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FORMAL GEOMETRIC QUANTIZATION

by Paul-Émile PARADAN

ABSTRACT. — Let K be a compact Lie group acting in a Hamiltonian way on a symplectic manifold (M,Ω) which is pre-quantized by a Kostant-Souriau line bundle. We suppose here that the moment map Φ is *proper* so that the reduced space $M_\mu := \Phi^{-1}(K\cdot\mu)/K$ is compact for all μ. Then, we can define the "formal geometric quantization" of M as

$$\mathcal{Q}_K^{-\infty}(M) := \sum_{\mu\in\widehat{K}} \mathcal{Q}(M_\mu)V_\mu^K.$$

The aim of this article is to study the functorial properties of the assignment $(M,K) \longrightarrow \mathcal{Q}_K^{-\infty}(M)$.

RÉSUMÉ. — Considérons l'action hamiltonienne d'un groupe de Lie compact K sur une variété symplectique (M,Ω) préquantifiée par un fibré en droites de Kostant-Souriau. On suppose que l'application moment Φ est *propre*, ainsi les réductions symplectiques $M_\mu := \Phi^{-1}(K\cdot\mu)/K$ sont compactes pour tout μ. On peut alors définir la quantification formelle de M comme

$$\mathcal{Q}_K^{-\infty}(M) := \sum_{\mu\in\widehat{K}} \mathcal{Q}(M_\mu)V_\mu^K.$$

Le but de ce travail est l'étude de certaines propriétés fonctorielles de l'application $(M,K) \longrightarrow \mathcal{Q}_K^{-\infty}(M)$.

The aim of this article is to study the functorial properties of the "formal geometric quantization" process, which is defined on *non-compact* Hamiltonian manifolds when the moment map is *proper*. For this purpose, we introduce a technique of symplectic cutting that uses the (wonderful) compactifications of de Concini-Procesi [14, 15] and Brion [11], and we prove an extension of the "quantization commutes with reduction" theorem to the *singular* setting (here the *singular* manifolds that we consider are those obtained by symplectic reduction).

Keywords: Geometric quantization, moment map, symplectic reduction, index, transversally elliptic.
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1. Introduction and statement of results

Let (M,Ω) be a symplectic manifold which is equipped with a Hamiltonian action of a compact connected Lie group K. Let us denote by $\mathfrak{k}^*$ the dual of the Lie algebra of K. Let $\Phi : M \to \mathfrak{k}^*$ be the moment map. We assume the existence of a K-equivariant line bundle L on M having a connection with curvature equal to $-i\,\Omega$. In other words M is pre-quantizable in the sense of [21] and we call L a Kostant-Souriau line bundle.

In the process of quantization one tries to associate a unitary representation of K to the data (M,Ω,Φ,L). When M is **compact** one associates to this data a virtual representation $\mathcal{Q}_K(M)\in R(K)$ of K defined as the equivariant index of a Dolbeault-Dirac operator: $\mathcal{Q}_K(M)$ is the geometric quantization of M.

This quantization process satisfies the following functorial properties:

[P1] When N and M are respectively pre-quantized compact Hamiltonian K_1 and K_2-manifolds, the product $M\times N$ is a pre-quantized compact Hamiltonian $K_1\times K_2$-manifold, and we have

$$\mathcal{Q}_{K_1\times K_2}(M\times N)=\mathcal{Q}_{K_1}(M)\otimes\mathcal{Q}_{K_2}(N) \quad \text{in} \quad R(K_1\times K_2)\simeq R(K_1)\otimes R(K_2). \tag{1.1}$$

[P2] If $H\subset K$ is a closed and connected Lie subgroup, then the restriction of $\mathcal{Q}_K(M)$ to H is equal to $\mathcal{Q}_H(M)$.

Note that **[P1]** and **[P2]** give the following functorial property:

[P3] When N and M are pre-quantized compact Hamiltonian K-manifolds, the product $M\times N$ is a pre-quantized compact Hamiltonian K-manifold, and we have $\mathcal{Q}_K(M\times N)=\mathcal{Q}_K(M)\cdot\mathcal{Q}_K(N)$, where $\cdot$ denotes the product in $R(K)$.

Another fundamental property is the behaviour of the K-multiplicities of $\mathcal{Q}_K(M)$ that is known as "quantization commutes with reduction".

Let T be a maximal torus of K. Let $\mathfrak{t}^*$ be the dual of the Lie algebra of T containing the weight lattice $\wedge^*$: $\alpha\in\wedge^*$ if $i\alpha:\mathfrak{t}\to i\mathbb{R}$ is the differential of a character of T. Let $C_K\subset\mathfrak{t}^*$ be a Weyl chamber, and let $\widehat{K}:=\wedge^*\cap C_K$ be the set of dominant weights. The ring of characters $R(K)$ has a $\mathbb{Z}$-basis $V_\mu^K,\mu\in\widehat{K}$: V_μ^K is the irreducible representation of K with highest weight μ.

For any $\mu\in\widehat{K}$ which is a regular value of Φ, the reduced space (or symplectic quotient) $M_\mu:=\Phi^{-1}(K\cdot\mu)/K$ is an orbifold equipped with a symplectic structure Ω_μ. Moreover $L_\mu:=(L|_{\Phi^{-1}(\mu)}\otimes\mathbb{C}_{-\mu})/K_\mu$ is a

Kostant-Souriau line orbibundle over (M_μ, Ω_μ). The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is defined. In [26], this is extended further to the case of singular symplectic quotients, using partial (or shift) de-singularization. So the integer $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is well defined for every $\mu \in \widehat{K}$: in particular $\mathcal{Q}(M_\mu) = 0$ if $\mu \notin \Phi(M)$.

The following theorem was conjectured by Guillemin-Sternberg [17] and is known as "quantization commutes with reduction" [25, 26, 31, 29]. For complete references on the subject the reader should consult [30, 33].

Theorem 1.1 (Meinrenken, Meinrenken-Sjamaar). — *We have the following equality in $R(K)$:*

$$\mathcal{Q}_K(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu)\, V_\mu^K \ .$$

Suppose now that M is **non-compact** but that the moment map $\Phi : M \to \mathfrak{k}^*$ is assumed to be **proper** (we will simply say "M is proper"). In this situation the geometric quantization of M as an index of an elliptic operator is not well defined. Nevertheless the integers $\mathcal{Q}(M_\mu), \mu \in \widehat{K}$ are well defined since the symplectic quotients M_μ are **compact**.

Following Weitsman [34], we introduce the following

Definition 1.2. — *The formal quantization of (M, Ω, Φ) is the element of $R^{-\infty}(K) := \hom_{\mathbb{Z}}(R(K), \mathbb{Z})$ defined by*

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu)\, V_\mu^K \ .$$

A representation E of K is admissible if it has finite K-multiplicities: $\dim(\hom_K(V_\mu^K, E)) < \infty$ for every $\mu \in \widehat{K}$. Here $R^{-\infty}(K)$ is the Grothendieck group associated to the K-admissible representations. We have a canonical inclusion $i : R(K) \hookrightarrow R^{-\infty}(K)$: to $V \in R(K)$ we associate the map $i(V) : R(K) \to \mathbb{Z}$ defined by $W \mapsto \dim(\hom_K(V, W))$. In order to simplify notation, $i(V)$ will be written V. Moreover the tensor product induces an $R(K)$-module structure on $R^{-\infty}(K)$ since $E \otimes V$ is an admissible representation when V and E are, respectively, a finite dimensional and an admissible representation of K.

It is an easy matter to see that **[P1]** holds for the formal quantization process $\mathcal{Q}^{-\infty}$. Let N and M be respectively pre-quantized *proper* Hamiltonian K_1 and K_2-manifolds: the product $M \times N$ is then a pre-quantized *proper* Hamiltonian $K_1 \times K_2$-manifold. For the reduced spaces we have

$(M\times N)_{(\mu_1,\mu_2)}\simeq M_{\mu_1}\times N_{\mu_2}$, for all $\mu_1\in\widehat{K}_1$, $\mu_2\in\widehat{K}_2$. It follows then that

$$\mathcal{Q}^{-\infty}_{K_1\times K_2}(M\times N)=\mathcal{Q}^{-\infty}_{K_1}(M)\widehat{\otimes}\,\mathcal{Q}^{-\infty}_{K_2}(N) \tag{1.2}$$

in $R^{-\infty}(K_1\times K_2)\simeq R^{-\infty}(K_1)\widehat{\otimes}R^{-\infty}(K_2)$.

The purpose of this article is to show that the functorial property **[P2]** still holds for the formal quantization process $\mathcal{Q}^{-\infty}$.

Theorem 1.3. — *Let M be a pre-quantized Hamiltonian K-manifold which is* proper. *Let $H\subset K$ be a closed connected Lie subgroup such that M is still* proper *as a Hamiltonian H-manifold. Then $\mathcal{Q}^{-\infty}_K(M)$ is H-admissible and we have the following equality in $R^{-\infty}(H)$:*

$$\mathcal{Q}^{-\infty}_K(M)|_H=\mathcal{Q}^{-\infty}_H(M). \tag{1.3}$$

For $\mu\in\widehat{K}$ and $\nu\in\widehat{H}$ we denote $N^\mu_\nu=\dim(\hom_H(V^H_\nu,V^K_\mu|_H))$ the multiplicity of V^H_ν in the restriction $V^K_\mu|_H$. In the situation of Theorem 1.3, the moment maps relative to the K and H-actions are Φ_K and $\Phi_H=\mathrm{p}\circ\Phi_K$, where $\mathrm{p}:\mathfrak{k}^*\to\mathfrak{h}^*$ is the canonical projection.

Corollary 1.4. — *In the situation of Theorem 1.3, we have for every $\nu\in\widehat{H}$:*

$$\mathcal{Q}\left(M_{\nu,H}\right)=\sum_{\mu\in\widehat{K}}N^\mu_\nu\mathcal{Q}\left(M_{\mu,K}\right). \tag{1.4}$$

Here $M_{\nu,H}=\Phi_H^{-1}(H\cdot\nu)/H$ and $M_{\mu,K}=\Phi_K^{-1}(K\cdot\mu)/K$ are respectively the symplectic reductions relative to the H and K-actions.

Since V^K_μ is equal to the K-quantization of $K\cdot\mu$, the "quantization commutes with reduction" theorem tells us that $N^\mu_\nu=\mathcal{Q}((K\cdot\mu)_{\nu,H})$: in particular $N^\mu_\nu\neq0$ implies that $\nu\in\mathrm{p}(K\cdot\mu)\iff\mu\in K\cdot\mathrm{p}^{-1}(\nu)$. Finally

$$N^\mu_\nu\mathcal{Q}\left(M_{\mu,K}\right)\neq0\implies\mu\in K\cdot\mathrm{p}^{-1}(\lambda)\quad\text{and}\quad\Phi_K^{-1}(\mu)\neq\emptyset.$$

These two conditions imply that we can restrict the sum of RHS of (1.4) to

$$\mu\in\widehat{K}\cap\Phi_K\left(K\cdot\Phi_H^{-1}(\nu)\right) \tag{1.5}$$

which is *finite* since Φ_H is proper.

Theorem 1.3 and (1.2) give the following extended version of **[P3]**.

Theorem 1.5. — *Let N and M be two pre-quantized Hamiltonian K-manifolds where N is* compact *and M is* proper. *The product $M\times N$ is then proper and we have the following equality in $R^{-\infty}(K)$:*

$$\mathcal{Q}^{-\infty}_K(M\times N)=\mathcal{Q}^{-\infty}_K(M)\cdot\mathcal{Q}_K(N). \tag{1.6}$$

For $\mu, \lambda, \theta \in \widehat{K}$ we denote $C^{\mu}_{\lambda,\theta} = \dim(\hom_K(V^K_\mu, V^K_\lambda \otimes V^K_\theta))$ the multiplicity of V^K_μ in the tensor product $V^K_\lambda \otimes V^K_\theta$. Since $V^K_\lambda \otimes V^K_\theta$ is equal to the quantization of the product $K\cdot\lambda \times K\cdot\theta$, the "quantization commutes with reduction" theorem tells us that $C^{\mu}_{\lambda,\theta} = \mathcal{Q}((K\cdot\lambda \times K\cdot\theta)_\mu)$: in particular $C^{\mu}_{\lambda,\theta} \neq 0$ implies that $(*)$ $\|\lambda\| \leqslant \|\theta\| + \|\mu\|$.

Corollary 1.6. — *In the situation of Theorem 1.5, we have for every $\mu \in \widehat{K}$:*

$$\mathcal{Q}\left((M \times N)_\mu\right) = \sum_{\lambda,\theta\in\widehat{K}} C^{\mu}_{\lambda,\theta}\, \mathcal{Q}\left(M_\lambda\right) \mathcal{Q}\left(N_\theta\right). \tag{1.7}$$

Since N is compact, $\mathcal{Q}(N_\theta) \neq 0$ for $(**)$ $\theta \in \{\text{finite set}\}$. Then $(*)$ and $(**)$ show that the sum in the RHS of (1.7) is *finite*.

Weitsman proved in [34] the validity of (1.6) in a particular case. The natural strategy to obtain Theorem 1.3 can be summarized as follows:

(1) Cut the non-compact manifold M at different levels "n" to obtain Hamiltonian K-manifolds $M^{(n)}$, possibly *singular*, but which are *compact* and *pre-quantized*. We require that the manifold M is the limit of the sequence $M^{(n)}$ in the following sense. Each $M^{(n)}$ contains an invariant and dense open subset $\mathcal{U}^n$ which is symplectomorphic to an invariant open subset $\widetilde{\mathcal{U}^n}$ of M. The sequence $\widetilde{\mathcal{U}^n}$ is increasing and we have $M = \bigcup_n \widetilde{\mathcal{U}^n}$.

(2) Compute $\mathcal{Q}_K(M^{(n)})$.

We then expect to have another definition of $\mathcal{Q}^{-\infty}_K(M)$ as the limit of $\mathcal{Q}_K(M^{(n)})$ when "n" goes to infinity. Then we can prove that "$\mathcal{Q}^{-\infty}$" satisfies **[P2]**.

Weitsman worked out point (1) in the case where K is the unitary group $\mathrm{U}(r)$. He defines the cut spaces $M^{(n)}$ via symplectic reductions of $M \times \mathrm{Mat}_r(\mathbb{C})$, where $\mathrm{Mat}_r(\mathbb{C})$ is the vector space of complex $r \times r$ matrices, viewed as a Hamiltonian $\mathrm{U}(r) \times \mathrm{U}(r)$-manifold. He could handle point (2) *under the hypothesis that all the cut spaces $M^{(n)}$ are smooth*. Under this strong *smoothness hypothesis*, Weitsman was then able to show Theorem 1.5.

A natural way to carry out point (1) for any compact connected Lie group is by using another version of symplectic cutting due to C. Woodward [36] (see also [26]): each *non-abelian* cut space $M^{(n)}_{\mathrm{CW}}$ is defined by patching together *abelian* cut spaces (made on each symplectic slice of M). But the cut spaces $M^{(n)}_{\mathrm{CW}}$ are either *singular* or *not* pre-quantized, hence the main difficulty is point (2).

Let $K_{\mathbb{C}}$ be the complexification of the Lie group K. In this article, a smooth projective compactification of $K_{\mathbb{C}}$ is a smooth projective complex variety $\mathcal{X}$ embedded in $\mathbb{P}(E)$ where

i) E is a $K_{\mathbb{C}} \times K_{\mathbb{C}}$-module,
ii) $\mathcal{X}$ is $K_{\mathbb{C}} \times K_{\mathbb{C}}$-stable,
iii) $\mathcal{X}$ contains an open and dense $K_{\mathbb{C}} \times K_{\mathbb{C}}$-orbit $\mathcal{O}$ isomorphic to $K_{\mathbb{C}}$.

In this paper, we work out point (1) for any compact connected Lie group K by introducing another method of symplectic cutting which uses projective compactifications of $K_{\mathbb{C}}$. Each cut space $M^{(n)}_{\mathrm{PEP}}$ is defined as the symplectic reduction of a Hamiltonian $K \times K$-manifold of the type $M \times \mathcal{X}$: here $\mathcal{X}$ is a smooth projective compactification of $K_{\mathbb{C}}$ viewed as a Hamiltonian $K\times K$-manifold. We make the reduction relatively to one copy of K, so that the reduced space $M^{(n)}_{\mathrm{PEP}}$ is a Hamiltonian K-manifold. These cut spaces are in general *singular*, but each of them contains an open and dense subset of smooth points which is symplectomorphic to an invariant open subset of M.

