{\gamma_n O\}$ is a sequence conically converging to ω, then by (7.1), $F_i(\gamma_n O) \to \mathrm{bar}(\overline{F}_i(\omega))$. By equivariance we have

$$F_i(\gamma_n O) = \rho(\gamma_n) F_i(O) \qquad i = 1, 2.$$

Up to pass to a sub-sequence, either $\rho(\gamma_n)$ converges to an isometry ψ of $\mathbb{H}^n$, or the limit of $\rho(\gamma_n) y$ belongs to $\partial\mathbb{H}^n$ and does not depend on $y \in \mathbb{H}^n$ (see for example [15]). In the former case, we get $\psi(F_i(O)) = \mathrm{bar}(\overline{F}_i(\omega))$; but also, for all $\gamma \in \Gamma$ we have

$$\psi(\rho(\gamma) F_i(O)) = \psi(F_i(\gamma O)) = \lim \rho(\gamma_n) F_i(\gamma O).$$

The sequence $\gamma_n \gamma O$ conically converges to ω, since $\gamma_n O$ does. Then, by equivariance and by(7.1), we get

$$\lim \rho(\gamma_n) F_i(\gamma O) = \lim F_i(\gamma_n(\gamma O)) = \mathrm{bar}(\overline{F}_i(\omega))$$

whence $\psi(\rho(\gamma) F_i(O)) = \psi(F_i(O))$ and thus $\rho(\gamma) F_i(O) = F_i(O)$, contradicting the fact that the image of ρ is non-elementary. Therefore, we are in the latter case and in particular

$$\mathrm{bar}(\overline{F}_1(\omega)) = \lim \rho(\gamma_n) F_i(O) = \lim \rho(\gamma_n) F_2(O) = \mathrm{bar}(\overline{F}_2(\omega)) \in \partial\mathbb{H}^n$$

but this is possible if and only if $\overline{F}_i(\omega) = \delta_{\mathrm{bar}(F_i(\omega))}$. So $\overline{F}_1$ and $\overline{F}_2$ are ordinary functions with values in $\partial\mathbb{H}^n$, and they coincide almost everywhere. $\square$

Proof of Theorem 7.2. — The map $x \mapsto \int_{\partial\mathbb{H}^k} f(\theta)\, d\mu_x(\theta)$ can be viewed as the harmonic extension of f, and one can prove its convergence to f along cones and almost everywhere by using standard techniques of harmonic analysis. We give a proof for completeness.

For the whole proof, we work in the half space model $\mathbb{H}^k = \mathbb{R}^{k-1} \times \mathbb{R}^+$, using the following notation. For a point $x \in \mathbb{H}^k$, we denote by $(x', x_k) \in$

$\mathbb{R}^{k-1} \times \mathbb{R}^+$ its coordinates in the half-space model, by the symbol $|x|$ we denote the Euclidean norm of x in the model, and $B(\omega, r)$ will denote the ball of centre ω and Euclidean radius r. In the half-space model, setting $\delta = \delta(\Gamma)$ we have

$$e^{-\delta B_K(x,\xi)} = \left(\frac{x_k(1+|\xi|^2)}{|\xi - x'|^2 + x_k^2}\right)^\delta$$

and for all $\gamma \in \Gamma$, if $x = \gamma O$

$$1 = ||\mu_O|| = ||\gamma_* \mu_O|| = ||\mu_{\gamma O}|| = \int_{\partial \mathbb{H}^k} \left(\frac{x_k(1+|\xi|^2)}{|\xi - x'|^2 + x_k^2}\right)^\delta .$$

We will work on a fixed ball of centre 0 and radius R of $\mathbb{R}^{k-1}$. This is not restrictive because proving convergence almost everywhere for all balls is equivalent to prove convergence almost everywhere.

