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CONGRUENCES AMONG MODULAR FORMS ON U(2,2) AND THE BLOCH-KATO CONJECTURE

by Krzysztof KLOSIN

Abstract. — Let k be a positive integer divisible by 4, $p > k$ a prime, f an elliptic cuspidal eigenform (ordinary at p) of weight $k-1$, level 4 and non-trivial character. In this paper we provide evidence for the Bloch-Kato conjecture for the motives $\mathrm{ad}^0 M(-1)$ and $\mathrm{ad}^0 M(2)$, where M is the motif attached to f. More precisely, we prove that under certain conditions the p-adic valuation of the algebraic part of the symmetric square L-function of f evaluated at k provides a lower bound for the p-adic valuation of the order of the Pontryagin dual of the Selmer group for the adjoint of the p-adic Galois representation attached to f restricted to the Gaussian field and twisted by the inverse of the cyclotomic character. Our method uses an idea of Ribet, in that we introduce an intermediate step and produce congruences between CAP and non-CAP modular forms on the unitary group $\mathrm{U}(2,2)$.

Résumé. — Soit k un entier strictement positif divisible par 4, soit $p > k$ un nombre premier, et soit f une forme modulaire elliptique de poids $k-1$, de niveau 4, et de caractère non trivial qui est propre pour les opérateurs de Hecke et ordinaire en p. Dans cet article, on donne des résultats sur la conjecture de Bloch-Kato pour les motifs $\mathrm{ad}^0 M(-1)$ et $\mathrm{ad}^0 M(2)$, où M est le motif associé à f. Plus précisément, on démontre que, sous certaines hypothèses, la valuation p-adique de la part algébrique de la valeur de la fonction L du carré symétrique de f en k est bornée par la valuation p-adique de la valeur de l'ordre du groupe de Selmer de l'adjoint de la représentation p-adique galoisienne associée à f restreinte au corps gaussien et tordue par l'inverse du caractère cyclotomique. Notre méthode suit une idée de Ribet, dans le sens où nous introduisons une étape intermédiaire et produisons des congruences entre formes CAP et formes non CAP sur le groupe unitaire $\mathrm{U}(2,2)$.

1. Introduction

Let $\ell > 2$ be a prime and let λ be a prime of $\overline{\mathbf{Q}}$ lying over ℓ. The idea of linking up λ-divisibility of an L-value with the existence of congruences

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among modular forms and using these congruences to construct elements in a Selmer group goes back to Ribet and his proof of the converse to Herbrand's theorem [45]. In that paper a special value $L(\chi,-1)$ of an even Dirichlet character χ is realized as a constant term of an Eisenstein series E_χ. If $L(\chi,-1)\equiv 0 \pmod{\lambda}$, one shows there exists a cuspidal Hecke eigenform f whose Hecke eigenvalues are congruent to those of E_χ (mod λ) and as a result of that the mod λ Galois representation $\overline{\rho}_f$ attached to f is reducible (a consequence of the congruence) but can be chosen to be non-semisimple (a consequence of the irreducibility of the λ-adic Galois representation ρ_f), thus giving rise to a non-split extension of one-dimensional Galois modules over $\overline{\mathbf{F}}_\ell$. This extension can be interpreted as a non-zero element in a certain piece (determined by χ) of the class group of $\mathbf{Q}(\mu_\ell)$.

This strategy can be phrased in the language of automorphic representations suggesting ways to generalize it to other situations. Let π be an automorphic representation of an algebraic group M over $\mathbf{Q}$. Realize M as a Levi subgroup in a maximal parabolic subgroup of a larger algebraic group G over $\mathbf{Q}$ and lift π (e.g., by inducing) to an automorphic representation Π of $G(\mathbf{A})$. Assuming one knows how to attach λ-adic Galois representations to automorphic representations of $G(\mathbf{A})$, the one attached to Π will be reducible and semisimple. If λ divides a certain L-value $L(\pi)$, construct a representation Π' of $G(\mathbf{A})$, whose Hecke eigenvalues are congruent to those of Π mod λ and whose λ-adic Galois representation is irreducible. These two conditions (respectively) ensure that the mod λ Galois representation attached to Π' is reducible and non-semisimple, thus giving rise to some non-split extension of Galois modules which one interprets as lying in an appropriate Selmer group related to π. In Ribet's case, $M=\mathrm{GL}_1\times\mathrm{GL}_1$, $G=\mathrm{GL}_2$, $\pi=\chi\otimes 1$, Π (resp. Π') is the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ attached to E_χ (resp. f).