Originally, projective compactifications of $K_{\mathbb{C}}$ were defined by de Concini-Procesi in the case of an adjoint group: these compactifications were *wonderful* [14, 15]. This construction was extended by Brion [11] to the case of a connected reductive group. In Section 3.1, we recall the construction of these compactifications and we study them from the Hamiltonian point of view. We show in particular that the open $K_{\mathbb{C}} \times K_{\mathbb{C}}$-orbit in $\mathcal{X}$ is $K \times K$-equivariantly symplectomorphic to an open subset of the cotangent bundle T^*K.

In order to work out point (2), we have to handle the non-smoothness of the cut spaces. For this purpose, we prove an extension of Theorem 1.1 to the *singular* setting.

Let N be a smooth pre-quantized Hamiltonian $K \times H$-manifold. Let us denote by $N/\!\!/_0 H$ the symplectic reduction of N at 0 relatively to the H-action: we assume that the moment map relatively to H is *proper* so that $N/\!\!/_0 H$ is a compact Hamiltonian K-manifold. Even if $N/\!\!/_0 H$ is *singular*, one can still define its geometric quantization $\mathcal{Q}_K(N/\!\!/_0 H) \in R(K)$. In Section 2, we prove the following

THEOREM 2.4 (Quantization commutes with reduction in the singular setting). — *We have the following equality in $R(K)$:*

$$\mathcal{Q}_K(N/\!\!/_0 H) = \sum_{\mu\in\widehat{K}} \mathcal{Q}\left((N/\!\!/_0 H)_\mu\right) V^K_\mu,$$

where the reduced space $(N/\!\!/_0 H)_\mu$ is equal to $(N \times \overline{K\cdot\mu})/\!\!/_{(0,0)} K \times H$.

Note that Theorem 2.4 applies naturally to the cut spaces $M^{(n)}_{\mathrm{PEP}}$, but a priori not to the cut spaces $M^{(n)}_{\mathrm{CW}}$.

In a forthcoming paper, we will exploit these results to compute the multiplicities of the holomorphic discrete series representations of a real semi-simple Lie group S relatively to a compact subgroup $H \subset S$.

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2. Quantization commutes with reduction

In this section we give the precise definition of the geometric quantization of a smooth and compact Hamiltonian manifold. We extend the definition to the case of a *singular* Hamiltonian manifold and we prove a "quantization commutes with reduction" theorem in the singular setting.

Let K be a compact connected Lie group, with Lie algebra $\mathfrak{k}$. In the Kostant-Souriau framework, a Hamiltonian K-manifold (M, Ω, Φ) is pre-quantized if there is an equivariant Hermitian line bundle L with an invariant Hermitian connection ∇ such that

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X\rangle \quad \text{and} \quad \nabla^2 = -i\Omega, \tag{2.1}$$

for every $X \in \mathfrak{k}$. Here X_M is the vector field on M defined by $X_M(m) = \frac{d}{dt} e^{-tX} m|_0$.

(L, ∇) is also called a Kostant-Souriau line bundle. Remark that conditions (2.1) imply via the equivariant Bianchi formula the relation

$$\iota(X_M)\Omega = -d\langle \Phi, X\rangle, \quad X \in \mathfrak{k}. \tag{2.2}$$

We will now recall the notion of geometric quantization.

2.1. Geometric quantization: the compact and smooth case

We suppose here that (M, Ω, Φ) is **compact** and is pre-quantized by a Hermitian line bundle L. Choose a K-invariant almost complex structure J on M which is compatible with Ω in the sense that the symmetric bilinear form $\Omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\overline{\partial}_L$ be the Dolbeault operator with coefficients in L, and let $\overline{\partial}_L^*$ be its (formal) adjoint. The *Dolbeault-Dirac operator* on M with coefficients in L is $D_L = \overline{\partial}_L + \overline{\partial}_L^*$, considered as an operator from $\mathcal{A}^{0,\mathrm{even}}(M, L)$ to $\mathcal{A}^{0,\mathrm{odd}}(M, L)$.

Definition 2.1. — *The geometric quantization of* (M,Ω,Φ) *is the element* $\mathcal{Q}_K(M)\in R(K)$ *which is defined as the equivariant index of the Dolbeault-Dirac operator* D_L*.*

Remark 2.2.

• We can define the Dolbeault-Dirac operator D^J_L for any invariant almost complex structure J. If J_0 and J_1 are equivariantly *homotopic* the indices of $D^{J_0}_L$ and $D^{J_1}_L$ coincide (see [29]).

• Since the set of *compatible* invariant almost complex structures on M is path-connected, the element $\mathcal{Q}_K(M)\in R(K)$ does not depend of the choice of J.

2.2. Geometric quantization: the compact and singular case

We are interested in defining the geometric quantization of *singular* compact Hamiltonian manifolds: here "singular" means that the manifold is obtained by symplectic reduction.

Let (N,Ω) be a smooth symplectic manifold equipped with a Hamiltonian action of $K\times H$: we denote $(\Phi_K,\Phi_H): N\to\mathfrak{k}^*\times\mathfrak{h}^*$ the corresponding moment map. We assume that N is pre-quantized by a $K\times H$-equivariant line bundle L and we suppose that the map Φ_H is **proper**. One wants to define the geometric quantization of the (compact) symplectic quotient

$$N/\!\!/_0 H := \Phi_H^{-1}(0)/H.$$

When 0 is a regular value of Φ_H, $N/\!\!/_0 H$ is a compact symplectic *orbifold* equipped with a Hamiltonian action of K: the corresponding moment map is induced by the restriction of Φ_K to $\Phi_H^{-1}(0)$. The symplectic quotient $N/\!\!/_0 H$ is pre-quantized by the line orbibundle

$$L_0 := \left(L|_{\Phi_H^{-1}(0)}\right)/H.$$

Definition 2.1 extends to the orbifold case, so one can still define the geometric quantization of $N/\!\!/_0 H$ as an element $\mathcal{Q}_K(N/\!\!/_0 H)\in R(K)$.

Suppose now that 0 is not a regular value of Φ_H. Let T_H be a maximal torus of H, and let $C_H\subset\mathfrak{t}^*_H$ be a Weyl chamber. Since Φ_H is proper, the convexity theorem says that the image of Φ_H intersects C_H in a closed locally polyhedral convex set, that we denote $\Delta_H(N)$, [23].

We consider an element $a\in\Delta_H(N)$ which is generic and sufficiently close to $0\in\Delta_H(N)$: we denote H_a the subgroup of H which stabilizes a. When $a\in\Delta_H(N)$ is generic, one can show (see [26]) that

$$N/\!\!/_a H := \Phi_H^{-1}(a)/H_a$$

is a compact Hamiltonian K-orbifold, and that

$$L_a := \left(L|_{\Phi_H^{-1}(a)}\right)/H_a.$$

is a K-equivariant line orbibundle over $N/\!\!/_a H$: we can then define, like in Definition 2.1, the element $\mathcal{Q}_K(N/\!\!/_a H) \in R(K)$ as the equivariant index of the Dolbeault-Dirac operator on $N/\!\!/_a H$ (with coefficients in L_a).

Proposition-Definition 2.3. — *The elements $\mathcal{Q}_K(N/\!\!/_a H) \in R(K)$ do not depend on the choice of the generic element $a \in \Delta_H(N)$, when a is sufficiently close to 0. Their common value will be taken as the geometric quantization of $N/\!\!/_0 H$, and still be denoted by $\mathcal{Q}_K(N/\!\!/_0 H)$.*

Proof. — When N is compact and $K = \{e\}$, the proof can be found in [26] and in [29]. The **K**-theoretic proof of [29] extends naturally to our case. □

2.3. Quantization commutes with reduction: the singular case

In Section 2.2, we have defined the geometric quantization $\mathcal{Q}_K(N/\!\!/_0 H) \in R(K)$ of a compact symplectic reduced space $N/\!\!/_0 H$. We will compute its K-multiplicities like in Theorem 1.1.

For every $\mu \in \widehat{K}$, we consider the co-adjoint orbit $K\cdot\mu \simeq K/K_\mu$ which is pre-quantized by the line bundle $\mathbb{C}_{[\mu]} \simeq K \times_{K_\mu} \mathbb{C}_\mu$. We consider the product[(1)] $N \times \overline{K\cdot\mu}$ which is a Hamiltonian $K\times H$-manifold pre-quantized by the $K\times H$-equivariant line bundle $L \otimes \mathbb{C}_{[\mu]}^{-1}$. The moment map $N \times \overline{K\cdot\mu} \to \mathfrak{k}^* \times \mathfrak{h}^*$, $(n,\xi) \mapsto (\Phi_K(n)-\xi, \Phi_H(n))$ is proper, so that the reduced space

$$(N/\!\!/_0 H)_\mu := (N \times \overline{K\cdot\mu})/\!\!/_{(0,0)} K \times H$$

is compact. Following Proposition 2.3, we can then define its quantization $\mathcal{Q}((N/\!\!/_0 H)_\mu) \in \mathbb{Z}$. The main result of this section is the

Theorem 2.4. — *We have the following equality in $R(K)$:*

$$\mathcal{Q}_K(N/\!\!/_0 H) = \sum_{\mu\in\widehat{K}} \mathcal{Q}\left((N/\!\!/_0 H)_\mu\right) V_\mu^K. \tag{2.3}$$

Proof. — The proof will occupy the remainder of this section. The starting point is to state another definition of the geometric quantization of a symplectic reduced space which uses the Atiyah-Singer theory of transversally elliptic operators. □

(1) $\overline{K\cdot\mu}$ denotes the co-adjoint orbit with the opposite symplectic form.

2.3.1. Transversally elliptic symbols

Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [6]. For an axiomatic treatment of the index morphism see Berline-Vergne [8, 9] and for a short introduction see [29].

Let $\mathcal{X}$ be a *compact* $K_1 \times K_2$-manifold. Let $p : \mathrm{T}\mathcal{X} \to \mathcal{X}$ be the projection, and let $(-,-)_{\mathcal{X}}$ be a $K_1 \times K_2$-invariant Riemannian metric. If E^0, E^1 are $K_1 \times K_2$-equivariant complex vector bundles over $\mathcal{X}$, a $K_1 \times K_2$-equivariant morphism $\sigma \in \Gamma(\mathrm{T}\mathcal{X}, \hom(p^*E^0, p^*E^1))$ is called a *symbol*. The subset of all $(x,v) \in \mathrm{T}\mathcal{X}$ where $\sigma(x,v) : E^0_x \to E^1_x$ is not invertible is called the *characteristic set* of σ, and is denoted by $\mathrm{Char}(\sigma)$.

Let $\mathrm{T}_{K_2}\mathcal{X}$ be the following subset of $\mathrm{T}\mathcal{X}$:

$$\mathrm{T}_{K_2}\mathcal{X} \ = \big\{(x,v) \in \mathrm{T}\mathcal{X},\ (v, X_{\mathcal{X}}(x))_{\mathcal{X}} = 0 \quad \text{for all } X \in \mathfrak{k}_2\big\}.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $\mathrm{T}\mathcal{X}$ (*i.e.* $\mathrm{Char}(\sigma)$ is compact), and is K_2*-transversally elliptic* if the restriction of σ to $\mathrm{T}_{K_2}\mathcal{X}$ is invertible outside a compact subset of $\mathrm{T}_{K_2}\mathcal{X}$ (*i.e.* $\mathrm{Char}(\sigma) \cap \mathrm{T}_{K_2}\mathcal{X}$ is compact). An elliptic symbol σ defines an element in the equivariant $\mathbf{K}$-theory of $\mathrm{T}\mathcal{X}$ with compact support, which is denoted by $\mathbf{K}_{K_1\times K_2}(\mathrm{T}\mathcal{X})$, and the index of σ is a virtual finite dimensional representation of $K_1 \times K_2$ [2, 3, 4, 5].

A K_2*-transversally elliptic* symbol σ defines an element of $\mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_2}\mathcal{X})$, and the index of σ is defined as a trace class virtual representation of $K_1 \times K_2$ (see [6] for the analytic index and [8, 9] for the cohomological one): in fact $\mathrm{Index}^{\mathcal{X}}(\sigma)$ belongs to the tensor product $R(K_1)\widehat{\otimes} R^{-\infty}(K_2)$.

Remark that any elliptic symbol of $\mathrm{T}\mathcal{X}$ is K_2-transversally elliptic, hence we have a restriction map $\mathbf{K}_{K_1\times K_2}(\mathrm{T}\mathcal{X}) \to \mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_2}\mathcal{X})$, and a commutative diagram

$$\begin{array}{ccc} \mathbf{K}_{K_1\times K_2}(\mathrm{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_2}\mathcal{X}) \\ \downarrow{\scriptstyle \mathrm{Index}^{\mathcal{X}}} & & \downarrow{\scriptstyle \mathrm{Index}^{\mathcal{X}}} \\ R(K_1)\otimes R(K_2) & \longrightarrow & R(K_1)\widehat{\otimes}\ R^{-\infty}(K_2)\ . \end{array} \tag{2.4}$$

Using the *excision property*, one can easily show that the index map $\mathrm{Index}^{\mathcal{U}} : \mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_2}\mathcal{U}) \to R(K_1)\widehat{\otimes} R^{-\infty}(K_2)$ is still defined when $\mathcal{U}$ is a $K_1 \times K_2$-invariant relatively compact open subset of a $K_1 \times K_2$-manifold (see [29], Section 3.1).

2.3.2. Quantization of singular spaces: second definition

Let $(\mathcal{X}, \Omega)$ be a Hamiltonian $K_1 \times K_2$-manifold pre-quantized by a $K_1 \times K_2$-equivariant line bundle L. The moment map $\Phi_2 : \mathcal{X} \to \mathfrak{k}_2^*$ relative to the K_2-action is supposed to be **proper**. Take a compatible $K_1 \times K_2$-invariant almost complex structure on $\mathcal{X}$. We choose a $K_1 \times K_2$-invariant Hermitian metric $\|v\|^2$ on the tangent bundle $\mathrm{T}\mathcal{X}$, and we identify the cotangent bundle $\mathrm{T}^*\mathcal{X}$ with $\mathrm{T}\mathcal{X}$. For $(x, v) \in \mathrm{T}\mathcal{X}$, the principal symbol of the Dolbeault-Dirac operator $\overline{\partial}_L + \overline{\partial}_L^*$ is the Clifford multiplication $\mathbf{c}_{\mathcal{X}}(v)$ on the complex vector bundle $\Lambda^\bullet \mathrm{T}_x\mathcal{X} \otimes L_x$. It is invertible for $v \neq 0$, since $\mathbf{c}_{\mathcal{X}}(v)^2 = -\|v\|^2$.

When $\mathcal{X}$ is compact, the symbol $\mathbf{c}_{\mathcal{X}}$ is elliptic and then defines an element of the equivariant $\mathbf{K}$-group of $\mathrm{T}\mathcal{X}$. The topological index of $\mathbf{c}_{\mathcal{X}} \in \mathbf{K}_{K_1\times K_2}(\mathrm{T}\mathcal{X})$ is equal to the analytical index of the Dolbeault-Dirac operator $\overline{\partial}_L + \overline{\partial}_L^*$:

$$\mathcal{Q}_{K_1\times K_2}(\mathcal{X}) = \mathrm{Index}^{\mathcal{X}}(\mathbf{c}_{\mathcal{X}}) \quad \text{in} \quad R(K_1)\otimes R(K_2). \tag{2.5}$$

When $\mathcal{X}$ is not compact the topological index of $\mathbf{c}_{\mathcal{X}}$ is not defined. In order to give a topological definition of $\mathcal{Q}_{K_1}(\mathcal{X}/\!\!/_0 K_2)$, we will deform the symbol $\mathbf{c}_{\mathcal{X}}$ as follows. Consider the identification $\mathfrak{k}_2^* \simeq \mathfrak{k}_2$ defined by a K_2-invariant scalar product on the Lie algebra $\mathfrak{k}_2$. From now on the moment map Φ_2 will take values in $\mathfrak{k}_2$, and we define the vector field on $\mathcal{X}$

$$\kappa_x = (\Phi_2(x))_{\mathcal{X}}(x), \quad x \in \mathcal{X}. \tag{2.6}$$

We consider now the symbol

$$\mathbf{c}_{\mathcal{X}}^{\kappa}(v) = \mathbf{c}(v - \kappa_x), \quad v \in \mathrm{T}_x\mathcal{X}.$$

Note that $\mathbf{c}_{\mathcal{X}}^{\kappa}(v)$ is invertible except if $v = \kappa_x$. If furthermore v belongs to the subset $\mathrm{T}_{K_2}\mathcal{X}$ of tangent vectors orthogonal to the K_2-orbits, then $v = 0$ and $\kappa_x = 0$. Indeed κ_x is tangent to $K_2 \cdot x$ while v is orthogonal.