For all $\omega \in \mathbb{R}^{k-1}$, we denote by $C_\omega(\alpha)$ the vertical cone in $\mathbb{H}^{k-1} \times \mathbb{R}^+$ of vertex ω and emi-angle α. For any non-negative $g \in L^1(\partial \mathbb{H}^k, \mu_O)$, $\omega \in \partial \mathbb{H}^k$, $\alpha \in (0, \pi/2)$ we define the maximal operator $M_\alpha g(\omega)$ by

$$M_\alpha g(\omega) = \sup_{\gamma \in \Gamma,\ \gamma O \in C_\omega(\alpha)} \int_{\partial \mathbb{H}^k} g(\xi)\, d\mu_{\gamma O}(\xi)$$

and the so-called Hardy Littlewood operator $Ng(\omega)$ by

$$Ng(\omega) = \sup_{r>0} \frac{1}{\mu_O(B(\omega, r))} \int_{B(\omega,r)} g(\xi)\, d\mu_O(\xi).$$

From now on, the symbol c will denote a generic constant, and different occurrences may denote different constants. If not specified, the constants do not depend on the other quantities we are considering.

Lemma 7.3. — *There exists a constant c such that for every point ω of the limit set of Γ and any $r > 0$*

$$\mu_O(B(\omega, r)) \leqslant c r^\delta.$$

Proof. — For all $x \in \mathbb{H}^k$

$$||\mu_x|| = \int_{\partial \mathbb{H}^k} \left(\frac{x_k(1+|\xi|^2)}{|\xi - x'|^2 + x_k^2}\right)^\delta \geqslant \int_{\partial \mathbb{H}^k} \left(\frac{x_k}{|\xi - x'|^2 + x_k^2}\right)^\delta .$$

Suppose now that x is of the form $x = \gamma O$, with

$$c_1 r \leqslant |x - \omega| \leqslant c_2 r. \tag{7.2}$$

Then $|\xi - x'| \leqslant |\xi - \omega| + |x' - \omega| \leqslant (1 + c_2) r$ for any $\xi \in B(\omega, r)$. Then

$$1 = ||\mu_x|| = \int_{\partial \mathbb{H}^k} \left(\frac{x_k}{|\xi - x'|^2 + x_k^2}\right)^\delta \geqslant c \frac{1}{r^\delta} \mu_O(B(\omega, r))$$

and the claim holds for such points. Let $a > 1$ and for $j \in \mathbb{Z}$, consider the set $A_j = \{x \in \overline{\mathbb{H}}^k : |x - \omega| \in [a^j, a^{j+1})\}$. If for a certain $j \in \mathbb{Z}$ the set A_j contains a point of the Γ-orbit of O, then for $r \in [a^j, a^{j+1}]$

$$\frac{r}{a} \leqslant \frac{a^{j+1}}{a} = a^j \leqslant x \leqslant a^{j+1} = a^j a \leqslant ra$$

and inequalities (7.2) hold with $c_2 = a = 1/c_1$. Let now $r > 0$. Since ω lies on the limit set, there exists j with $a^j \leqslant r$ and such that A_j intersects the Γ-orbit of O. Let j_0 be the maximum of such j's. If $a^{j_0+1} \geqslant r$, we have finished. Otherwise, since μ_O is concentrated on the limit set

$$\mu_0(B(\omega, r)) = \mu(B(\omega, a^{j_0+1})) \leqslant c(a^{j_0+1})^\delta \leqslant cr^\delta.$$

This completes the proof of Lemma 7.3. □

Lemma 7.4. — *There exists a constant c, depending only on α, such that for any non-negative $g \in L^1(\partial\mathbb{H}^k, \mu_O)$, $\omega \in \partial\mathbb{H}^k$, $\alpha \in (0, \pi/2)$, we have*

$$M_\alpha g(\omega) \leqslant cNg(\omega).$$

Proof. — Let $x = \gamma O$ with $\gamma \in \Gamma$ and $\gamma O \in C_\omega(\alpha)$. As noticed above, it is not restrictive to work in the ball $B(0, R)$.