Some versions of this approach have been applied by several authors: Mazur, Wiles, Bellaïche-Chenevier, Skinner-Urban, Brown, Berger ([41, 60, 2, 9, 3])) to give lower bounds in terms of L-values on the orders of Selmer groups. The difficult point is the construction of Π' and different cases require different methods to tackle that point.

Let $K=\mathbf{Q}(i)$, $f\in S_{k-1}\left(4,\left(\frac{-4}{\cdot}\right)\right)$, a normalized cuspidal Hecke eigenform, $\ell>k$ a prime such that f is ordinary at ℓ. Write $L^{\mathrm{int}}(\mathrm{Symm}^2 f,k)$ for the value at k of the symmetric square L-function of f divided by a suitable "integral" period. In this article we implement the above strategy with $M=\mathrm{Res}_{K/\mathbf{Q}}(\mathrm{GL}_{2/K})$, $G=\mathrm{U}(2,2)$ - a quasi-split unitary group associated with the extension $K/\mathbf{Q}$, π = base change to K of the automorphic representation associated to f, and $L(\pi) = L^{\mathrm{int}}(\mathrm{Symm}^2 f, k)$. This will allow us to construct elements in the Selmer group of $V := \mathrm{ad}^0 \rho_f|_{G_K}(-1)$, where ρ_f is the λ-adic Galois representation attached to f, $G_K = \mathrm{Gal}(\overline{K}/K)$, and -1 denotes a Tate twist. We will now describe the construction of Π = the lift of π to $G(\mathbf{A})$ and of the representation Π'.

The representation Π is obtained by lifting f to a Hecke eigenform $F_f \in \mathcal{S}_k$, where $\mathcal{S}_k$ is the space of (weight k and level 1) hermitian modular forms as defined by Braun [6, 7, 8], using the *Maass lifting* constructed by Kojima, Gritsenko and Krieg [36, 23, 37]. Denote by $\mathcal{S}_k^{\mathrm{M}} \subset \mathcal{S}_k$ the image of the Maass lift. It is known that the eigenvalues of eigenforms in $\mathcal{S}_k$ lie in a number field. In fact we always choose a sufficiently large finite extension E of $\mathbf{Q}_\ell$ and fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$, so we can view all the algebraic numbers of our interest as lying inside the same field E. From now on λ will denote a uniformizer of E and $\mathcal{O}$ its valuation ring. Assuming $L^{\mathrm{int}}(\mathrm{Symm}^2 f, k) \equiv 0 \pmod{\lambda}$, we need to construct a hermitian modular eigenform $F' \in \mathcal{S}_k$ orthogonal to the Maass space $\mathcal{S}_k^{\mathrm{M}}$ whose Hecke eigenvalues are congruent to those of $F_f \pmod{\lambda}$. The form F' will give rise to the representation Π' as above. Indeed, the λ-adic Galois representation attached to F_f is reducible and semisimple of the form $\rho_f \oplus (\rho_f \otimes \epsilon)$, where ϵ is the ℓ-adic cyclotomic character, while it is conjectured that the λ-adic Galois representation attached to an eigenform orthogonal to the Maass space is irreducible. In proving a bound on the Selmer group we will need to assume this conjecture (see Theorem 1.2).

The construction of F' is carried out in several steps. Note that, unlike in Ribet's case, our lift F_f is a cusp form, so there is no "constant term" which would "naturally" contain the L-value $L^{\mathrm{int}}(\mathrm{Symm}^2 f, k)$. Let χ be a Hecke character of K of infinity type $(z/|z|)^{-t}$ with $-k \leqslant t \leqslant -6$. We first define a "nice" hermitian modular form Ξ which is essentially a product of a hermitian Siegel Eisenstein series D and a hermitian theta series θ_χ depending on the character χ. Using some results of Shimura [49, 50, 51] on algebraicity of Fourier coefficients of hermitian Eisenstein series we show that Ξ has λ-integral Fourier coefficients. We write

$$\Xi = C_{F_f} F_f + F \quad \text{with} \quad C_{F_f} = (*)\pi^{-3} \frac{\langle D\theta_\chi, F_f\rangle}{\langle F_f, F_f\rangle}, \tag{1.1}$$

where $(*)$ is a λ-adic unit, F is orthogonal to F_f, i.e., $\langle F_f, F\rangle = 0$, and the inner products are the Petersson inner products on $\mathcal{S}_k$. Using results of Shimura [51] and some formulae due to Raghavan and Sengupta [44] we are able to express the inner products by L-functions related to f. More precisely, we get