Since κ is the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi_2\|^2$, the set of zeros of κ coincides with the set $\mathrm{Cr}(\|\Phi_2\|^2)$ of critical points of $\|\Phi_2\|^2$.

Let $\mathcal{U} \subset \mathcal{X}$ be a $K_1 \times K_2$-invariant open subset which is *relatively compact*. If the boundary $\partial\mathcal{U}$ does not intersect $\mathrm{Cr}(\|\Phi_2\|^2)$, then the restriction $\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}}$ defines a class in $\mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_2}\mathcal{U})$ since

$$\mathrm{Char}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}}) \cap \mathrm{T}_{K_2}\mathcal{U} \simeq \mathrm{Cr}(\|\Phi_2\|^2) \cap \mathcal{U}$$

is compact. In this situation the index of $\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}}$ is defined as an element $\mathrm{Index}^{\mathcal{U}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}}) \in R(K_1)\widehat{\otimes} R^{-\infty}(K_2)$.

Theorem 2.5. — *The K_2-invariant part of* $\mathrm{Index}^{\mathcal{U}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}})$ *is equal to:*

- $\mathcal{Q}_{K_1}(\mathcal{X}/\!\!/_0 K_2)$ *when* $\Phi_2^{-1}(0)\subset\mathcal{U}$,
- 0 *in the other case.*

Proof. — When $K_1=\{e\}$, the proof is done in [29] (see Section 7). This proof works equally well in the general case. □

Remark 2.6. — If $\mathcal{X}$ is compact we can take $\mathcal{U}=\mathcal{X}$ in the last theorem. In this case the symbols $\mathbf{c}^{\kappa}_{\mathcal{X}}$ and $\mathbf{c}_{\mathcal{X}}$ define the same class in $\mathbf{K}_{K_1\times K_2}(\mathrm{T}\mathcal{X})$ so they have the same index. Theorem 2.5 corresponds then to the traditional "quantization commutes with reduction" phenomenon: $[\mathcal{Q}_{K_1\times K_2}(\mathcal{X})]^{K_2}=\mathcal{Q}_{K_1}(\mathcal{X}/\!\!/_0 K_2)$.

>From now one we will work with this (topological) definition of the geometric quantization of the reduced K_1-Hamiltonian manifold $\mathcal{X}/\!\!/_0 K_2$ (which is possibly singular):

$$\mathcal{Q}_{K_1}(\mathcal{X}/\!\!/_0 K_2)=[\mathrm{Index}^{\mathcal{U}}(\mathbf{c}^{\kappa}_{\mathcal{X}}|_{\mathcal{U}})]^{K_2}, \tag{2.7}$$

where $\mathcal{U}$ is any relatively compact neighborhood of $\Phi_2^{-1}(0)$ such that $\partial\mathcal{U}\cap\mathrm{Cr}(\|\Phi_2\|^2)=\emptyset$.

Remark 2.7. — In this topological definition of $\mathcal{Q}_{K_1}(\mathcal{X}/\!\!/_0 K_2)$ one has to check that such open subset $\mathcal{U}$ exists. Take $\mathcal{U}=\{\|\Phi_2\|^2<\epsilon\}$ for $\epsilon>0$: one can check that $\partial\mathcal{U}=\{\|\Phi_2\|^2=\epsilon\}$ does not intersect $\mathrm{Cr}(\|\Phi_2\|^2)$ for ϵ small enough.

The functorial properties still hold in this singular setting. In particular:

[P2] If $H\subset K_1$ is a closed and connected Lie subgroup, then the restriction of $\mathcal{Q}_{K_1}(\mathcal{X}/\!\!/_0 K_2)$ to H is equal to $\mathcal{Q}_H(\mathcal{X}/\!\!/_0 K_2)$.

2.3.3. Proof of Theorem 2.4

We go back to the situation of Sections 2.2 and 2.3 .

First we apply Theorem 2.5 to $\mathcal{X}=N$, $K_1=K$ and $K_2=H$. (2.3) is trivially true when $0\notin\mathrm{Image}(\Phi_H)$. So we suppose now that $0\in\mathrm{Image}(\Phi_H)$, and we consider a $K\times H$-invariant open subset $\mathcal{U}\subset N$ which is *relatively compact* and such that

$$\Phi_H^{-1}(0)\subset\mathcal{U}\quad\text{and}\quad\partial\mathcal{U}\cap\mathrm{Cr}(\|\Phi_H\|^2)=\emptyset.$$

We have $\mathcal{Q}_K(N/\!\!/_0 H)=[\mathrm{Index}^{\mathcal{U}}(\mathbf{c}^{\kappa^H}_N|_{\mathcal{U}})]^H$ and one want to compute its K- multiplicities $m_\mu,\mu\in\widehat{K}$. Here κ^H is the vector field on N associated to the moment map Φ_H (see (2.6)).

Take $\mu \in \widehat{K}$. We denote $\mathbf{c}_{-\mu}$ the principal symbol of the Dolbeault-Dirac operator on $\overline{K\cdot\mu}$ with values in the line bundle $\mathbb{C}_{[-\mu]}$: we have $\mathrm{Index}^{K\cdot\mu}(\mathbf{c}_{-\mu}) = (V_\mu^K)^*$.

We know then that the multiplicity of $[\mathrm{Index}^{\mathcal{U}}(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}})]^H$ relatively to V_μ^K is equal to

$$m_\mu := \Big[\mathrm{Index}^{\mathcal{V}}\left(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}} \odot \mathbf{c}_{-\mu}\right)\Big]^{K\times H}, \tag{2.8}$$

with $\mathcal{V} = \mathcal{U}\times K\cdot\mu$. This identity is due to the fact that we have a "multiplication"

$$\begin{aligned}\mathbf{K}_{K\times H}(\mathrm{T}_H\mathcal{U})\times\mathbf{K}_K(\mathrm{T}(K\cdot\mu)) &\longrightarrow \mathbf{K}_{K\times H}(\mathrm{T}_{K\times H}(\mathcal{U}\times K\cdot\mu))\\ (\sigma_1,\sigma_2) &\longmapsto \sigma_1\odot\sigma_2\ .\end{aligned}$$

so that $\mathrm{Index}^{\mathcal{U}\times K\cdot\mu}(\sigma_1\odot\sigma_2) = \mathrm{Index}^{\mathcal{U}}(\sigma_1)\cdot\mathrm{Index}^{K\cdot\mu}(\sigma_2)$ in $R^{-\infty}(K\times H)$. See [6].

Consider now the case where $\mathcal{X} = N\times\overline{K\cdot\mu}$, $K_1 = \{e\}$ and $K_2 = K\times H$. By Theorem 2.5, we know that

$$\mathcal{Q}((N/\!\!/_0 H)_\mu) = \Big[\mathrm{Index}^{\mathcal{V}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{V}})\Big]^{K\times H}, \tag{2.9}$$

where κ is the vector field on $N\times\overline{K\cdot\mu}$ associated to the moment map

$$\begin{aligned}\Phi : N\times\overline{K\cdot\mu} &\longrightarrow \mathfrak{k}^*\times\mathfrak{h}^*\\ (x,\xi) &\longmapsto (\Phi_K(x)-\xi, \Phi_H(n)).\end{aligned} \tag{2.10}$$

Note that $\mathcal{V} = \mathcal{U}\times K\cdot\mu$ is a neighborhood of $\Phi^{-1}(0)\subset(\Phi_H)^{-1}(0)$.

Our aim now is to prove that the quantities (2.8) and (2.9) are equal.

Since the definition of κ requires the choice of an invariant scalar product on the Lie algebra $\mathfrak{k}\times\mathfrak{h}$, we give a precise definition of it. Let $\|\cdot\|_K$ and $\|\cdot\|_H$ be two invariant Euclidean norms respectively on $\mathfrak{k}$ and $\mathfrak{h}$. For any $r>0$, we consider on $\mathfrak{k}\times\mathfrak{h}$ the invariant Euclidean norm $\|(X,Y)\|_r^2 = r^2\|X\|_K^2 + \|Y\|_H^2$.

Let κ^K be the vector field on $N\times\overline{K\cdot\mu}$ associated to the map $N\times\overline{K\cdot\mu}\to\mathfrak{k}^*, (x,\xi)\mapsto\Phi_K(x)-\xi$, and where the identification $\mathfrak{k}\simeq\mathfrak{k}^*$ is made via the Euclidean norm $\|\cdot\|_K$ (see (2.6)). For $(x,\xi)\in N\times\overline{K\cdot\mu}$, we have the decomposition

$$\kappa^K(x,\xi) = (\kappa_1(x,\xi),\kappa_2(x,\xi))\ \in\ \mathrm{T}_xN\times\mathrm{T}_\xi(K\cdot\mu).$$

Let κ^H be the vector field on $N\times\overline{K\cdot\mu}$ associated to the map $N\times\overline{K\cdot\mu}\to\mathfrak{k}^*, (x,\xi)\mapsto\Phi_H(x)$, and where the identification $\mathfrak{h}\simeq\mathfrak{h}^*$ is made via the

Euclidean norm $\|\cdot\|_H$. For $(x,\xi)\in N\times\overline{K\cdot\mu}$, we have the decomposition

$$\kappa^H(x,\xi)=(\kappa^H(x),0)\ \in\ \mathrm{T}_xN\times\mathrm{T}_\xi(K\cdot\mu).$$

For any $r>0$, we denote by κ_r the vector field on $N\times\overline{K\cdot\mu}$ associated to the map (2.10), and where the identification $\mathfrak{k}\times\mathfrak{h}\simeq\mathfrak{k}^*\times\mathfrak{h}^*$ is made via the Euclidean norm $\|\cdot\|_r$. We have then

$$\begin{aligned}\kappa_r&=\kappa^H+r\,\kappa^K\\&=(\kappa^H+r\,\kappa_1,r\,\kappa_2).\end{aligned}$$

Now we can specify (2.9). Take an invariant relatively compact neighborhood $\mathcal{U}$ of $\Phi_H^{-1}(0)$ such that $\partial\mathcal{U}\cap\{\text{zeros of }\kappa^H\}=\emptyset$. With the help of a invariant Riemannian metric on $\mathcal{X}$ we define

$$\varepsilon_H=\inf_{x\in\partial\mathcal{U}}\|\kappa^H(x)\|\ >0\quad\text{and}\quad\varepsilon_K=\sup_{(x,\xi)\in\partial\mathcal{U}\times K\mu}\|\kappa_1(x,\xi)\|.$$

Note that for any $0\leqslant r<\frac{\varepsilon_H}{\varepsilon_K}$, we have $\partial\mathcal{U}\times K\cdot\mu\cap\{\text{zeros of }\kappa^H+r\kappa_1\}=\emptyset$, and then $\partial\mathcal{V}\cap\{\text{zeros of }\kappa_r\}=\emptyset$ for the neighborhood $\mathcal{V}:=\mathcal{U}\times K\cdot\mu$ of $\Phi^{-1}(0)$. We can then use Theorem 2.5: for $0<r<\frac{\varepsilon_H}{\varepsilon_K}$ we have

$$\mathcal{Q}((N/\!\!/_0H)_\mu)=\left[\mathrm{Index}^{\mathcal{V}}(\mathbf{c}_{\mathcal{X}}^{\kappa_r}|_{\mathcal{V}})\right]^{K\times H}.$$

We are now close to the end of the proof. Let us compare the symbols $\mathbf{c}_{\mathcal{X}}^{\kappa_r}|_{\mathcal{V}}$ and $\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}}\odot\mathbf{c}_{-\mu}$ in $\mathbf{K}_{K\times H}(\mathrm{T}_{K\times H}(\mathcal{U}\times K\cdot\mu))$. First one sees that the symbol $\mathbf{c}_{\mathcal{X}}$ is equal to the product $\mathbf{c}_N\odot\mathbf{c}_{-\mu}$ hence the symbol $\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}}\odot\mathbf{c}_{-\mu}$ is equal to $\mathbf{c}_{\mathcal{X}}^{\kappa_r}|_{\mathcal{V}}$ when $r=0$. Since for $r<\frac{\varepsilon_H}{\varepsilon_K}$ the path $s\in[0,r]\to\mathbf{c}_{\mathcal{X}}^{\kappa_s}|_{\mathcal{V}}$ defines a homotopy of $K\times H$-transversally elliptic symbols on $\mathcal{V}$, we get

$$\mathrm{Index}^{\mathcal{V}}(\mathbf{c}_{\mathcal{X}}^{\kappa_r}|_{\mathcal{V}})=\mathrm{Index}^{\mathcal{V}}(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}}\odot\mathbf{c}_{-\mu})$$

and then $m_\mu=\mathcal{Q}((N/\!\!/_0H)_\mu)$. □

3. Wonderful compactifications and symplectic cutting

In this section we use projective compactifications of $K_{\mathbb{C}}$ “à la de Concini-Procesi” [14, 15] to perform symplectic cutting. These compactifications are special cases of Spherical varieties, see [10].

3.1. Wonderful compactifications: definitions

Here we study the projective compactifications of $K_{\mathbb{C}}$ defined by Brion [11] from the Hamiltonian point of view. This construction generalizes previous work of de Concini-Procesi [14, 15], where *wonderful* compactifications of an adjoint group were defined.

We consider a compact connected Lie group K and its complexification $K_{\mathbb{C}}$. Let T be a maximal torus of K, and let $W := N(T)/T$ be the Weyl group. Let $\mathfrak{t}^*$ be the dual of the Lie algebra of T containing the lattice of weights $\wedge^*$. Let $C_K \subset \mathfrak{t}^*$ be a Weyl chamber and let $\widehat{K} := \wedge^* \cap C_K$ be the set of dominant weights. An element $\xi \in \mathfrak{t}^*$ is called *regular* if its stabilizer subgroup K_ξ is equal to T.

We recall the notion of Delzant polytope [28]. Let P be a convex polytope in $\mathfrak{t}^*$.

Definition 3.1. — *P is a Delzant polytope (relatively to $\wedge^*$) if:*

i) *the vertices of P belong to $\wedge^*$,*
ii) *P is simple: there are exactly* $\dim(\mathfrak{t}^*)$ *edges through each vertex,*
iii) *at each vertex ξ, the tangent cone to P at $\{\xi\}$ is generated by a $\mathbb{Z}$-basis of the lattice $\wedge^*$.*

We need the following refinement of the notion of Delzant polytope.

Definition 3.2. — *A convex polytope P in $\mathfrak{t}^*$ is K-adapted if:*

i) *P is a Delzant polytope (relatively to $\wedge^*$),*
ii) *the vertices of P are regular elements of $\mathfrak{t}^*$,*
iii) *P is W-invariant.*

Example 1. — When $K = T$ is a torus, a T-adapted polytope is just a Delzant polytope.

Example 2. — We consider the Lie groups SU(3) and PSU(3) $:=$ SU(3)$/Z$, where $Z \simeq \mathbb{Z}/3\mathbb{Z}$ is the center of SU(3). Note that PSU(3) has a trivial center. In Figures 3.1 and 3.2, the lattice $\wedge^*_{\mathrm{PSU}}$ of weights for PSU(3) is formed by the black dots and the lattice $\wedge^*_{\mathrm{SU}}$ of weights for SU(3) is formed by all the dots (grey and black). In Figure 3.1, the polytope is a Delzant polytope relatively to $\wedge^*_{\mathrm{PSU}}$, but it is not a Delzant polytope relatively to $\wedge^*_{\mathrm{SU}}$: hence the polytope is PSU(3)-adapted but not SU(3)-adapted.

Example 3. — When K has trivial center, the convex hull of $W \cdot \mu$ is a K-adapted polytope for any regular dominant weight μ. Figure 3.1 is an example of this case for the Lie group PSU(3).