$$\begin{aligned}
&\int_{\partial \mathbb{H}^k} g(\xi)\, d\mu_x(\xi) \leqslant (1+R^2)^\delta \int_{\partial \mathbb{H}^k} g(\xi) \left(\frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^\delta d\mu_O(\xi) \\
&= c\Bigg[\int_{B(\omega, x_k)} g(\xi) \left(\frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^\delta d\mu_O(\xi) + \\
&+ \sum_{j \geqslant 0} \int_{B(\omega, 2^{j+1}x_k) \setminus B(\omega, 2^j x_k)} g(\xi) \left(\frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^\delta d\mu_O(\xi) \Bigg] \\
&\leqslant c\Bigg[\frac{1}{x_k^\delta} \int_{B(\omega, x_k)} g(\xi)\, d\mu_O(\xi) + \\
&\qquad\qquad + \sum_{j \geqslant 0} \frac{1}{(c 2^{2j} x_k^2)^\delta} \int_{B(\omega, 2^{j+1}x_k) \setminus B(\omega, 2^j x_k)} g(\xi)\, d\mu_O(\xi) \Bigg] \\
&\leqslant c\Bigg[\frac{1}{\mu_O(B(\omega, x_k))} \int_{B(\omega, x_k)} g(\xi)\, d\mu_O(\xi) + \\
&\qquad\qquad + \sum_{j \geqslant 0} \frac{2^{-j}}{\mu_O(B(\omega, 2^{j+1}x_k))} \int_{B(\omega, 2^{j+1}x_k)} g(\xi)\, d\mu_O(\xi) \Bigg] \\
&\leqslant cNg(\omega) \left(1 + \sum_{j \geqslant 0} 2^{-j} \right) \leqslant cNg(\omega).
\end{aligned}$$

□

The constant c actually depends on α because we used that for $x \in C_\alpha(\omega)$ and $\xi \in B(\omega, 2^{j+1}x_k) \setminus B(\omega, 2^j x_k)$ we have $|\xi - x'|^2 + x_k^2 \geqslant c 2^{2j} x_k^2$. It can be shown that c is bounded by $(\tan \alpha)^{2\delta}$.

We can now finish the proof of Theorem 7.2. Since $||\mu_{\gamma O}|| = 1$ and $\lim_{x \to \omega} e^{-\delta B^K(x,z)} = 0$ for all $z \neq \omega$, the claim is true for continuous functions. Suppose now $f \in L^1(\partial \mathbb{H}^k, \mu_O)$, and let $f_j \to f$ be a sequence of continuous functions converging to f μ_O-almost everywhere and in L^1. We have

$$\begin{aligned}
\left| \int_{\partial \mathbb{H}^k} f(\xi)\, d\mu_{\gamma O}(\xi) - f(\omega) \right| \leqslant &\left| \int_{\partial \mathbb{H}^k} f(\xi) - f_j(\xi)\, d\mu_{\gamma O}(\xi) \right| \\
&+ \left| \int_{\partial \mathbb{H}^k} f_j(\xi)\, d\mu_{\gamma O}(\xi) - f_j(\omega) \right| \\
&+ |f_j(\omega) - f(\omega)|.
\end{aligned}$$

The second summand of the second term goes to zero as $\gamma O \to \omega$ because f_j is continuous. For μ_O-almost all ω the last summand can be chosen arbitrarily small because $f_j \to f$ μ_O-almost everywhere. By Lemma 7.4, the first summand of the second term is bounded by $cN(f_j - f)(\omega)$ on each cone $C_\alpha(\omega)$, the constant c depending on α. The Hardy Littlewood operator is bounded from L^1 to $L^{1,\infty}$ (see for example [26]), that is, for all $g \in L^1$ and $\epsilon > 0$

$$\mu_O\left(\{\omega \in \partial\mathbb{H}^k : \ |Ng(\omega)| \geqslant \epsilon\}\right) \leqslant \frac{c||g||_{L^1}}{\epsilon}.$$

Let $A_j^\epsilon = \{\omega \in \partial\mathbb{H}^k : \ |N(f_j - f)(\omega)| \geqslant \epsilon\}$. Since $f_j \to f$ in L^1, for all ε the measure $\mu_O(A_j^\epsilon)$ goes to zero. This is equivalent to say that the characteristic function $\chi_{A_j^\epsilon} \to 0$ in L^1. Then, up to pass to sub-sequences, $\chi_{A_j^\epsilon} \to 0$ μ_O-almost everywhere, that is, for μ_O-almost all ω the quantity $|N(f_j - f)(\omega)| \leqslant \epsilon$ eventually on j. We have so proved that, for each $\alpha \in (0, \pi/2)$ there exist a negligible set Z_α^ϵ such that, for all $\omega \in \partial\mathbb{H}^k \setminus Z_\alpha^\epsilon$, the quantity $|\int_{\partial\mathbb{H}^k} f(\xi)\, d\mu_{\gamma O}(\xi) - f(\omega)|$ is small than or equal to ϵ as γO converges to ω through $C_\alpha(\omega)$. The thesis now follows setting

$$Z = \bigcup_{\substack{\alpha \in \mathbb{Q} \cap (0,\pi/2) \\ 0 < \epsilon \in \mathbb{Q}}} Z_\alpha^\epsilon.$$