(1.2)
$$\frac{\langle D\theta_\chi, F_f\rangle}{\langle F_f, F_f\rangle} = (*)\,\eta^{-1}\,\frac{L^{\rm int}(\mathrm{BC}(f), 1+\frac{t+k}{2}, \overline{\chi\omega})L^{\rm int}(\mathrm{BC}(f), 2+\frac{t+k}{2}, \overline{\chi\omega})}{L^{\rm int}(\mathrm{Symm}^2 f, k)},$$

where $(*) \in E$ with $\mathrm{ord}_\lambda((*)) \leqslant 0$, η is the Hida congruence ideal of f, ω is the unramified Hecke character of K of infinity type $(z/|z|)^{-k}$, and $L^{\rm int}(\mathrm{BC}(f), s, \psi)$ is the L-function of the base change of f to K twisted by the Hecke character ψ. Note that (ignoring the factor η^{-1} for the moment) if one can find χ which makes the numerator of the right-hand side of (1.2) a λ-adic unit, then using (1.1) and λ-integrality of the Fourier coefficients of Ξ and of F_f we see that if $\lambda^n \mid L^{\rm int}(\mathrm{Symm}^2 f, k)$ then we can write $C_{F_f} = a\lambda^{-n}$ with a a λ-adic unit and hence $F_f \equiv -a^{-1}\lambda^n F \pmod{\lambda^n}$. However $F'' := -a^{-1}\lambda^n F$ need not be orthogonal to the Maass space. To achieve this last property we modify F'' appropriately (to obtain F') using some results of Hida as well as deformation theory of Galois representations.

This way we obtain the first main result of the paper. To simplify the exposition here we omit a certain technical hypothesis on the character χ. For the full statement see Theorem 7.12.

Theorem 1.1. — *With notation as before, assume that $k \in \mathbf{Z}_+$, $4 \mid k$, $\ell > k$, f is ordinary at ℓ and the mod λ representation $\overline{\rho}_f$ restricted to G_K is absolutely irreducible. Assume there exists a Hecke character χ of K of conductor prime to ℓ, infinity type $(z/|z|)^t$, $-k \leqslant t \leqslant -6$, such that the numerator of the right-hand side of* (1.2) *is a λ-adic unit. If $\mathrm{ord}_\lambda(L^{\rm int}\,\mathrm{Symm}^2 f, k) = n > 0$, then there exists $F' \in \mathcal{S}_k$, orthogonal to the Maass space, such that $F' \equiv F_f \pmod{\lambda^n}$.*

To be precise, the form F' in Theorem 1.1 need not be a Hecke eigenform. To measure congruences between F_f and eigenforms orthogonal to the Maass space we introduce the notion of a CAP ideal, which is a simple modification of the Eisenstein ideal introduced by Mazur [39]. Theorem 1.1 implies that

$$\mathrm{ord}_\ell(I_f) \geqslant \mathrm{ord}_\ell(\#\mathcal{O}/\lambda^n), \tag{1.3}$$

where I_f is the index of the CAP ideal of F_f inside the hermitian Hecke $\mathcal{O}$-algebra acting on the orthogonal complement of $\mathcal{S}_k^{\rm M}$ localized at the maximal ideal corresponding to F_f.

We emphasize that the ordinarity assumption on f is essential to our method. It is used to ensure that F' is orthogonal to the Maass space (see section 8). For a given f it is unknown for how many primes f is ordinary, although one conjectures that for a non-CM form (which is the case here) this set of primes has Dirichlet density one. An analogous statement for elliptic curves is due to Serre [48].