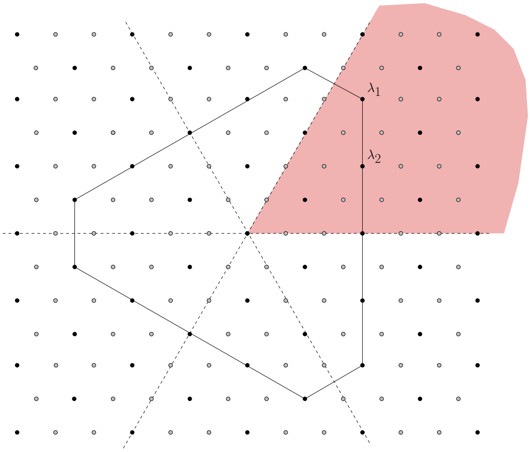

Figure 3.1. PSU(3)-*adapted polytope*

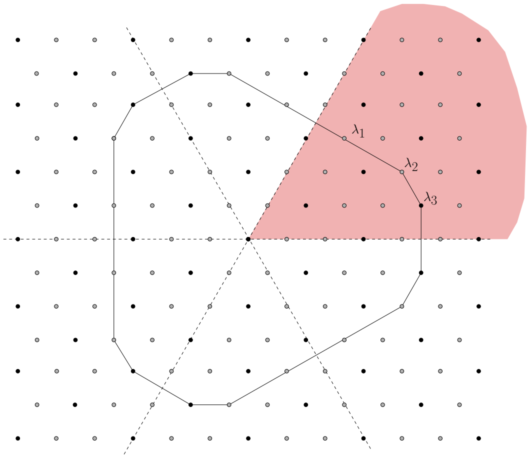

Figure 3.2. SU(3)-*adapted polytope*

PROPOSITION 3.3. — *For any compact connected Lie group K, there exist K-adapted polytopes in $\mathfrak{t}^*$.*

Proof. — Let us use the dictionary between polytopes and projective fans [28]. Conditions *i)* and *iii)* of Definition 3.2 means that we are looking after a smooth projective W-invariant fan $\mathcal{F}$ in $\mathfrak{t}$. Condition *ii)* means that each cone of $\mathcal{F}$ of maximal dimension should not be fixed by any element of $W \smallsetminus \{\mathrm{Id}\}$. For a proof of the existence of such a fan, see [12, 13]. In particular condition $(*)$ in Proposition 2 of [13] implies *ii)*. □

In the rest of this section, we consider a K-adapted polytope P. Let

$$\{\lambda_1, \ldots, \lambda_N\} \tag{3.1}$$

be the set of *regular* dominant weights which are on the edges of P (*i.e.* on the 1-dimensional faces of P). Note that some of the λ_i are the vertices of P which belong to the (interior) of the Weyl chamber.

Let V_{λ_i} be an irreducible representation of K with highest weight λ_i: this representation extends canonically to the complexification $K_{\mathbb{C}}$. We denote $\rho : K_{\mathbb{C}} \to \Pi_{i=1}^N GL(V_{\lambda_i})$ the representation of $K_{\mathbb{C}}$ on

$$V := \oplus_{i=1}^N V_{\lambda_i}.$$

Let $T_{\mathbb{C}} \subset K_{\mathbb{C}}$ be the complexification of the (compact) torus $T \subset K$. Let $\Delta(T_{\mathbb{C}}, V)$ be the set of weights relative to the action of $T_{\mathbb{C}}$ on V. Let us sum up the basic but essential properties concerning the set $\Delta(T_{\mathbb{C}}, V)$

Lemma 3.4.

- *We have* $W \cdot \{\lambda_1, \ldots, \lambda_N\} \subset \Delta(T_{\mathbb{C}}, V) \subset P$.
- P *is equal to the convex hull of* $W \cdot \{\lambda_1, \ldots, \lambda_N\}$.
- *For any vertex* λ *of* P, *the* $\mathbb{Z}$*-basis of the lattice* $\wedge^*$ *which generates the tangent cone to* P *at* $\{\lambda\}$ *is of the form:* $\alpha_1 - \lambda, \ldots, \alpha_r - \lambda$ *where* $\alpha_k \in \Delta(T_{\mathbb{C}}, V)$.

Proof. — Since each λ_i is a weight for the action of $T_{\mathbb{C}}$ on V_{λ_i}, we have $\lambda_i \in \Delta(T_{\mathbb{C}}, V)$. Using the W-invariance of $\Delta(T_{\mathbb{C}}, V)$, we get one inclusion of the first point. The other inclusion follows from the fact that the set of weights relative to the action of $T_{\mathbb{C}}$ on V_{λ_i} is contained in the convex hull of $W \cdot \lambda_i$. The second point is due to the fact that all the vertices of P belong to $W \cdot \{\lambda_1, \ldots, \lambda_N\}$.

Let us prove the last point for a vertex λ which is dominant. Since P is a Delzant polytope, the tangent cone to P at $\{\lambda\}$ is generated by a $\mathbb{Z}$-basis of the lattice $\wedge^*$ that we denote $\alpha_1 - \lambda, \ldots, \alpha_r - \lambda$. Let us show that all the α_k belong to $\Delta(T_{\mathbb{C}}, V)$. We consider the segment $[\lambda, \alpha_k] \subset \mathfrak{t}^*$ which is part of an edge of P. If $[\lambda, \alpha_k]$ is included in the interior of the Weyl chamber, we have then $\alpha_k \in \{\lambda_1, \ldots, \lambda_N\} \subset \Delta(T_{\mathbb{C}}, V)$. Suppose now that the segment $[\lambda, \alpha_k]$ intersects the wall Π_α of the Weyl chamber defined by a simple root α. Let $s_\alpha \in W$ be the symmetry relative to the wall Π_α. Since P is W-invariant, the segment

$$[s_\alpha(\lambda), s_\alpha(\alpha_k)] = s_\alpha\left([\lambda, \alpha_k]\right)$$

is also part of an edge of P, and it intersects $[\lambda, \alpha_k]$. Since two distinct edges can intersect only at the vertices, the line (λ, α_k) must be invariant under s_α.

Let us sum up the properties of the weight α_k: the segment $[\lambda, \alpha_k]$ intersects the wall Π_α orthogonally and $\lambda - \alpha_k$ is part of a $\mathbb{Z}$-basis of $\wedge^*$. There are only two possibilities: either $\alpha_k \in \Pi_\alpha$ or $\alpha_k = s_\alpha(\lambda)$. Both of them implies that

$$\alpha_k = \lambda - \alpha.$$

Finally, it is a standard fact of representation theory that, for any simple root α and any regular dominant weight λ, $\lambda - \alpha$ is a weight relative to the action of $T_\mathbb{C}$ on V_λ. We have proved that $\alpha_k = \lambda - \alpha \in \Delta(T_\mathbb{C}, V)$. □

We consider now the vector space

$$E = \oplus_{i=1}^N \mathrm{End}(V_{\lambda_i})$$

equipped with the action of $K_\mathbb{C} \times K_\mathbb{C}$ given by: $(g_1, g_2) \cdot f = \rho(g_1) \circ f \circ \rho(g_2)^{-1}$. Let $\mathbb{P}(E)$ be the projective space associated to E: it comes equipped with an algebraic action of the reductive group $K_\mathbb{C} \times K_\mathbb{C}$. We consider the map $g \mapsto [\rho(g)]$ from $K_\mathbb{C}$ into $\mathbb{P}(E)$, and we denote it $\bar{\rho}$.

Lemma 3.5. — *The map $\bar{\rho} : K_\mathbb{C} \to \mathbb{P}(E)$ is an embedding.*

Proof. — Let $g \in K_\mathbb{C}$ such that $\bar{\rho}(g) = [\mathrm{Id}]$: there exists $a \in \mathbb{C}^*$ such that $\rho(g) = a\,\mathrm{Id}$. The Cartan decomposition gives

$$\rho(k) = \frac{a}{|a|}\,\mathrm{Id} \quad \text{and} \quad \rho(e^{iX}) = |a|\,\mathrm{Id} \tag{3.2}$$

for $g = ke^{iX}$ with $k \in K$ and $X \in \mathfrak{k}$. Since there exist $Y, Y' \in \mathfrak{t}$ and $u, u' \in K$ such that $k = ue^Y u^{-1}$ and $X = u' \cdot Y'$, (3.2) gives

$$\rho(e^Y) = \frac{a}{|a|}\,\mathrm{Id} \quad \text{and} \quad \rho(e^{iY'}) = |a|\,\mathrm{Id}\,.$$

and then

$$e^{i\langle \alpha - \alpha', Y\rangle} = 1 \quad \text{and} \quad e^{\langle \alpha - \alpha', Y'\rangle} = 1, \tag{3.3}$$

for every $\alpha, \alpha' \in \Delta(T_\mathbb{C}, V)$. Using now the last point of Lemma 3.4, we see that (3.3) implies $Y' = 0$ and $Y \in \ker(Z \in \mathfrak{t} \to e^Z)$. We have proved that $g = e$. □

We can now define the projective compactification $\mathcal{X}_P$ of $K_\mathbb{C}$.

Definition 3.6. — *Let P be a K-adapted polytope in $\mathfrak{t}^*$. Let $\{\lambda_1, \ldots, \lambda_N\}$ be the set of* regular *dominant weights which are on the edges of P. Let $E := \oplus_{i=1}^N \mathrm{End}(V_{\lambda_i})$. We define the varieties:*

- *$\mathcal{X}_P$ which is the Zariski closure of $\bar{\rho}(K_\mathbb{C})$ in $\mathbb{P}(E)$,*

- *$\mathcal{Y}_P \subset \mathcal{X}_P$ which is the Zariski closure of $\bar{\rho}(T_{\mathbb{C}})$ in $\mathbb{P}(E)$.*

Since $\bar{\rho}(K_{\mathbb{C}}) = K_{\mathbb{C}} \times K_{\mathbb{C}} \cdot [\mathrm{Id}]$ and $\bar{\rho}(T_{\mathbb{C}}) = T_{\mathbb{C}} \times T_{\mathbb{C}} \cdot [\mathrm{Id}]$ are orbits of algebraic group actions their Zariski closures coincide with their closures for the Euclidean topology.

THEOREM 3.7. — *The varieties $\mathcal{X}_P$ and $\mathcal{Y}_P$ are smooth.*

The proof will be given in the next section.

Remark 3.8. — In the definition of $\mathcal{X}_P$, we work with the representation $V = \oplus_{i=1}^{N} V_{\lambda_i}$, where the λ_i run over the set of regular dominant weights that belong to the edges of P. We can be interested to work with a subset $\Delta \subset \{\lambda_1, \dots, \lambda_N\}$. We consider then the representations $V(\Delta) := \oplus_{\lambda\in\Delta} V_\lambda$ and $E(\Delta) := \oplus_{\lambda\in\Delta} \mathrm{End}(V_\lambda)$. We define the variety $\mathcal{X}(\Delta)$ as the Zariski closure of $\bar{\rho}(K_{\mathbb{C}})$ in $\mathbb{P}(E(\Delta))$.

Suppose now that Δ contains all the vertices of P which are in the Weyl chamber: the first two points of Lemma 3.4 apply to $\Delta(T_{\mathbb{C}}, V(\Delta))$. One can show by the method described in Section 3.2 that $\mathcal{X}(\Delta)$ is *smooth* if $\Delta(T_{\mathbb{C}}, V(\Delta))$ satisfies the third point of Lemma 3.4. In other words we have the following

Smoothness criterion for $\mathcal{X}(\Delta)$: for any vertex λ of P, the $\mathbb{Z}$-basis of the lattice $\wedge^*$ which generates the tangent cone to P at $\{\lambda\}$ is of the form: $\alpha_1 - \lambda, \dots, \alpha_r - \lambda$ where $\alpha_k \in \Delta(T_{\mathbb{C}}, V(\Delta))$.

When K has trivial center (see Figure 3.1) one can work with the polytope equal to the convex hull of $W \cdot \mu$, with μ a regular dominant weight. In this case one can take $\Delta := \{\mu\}$: the variety $\mathcal{X}(\Delta) \subset \mathbb{P}(\mathrm{End}(V_\mu))$ is a smooth compactification of $K_{\mathbb{C}}$. This was the situation studied originally by de Concini-Procesi [14].

In the example of Figure 3.1, if one takes $\Delta := \{\lambda_2, \lambda_3\}$, the variety $\mathcal{X}(\Delta)$ is a smooth compactification of $\mathrm{SL}(3, \mathbb{C})$.

3.2. Smoothness of $\mathcal{X}_P$ and $\mathcal{Y}_P$

Let E be a complex vector space equipped with a linear action of a reductive group G. Let $\mathcal{Z} \subset \mathbb{P}(E)$ be a projective variety which is G-stable. We have the classical fact

LEMMA 3.9.

- *$\mathcal{Z}$ has closed G-orbits.*
- *$\mathcal{Z}$ is smooth if $\mathcal{Z}$ is smooth near its closed G-orbits.*

- *$\mathcal{Z}$ is smooth near an orbit $G \cdot z$ if $\mathcal{Z}$ is smooth near z.*

We are interested here respectively in

- the $K_{\mathbb{C}} \times K_{\mathbb{C}}$-variety $\mathcal{X}_P \subset \mathbb{P}(E) \subset \mathbb{P}(\mathrm{End}(V))$,
- the $T_{\mathbb{C}} \times T_{\mathbb{C}}$-variety $\mathcal{Y}_P \subset \mathbb{P}(E)$.

Since the diagonal $Z_{\mathbb{C}} = \{(t,t) | t \in T_{\mathbb{C}}\}$ stabilizes [Id], its action on $\mathcal{Y}_P$ is trivial. Hence we will restrict ourselves to the action of $T_{\mathbb{C}} \times T_{\mathbb{C}}/Z_{\mathbb{C}} \simeq T_{\mathbb{C}}$ on $\mathcal{Y}_P$: for $t \in T_{\mathbb{C}}$ and $[y] \in \mathcal{Y}_P$ we take $t \cdot [y] = [\rho(t) \circ y]$.

3.2.1. The case of $\mathcal{Y}_P$

We apply Lemma 3.9 to the $T_{\mathbb{C}}$-variety $\mathcal{Y}_P = \overline{T_{\mathbb{C}} \cdot [\mathrm{Id}]}$ in $\mathbb{P}(E)$. Let $\{\alpha_j, j \in J\}$ be the $T_{\mathbb{C}}$-weights on $V = \oplus_{i=1}^N V_{\lambda_i}$, counted with their multiplicities. We suppose that a K-invariant Hermitian metric is fixed on each representation V_{λ_i}.

Their exists an orthonormal basis $\{v_j, j \in J\}$ of $V = \oplus_{i=1}^N V_{\lambda_i}$ such that $\mathrm{Id} = \sum_{j\in J} v_j \otimes v_j^*$ and

$$\rho(e^Z) = \sum_{j \in J} e^{i\langle \alpha_j, Z\rangle} v_j \otimes v_j^*, \quad Z \in \mathfrak{t}_{\mathbb{C}}. \tag{3.4}$$

So the action of $e^Z \in T_{\mathbb{C}}$ on $[\mathrm{Id}] \in \mathbb{P}(E)$ is $e^Z \cdot [\mathrm{Id}] = \left[\sum_{j\in J} e^{i\langle \alpha_j, Z\rangle} v_j \otimes v_j^*\right]$. We introduce a subset J' of J such that for every $j \in J$ there exists a *unique* $j' \in J'$ such that $\alpha_j = \alpha_{j'}$. So the variety $\mathcal{Y}_P$ lives into $\mathbb{P}(E')$ where $E' = \oplus_{j' \in J'} \mathbb{C} m_{j'}$ with $m_{j'} = \sum_{j \in J, \alpha_j = \alpha_{j'}} v_j \otimes v_j^*$. The closed $T_{\mathbb{C}}$-orbits in $\mathbb{P}(E')$ are the fixed points $[m_{j'}]$, $j' \in J'$.

Lemma 3.10. — *$[m_{j_o}] \in \mathcal{Y}_P$ if and only if α_{j_o} is a vertex of the polytope P.*

Proof. — If α_{j_o} is a vertex of P, there exists $X \in \mathfrak{t}$ such that $\langle \alpha_{j_o}, X\rangle > \langle \alpha_j, X\rangle$ whenever $\alpha_{j_o} \neq \alpha_j$. Hence $e^{-isX} \cdot [\mathrm{Id}]$ tends to $[m_{j_o}]$ when $s \to +\infty$. If α_{j_o} is not a vertex of P, we can find $L \subset J' \setminus \{j_o\}$ such that $\alpha_{j_o} = \sum_{l \in L} a_l \alpha_l$ with $0 < a_l < 1$ and $\sum_l a_l = 1$. So $\mathcal{Y}_P$ is included into the closed subset defined by

$$\left\{ \Big[\sum_{j' \in J'} \delta_{j'} m_{j'}\Big] \in \mathbb{P}(E') \colon \prod_{l \in L} |\delta_l|^{a_l} = |\delta_{j_o}| \right\}.$$

Hence $[m_{j_o}] \notin \mathcal{Y}_P$. □

Remark 3.11. — When α_j is a vertex of the polytope P, the multiplicity of α_j in $\oplus_{i=1}^N V_{\lambda_i}$ is one, so $m_j = v_j \otimes v_j^*$.