□

8. Measurable Cannon-Thurston maps

In this sections we study existence, uniqueness and convergence of measurable Cannon-Thurston maps.

We keep here the notations of previous sections. In particular, if not specified, Γ is a discrete group of $\mathrm{Isom}(\mathbb{H}^k)$, $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ is a representation whose image is not elementary, $\{\mu_x\}$ is the family of Patterson-Sullivan measure, and $\{\lambda_z\}_{\mathbf{z} \in \partial\mathbb{H}^k}$ is a family of developing measures for ρ.

The following lemma collects some ergodic properties of Γ that we need in the sequel (see [31, Theorem A], [22, Theorem 6.3.6], [27, 23] for the proof).

Lemma 8.1. — *Any non-elementary discrete group Γ acts ergodically on Λ w.r.t. the measure μ_O. Therefore, Λ_c has either zero or full μ_O-measure. Moreover, the following are equivalent:*

(1) *Λ_c has full measure.*

(2) Γ *diverges at* $\delta(\Gamma)$.
(3) *The geodesic flow is ergodic.*
(4) Γ *acts ergodically on* $\partial\mathbb{H}^k \times \partial\mathbb{H}^k$ *w.r.t.* $\mu_O \times \mu_O$.

To begin with, we prove a couple of lemmas we need in the sequel.

LEMMA 8.2. — *The subset of* $\partial\mathbb{H}^k$ *consisting of the points* z *such that* λ_z *is the sum of two Dirac deltas with equal weights is* μ_O*-measurable.*

Proof. — Let $B(\theta, r)$ denotes the ball of centre $\theta \in \partial\mathbb{H}^n$ and radius r in some metric of $\partial\mathbb{H}^n$, and let Q be a countable, dense subset of $\partial\mathbb{H}^n$.

For any open set $B \subset \partial\mathbb{H}^n$, the function $z \mapsto \lambda_z(B)$ is μ_O-measurable. It follows that the function

$$z \mapsto \inf_{0<r_1,r_2\in\mathbb{Q}} \left(\sup_{\theta_1,\theta_2\in Q} \lambda_z \left(B(\theta_1, r_1) \cup B(\theta_2, r_2)\right)\right)$$

is μ_O-measurable. The pre-image of 1, which therefore is a μ_O-measurable set, is the set of points z such that the support of λ_z contains at most two points. Similarly, the following sets are μ_O-measurable

$$\{z \in \partial\mathbb{H}^k : \lambda_z \text{ is concentrated on one point}\}$$

$$\{z \in \partial\mathbb{H}^k : \lambda_z \text{ has an atom of weight } \frac{1}{2}\}$$

and the claim follows. □

LEMMA 8.3. — *For all* μ_O*-measurable set* $A \subset \partial\mathbb{H}^k$ *the set*

$$\mathcal{O}(A) = \{(x, y) \in \partial\mathbb{H}^k \times \partial\mathbb{H}^k : \ \exists\gamma \in \Gamma : \gamma(x), \gamma(y) \in A\}$$

is $\mu_O \times \mu_O$*-measurable.*

Proof. — Clearly, it is sufficient to show that the function

$$(x, y) \mapsto \#\{\gamma \in \Gamma : \gamma(x), \gamma(y) \in A\}$$

is $\mu_O \times \mu_O$-measurable. The pre-image of $(n, \infty]$ is the set

$$\bigcup_{\substack{\gamma_1,\dots,\gamma_n \in \Gamma \\ \gamma_1 \neq \dots \neq \gamma_n}} \Big(\big((\gamma_1(A) \times \gamma_1(A)\big) \cap \dots \cap \big((\gamma_n(A) \times \gamma_n(A)\big)\Big)$$

which is a countable union of countable measurable sets, and therefore it is measurable. □