Let $V_{\mathbf{Q}}$ be the $E[G_{\mathbf{Q}}]$-module $\mathrm{ad}^0 \rho_f(-1)$. Fix a $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice $T_{\mathbf{Q}} \subset V_{\mathbf{Q}}$ and set $W_{\mathbf{Q}} := V_{\mathbf{Q}}/T_{\mathbf{Q}}$. For $A = V, T, W$, write A_K for $A_{\mathbf{Q}}$ regarded as an $E[G_K]$-module. Write Σ_ℓ for the set of primes of K lying over ℓ. Let $\mathrm{Sel}_{\Sigma_\ell}(W_K)$ be the Selmer group of W_K in the sense of Bloch-Kato (for definition see section 9) and write $S_{\Sigma_\ell}(W_K)$ for the Pontryagin dual of $\mathrm{Sel}_{\Sigma_\ell}(W_K)$. Then our second main theorem (which uses a result of Urban [55]) is the following (again we omit some mild hypotheses on f - see section 9):

Theorem 1.2. — *Assume that for every eigenform $F \in \mathcal{S}_k$ orthogonal to $\mathcal{S}_k^{\mathrm{M}}$ the λ-adic Galois attached to F is absolutely irreducible. Then*

$$\mathrm{ord}_\ell(\#S_{\Sigma_\ell}(W_K)) \geqslant \mathrm{ord}_\ell(I_f).$$

The existence of λ-adic Galois representations attached to general automorphic forms on G is at this point only conjectural, so Theorem 1.2 is conditional upon that conjecture. In fact once this conjecture is known in full strength, one should be able to remove the irreducibility assumption from Theorem 1.2, as it is expected that Galois representations attached to non-Maass cuspidal eigenforms are irreducible.

Combining Theorem 1.2 with (1.3) which is a consequence of Theorem 1.1, we obtain the following corollary.

Corollary 1.3. — *With the same assumptions as in Theorems 1.1 and 1.2 one has*

$$\mathrm{ord}_\ell(\#S_{\Sigma_\ell}(W_K)) \geqslant \mathrm{ord}_\ell(\#\mathcal{O}/L^{\mathrm{int}}(\mathrm{Symm}^2 f, k)).$$

Let us briefly explain the relation of Corollary 1.3 to the Bloch-Kato conjecture for $M = \mathrm{ad}^0 M_0(-1)$, where M_0 is the motif attached to f. Let $\Sigma = \{2, \ell\}$ and write $S_\Sigma(W_{\mathbf{Q}})$ for the Pontryagin dual of $\mathrm{Sel}_\Sigma(W_{\mathbf{Q}})$ (where we require that the classes be unramified away from ℓ and crystalline at ℓ). Let $L(M,s) = \prod_p L_p(s-1)$ be the L-function of M defined by
(1.4)

$$L_p(s) := \begin{cases} (1-\alpha_{p,1}\alpha_{p,2}^{-1}p^{-s})^{-1}(1-p^{-s})^{-1}(1-\alpha_{p,1}^{-1}\alpha_{p,2}p^{-s})^{-1} & p \neq 2 \\ (1-p^{-s})^{-1} & p = 2, \end{cases}$$

where $\alpha_{p,1}$, $\alpha_{p,2}$ are the p-Satake parameters of f. Then the Bloch-Kato conjecture for M can be phrased in the following way:

Conjecture 1.4 (Bloch-Kato). — *One has*

$$\#S_{\Sigma}(W_{\mathbf{Q}})\cdot \mathrm{Tam}_{\omega}(T_{\mathbf{Q}})=\frac{L(M,0)}{\Omega_{\omega}(T_{\mathbf{Q}})}\mathcal{O} \tag{1.5}$$

as fractional ideals of E, where $\mathrm{Tam}_{\omega}(T_{\mathbf{Q}})$ is the ℓ-Tamagawa factor and $\Omega_{\omega}(V_{\mathbf{Q}})$ is a certain period defined with respect to the "integral structures" $T_{\mathbf{Q}}$ and ω (for precise definitions see section 9.3).