Consider now a vertex α_{j_o} of P (for $j_o \in J'$). We consider the open neighborhood $\mathcal{V} \subset \mathbb{P}(E')$ of $[m_{j_o}]$ defined by $[\sum_{j'\in J'} \delta_{j'} m_{j'}] \in \mathcal{V} \Leftrightarrow \delta_{j_o} \neq 0$, and the diffeomorphism $\psi : \mathcal{V} \to \mathbb{C}^{J' \smallsetminus \{j_o\}}$, $[\sum_{j'\in J'} \delta_{j'} m_{j'}] \mapsto (\frac{\delta_{j'}}{\delta_{j_o}})_{j'\neq j_o}$. The map ψ realizes a diffeomorphism between $\mathcal{Y}_P \cap \mathcal{V}$ and the affine subvariety

$$\mathcal{Z} := \overline{\{(t^{\alpha_{j'}-\alpha_{j_o}})_{j'\neq j_o} \mid t \in T_{\mathbb{C}}\}} \subset \mathbb{C}^{J' \smallsetminus \{j_o\}}.$$

The set of weights $\{\alpha_j, j \in J\}$ contains all the lattice points that belong to the edges of P. Since the polytope P is K-adapted, there exists a subset $L_{j_o} \subset J'$ such that $\alpha_l - \alpha_{j_o}, l \in L_{j_o}$ is a $\mathbb{Z}$-basis of the group of weights $\wedge^*$. And for every $j' \neq j_o$ we have

$$\alpha_{j'} - \alpha_{j_o} = \sum_{l\in L_{j_o}} n^l_{j'}(\alpha_l - \alpha_{j_o}) \quad \text{with} \quad n^l_{j'} \in \mathbb{N}. \tag{3.5}$$

We define on $\mathbb{C}^{L_{j_o}}$ the monomials $P_{j'}(Z) = \Pi_{l\in L_{j_o}}(Z_l)^{n^l_{j'}}$. Note that $P_{j'}(Z) = Z_l$ when $j' = l \in L_{j_o}$. Now it is not difficult to see that the map

$$\begin{aligned} \mathbb{C}^{L_{j_o}} &\longrightarrow \mathbb{C}^{J' \smallsetminus \{j_o\}} \\ Z &\longmapsto (P_{j'}(Z))_{j'\neq j_o} \end{aligned}$$

realizes a diffeomorphism between $\mathbb{C}^{L_{j_o}}$ and $\mathcal{Z}$.

We have shown that $\mathcal{Y}_P$ is smooth near $[m_{j_o}]$: hence $\mathcal{Y}_P$ is a smooth subvariety of $\mathbb{P}(E)$. Since $T_{\mathbb{C}}$ acts on $\mathcal{Y}_P$ with a dense orbit, $\mathcal{Y}_P$ is a smooth projective toric variety.

3.2.2. The case of $\mathcal{X}_P$

Recall that $E := \oplus_{i=1}^N \mathrm{End}(V_{\lambda_i})$. The closed $K_{\mathbb{C}} \times K_{\mathbb{C}}$-orbits in $\mathbb{P}(E)$ are those passing through $[v_{\lambda_i} \otimes v^*_{\lambda_i}]$ where $v_{\lambda_i} \in V_{\lambda_i}$ is a highest weight vector. Recall that all the λ_i are regular elements of $\mathfrak{t}^*$.

Lemma 3.12. — *$[v_{\lambda_i} \otimes v^*_{\lambda_i}] \in \mathcal{X}_P$ if and only if λ_i is a vertex of the polytope P.*

Proof. — If λ_i is a vertex of P, we have proved in Lemma 3.10 that $[v_{\lambda_i} \otimes v^*_{\lambda_i}]$ belongs to $\mathcal{Y}_P$ and so belongs to $\mathcal{X}_P$. We shall prove the converse in Corollary 3.17. □

For the remainder of this section we consider a vertex $\lambda_{i_o} \in \widehat{K}$ of the polytope P. Let B^+, B^- be the subgroups fixing respectively the elements $[v_{\lambda_{i_o}}] \in \mathbb{P}(V_{\lambda_i})$ and $[v^*_{\lambda_{i_o}}] \in \mathbb{P}(V^*_{\lambda_i})$: since λ_{i_o} is regular, B^+ and B^- are opposite Borel subgroups of $K_{\mathbb{C}}$. Consider also the maximal unipotent subgroups $N^{\pm} \subset B^{\pm}$.

We consider the open subset $\mathcal{V}_{\mathrm{End}} \subset \mathbb{P}(E)$ of elements $[f]$ such that $\langle v^*_{\lambda_{i_o}}, f(v_{\lambda_{i_o}})\rangle \neq 0$: $\mathcal{V}_{\mathrm{End}}$ is a $B^- \times B^+$-stable neighborhood of $[v_{\lambda_{i_o}} \otimes v^*_{\lambda_{i_o}}]$. Consider the open subsets $\mathcal{V} \subset \mathbb{P}(V_{\lambda_{i_o}})$ and $\mathcal{V}^* \subset \mathbb{P}(V^*_{\lambda_{i_o}}))$ defined by:

- $[v] \in \mathcal{V} \;\Leftrightarrow\; \langle v^*_{\lambda_{i_o}}, v\rangle \neq 0$: $\mathcal{V}$ is B^- stable,
- $[\xi] \in \mathcal{V}^* \;\Leftrightarrow\; \langle \xi, v_{\lambda_{i_o}}\rangle \neq 0$: $\mathcal{V}^*$ is B^+ stable.

We define now the rational maps $l : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_{i_o}}), [f] \mapsto [f(v_{\lambda_{i_o}})]$ and $r : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V^*_{\lambda_{i_o}}), [f] \mapsto [v^*_{\lambda_{i_o}} \circ f]$. The maps l and r are defined on $\mathcal{V}_{\mathrm{End}}$: they define respectively a B^--equivariant map from $\mathcal{V}_{\mathrm{End}}$ to $\mathcal{V}$, and a B^+-equivariant map from $\mathcal{V}_{\mathrm{End}}$ to $\mathcal{V}^*$.

The orbits $K_{\mathbb{C}} \cdot [v_{\lambda_{i_o}}] \subset \mathbb{P}(V_{\lambda_{i_o}})$ and $K_{\mathbb{C}} \cdot [v^*_{\lambda_{i_o}}] \subset \mathbb{P}(V^*_{\lambda_{i_o}})$ are closed and we have

$$K_{\mathbb{C}} \cdot [v_{\lambda_{i_o}}] \cap \mathcal{V} = N^- \cdot [v_{\lambda_{i_o}}] \simeq N^-$$
$$K_{\mathbb{C}} \cdot [v^*_{\lambda_{i_o}}] \cap \mathcal{V}^* = N^+ \cdot [v^*_{\lambda_{i_o}}] \simeq N^+.$$

The rational map $(l, r) : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_{i_o}}) \times \mathbb{P}(V^*_{\lambda_{i_o}})$ then induces a map

$$q : \mathcal{V}_{\mathrm{End}} \cap \mathcal{X}_P \to N^- \times N^+$$

which is $N^- \times N^+$-equivariant: $q\left((n^-, n^+) \cdot x\right) = (n^-, n^+) \cdot q(x)$ for $x \in \mathcal{V}_{\mathrm{End}} \cap \mathcal{X}_P$, and $n^{\pm} \in N^{\pm}$.

We can now finish the proof. The set $N^- T_{\mathbb{C}} N^+ \subset K_{\mathbb{C}}$ is dense in $K_{\mathbb{C}}$, so it is now easy to see that the map

$$\begin{aligned} N^- \times N^+ \times (\mathcal{Y}_P \cap \mathcal{V}_{\mathrm{End}}) &\longrightarrow \mathcal{X}_P \cap \mathcal{V}_{\mathrm{End}} \\ (n^-, n^+, y) &\longmapsto (n^-, n^+) \cdot y \end{aligned}$$

is a diffeomorphism. We proved above that $\mathcal{Y}_P \cap \mathcal{V}_{\mathrm{End}}$ is a smooth affine variety, hence $\mathcal{X}_P$ is smooth near $[v_{\lambda_{i_o}} \otimes v^*_{\lambda_{i_o}}] \in \mathcal{X}_P \cap \mathcal{V}_{\mathrm{End}}$. Lemma 3.9 then tells us that $\mathcal{X}_P$ is a smooth variety.

3.3. Hamiltonian actions

First consider a Hermitian vector space V. The Hermitian structure on $\mathrm{End}(V)$ is $(A, B) := \mathrm{Tr}(AB^*)$, hence the associated symplectic structure on $\mathrm{End}(V)$ is defined by the relation $\Omega_{\mathrm{End}}(A, B) := -\mathrm{Im}(\mathrm{Tr}(AB^*))$.

Let $U(V)$ be the unitary group and $\mathfrak{u}(V)$ its Lie algebra. We will use the identification $\epsilon : \mathfrak{u}(V) \simeq \mathfrak{u}(V)^*$, $X \mapsto \epsilon_X$ where $\epsilon_X(Y) = -\mathrm{Tr}(XY)$. The action of $U(V)\times U(V)$ on $\mathrm{End}(V)$ is $(g,h)\cdot A = gAh^{-1}$. The moment map relative to this action is

$$\begin{aligned}\mathrm{End}(V) &\longrightarrow \mathfrak{u}(V)^*\times\mathfrak{u}(V)^*\\ A &\longmapsto \frac{-1}{2}\left(iAA^*, -iA^*A\right).\end{aligned}$$

We now consider the projective space $\mathbb{P}(\mathrm{End}(V))$ equipped with the Fubini-Study symplectic form Ω_{FS}. Here the action of $U(V)\times U(V)$ on $\mathbb{P}(\mathrm{End}(V))$ is Hamiltonian with moment map

$$\begin{aligned}\mathbb{P}(\mathrm{End}(V)) &\longrightarrow \mathfrak{u}(V)^*\times\mathfrak{u}(V)^*\\ [A] &\longmapsto \left(\frac{iAA^*}{\|A\|^2}, \frac{-iA^*A}{\|A\|^2}\right)\end{aligned}$$

where $\|A\|^2 = \mathrm{Tr}(AA^*)$ (see [27], Section 7). If $\rho : K \hookrightarrow U(V)$ is a closed connected Lie subgroup, we can consider the action of $K\times K$ on $\mathbb{P}(\mathrm{End}(V))$. Let $\pi_K : \mathfrak{u}(V)^* \to \mathfrak{k}^*$ be the projection which is dual to the inclusion $\rho : \mathfrak{k}\hookrightarrow \mathfrak{u}(V)$. The moment map for the action of $K\times K$ on $(\mathbb{P}(\mathrm{End}(V)),\Omega_{\mathrm{FS}})$ is then

$$\begin{aligned}\mathbb{P}(\mathrm{End}(V)) &\longrightarrow \mathfrak{k}^*\times\mathfrak{k}^*\\ [A] &\longmapsto \frac{1}{\|A\|^2}(\pi_K(iAA^*), -\pi_K(iA^*A)).\end{aligned}\tag{3.6}$$

Here we are interested in

- the projective variety $\mathcal{X}_P \subset \mathbb{P}(\mathrm{End}(V))$ with the action of $K\times K$,
- the projective variety $\mathcal{Y}_P \subset \mathbb{P}(\mathrm{End}(V))$ with the action of $T\times T$,

where $V = \oplus_{i=1}^N V_{\lambda_i}$. The Fubini-Study two-form restricts to symplectic forms on $\mathcal{X}_P$ and $\mathcal{Y}_P$. The action of $K\times K$ on $\mathcal{X}_P$ is Hamiltonian with moment map

$$\begin{aligned}\Phi_{K\times K} : \mathcal{X}_P &\longrightarrow \mathfrak{k}^*\times\mathfrak{k}^*\\ [x] &\longmapsto \frac{1}{\|x\|^2}(\pi_K(ixx^*), -\pi_K(ix^*x)).\end{aligned}\tag{3.7}$$

Since the diagonal $Z = \{(t,t)|t\in T\}$ acts trivially on $\mathcal{Y}_P$ we restrict ourselves to the action of $T\times T/Z\simeq T$ on $\mathcal{Y}_P$. Let us compute the moment map $\Phi_T : \mathcal{Y}_P \to \mathfrak{t}^*$ associated to this action. First we have

$$\Phi_T([y]) = \frac{\pi_T(iy^*y)}{\|y\|^2} = \frac{\pi_T(iyy^*)}{\|y\|^2}\tag{3.8}$$

where $\pi_T : \mathfrak{u}(V)^* \to \mathfrak{t}^*$ is the projection which is dual to $\rho : \mathfrak{t}\to\mathfrak{u}(V)$. Since $\rho(X) = i\sum_{j\in J}\alpha_j(X)v_j\otimes v_j^*$, a small computation shows that for

$B \in \mathfrak{u}(V) \simeq \mathfrak{u}(V)^*$ we have $\pi_T(B) = -i\sum_{j\in J}(Bv_j, v_j)\alpha_j$. Finally for any $[y] \in \mathcal{Y}_P$ we get

$$\Phi_T([y]) = \sum_{j\in J} \frac{\|yv_j\|^2}{\|y\|^2}\alpha_j.$$

Together with the action of T, we also have an action of the Weyl group $W = N(T)/T$ on $\mathcal{Y}_P$: for $\bar{w} \in W$ we take

$$\bar{w}\cdot[y] = [\rho(w)\circ y\circ\rho(w)^{-1}], \quad [y] \in \mathcal{Y}_P. \tag{3.9}$$

This action is well defined since the diagonal $Z \subset T \times T$ acts trivially on $\mathcal{Y}_P$. The set of weights $\{\alpha_j, j \in J\}$ is stable under the action of W, hence it is easy to verify that the map Φ_T is W-equivariant.

A dense part of $\mathcal{Y}_P$ is formed by the elements $e^Z \cdot [\mathrm{Id}] = [\rho(e^Z)]$ with $Z = X + iY \in \mathfrak{t}_{\mathbb{C}}$. We have $\Phi_T(e^Z \cdot [\mathrm{Id}]) = \psi_T(Y) \in \mathfrak{t}^*$ with

$$\psi_T(Y) = \frac{1}{\sum_{j\in J} e^{-2\langle\alpha_j, Y\rangle}} \sum_{j\in J} e^{-2\langle\alpha_j,Y\rangle}\alpha_j. \tag{3.10}$$

Hence the image of the moment map $\Phi_T : \mathcal{Y}_P \to \mathfrak{t}^*$ is equal to the closure of the image of the map $\psi_T : \mathfrak{t} \to \mathfrak{t}^*$.

Proposition 3.13. — *The map ψ_T realizes a diffeomorphism between $\mathfrak{t}$ and the interior of the polytope $P \subset \mathfrak{t}^*$.*

Proof. — Consider the function $F_T : \mathfrak{t} \to \mathbb{R}$, $F_T(Y) = \ln\left(\sum_j e^{\langle\alpha_j,Y\rangle}\right)$, and let $L_T : \mathfrak{t} \to \mathfrak{t}^*$ be its Legendre transform: $L_T(X) = dF_T|_X$. Note that we have $L_T(-2Y) = \psi_T(Y)$.

We see that F_T is strictly convex. So, it is a classical fact that L_T realizes a diffeomorphism of $\mathfrak{t}$ onto its image, and for $\xi \in \mathfrak{t}^*$ we have

$$\begin{aligned}\xi \in \mathrm{Image}(L_T) &\iff \lim_{Y\to\infty} F_T(Y) - \langle\xi, Y\rangle = \infty\\ &\iff \lim_{Y\to\infty}\sum_{j\in J} e^{\langle\alpha_j-\xi,Y\rangle} = \infty.\end{aligned}$$

□

In order to conclude we need the following

Lemma 3.14. — *Let $\{\beta_j, j \in J\}$ be a sequence of elements of $\mathfrak{t}^*$, and let Q be its convex hull. We have*

$$\lim_{Y\to\infty}\sum_{j\in J} e^{\langle\beta_j,Y\rangle} = \infty \iff 0 \in \mathrm{Interior}(Q).$$

Proof. — First we see that $0 \notin \mathrm{Interior}(Q)$ if and only there exists $v \in \mathfrak{t} - \{0\}$ such that $\langle \beta_j, v\rangle \leqslant 0$ for all j: for such a vector v, the map $t \to \sum_{j\in J} e^{t\langle \beta_j, v\rangle}$ is bounded for $t \geqslant 0$. Suppose now that $\lim_{Y\to\infty} \sum_{j\in J} e^{\langle \beta_j, Y\rangle} \neq \infty$. Then there exists a sequence $(X_k)_k \in \mathfrak{t}$ such that $\lim_k |X_k| = \infty$ and for all j the sequence $(\langle \beta_j, X_k\rangle)_k$ remains bounded from above. If v is a limit of a sub-sequence of $(\frac{X_k}{|X_k|})_k$ we have then $\langle \beta_j, v\rangle \leqslant 0$ for all j. □

Lemma 3.15. — *For* $[y]{\in}\mathcal{Y}_P$ *we have* $\Phi_{K\times K}([y]) = (\Phi_T([y]), -\Phi_T([y]))$.