LEMMA 8.4. — *Suppose that* Γ *diverges at* $\delta(\Gamma)$*. If* $\mu_O(A) > 0$*, then* $\mathcal{O}(A)$ *has full measure.*

Proof. — Since $\mu_O(A) > 0$, then $A \times A$ has positive measure. Thus, $\mathcal{O}(A)$ has positive measure because it contains $A \times A$. Moreover, it is readily checked that $\mathcal{O}(A)$ is Γ-invariant. By Lemma 8.1 the action of Γ on $\partial\mathbb{H}^k \times \partial\mathbb{H}^k$ is ergodic, whence $\mathcal{O}(A)$ has full measure. □

THEOREM 8.5. — *Let $\Gamma < \mathrm{Isom}(\mathbb{H}^k)$ be a discrete group which diverges at $\delta(\Gamma)$. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation whose image is not elementary and let $\{\lambda_z\}_{z \in \partial\mathbb{H}^k}$ be a family of developing measures for ρ. Then, for almost all z, the measure λ_z is not the sum of two Dirac deltas with equal weights.*

Proof. — By Lemma 8.1, the conical limit set has full measure in $\partial\mathbb{H}^k$ and Γ acts ergodically on $\partial\mathbb{H}^k$. Let E be the set of points $z \in \partial\mathbb{H}^k$ such that λ_z is the sum of two deltas with equal weights. We have to prove that $\mu_O(E) = 0$. Clearly, the set E is Γ-invariant, and by Lemma 8.2 it is measurable. Therefore it has either zero or full μ_O-measure.

Suppose that E has full μ_O-measure. Then, a μ_O-measurable map is well defined by

$$\begin{aligned} f : \partial\mathbb{H}^k &\to \big(\partial\mathbb{H}^n \times \partial\mathbb{H}^n\big)/\mathfrak{S}_2 \\ z &\mapsto \text{support of } \lambda_z. \end{aligned}$$

By Lusin theorem (see for example [1]) for all $\varepsilon > 0$ there exists a compact set $A \subset \partial\mathbb{H}^k$, with $\mu_O(\partial\mathbb{H}^k \setminus A) < \varepsilon$ and such that the restriction of f to A is continuous. By Lemma 8.4, for any density-point x of A, for $\mu_O \times \mu_O$-almost all $(z_1, z_2) \in \partial\mathbb{H}^k \times \partial\mathbb{H}^k$ there exists a sequence $\{\gamma_i\} \subset \Gamma$ such that for $j = 1, 2$

$$\gamma_i(z_j) \in A \qquad \text{and} \qquad \gamma_i(z_j) \to x.$$

Therefore, for $j = 1, 2$

$$f(\gamma_i(z_j)) \to f(x).$$

Up to pass to sub-sequences, $\rho(\gamma_i)$ converges either to an isometry of $\mathbb{H}^n$ or to a quasi-constant C_a^b - a quasi-constant is a map such that $C_a^b(p) = b$ for all points $p \neq a$ of $\overline{\mathbb{H}}^n$ - where the convergence is uniform on compact sets not containing a (see for example [15]).

Let $\{\theta, \omega\}$ be the support of λ_x, that is $\lambda_x = \frac{\delta_\theta + \delta_\omega}{2}$, and for $j = 1, 2$ let $\{\xi_j, \zeta_j\}$ be the support of λ_{z_j}.

If $\rho(\gamma_i)$ converges to an isometry ψ, then for $j = 1, 2$

$$f(\gamma_i(z_j)) = \rho(\gamma_i)\{\xi_j, \zeta_j\} \to \{\psi(\xi_j), \psi(\zeta_j)\}.$$