Given our assumptions (which in particular include the ordinarity assumption on f) Corollary 1.3 falls short of proving that the left-hand side of (1.5) is contained in the right-hand side of (1.5), but provides some evidence for it. In fact (for an appropriate choice of $T_{\mathbf{Q}}$ and ω) the right-hand side of (1.5) equals $L^{\mathrm{int}}(\mathrm{Symm}^2 f,k)\cdot\mathcal{O}$. However, the Selmer group on the left-hand side of (1.5) could potentially be smaller than $S_{\Sigma_\ell}(W_K)$ and we do not know if $\mathrm{Tam}(T_{\mathbf{Q}})=\mathcal{O}$. For a more detailed discussion see section 9.3. In that section we also explain the relation of Corollary 1.3 to the Bloch-Kato conjecture for the motif $\mathrm{ad}^0 M_0(2)$ which is "dual" to $\mathrm{ad}^0 M_0(-1)$.

The Bloch-Kato conjecture is currently known only for a few motives - see [33] for a survey of known cases. The most recent result is due to Diamond, Flach and Guo [14] and it concerns the motives $\mathrm{ad}^0 M_0$ and $\mathrm{ad}^0 M_0(1)$, while our result concerns the motives $\mathrm{ad}^0 M_0(-1)$ and $\mathrm{ad}^0 M_0(2)$. The method used in [14] is related to the method employed by Wiles and Taylor to prove the Taniyama-Shimura conjecture [61, 54] and so is different from ours.

Let us briefly discuss the organization of the paper. In section 2 we introduce notation which is used throughout the paper. In section 4 we summarize the basic facts concerning the Maass lifting $f\mapsto F_f$ and compute the Petersson inner product $\langle F_f,F_f\rangle$ in terms of $L(\mathrm{Symm}^2 f,k)$. To carry out the calculations we need to first compute the residue of the hermitian Klingen Eisenstein series and this is done in section 3. The inner product $\langle D\theta_\chi,F_f\rangle$ is computed in section 6. In section 5 we gather the necessary facts concerning the hermitian Hecke algebra which are later used in section 7, where the first main theorem (Theorem 1.1) is proved assuming existence of a certain Hecke operator which allows one to "kill" the "Maass part" of the form F'' as above, i.e., obtain a form F' that would be orthogonal to $\mathcal{S}_k^{\mathrm{M}}$. The existence of such a Hecke operator is proved in section 8 using methods of deformation theory of Galois representations. Finally in section 9 we prove Theorem 1.2 and Corollary 1.3 and discuss the relation of the latter to the Bloch-Kato conjecture.

We also want to mention that it seems possible to extend our result to an arbitrary imaginary quadratic field K provided one knows how to construct a Hecke-equivariant Maass lifting in that setting. Such a construction has recently been carried out by the author [35] for K with prime discriminant (see also [30]). We hope to use that construction to prove Theorems 1.1 and 1.2 for such a K in a subsequent paper.

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2. Notation and Terminology

In this section we introduce some basic concepts and establish notation which will be used throughout this paper unless explicitly indicated otherwise.

2.1. Number fields and Hecke characters

Throughout this paper ℓ will always denote an odd prime. Let $i = \sqrt{-1}$, $K = \mathbf{Q}(i)$ and let $\mathcal{O}_K$ be the ring of integers of K. For $\alpha \in K$, denote by $\overline{\alpha}$ the image of α under the non-trivial automorphism of K. Set $N\alpha := N(\alpha) := \alpha\overline{\alpha}$, and for an ideal $\mathfrak{n}$ of $\mathcal{O}_K$, set $N\mathfrak{n} := \#(\mathcal{O}_K/\mathfrak{n})$. As remarked below we will always view K as a subfield of $\mathbf{C}$. For $\alpha \in \mathbf{C}$, $\overline{\alpha}$ will denote the complex conjugate of α and we set $|\alpha| := \sqrt{\alpha\overline{\alpha}}$.