Proof. — It is sufficient to consider the case

$$y = \rho(e^Z) = \sum_{j\in J} e^{i\langle \alpha_j, Z\rangle} v_j \otimes v_j^*, \text{ for } Z = X + iY \in \mathfrak{t}_{\mathbb{C}}.$$

Then $yy^* = y^*y = \sum_j e^{-2\langle \alpha_j, Y\rangle} v_j \otimes v_j^* = \rho(e^{2iY})$. So what remains to prove is that $\pi_K(iyy^*) = \pi_T(iyy^*)$. We have to check that $\langle \pi_K(iyy^*), [U,V]\rangle = 0$ for $U \in \mathfrak{t}$ and $V \in \mathfrak{k}$. We have

$$\begin{aligned}\Big\langle \pi_K(iyy^*), [U,V]\Big\rangle &= -i\ \mathrm{Tr}\Big(yy^*\rho([U,V])\Big)\\ &= -i\ \mathrm{Tr}\Big(\rho(e^{2iY})[\rho(U),\rho(V)]\Big)\\ &= -i\ \mathrm{Tr}\Big([\rho(e^{2iY}),\rho(U)]\rho(V)\Big) = 0.\end{aligned}$$

□

Theorem 3.16. — *We have*

- $\mathrm{Image}(\Phi_T) = P$,
- $\mathrm{Image}(\Phi_{K\times K}) = \big\{(k_1\cdot\xi, -k_2\cdot\xi) \mid \xi \in P \text{ and } k_1, k_2 \in K\big\}$,
- $\mathcal{Y}_P \subset \Phi^{-1}_{K\times K}(\mathfrak{t}^* \times \mathfrak{t}^*)$,
- $\Phi^{-1}_{K\times K}(\mathrm{interior}(\mathcal{C})) \subset \mathcal{Y}_P$, *where* $\mathcal{C} = C_K \times -C_K$.

Proof. — The first point follows from Proposition 3.13. Since the map $(k_1, t, k_2) \mapsto k_1 t k_2$ from $K \times T_{\mathbb{C}} \times K$ to $K_{\mathbb{C}}$ is onto, we have

$$\mathcal{X}_P = (K\times K)\cdot \mathcal{Y}_P. \tag{3.11}$$

So if $[x] \in \mathcal{X}_P$, there exist $[y] \in \mathcal{Y}$ and $k_1, k_2 \in K$ such that $[x] = (k_1,k_2)\cdot[y]$, hence

$$\begin{aligned}\Phi_{K\times K}([x]) &= (k_1,k_2)\cdot \Phi_{K\times K}([y])\\ &= (k_1\cdot\Phi_T([y]), -k_2\cdot\Phi_T([y]))\,. \end{aligned}\tag{3.12}$$

The second point is proved. The third point follows also from the identity (3.12) when $k_1 = k_2 = e$. Consider now $[x] = (k_1,k_2)\cdot[y]$ such that $\Phi_{K\times K}([x])$ belongs to the interior of the cone $C_K \times -C_K$. Then $k_1\cdot\Phi_T([y])$

and $k_2 \cdot \Phi_T([y])$ are regular points of C_K. This implies that $k_1, k_2 \in N(T)$ and $k_2 k_1^{-1} \in T$. So

$$\begin{aligned}[x] &= (k_1, k_2) \cdot [y] \\ &= (e, k_2 k_1^{-1}) \cdot \Big((k_1, k_1) \cdot [y]\Big) \in \mathcal{Y}_P\end{aligned}$$

since $\mathcal{Y}_P$ is stable under the actions of $T \times T$ and W. □

Let $\mathcal{O}_i$ be the closed $K_{\mathbb{C}} \times K_{\mathbb{C}}$-orbit in $\mathbb{P}(E)$ passing through $[v_{\lambda_i} \otimes v_{\lambda_i}^*]$, where $v_{\lambda_i} \in V_{\lambda_i}$ is a highest weight vector and λ_i is regular dominant weight.

Corollary 3.17. — *If $\mathcal{O}_i \subset \mathcal{X}_P$ then λ_i is a vertex of the polytope P.*

Proof. — Let $x = v_{\lambda_i} \otimes v_{\lambda_i}^*$, and suppose that $[x]$ belongs to $\mathcal{X}_P$. In order to show that $[x] \in \mathcal{Y}_P$, we compute $\Phi_{K\times K}([x])$. We see that $xx^* = x^*x = x$ and $\|x\| = 1$, so $\Phi_{K\times K}([x]) = (\pi_K(ix), -\pi_K(ix))$. For $X \in \mathfrak{k}$ we have

$$\begin{aligned}\langle \pi_K(ix), X\rangle &= -i \ \mathrm{Tr}\Big(v_{\lambda_i} \otimes v_{\lambda_i}^* \rho(X)\Big) \\ &= -i \ (\rho(X) v_{\lambda_i}, v_{\lambda_i}) \\ &= \langle \lambda_i, X\rangle.\end{aligned}$$

We then have $\Phi_{K\times K}([x]) = (\lambda_i, -\lambda_i)$ with λ_i being a *regular* point of C_K: then the last point of Theorem 3.16 shows that $[x] \in \mathcal{Y}_P$. Now we can conclude with the help of Lemma 3.10. Since $[v_{\lambda_i} \otimes v_{\lambda_i}^*]$ belongs to $\mathcal{Y}_P$, the weight λ_i is a vertex of the polytope P. □

Remark 3.18. — In this section, Theorem 3.16 was obtained without using the fact that the varieties $\mathcal{X}_P$ and $\mathcal{Y}_P$ are smooth. Hence Corollary 3.17 can be used to prove the smoothness of $\mathcal{X}_P$ (*cf.* Lemma 3.12).

3.4. Symplectic cutting

Let (M, Ω_M, Φ_M) be a Hamiltonian K-manifold. At this stage the moment map Φ_M is *not assumed to be proper*. We also consider the Hamiltonian $K \times K$-manifold $\mathcal{X}_P$ associated to a K-adapted polytope P.

The purpose of this section is to define a *symplectic cutting* of M which uses $\mathcal{X}_P$. The notion of symplectic cutting was introduced by Lerman in [22] in the case of a torus action. Later Woodward [36] extended this procedure to the case of a non-abelian group action (see also [25, 26]). The method of symplectic cutting that we define in this section is different from that of Woodward.

We have two actions of K on $\mathcal{X}_P$: the action from the left (resp. right), denoted $\cdot_l$ (resp. $\cdot_r$), with moment map $\Phi_l : \mathcal{X}_P \to \mathfrak{k}^*$ (resp. Φ_r). We consider now the product $M \times \mathcal{X}_P$ with

- the action $k \cdot_1 (m, x) = (k \cdot m, k \cdot_r x)$: the corresponding moment map is $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$,
- the action $k \cdot_2 (m, x) = (m, k \cdot_l x)$: the corresponding moment map is $\Phi_2(m, x) = \Phi_l(x)$.

Definition 3.19. — *We denote M_P the symplectic reduction at 0 of $M \times \mathcal{X}_P$ for the action $\cdot_1$: $M_P := (\Phi_1)^{-1}(0)/K$.*

Note that M_P is compact when Φ_M is proper. The action $\cdot_2$ on $M \times \mathcal{X}_P$ induces an action of K on M_P. The moment map Φ_2 induces an equivariant map $\Phi_{M_P} : M_P \to \mathfrak{k}^*$. Let $\mathcal{Z} \subset (\Phi_1)^{-1}(0)$ be the set of points where $(K, \cdot_1)$ has a trivial stabilizer.

Definition 3.20. — *We denote M'_P the quotient $\mathcal{Z}/K \subset M_P$.*

M'_P is an open subset of smooth points of M_P which is invariant under the K-action. The symplectic structure of $M \times \mathcal{X}_P$ induces a canonical symplectic structure on M'_P that we denote $\Omega_{M'_P}$. The action of K on $(M'_P, \Omega_{M'_P})$ is Hamiltonian with moment map equal to the restriction of $\Phi_{M_P} : M_P \to \mathfrak{k}^*$ to M'_P.

We start with the easy

Lemma 3.21. — *The image of $\Phi_{M_P} : M_P \to \mathfrak{k}^*$ is equal to the intersection of the image of $\Phi_M : M \to \mathfrak{k}^*$ with $K \cdot P$.*

Let $\mathcal{U}_P = K \cdot \mathrm{Interior}(P) \subset K \cdot P$. We will show now that the open and dense subset $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ of M_P is contained in M'_P. Afterwards we will prove that $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ is quasi-symplectomorphic to the open subset $\Phi_M^{-1}(\mathcal{U}_P)$ of M.

We consider the open and dense subset of $\mathcal{X}_P$ which is equal to the open orbit $\bar{\rho}(K_{\mathbb{C}})$. From Lemma 3.5, we know that

$$\begin{aligned} \Theta : K \times \mathfrak{k} &\longrightarrow \bar{\rho}(K_{\mathbb{C}}) \\ (k, X) &\longmapsto [\rho(ke^{iX})] \end{aligned} \tag{3.13}$$

is a diffeomorphism. Via Θ, the action of $K \times K$ on $K \times \mathfrak{k}$ is $k \cdot_l (a, X) = (ka, X)$ for the action "from the left" and $k \cdot_r (a, X) = (ak^{-1}, k \cdot X)$ for the action "from the right".

We now consider the map $\psi_K : \mathfrak{k} \to \mathfrak{k}^*$ defined by $\psi_K(X) = \Phi_l([\rho(e^{iX})])$. In other words,

$$\psi_K(X) = \frac{\pi_K(i\rho(e^{i2X}))}{\mathrm{Tr}(\rho(e^{i2X}))}.$$

Consider the function $F_K : \mathfrak{k} \to \mathbb{R}$, $F_K(X) = \ln(\mathrm{Tr}(\rho(e^{-iX}))$. Let $L_K : \mathfrak{k} \to \mathfrak{k}^*$ be its Legendre transform.

Proposition 3.22.

- *We have $\psi_K(X) = L_K(-2X)$, for $X \in \mathfrak{k}$.*
- *The function F_K is strictly convex.*
- *The map ψ_K realizes an equivariant diffeomorphism between $\mathfrak{k}$ and $\mathcal{U}_P$.*
- *The image of $\Phi_l : \mathcal{X}_P \to \mathfrak{k}^*$ is equal to the closure of $\mathcal{U}_P$.*
- *$\Phi_l^{-1}(\mathcal{U}_P) = \bar{\rho}(K_{\mathbb{C}})$.*

Proof. — For $X, Y \in \mathfrak{k}$ we consider the function $\tau(s) = F_K(X + sY)$. Since F_K is K-invariant we can restrict our computation to the case where $X \in \mathfrak{t}$. We will use the decomposition of $Y \in \mathfrak{k}$ relatively to the T-weights on $\mathfrak{k}_{\mathbb{C}}$: $Y = \sum_\alpha Y_\alpha$ where $\mathrm{ad}(Z)Y_\alpha = i\alpha(Z)Y_\alpha$ for any $Z \in \mathfrak{t}$, and $Y_0 \in \mathfrak{t}$. We have

$$\begin{aligned}\tau'(s) &= \frac{-i}{\mathrm{Tr}(\rho(e^{-iX_s}))}\,\mathrm{Tr}\left(\rho(e^{-iX_s})\rho\left(\frac{e^{i\,\mathrm{ad}(X_s)}-1}{i\,\mathrm{ad}(X_s)}Y\right)\right)\\ &= \frac{-i}{\mathrm{Tr}(\rho(e^{-iX_s}))}\,\mathrm{Tr}\left(\rho(e^{-iX_s})\rho(Y)\right)\\ &= \frac{1}{\mathrm{Tr}(\rho(e^{-iX_s}))}\langle \pi_K(i\rho(e^{-iX_s})), Y\rangle\end{aligned}$$

where $X_s = X + sY$. Since by definition $\tau'(0) = \langle L_K(X), Y\rangle$, the first point is proved. For the second derivative we have

$$\begin{aligned}\tau''(0) &= -\left(\frac{\mathrm{Tr}(\rho(e^{-iX})\rho(iY))}{\mathrm{Tr}(\rho(e^{-iX}))}\right)^2 + \frac{\mathrm{Tr}\left(\rho(e^{-iX})\rho(\frac{e^{i\,\mathrm{ad}(X)}-1}{i\,\mathrm{ad}(X)}iY)\rho(iY)\right)}{\mathrm{Tr}(\rho(e^{-iX}))}\\ &= R_1 + R_2\end{aligned}$$

where

$$\begin{aligned}R_1 &= \frac{\mathrm{Tr}\left(\rho(e^{-iX})\rho(iY_0)\rho(iY_0)\right)}{\mathrm{Tr}(\rho(e^{-iX}))} - \left(\frac{\mathrm{Tr}(\rho(e^{-iX})\rho(iY_0))}{\mathrm{Tr}(\rho(e^{-iX}))}\right)^2\\ &= \frac{\sum_j e^{-\langle\alpha_j, X\rangle}\langle\alpha_j, Y_0\rangle^2}{\sum_j e^{-\langle\alpha_j, X\rangle}} - \left(\frac{\sum_j e^{-\langle\alpha_j, X\rangle}\langle\alpha_j, Y_0\rangle}{\sum_j e^{-\langle\alpha_j, X\rangle}}\right)^2\end{aligned}$$

and

$$R_2 = \frac{1}{\mathrm{Tr}(\rho(e^{-iX}))} \sum_{\alpha\neq 0,\beta\neq 0} \frac{e^{-\langle\alpha,X\rangle}-1}{-\langle\alpha,X\rangle} \mathrm{Tr}\left(\rho(e^{-iX})\rho(iY_\alpha)\rho(iY_\beta)\right)$$
$$= \frac{1}{\mathrm{Tr}(\rho(e^{-iX}))} \sum_{\alpha\neq 0,j} \frac{e^{-\langle\alpha,X\rangle}-1}{-\langle\alpha,X\rangle} e^{-\langle\alpha_j,X\rangle}\|\rho(Y_\alpha)v_j\|^2.$$

It is now easy to see that R_1 and R_2 are nonnegative and that $R_1+R_2>0$ if $Y\neq 0$. We have proved that F_K is strictly convex. So, its Legendre transform L_K realizes a diffeomorphism of $\mathfrak{k}$ onto its image. Using the first point we know that ψ_K realizes a diffeomorphism of $\mathfrak{k}$ onto its image. The map ψ_K is equivariant and coincides with ψ_T on $\mathfrak{t}$. We have proved in Proposition 3.13 that the image of ψ_T is equal to the interior of P, hence the image of ψ_K is $\mathcal{U}_P$.

For the last two points we first remark that

$$\Phi_l\big([\rho(ke^{iX})]\big) = k\cdot\psi_K(X) \tag{3.14}$$

hence the image of Φ_l is the closure of $\mathcal{U}_P$. If we use the fact that ψ_K is a diffeomorphism from $\mathfrak{k}$ onto $\mathcal{U}_P$, (3.14) shows that $\Phi_l^{-1}(K\cdot\xi)\cap\bar\rho(K_\mathbb{C})$ is a non empty and closed subset of $\Phi_l^{-1}(K\cdot\xi)$ for any $\xi\in\mathcal{U}_P$ (in fact it is a $K\times K$-orbit). On the other hand $\Phi_l^{-1}(K\cdot\xi)\cap(\mathcal{X}_P\setminus\bar\rho(K_\mathbb{C}))$ is also a closed subset of $\Phi_l^{-1}(K\cdot\xi)$ since $\bar\rho(K_\mathbb{C})$ is open in $\mathcal{X}_P$. Since $\Phi_l^{-1}(K\cdot\xi)$ is connected the second subset is empty: in other words $\Phi_l^{-1}(K\cdot\xi)\subset\bar\rho(K_\mathbb{C})$. □

We introduce now the equivariant diffeomorphism

$$\begin{aligned}\Upsilon : K\times\mathcal{U}_P &\longrightarrow \bar\rho(K_\mathbb{C})\\ (k,\xi) &\longmapsto \Theta(k,\psi_K^{-1}(\xi)).\end{aligned} \tag{3.15}$$

We now consider $K\times\mathcal{U}_P$ equipped with the symplectic structure $\Upsilon^*(\Omega_{\mathcal{X}_P})$, and the Hamiltonian action of $K\times K$: the moment maps satisfy

$$\Upsilon^*(\Phi_l)(k,\xi) = k\cdot\xi \text{ and } \Upsilon^*(\Phi_r)(k,\xi) = -\xi. \tag{3.16}$$

Proposition 3.23. — *We have*

$$\Upsilon^*(\Omega_{\mathcal{X}_P}) = d\lambda + d\eta$$

where λ is the Liouville 1-form on $K\times\mathfrak{k}^*\simeq \mathrm{T}^*K$ and η is an invariant 1-form on $\mathcal{U}_P\subset\mathfrak{k}^*$ which is killed by the vectors tangent to the K-orbits.