Since $f(\gamma_i(z_j)) \to f(x) = \{\theta, \omega\}$, we get $f(z_1) = f(z_2)$. On the other hand, if $\rho(\gamma_i) \to C_a^b$ and if $\xi_j \neq a \neq \zeta_j$, we get

$$f(\gamma_i(z_j)) \to \{b, b\} \neq f(x).$$

We have so proved that for $\mu_O \times \mu_O$-almost all $(z_1, z_2) \in \partial\mathbb{H}^k \times \partial\mathbb{H}^k$ the supports of $\lambda_{\mathbf{z}_1}$ and λ_{z_2} share at least one point. Whence, using Fubini's theorem and Lemma 8.4, it follows that there exists $\zeta \in \partial\mathbb{H}^n$ such that for μ_O-almost all $z \in \partial\mathbb{H}^k$, the support of λ_z contains ζ. For $z \in \partial\mathbb{H}^k$, let $\xi(z)$ denote the other point of the support of λ_z, that is

$$f(z) = \{\zeta, \xi(z)\}.$$

For each $\gamma \in \Gamma$, for μ_O-almost all z we have $f(z) = \{\zeta, \xi(z)\}$ and

$$\{\rho(\gamma)\zeta, \rho(\gamma)\xi(z)\} = \rho(\gamma)(f(z)) = f(\gamma z) = \{\zeta, \xi(\gamma z)\}.$$

Whence we have

Either: the set $\{\zeta\}$ is $\rho(\Gamma)$-invariant.

Or: there exists $\gamma \in \Gamma$ such that $\rho(\gamma)\zeta \neq \zeta$, which implies that $\xi(z) = \rho(\gamma)^{-1}(\zeta)$ does not depend on z. Therefore the set $f(z)$ does not depend on z and it is $\rho(\Gamma)$-invariant.

In both cases the image of ρ is elementary, which contradicts the hypotheses. It follows that the set E cannot have full μ_O-measure. □

Theorems 1.6 and 1.7 now easily follow.

Proof of Theorem 1.6. — It is well-known (see for example [15]) that if $\mathbb{H}^k/\Gamma$ is a complete hyperbolic manifold of finite volume, then $\delta(\Gamma) = k-1$ and Λ_c has full-measure. Then, by Lemma 8.1, the hypotheses of Theorem 8.5 are satisfied, and the claim follows from Theorem 1.5. □

Proof of Theorem 1.7. — The first claim is an immediate corollary of Theorems 1.5 and 8.5. Now, let $\varphi = \overline{F} \circ \overline{G} : \partial\mathbb{H}^k \to \partial\mathbb{H}^k$. The map φ is clearly Γ-equivariant and μ_O-measurable. Since the identity is also Γ-equivariant and measurable, by uniqueness it follows that $\overline{F} \circ \overline{G} = \mathrm{Id}$ μ_0-almost everywhere. The same holds replacing $\overline{F} \circ \overline{G}$ with $\overline{G} \circ \overline{F}$. □

In particular this extends (and provides a new proof of) the following result.

THEOREM 8.6 ([9, 20, 25]). — *Let M be a compact hyperbolic 3-manifold fibred over the circle with fibre a surface F. Let $\pi_1(F) \simeq \Gamma < \mathrm{Isom}(\mathbb{H}^2)$ and let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ be the representation induced by the inclusion $\pi_1(F) \hookrightarrow \pi_1(M)$. Then, there exist measurable maps $\overline{F} : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ and $\overline{G} : \partial\mathbb{H}^3 \to \partial\mathbb{H}^2$ which are respectively ρ and ρ^{-1} equivariant. Moreover, almost everywhere*

$$\overline{F} \circ \overline{G} = \mathrm{Id}_{\mathbb{H}^2} \qquad \overline{G} \circ \overline{F} = \mathrm{Id}_{\mathbb{H}^3}.$$

Proof. — Clearly, Γ satisfies the hypotheses of Theorem 1.7. Moreover, by Lemma 8.1 and [8, Corollary 2], also $\pi_1(F) < \pi_1(M) < \mathrm{Isom}(\mathbb{H}^3)$ satisfies the hypotheses of Theorem 1.7. $\square$

9. Convergence of Cannon-Thurston maps

We begin this section by describing an example of a converging sequence of representations whose Cannon-Thurston maps have no Cauchy subsequence with respect to the uniform convergence. Then, we prove Theorem 1.8 (point-wise convergence almost everywhere.)