Let L be a number field with ring of integers $\mathcal{O}_L$. For a place v of L, denote by L_v the completion of L at v and by $\mathcal{O}_{L,v}$ the valuation ring of L_v. If p is a place of $\mathbf{Q}$, we set $L_p := \mathbf{Q}_p \otimes_{\mathbf{Q}} L$ and $\mathcal{O}_{L,p} := \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathcal{O}_L$. The letter v will be used to denote places of number fields (including $\mathbf{Q}$ and K), while the letter p will be reserved for a (finite or infinite) place of $\mathbf{Q}$. For a finite p, let ord_p denote the p-adic valuation on $\mathbf{Q}_p$. For notational convenience we also define $\mathrm{ord}_p(\infty) := \infty$. If $\alpha \in \mathbf{Q}_p$, then $|\alpha|_{\mathbf{Q}_p} := p^{-\mathrm{ord}_p(\alpha)}$ denotes the p-adic norm of α. For $p = \infty$, $|\cdot|_{\mathbf{Q}_\infty} = |\cdot|_{\mathbf{R}} = |\cdot|$ is the usual absolute value on $\mathbf{Q}_\infty = \mathbf{R}$.

In this paper we fix once and for all an algebraic closure $\overline{\mathbf{Q}}$ of the rationals and algebraic closures $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, as well as compatible embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ for all finite places p of $\mathbf{Q}$. We extend ord_p to a function from $\overline{\mathbf{Q}}_p$ into $\mathbf{Q}$. Let L be a number field. We write G_L for $\mathrm{Gal}(\overline{L}/L)$. If $\mathfrak{p}$ is a prime of L, we also write $D_{\mathfrak{p}} \subset G_L$ for the decomposition group of $\mathfrak{p}$ and $I_{\mathfrak{p}} \subset D_{\mathfrak{p}}$ for the inertia group of $\mathfrak{p}$. The chosen embeddings allow us to identify $D_{\mathfrak{p}}$ with $\mathrm{Gal}(\overline{L}_{\mathfrak{p}}/L_{\mathfrak{p}})$.

For a number field L let $\mathbf{A}_L$ denote the ring of adeles of L and put $\mathbf{A} := \mathbf{A}_{\mathbf{Q}}$. Write $\mathbf{A}_{L,\infty}$ and $\mathbf{A}_{L,\mathrm{f}}$ for the infinite part and the finite part of $\mathbf{A}_L$ respectively. For $\alpha = (\alpha_p) \in \mathbf{A}$ set $|\alpha|_{\mathbf{A}} := \prod_p |\alpha|_{\mathbf{Q}_p}$. By a *Hecke character* of $\mathbf{A}_L^\times$ (or of L, for short) we mean a continuous homomorphism

$$\psi : L^\times \setminus \mathbf{A}_L^\times \to \mathbf{C}^\times$$

whose image is contained inside $\{z \in \mathbf{C} \mid |z| = 1\}$. The trivial Hecke character will be denoted by $\mathbf{1}$. The character ψ factors into a product of local characters $\psi = \prod_v \psi_v$, where v runs over all places of L. If $\mathfrak{n}$ is the ideal of the ring of integers $\mathcal{O}_L$ of L such that

- $\psi_v(x_v) = 1$ if v is a finite place of L, $x_v \in \mathcal{O}_{L,v}^\times$ and $x - 1 \in \mathfrak{n}\mathcal{O}_{L,v}$
- no ideal $\mathfrak{m}$ strictly containing $\mathfrak{n}$ has the above property,

then $\mathfrak{n}$ will be called the *conductor of* ψ. If $\mathfrak{m}$ is an ideal of $\mathcal{O}_L$, then we set $\psi_{\mathfrak{m}} := \prod \psi_v$, where the product runs over all the finite places of L such that $v \mid \mathfrak{m}$. For a Hecke character ψ of $\mathbf{A}_L^\times$, denote by ψ^* the associated ideal character. Let ψ be a Hecke character of $\mathbf{A}_K^\times$. We will sometimes think of ψ as a character of $(\mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_1)(\mathbf{A})$. We have a factorization $\psi = \prod_p \psi_p$ into local characters $\psi_p : \left(\mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_1\right)(\mathbf{Q}_p) \to \mathbf{C}^\times$. For $M \in \mathbf{Z}$, we set $\psi_M := \prod_{p \neq \infty,\, p \mid M} \psi_p$. If ψ is a Hecke character of $\mathbf{A}_K^\times$, we set $\psi_{\mathbf{Q}} = \psi|_{\mathbf{A}^\times}$.