Proof. — Let $E_1,\dots,E_r$ be a basis of $\mathfrak{k}$, with dual basis $\xi^1,\dots,\xi^r$. Let ω^i the 1-form on K, invariant by left translation and equal to ξ^i at the identity. The Liouville 1-form is $\lambda = -\sum_i\omega^i\otimes E_i$. For $X\in\mathfrak{k}$ we denote $X_l(k,\xi) = \frac{d}{dt}|_0 e^{-tX}\cdot_l (k,\xi)$ and $X_r(k,\xi) = \frac{d}{dt}|_0 e^{-tX}\cdot_r (k,\xi)$ the vector fields generated by the action of $K\times K$. Since $\iota(X_l)d\lambda = -d\langle\Phi_l, X\rangle$ and $\iota(X_r)d\lambda = -d\langle\Phi_r, X\rangle$, the closed invariant 2-form $\beta = \Upsilon^*(\Omega_{\mathcal{X}_P}) - d\lambda$ is $K\times K$ invariant and is killed by the vectors tangent to the orbits: $(*)$ $\iota(X_l)\beta = \iota(X_r)\beta = 0$ for all $X\in\mathfrak{k}$. We have $\beta = \beta_2+\beta_1+\beta_0$ where $\beta_2 = \sum_{i,j} a_{ij}(\xi)\omega^i\wedge\omega^j$, $\beta_1 = \sum_{i,j} b_{ij}(\xi)\omega^i\wedge dE_j$, and β_0 is an invariant 2-form on $\mathcal{U}_P$. The equalities $(*)$ gives $\iota(X_l)\beta_2 = \iota(X_l)\beta_1 = 0$ which imply that $\beta_2=\beta_1=0$. So $\beta=\beta_0$ is a closed invariant 2-form on $\mathcal{U}_P$ which is killed by the vectors tangent to the K-orbits. Since $\mathcal{U}_P$ admits a retraction to $\{0\}$, $\beta = d\eta$ where η is an invariant 1-form on $\mathcal{U}_P$ which is killed by the vectors tangent to the K-orbits. □

If $(m,x)\in M\times\mathcal{X}_P$ belongs to $\Phi_1^{-1}(0)$, we denote $[m,x]$ the corresponding element in M_P. By definition we have $\Phi_{M_P}([m,x]) = \Phi_l(x)$ for $[m,x]\in M_P$, hence the image of Φ_{M_P} is included in the closure of $\mathcal{U}_P$. We see also that $[m,x]\in\Phi_{M_P}^{-1}(\mathcal{U}_P)$ if and only if $x\in\Phi_l^{-1}(\mathcal{U}_P) = \bar\rho(K_{\mathbb{C}})$. Since $(K,\cdot_r)$ acts freely on $\bar\rho(K_{\mathbb{C}})$, we see that $(K,\cdot_1)$ acts freely on $\Phi_{M_P}^{-1}(\mathcal{U}_P)$: the open and dense set $\Phi_{M_P}^{-1}(\mathcal{U}_P)\subset M_P$ is then contained in M_P'.

Now, we can state our main result which compares the open invariant subsets $\Phi_M^{-1}(\mathcal{U}_P)\subset M$ and $\Phi_{M_P}^{-1}(\mathcal{U}_P)\subset M_P$ equipped respectively with the symplectic structures Ω_M and $\Omega_{M_P'}$.

Theorem 3.24. — *$\Phi_{M_P}^{-1}(\mathcal{U}_P)$ is an open and dense subset of smooth points in M_P. There exists an equivariant diffeomorphism $\Psi:\Phi_M^{-1}(\mathcal{U}_P)\to\Phi_{M_P}^{-1}(\mathcal{U}_P)$ such that*

$$\Psi^*(\Omega_{M_P'}) = \Omega_M + d\Phi_M^*\eta.$$

Here η is an invariant 1-form on $\mathcal{U}_P$ which is killed by the vectors tangent to the K-orbits. Moreover the path $\Omega^t = \Omega_M + td\Phi_M^*\eta$, defines a homotopy of symplectic 2-forms between Ω_M and $\Psi^*(\Omega_{M_P'})$.

Remark 3.25. — The map Ψ will be called a *quasi*-symplectomorphism.

Proof. — Consider the immersion

$$\begin{aligned}\psi:\Phi_M^{-1}(\mathcal{U}_P) &\longrightarrow M\times\mathcal{X}_P\\ m &\longmapsto (m,\Upsilon(e,\Phi_M(m))).\end{aligned}$$

We have $\Phi_1(\psi(m)) = \Phi_M(m)+\Upsilon^*\Phi_r(e,\Phi_M(m)) = 0$, and $\Phi_2(\psi(m)) = \Upsilon^*\Phi_l(e,\Phi_M(m)) = \Phi_M(m)\in\mathcal{U}_P$ (see (3.16)). Hence for all $m\in\Phi_M^{-1}(\mathcal{U}_P)$, we have $\psi(m)\in\Phi_1^{-1}(0)$, and its class $[\psi(m)]\in M_P$ belongs to $\Phi_{M_P}^{-1}(\mathcal{U}_P)$.

We denote $\Psi:\Phi_M^{-1}(\mathcal{U}_P)\to\Phi_{M_P}^{-1}(\mathcal{U}_P)$ the map $m\mapsto[\psi(m)]$. Let us show that it defines a diffeomorphism. If $\Psi(m)=\Psi(m')$, there exists $k\in K$ such that

$$\begin{aligned}(m,\Upsilon(e,\Phi_M(m))) &= k\cdot_1 (m',\Upsilon(e,\Phi_M(m')))\\ &= (k\cdot m', k\cdot_r \Upsilon(e,\Phi_M(m')))\\ &= (k\cdot m', \Upsilon(k^{-1}, k\cdot\Phi_M(m'))).\end{aligned}$$

Since Υ is a diffeomorphism, we must have $k=e$ and $m=m'$: the map Ψ is one to one. Consider now $(m,x)\in\Phi_1^{-1}(0)$ such that $\Phi_{M_P}([m,x]) = \Phi_l(x)\in\mathcal{U}_P$: then $x\in\Phi_l^{-1}(\mathcal{U}_P)=\bar{\rho}(K_{\mathbb{C}})=\mathrm{Image}(\Upsilon)$. We have $x=\Upsilon(k,\xi)$ where $\xi=-\Phi_r(x)=\Phi_M(m)$. Finally

$$\begin{aligned}(m,x) &= (m,\Upsilon(k,\Phi_M(m)))\\ &= k^{-1}\cdot_1 (k\cdot m,\Upsilon(e,k\cdot\Phi_M(m)))\\ &= k^{-1}\cdot_1 \psi(k\cdot m).\end{aligned}$$

We have proved that Ψ is onto.

In order to show that Ψ is a submersion we must show that for $m\in\Phi_M^{-1}(\mathcal{U}_P)$

$$\mathrm{Image}(\mathrm{T}_m\psi)\oplus\mathrm{T}_{\psi(m)}(K\cdot_1\psi(m)) = \mathrm{T}_{\psi(m)}\Phi_1^{-1}(0).$$

Here $\mathrm{T}_m\psi:\mathrm{T}_mM\to\mathrm{T}_{\psi(m)}(M\times\mathcal{X}_P)$ is the tangent map, and $\mathrm{T}_{\psi(m)}(K\cdot_1\psi(m))$ denotes the tangent space at $\psi(m)$ of the $(K,\cdot_1)$-orbit. We have $\dim(\mathrm{Image}(\mathrm{T}_m\psi))+\dim(\mathrm{T}_{\psi(m)}(K\cdot_1\psi(m)))=\dim(\mathrm{T}_{\psi(m)}\Phi_1^{-1}(0))$ so it is sufficient to prove that

$$\mathrm{Image}(\mathrm{T}_m\psi)\cap\mathrm{T}_{\psi(m)}(K\cdot_1\psi(m))=\{0\}.$$

Consider $(v,w)\in\mathrm{Image}(\mathrm{T}_m\psi)\cap\mathrm{T}_{\psi(m)}(K\cdot_1\psi(m))$. There exists $X\in\mathfrak{k}$ such $(v,w)=\frac{d}{dt}|_0 e^{tX}\cdot_1\psi(m)$: $v=\frac{d}{dt}_{|_0}e^{tX}\cdot m$ and $w=\frac{d}{dt}_{|_0}e^{tX}\cdot_r\Upsilon(e,\Phi_M(m))$. On the other hand, since $(v,w)\in\mathrm{Image}(\mathrm{T}_m\psi)$, we have

$$w=\frac{d}{dt}_{|_0}\Upsilon(e,\Phi_M(e^{tX}\cdot m)).$$

Since $e^{tX}\cdot_r\Upsilon(e,\Phi_M(m))=\Upsilon(e^{-tX},\Phi_M(e^{tX}\cdot m))$ we obtain that

$$\frac{d}{dt}_{|_0}\Upsilon(e^{-tX},\Phi_M(e^{tX}\cdot m))=\frac{d}{dt}_{|_0}\Upsilon(e,\Phi_M(e^{tX}\cdot m)),$$

or in other words $\frac{d}{dt}_{|_0}\Upsilon(e^{-tX},\Phi_M(m))=0$. Since Υ is a diffeomorphism we have $X=0$, and then $(v,w)=0$.

We can now compute the pull-back by Ψ of the symplectic form $\Omega_{M'_P}$. We have

$$\begin{aligned}\Psi^*(\Omega_{M'_P}) &= \psi^*(\Omega_M + \Omega_{\mathcal{X}_P})\\ &= \Omega_M + \Phi_M^*\Upsilon^*(\Omega_{\mathcal{X}_P})\\ &= \Omega_M + d\Phi_M^*\eta.\end{aligned}$$

It remains to prove that for every $t \in [0,1]$, the 2-form $\Omega^t = \Omega_M + td\Phi_M^*\eta$ is non-degenerate. Take $t \neq 0$, $m \in \Phi_M^{-1}(\mathcal{U}_P)$ and suppose that the contraction of $\Omega^t|_m$ by $v \in \mathrm{T}_m M$ is equal to 0. For every $X \in \mathfrak{k}$ we have

$$\begin{aligned}0 &= \Omega^t(X_M(m), v)\\ &= -\iota(v)d\langle\Phi_M, X\rangle_{|m} + t\iota(v)\iota(X_M)d\Phi_M^*\eta_{|m}\\ &= -\iota(v)d\langle\Phi_M, X\rangle_{|m}\end{aligned}$$

since $\iota(X_M)d\Phi_M^*\eta = d\Phi_M^*(\iota(X_{\mathfrak{k}^*})\eta) = 0$. Thus we have $\mathrm{T}_m\Phi_M(v) = 0$, and then $\iota(v)d\Phi_M^*\eta = 0$. Finally we have that $0 = \iota(v)\Omega^t|_m = \iota(v)\Omega_M|_m$. But Ω_M is non-degenerate, so $v = 0$. □

3.5. Formal quantization: second definition

We suppose here that the Hamiltonian K-manifold (M, Ω_M, Φ_M) is *proper* and admits a Kostant-Souriau line bundle L. Now we consider the complex $K \times K$-submanifold $\mathcal{X}_P$ of $\mathbb{P}(E)$. Since $\mathcal{O}(-1)$ is a $K \times K$-equivariant Kostant-Souriau line bundle on the projective space $\mathbb{P}(E)$ the restriction

$$L_P = \mathcal{O}(-1)|_{\mathcal{X}_P} \tag{3.17}$$

is a Kostant-Souriau line bundle on $\mathcal{X}_P$. Hence $L \boxtimes L_P$ is a Kostant-Souriau line bundle on the product $M \times \mathcal{X}_P$. In Section 2.2 we have defined the quantization $\mathcal{Q}_K(M_P)$ of the (singular) reduced space $M_P := (M \times \mathcal{X}_P)/\!\!/_0(K, \cdot_1)$.

Notation. — $O_K(r)$ will be any element $\sum_{\mu\in\widehat{K}} m_\mu V_\mu^K$ of $R^{-\infty}(K)$ where $m_\mu = 0$ if $\|\mu\| < r$. The limit $\lim_{r\to+\infty} O_K(r) = 0$ defines the notion of convergence in $R^{-\infty}(K)$.

Proposition 3.26. — *Let $\varepsilon_P > 0$ be the radius of the biggest ball center at $0 \in \mathfrak{t}^*$ which is contained in the polytope P. We have*

$$\mathcal{Q}_K(M_P) = \sum_{\|\mu\|<\varepsilon_P} \mathcal{Q}(M_\mu)V_\mu^K \;+\; O_K(\varepsilon_P). \tag{3.18}$$

Proof. — Theorem 2.4 – "Quantization commutes with reduction in the singular setting" – tells us that $\mathcal{Q}_K(M_P) = \sum_{\mu\in\widehat{K}} \mathcal{Q}((M_P)_\mu)V_\mu^K$ where $(M_P)_\mu$ is the symplectic reduction $(M_P \times \overline{K\cdot\mu})/\!\!/_0 K$.

Since the image of Φ_{M_P} is equal to the intersection of $K\cdot P = \overline{\mathcal{U}_P}$ with the image of Φ_M, we have $\mathcal{Q}((M_P)_\mu) = 0$ if $\mu \notin P \cap \mathrm{Image}(\Phi_M)$. We will now exploit Theorem 3.24 to show that $\mathcal{Q}((M_P)_\mu) = \mathcal{Q}(M_\mu)$ if μ belongs to the interior of P.

There exists a quasi-symplectomorphism Ψ between the open subset $\Phi_M^{-1}(\mathcal{U}_P)$ of M and the open and dense subset $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ of M_P. Moreover one can see easily that the restriction of the Kostant-Souriau line bundle $L_P \to \mathcal{X}_P$ to the open subset $\bar\rho(K_{\mathbb{C}})$ is *trivial.* If L_{M_P} is the Kostant-Souriau line bundle on M_P induced by $L \boxtimes L_P$, then the pull-back of the restriction $L_{M_P}|_{\Phi_M^{-1}(\mathcal{U}_P)}$ by Ψ is equivariantly diffeomorphic to the restriction of L to $\Phi_M^{-1}(\mathcal{U}_P)$.

Take now $\mu\in\widehat{K}$ that belongs to the interior of the polytope P. The element $\mathcal{Q}((M_P)_\mu)\in\mathbb{Z}$ is given by the index of a transversally elliptic symbol defined in a (small) neighborhood of $\Phi_{M_P}^{-1}(\mu)\subset M_P$. This symbol is defined through two auxiliary data: the Kostant-Souriau line bundle L_{M_P} and a compatible almost complex structure J which is defined in a neighborhood of $\Phi_{M_P}^{-1}(\mu)$. If we pull back everything by Ψ, we get a transversally elliptic symbol living in a (small) neighborhood of $\Phi_M^{-1}(\mu)\subset M$ which is defined by the Kostant-Souriau line bundle L and an almost complex structure J_1 *compatible with the symplectic structure* $\Omega_1 := \Omega_M + d\Phi_M^*\eta$. But since $\Omega_t = \Omega_M + td\Phi_M^*\eta$ defines a homotopy of symplectic structures, any almost complex structure compatible with Ω_M is homotopic to J_1. We have then shown that $\mathcal{Q}(M_\mu) = \mathcal{Q}((M_P)_\mu)$ for any μ belonging to the interior of P. So we have

$$\mathcal{Q}_K(M_P) = \sum_{\mu\in\mathrm{Interior}(P)} \mathcal{Q}(M_\mu)V_\mu^K + \sum_{\nu\in\partial P}\mathcal{Q}((M_P)_\nu)V_\nu^K.$$

Since for $\nu\in\partial P$ we have $\|\nu\|\geqslant\varepsilon_P$, the last equality proves (3.18). □

We work now with the dilated polytope nP, for any integer $n\geqslant 1$. The polytope nP is still K-adapted, so one can consider the reduced space(2) M_{nP} and Proposition 3.26 gives that

$$\mathcal{Q}_K(M_{nP}) = \sum_{\|\mu\|<n\varepsilon_P}\mathcal{Q}(M_\mu)V_\mu^K \;+\; O_K(n\varepsilon_P). \tag{3.19}$$

(2) These are the cut spaces denoted $M_{\mathrm{PEP}}^{(n)}$ in the introduction.

for any integer $n \geqslant 1$. We can summarize the result of this section in the following

PROPOSITION 3.27. — *Let (M, Ω_M) be a pre-quantized Hamiltonian K-manifold, with a proper moment map Φ_M.*

• *For any integer $n \geqslant 1$, the (singular) compact Hamiltonian manifold M_{nP} contains as an open and dense subset, the open subset $\Phi_M^{-1}(n\mathcal{U}_P)$ of M.*

• *We have $\mathcal{Q}_K^{-\infty}(M) = \lim_{n\to\infty} \mathcal{Q}_K(M_{nP})$.*

4. Functorial properties: Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We will use in a crucial way the characterisation of $\mathcal{Q}_K^{-\infty}$ given in Proposition 3.27.