Example 9.1 (Souto). — Let Γ be the fundamental group of a compact hyperbolic surface. Then, there exists a sequence of discrete and faithful representations $\rho_n : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ such that, if f_n and f denote the corresponding Cannon-Thurston maps, then:

- $\rho_n(\Gamma)$ is quasi-Fuchsian
- $\rho_n \to \rho$
- $\rho(\Gamma)$ is geometrically finite
- There is a converging sequence of points $x_n \to x \in \partial\mathbb{H}^2$ such that $f_n(x_n) \to y \neq f(x)$. In particular, no sub-sequence of $\{f_n\}$ converges uniformly to f.

To see that, let AH denote set of discrete and faithful representations $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ with the topology of algebraic convergence. Let ρ be a geometrically finite representation in the closure of AH (whence ρ has accidental parabolics.) The domain of discontinuity of $\rho(\Gamma)$ is therefore non-empty, and we can thus pick a point y in it. Since the doubly degenerate representations are dense in the closure of AH, there exists a sequence $\psi_n \to \rho$ with the property that the limit set of $\psi_n(\Gamma)$ is the whole sphere $\partial\mathbb{H}^3$. Let f_n and f be the Cannon-Thurston maps corresponding to ψ_n and ρ. By equivariance, there exists a sequence of points $x_n \in \Lambda(\Gamma)$ such that $x_n \to x \in \Lambda(\Gamma)$ and $f_n(x_n) \to y$. Since $y \notin \Lambda(\rho(\Gamma))$ we cannot have $f_n(x_n) \to f(x)$. Finally, one can approximate the ψ_n's with quasi-Fuchsian representations ρ_n getting the requested properties.

Proof of Theorem 1.8. — For $z \in \Lambda(\Gamma)$ let $\delta_{f_i(z)}$ be the Dirac measure concentrated on $f_i(z)$. For $x \in \mathbb{H}^k$ consider the measures on $\partial\mathbb{H}^k \times \partial\mathbb{H}^n$ defined by

$$\eta_{x,i} = \mu_x \times \{\delta_{f_i(z)}\}.$$

Up possibly to pass to sub-sequences, the measure $\eta_{O,i}$ have a weak limit η_O, which we disintegrate as

$$\eta_O = \mu_O \times \{\beta_z\}$$

via some family of probability measures $\{\beta_z\}_{z\in\Lambda(\Gamma)}$. Let now

$$\eta_x = \mu_x \times \{\beta_z\}.$$

It is readily checked that $\eta_{x,i}$ weakly converges to η_x (because $d\mu_x(\theta) = e^{-\delta(\Gamma)B_O(x,\theta)}d\mu_O(\theta)$.) It follows that the measures η_x are ρ-equivariant. Indeed, for any fixed $\gamma \in \Gamma$ we have $\eta_{\gamma x,i} = (\gamma, \rho_i(\gamma))_*\eta_{x,i}$, and $\rho_i(\gamma)$ converges to $\rho(\gamma)$ uniformly on $\partial\mathbb{H}^n$.

As in Lemma 4.4 this implies that the family β_z is ρ-equivariant, and uniqueness part of Theorem 1.5 implies that in fact $\beta_z = \delta_{f(z)}$.

Now that we know that $\eta_x = \mu_x \times \{\delta_{f(z)}\}$, the proof is completed by the fact that the weak convergence of $\eta_{x,i}$ to η_x is equivalent to the convergence of f_i to f almost everywhere (see for example [2, Lemma 2.3].) □

BIBLIOGRAPHY

[1] L. Ambrosio, N. Fusco & D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[2] L. Ambrosio, S. Lisini & G. Savaré, "Stability of flows associated to gradient vector fields and convergence of iterated transport maps", 2005, Preprint SNS, available version on http://cvgmt.sns.it/papers/amblissav051.

[3] R. Benedetti & C. Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992.

[4] G. Besson, G. Courtois & S. Gallot, "Entropies et rigidités des espaces localement symétriques de courbure strictement négative", *Geom. Funct. Anal.* **5** (1995), p. 731-799.

[5] ———, "Minimal entropy and Mostow's rigidity theorems", *Ergodic Theory Dynam. Systems* **16** (1996), no. 4, p. 623-649.

[6] ———, "Lemme de Schwarz réel et applications géométriques", *Acta Mathematica* **183** (1999), no. 2, p. 145-169.