Let $H \subset K$ be a closed and connected Lie subgroup. Here we consider a pre-quantized Hamiltonian K-manifold M which is *proper* as a Hamiltonian H-manifold. We want to compare $\mathcal{Q}_K^{-\infty}(M)$ and $\mathcal{Q}_H^{-\infty}(M)$. For $\mu \in \widehat{K}$ and $\nu \in \widehat{H}$ we denote N_ν^μ the multiplicity of V_ν^H in the restriction $V_\mu^K|_H$. We have seen in the introduction that $N_\nu^\mu \mathcal{Q}\left(M_{\mu,K}\right) \neq 0$ only for the μ belonging to *finite* subset $\widehat{K} \cap \Phi_K\left(K \cdot \Phi_H^{-1}(\nu)\right)$. Then $\mathcal{Q}_K^{-\infty}(M)$ is H-admissible and we have the following equality in $R^{-\infty}(H)$:

$$\mathcal{Q}_K^{-\infty}(M)|_H = \sum_{\nu \in \widehat{H}} m_\nu V_\nu^H \tag{4.1}$$

with $m_\nu = \sum_\mu N_\nu^\mu \mathcal{Q}\left(M_{\mu,K}\right)$. We will now prove that

$$\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M).$$

LEMMA 4.1. — *The restriction $\mathcal{Q}_K^{-\infty}(M)|_H$ is equal to $\lim_{n\to\infty} \mathcal{Q}_K(M_{nP})|_H$.*

Proof. — Let us denote by P^o and ∂P respectively the interior and the boundary of the K-adapted polytope P. We write

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in nP^o} \mathcal{Q}(M_{\mu,K}) V_\mu^K + \sum_{\mu \notin nP^o} \mathcal{Q}(M_{\mu,K}) V_\mu^K.$$

On the other side

$$\mathcal{Q}_K(M_{nP}) = \sum_{\mu \in nP^o} \mathcal{Q}(M_{\mu,K}) V_\mu^K + \sum_{\mu \in n\partial P} \mathcal{Q}((M_{nP})_{\mu,K}) V_\mu^K.$$

So the difference $D(n) = \mathcal{Q}_K^{-\infty}(M) - \mathcal{Q}_K(M_{nP})$ is equal to

$$D(n) = -\sum_{\mu' \in n\partial P} \mathcal{Q}((M_{nP})_{\mu',K}) V_\mu^K + \sum_{\mu \notin nP^o} \mathcal{Q}(M_{\mu,K}) V_\mu^K.$$

We show now that the restriction $D(n)|_H$ tends to 0 in $R^{-\infty}(H)$ as n goes to infinity. For this purpose, we will prove that for any $c>0$ there exists $n_c\in\mathbb{N}$ such that $D(n)|_H=O_H(c)$ for any $n\geqslant n_c$.

For $c>0$ we consider the compact subset of $\mathfrak{k}^*$ defined by

$$\mathcal{K}_c=\Phi_K\left(K\cdot\Phi_H^{-1}(\xi\in\mathfrak{h}^*,\|\xi\|\leqslant c)\right). \tag{4.2}$$

Let $n_c\in\mathbb{N}$ such that $\mathcal{K}_c$ is included in $K\cdot(n_cP^o)$: hence $\mathcal{K}_c\subset K\cdot(nP^o)$ for any $n\geqslant n_c$. We know that for $\mu\in\widehat{K}$, we have $N_\nu^\mu\mathcal{Q}\left(M_{\mu,K}\right)\neq 0$ only for $\mu\in\Phi_K\left(K\cdot\Phi_H^{-1}(\nu)\right)$, and for $\mu'\in\widehat{K}$, we have $N_\nu^{\mu'}\mathcal{Q}\left((M_{nP})_{\mu',K}\right)\neq 0$ only for $\mu'\in nP\cap\Phi_K\left(K\cdot\Phi_H^{-1}(\nu)\right)$.

Then if $n\geqslant n_c$, we have

$$N_\nu^\mu\mathcal{Q}\left(M_{\mu,K}\right)=N_\nu^{\mu'}\mathcal{Q}\left((M_{nP})_{\mu',K}\right)=0$$

for any $\nu\in\widehat{H}\cap\{\xi\in\mathfrak{h}^*,\|\xi\|\leqslant c\}$, $\mu\notin nP^o$ and $\mu'\in n\partial P$. This means that $D(n)|_H=O_H(c)$ for any $n\geqslant n_c$. □

Since $\mathcal{Q}_K(M_{nP})|_H=\mathcal{Q}_H(M_{nP})$, we are led to the

Lemma 4.2. — *The limit* $\lim_{n\to\infty}\mathcal{Q}_H(M_{nP})$ *is equal to* $\mathcal{Q}_H^{-\infty}(M)$.

Proof. — Theorem 2.4 – "Quantization commutes with reduction in the singular setting" – tells us that $\mathcal{Q}_H(M_{nP})=\sum_{\nu\in\widehat{H}}\mathcal{Q}((M_{nP})_{\nu,H})V_\nu^H$ where $(M_{nP})_{\nu,H}$ is the symplectic reduction

$$(M_{nP}\times\overline{H\cdot\nu})/\!\!/_0H\cong(M\times\mathcal{X}_{nP}\times\overline{H\cdot\mu})/\!\!/_{(0,0)}H\times K.$$

For $c>0$ we consider the compact subset of $\mathcal{K}_c$ defined in (4.2). Let $n_c\in\mathbb{N}$ such that $\mathcal{K}_c\subset K\cdot(nP^o)$ for any $n\geqslant n_c$. This implies that

$$\Phi_H^{-1}\left(\xi\in\mathfrak{h}^*,\|\xi\|\leqslant c\right)\subset\Phi_K^{-1}(K\cdot(nP^o))$$

for $n\geqslant n_c$. Since M_{nP} "contains" the open subset $\Phi_K^{-1}(K\cdot(nP^o))$, arguments similar to those used in the proof of Proposition 3.26 show that $\mathcal{Q}((M_{nP})_{\nu,H})=\mathcal{Q}(M_{\nu,H})$ for $\|\nu\|\leqslant c$ and $n\geqslant n_c$. This means that

$$\mathcal{Q}_H(M_{nP})=\sum_{\|\nu\|\leqslant c}\mathcal{Q}(M_{\nu,H})V_\nu^H+O_H(c)\quad\text{when}\quad n\geqslant n_c.$$

It follows that $\lim_{n\to\infty}\mathcal{Q}_H(M_{nP})=\sum_{\nu\in\widehat{H}}\mathcal{Q}(M_{\nu,H})V_\nu^H=\mathcal{Q}_H^{-\infty}(M)$. □

5. The case of a Hermitian vector space

Let (E,h) be a Hermitian vector space of dimension n.

5.1. The quantization of E

Let $\mathrm{U} := \mathrm{U}(E)$ be the unitary group with Lie algebra $\mathfrak{u}$. We use the isomorphism $\epsilon : \mathfrak{u} \to \mathfrak{u}^*$ defined by $\langle \epsilon(X), Y\rangle = -\operatorname{Tr}(XY) \in \mathbb{R}$. For $v, w \in E$, let $v \otimes w^* : E \to E$ be the linear map $x \mapsto \mathrm{h}(x, w)v$.

Let $E_{\mathbb{R}}$ be the space E view as a real vector space. Let Ω be the imaginary part of $-\mathrm{h}$, and let J the complex structure on $E_{\mathbb{R}}$. Then on $E_{\mathbb{R}}$, Ω is a (constant) symplectic structure and $\Omega(-, J-)$ defines a scalar product. The action of U on $(E_{\mathbb{R}}, \Omega)$ is Hamiltonian with moment map $\Phi : E \to \mathfrak{u}^*$ defined by $\langle \Phi(v), X\rangle = \frac{1}{2}\Omega(Xv, v)$. Via ϵ, the moment map Φ is defined by

$$\Phi(v) = \frac{1}{2i} v \otimes v^*. \tag{5.1}$$

The pre-quantization data $(L, \langle -, -\rangle, \nabla)$ on the Hamiltonian U-manifold $(E_{\mathbb{R}}, \Omega, \Phi)$ is a trivial line bundle L with a trivial action of U equipped with the Hermitian structure $\langle s, s'\rangle_v = e^{\frac{-\mathrm{h}(v,v)}{2}} s\overline{s'}$ and the Hermitian connexion $\nabla = d - i\theta$ where θ is the 1-form on E defined by $\theta = \frac{1}{2}\Omega(v, dv)$.

The traditional quantization of the Hamiltonian U-manifold $(E_{\mathbb{R}}, \Omega, \Phi)$, that we denote $\mathcal{Q}_U^{\mathrm{L}^2}(E)$, is the *Bargman space* of entire holomorphic functions on E which are L^2 integrable with respect to the Gaussian measure $e^{\frac{-\mathrm{h}(v,v)}{2}}\Omega^n$. The representation $\mathcal{Q}_U^{\mathrm{L}^2}(E)$ of U is admissible. The irreducible representations of U that occur in $\mathcal{Q}_U^{\mathrm{L}^2}(E)$ are the vector subspaces $S^j(E^*)$ formed by the homogeneous polynomials on E of degree $j \geqslant 0$.

On the other hand, the moment map Φ is proper (see (5.1)). Hence we can consider the formal quantization $\mathcal{Q}_{\mathrm{U}}^{-\infty}(E) \in R^{-\infty}(\mathrm{U})$ of the U-action on E.

LEMMA 5.1. — *The two quantizations of* (E, Ω, Φ), $\mathcal{Q}_U^{\mathrm{L}^2}(E)$ *and* $\mathcal{Q}_{\mathrm{U}}^{-\infty}(E)$ *coincide in* $R^{-\infty}(\mathrm{U})$. *In other words, we have*

$$\mathcal{Q}_{\mathrm{U}}^{-\infty}(E) = S^\bullet(E^*) := \sum_{j \geqslant 0} S^j(E^*) \quad \text{in} \quad R^{-\infty}(U). \tag{5.2}$$

Proof. — Let $T \subset \mathrm{U}$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{u}$. There exist an orthonormal basis $(e_k)_{k=1,\ldots,n}$ of E and characters $(\chi_k)_{k=1,\ldots,n}$ of T such that $t \cdot e_k = \chi_k(t)e_k$ for all k. The family $(ie_k \otimes e_k^*)_{k=1,\ldots,n}$ is then a basis of $\mathfrak{t}$ such that $\frac{1}{i}d\chi_l(ie_k \otimes e_k^*) = \delta_{l,k}$. The set $\widehat{U} \subset \mathfrak{t}^* \subset \mathfrak{u}^*$ of dominant weights is composed, via ϵ, by the elements

$$\underline{\lambda} = i\sum_{k=1}^n \lambda_k e_k \otimes e_k^*,$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a *decreasing* sequence of integers.

The formal quantization $\mathcal{Q}_{\mathrm{U}}^{-\infty}(E) \in R^{-\infty}(\mathrm{U})$ is defined by

$$\mathcal{Q}_{\mathrm{U}}^{-\infty}(E) = \sum_{\lambda_1 \geqslant \ldots \geqslant \lambda_n} \mathcal{Q}(E_{\underline{\lambda}}) V_{\underline{\lambda}}$$

where $E_{\underline{\lambda}} = \Phi^{-1}(U \cdot \underline{\lambda})/U$ is the reduced space and $V_{\underline{\lambda}}$ is the irreducible representation of U with highest weight $\underline{\lambda}$.

It is now easy to check that

$$E_{\underline{\lambda}} = \begin{cases} \{\mathrm{pt}\} & \text{if} \quad \lambda = (0, \ldots, 0, -j) \text{ with } j \geqslant 0, \\ \emptyset & \text{in the other cases,} \end{cases}$$

and then

$$\mathcal{Q}(E_{\underline{\lambda}}) = \begin{cases} 1 & \text{if} \quad \lambda = (0, \ldots, 0, -j) \text{ with } j \geqslant 0, \\ 0 & \text{in the other cases.} \end{cases}$$

Finally (5.2) follows from the fact that $V_{\underline{(0,\ldots,0,-j)}} = S^j(E^*)$. □

5.2. The quantization of E restricted to a subgroup of U

Let $K \subset \mathrm{U}$ be a closed connected Lie subgroup with Lie algebra $\mathfrak{k}$. Let $K_{\mathbb{C}} \subset \mathrm{GL}(E)$ be its complexification. The moment map relative to the K-action on $(E_{\mathbb{R}}, \Omega)$ is the map

$$\Phi_K : E \to \mathfrak{k}^*$$

equal to the composition of Φ with the projection $\mathfrak{u}^* \to \mathfrak{k}^*$.

Lemma 5.2. — *The following conditions are equivalent:*

(a) *the map Φ_K is proper,*
(b) $\Phi_K^{-1}(0) = \{0\}$,
(c) $\{0\}$ *is the only closed $K_{\mathbb{C}}$-orbit in E,*
(d) *for every $v \in E$ we have $0 \in \overline{K_{\mathbb{C}} \cdot v}$,*
(e) $S^\bullet(E^*)$ *is an admissible representation of K,*
(f) *the K-invariant polynomials on E are the constant polynomials.*

Proof. — The equivalence $(a) \iff (b)$ is due to the fact that Φ_K is quadratic.

Let $\mathcal{O}$ be a $K_{\mathbb{C}}$-orbit in E. Classical results of Geometric Invariant Theory [27, 19] assert that $\overline{\mathcal{O}} \cap \Phi_K^{-1}(0) \neq \emptyset$ and that $\mathcal{O}$ is closed if and only if $\mathcal{O} \cap \Phi_K^{-1}(0) \neq \emptyset$. Hence $(b) \iff (c) \iff (d)$.

>From Lemma 5.1 we know that $\mathcal{Q}_{\mathrm{U}}^{-\infty}(E) = S^\bullet(E^*)$. Since $\mathcal{Q}_{\mathrm{U}}^{-\infty}(E)$ is K-admissible when Φ_K is proper (see Section 4), we have $(a) \Longrightarrow (e)$.

For every $\mu\in\widehat{K}$, the μ-isotopic component $[S^\bullet(E^*)]_\mu$ is a module over $[S^\bullet(E^*)]_0=[S^\bullet(E^*)]^K$. Hence $\dim[S^\bullet(E^*)]_\mu<\infty$ implies that $[S^\bullet(E^*)]^K=\mathbb{C}$. We have $(e) \Longrightarrow (f)$.

Finally $(f) \Longrightarrow (d)$ follows from the following fundamental fact. For any $v, w \in E$ we have $\overline{K_\mathbb{C}\cdot v}\cap\overline{K_\mathbb{C}\cdot w}\neq\emptyset$ if and only if $P(v)=P(w)$ for all $P\in[S^\bullet(E^*)]^K$. □

Theorem 1.3 implies the following

Proposition 5.3. — *Let $K\subset \mathrm{U}(E)$ be a closed connected subgroup such that $S^\bullet(E^*)$ is an admissible representation of K. For every $\mu\in\widehat{K}$, we have*

$$\dim\left([S^\bullet(E^*)]_\mu\right)=\mathcal{Q}(E_{\mu,K})$$

where $[S^\bullet(E^*)]_\mu$ is the μ-isotopic component of $S^\bullet(E^*)$ and $E_{\mu,K}$ is the reduced space $\Phi_K^{-1}(K\cdot\mu)/K$.

In the following examples the condition $\Phi_K^{-1}(0)=\{0\}$ is easy to check.

1) the subgroup $K\subset \mathrm{U}(E)$ contains the center of $\mathrm{U}(E)$,
2) $E=\wedge^2\mathbb{C}^n$ or $E=S^2(\mathbb{C}^n)$ and $K=\mathrm{U}(n)\subset\mathrm{U}(E)$,
3) $E=\mathrm{M}_{n,k}$ is the vector space of $n\times k$-matrices and $K=\mathrm{U}(n)\times\mathrm{U}(k)\subset\mathrm{U}(E)$.

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