[7] C. J. Bishop & P. W. Jones, "Hausdorff dimension and Kleinian groups", *Acta Mathematica* **179** (1997), no. 1, p. 1-39.

[8] R. D. Canary, "Ends of hyperbolic 3-manifolds", *J. Amer. Math. Soc.* **6** (1993), no. 1, p. 1-35.

[9] J. W. Cannon & W. P. Thurston, "Group invariant Peano curves", Preprint, 1989.

[10] C. Dellacherie & P.-A. Meyer, *Probabilities and potential*, North-Holland Mathematics Studies, vol. 29, North-Holland Publishing Co., Amsterdam, 1978, viii+189 pages.

[11] A. Douady & C. J. Earle, "Conformally natural extension of homeomorphisms of the circle", *Acta Math.* **157** (1986), no. 1-2, p. 23-48.

[12] N. M. Dunfield, "Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds", *Invent. Math.* **136** (1999), no. 3, p. 623-657.

[13] S. Francaviglia, “Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds”, *Int. Math. Res. Not.* (2004), no. 9, p. 425-459.

[14] S. Francaviglia & B. Klaff, “Maximal volume representations are Fuchsian”, *Geom. Dedicata* **117** (2006), p. 111-124.

[15] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics, vol. 183, Birkhäuser Boston Inc., Boston, MA, 2001, xxvi+467 pages.

[16] B. Klaff, “Boundary slopes of knots in closed 3-manifolds with cyclic fundamental group”, PhD Thesis, University Illinois-Chicago, 2003.

[17] M. Mahan, “Cannon-Thurston Maps and Bounded Geometry”, arXiv:math.GT/0701725.

[18] ———, “Ending Laminations and Cannon-Thurston Maps”, arXiv:math.GT/07017 25.

[19] C. T. McMullen, “Local connectivity, Kleinian groups and geodesics on the blowup of the torus.”, *Invent. Math.* **146** (2001), no. 1, p. 35-91.

[20] Y. N. Minsky, “On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds”, *J. Amer. Math. Soc.* **7** (1994), no. 3, p. 539-588.

[21] H. Miyachi, “Moduli of continuity of Cannon-Thurston maps”, in *Spaces of Kleinian groups*, London Math. Soc. Lecture Note Ser., vol. 329, Cambridge Univ. Press, Cambridge, 2006, p. 121-149.

[22] P. J. Nicholls, *The ergodic theory of discrete groups*, London Mathematical Society Lecture Note Series, vol. 143, Cambridge University Press, Cambridge, 1989, xii+221 pages.

[23] T. Roblin, “Sur l’ergodicité rationnelle et les propriétés ergodiques du flot géodésique dans les variétés hyperboliques”, *Ergodic Theory Dynam. Systems* **20** (2000), no. 6, p. 1785-1819.

[24] I. Scorza, “Fractal curves in the limit sets of simply degenerate once punctured torus groups”, Preprint.

[25] T. Soma, “Equivariant, almost homeomorphic maps between S^1 and S^2”, *Proc. Amer. Math. Soc.* **123** (1995), no. 9, p. 2915-2920.

[26] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970, xiv+290 pages.

[27] D. Sullivan, “The density at infinity of a discrete group of hyperbolic motions”, *Inst. Hautes Études Sci. Publ. Math.* (1979), no. 50, p. 171-202.

[28] ———, “Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups”, *Acta Math.* **153** (1984), no. 3-4, p. 259-277.

[29] W. P. Thurston, *The geometry and topology of 3-manifolds*, Mimeographed notes, Princeton University Mathematics Department, 1979.

[30] C. Yue, “Dimension and rigidity of quasi-Fuchsian representations”, *Ann. of Math. (2)* **143** (1996), no. 2, p. 331-355.

[31] ———, “The ergodic theory of discrete isometry groups on manifolds of variable negative curvature”, *Trans. Amer. Math. Soc.* **348** (1996), no. 12, p. 4965-5005.

Manuscrit reçu le 16 mars 2007,
accepté le 8 novembre 2007.

Stefano FRANCAVIGLIA
Dipartimento di Matematica Applicata “U.Dini”
via Buonarroti 1/c
56127 Pisa (Italy)
s.francaviglia@sns.it