2.2. The unitary group

To the imaginary quadratic extension $K/\mathbf{Q}$ one associates the unitary similitude group

$$\mathrm{GU}(n,n) = \{A \in \mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_n \mid AJ\bar{A}^t = \mu(A)J\},$$

where $J = \begin{bmatrix} & -I_n \\ I_n & \end{bmatrix}$, with I_n denoting the $n \times n$ identity matrix, the bar over A standing for the action of the non-trivial automorphism of $K/\mathbf{Q}$ and $\mu(A) \in \mathrm{GL}_1$. For a matrix (or scalar) A with entries in a ring affording an action of $\mathrm{Gal}(K/\mathbf{Q})$, we will sometimes write A^* for $\bar{A}^t$ and $\hat{A}$ for $(A^*)^{-1}$. We will also make use of the groups

$$\mathrm{U}(n,n) = \{A \in \mathrm{GU}(n,n) \mid \mu(A) = 1\},$$

and

$$\mathrm{SU}(n,n) = \{A \in \mathrm{U}(n,n) \mid \det A = 1\}.$$

Since the case $n = 2$ will be of particular interest to us we set $G = \mathrm{U}(2,2)$, $G_1 = \mathrm{SU}(2,2)$ and $G_\mu = \mathrm{GU}(2,2)$.

For a $\mathbf{Q}$-subgroup H of G write H_1 for $H \cap G_1$. Denote by $\mathbf{G}_a$ the additive group. In G we choose a maximal torus

$$T = \left\{ \begin{bmatrix} a &&& \\ & b && \\ && \hat{a} & \\ &&& \hat{b} \end{bmatrix} \mid a, b \in \mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_1 \right\},$$

and a Borel subgroup $B = TU_B$ with unipotent radical

$$U_B = \left\{ \begin{bmatrix} 1 & \alpha & \beta & \gamma \\ & 1 & \bar{\gamma} - \bar{\alpha}\phi & \phi \\ && 1 & \\ && -\bar{\alpha} & 1 \end{bmatrix} \;\Big|\; \alpha, \beta, \gamma \in \mathrm{Res}_{K/\mathbf{Q}} \mathbf{G}_a, \;\; \phi \in \mathbf{G}_a, \;\; \beta + \gamma\bar{\alpha} \in \mathbf{G}_a \right\}.$$

Let

$$T_{\mathbf{Q}} = \left\{ \begin{bmatrix} a &&& \\ & b && \\ && a^{-1} & \\ &&& b^{-1} \end{bmatrix} \mid a, b \in \mathrm{GL}_1 \right\}$$

denote the maximal $\mathbf{Q}$-split torus contained in T. Let $R(G)$ be the set of roots of $T_{\mathbf{Q}}$, and denote by e_j, $j = 1, 2$, the root defined by

$$e_j : \begin{bmatrix} a_1 &&& \\ & a_2 && \\ && a_1^{-1} & \\ &&& a_2^{-1} \end{bmatrix} \mapsto a_j.$$

The choice of B determines a subset $R^+(G) \subset R(G)$ of positive roots. We have

$$R^+(G) = \{e_1 + e_2, e_1 - e_2, 2e_1, 2e_2\}.$$

We fix a set $\Delta(G) \subset R^+(G)$ of simple roots

$$\Delta(G) := \{e_1 - e_2, 2e_2\}.$$

If $\theta \subset \Delta(G)$, denote the parabolic subgroup corresponding to θ by P_θ. We have $P_{\Delta(G)} = G$ and $P_\emptyset = B$. The other two possible subsets of $\Delta(G)$ correspond to maximal $\mathbf{Q}$-parabolics of G:

- the Siegel parabolic $P := P_{\{e_1 - e_2\}} = M_P U_P$ with Levi subgroup

$$M_P = \left\{ \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} \mid A \in \mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_2 \right\},$$

and (abelian) unipotent radical

$$U_P = \left\{ \begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & \bar{b}_2 & b_4 \\ & & 1 & \\ & & & 1 \end{bmatrix} \mid b_1, b_4 \in \mathbf{G}_a,\ \ b_2 \in \operatorname{Res}_{K/\mathbf{Q}} \mathbf{G}_a \right\}$$

- the Klingen parabolic $Q := P_{\{2e_2\}} = M_Q U_Q$ with Levi subgroup

$$M_Q = \left\{ \begin{bmatrix} x & & & \\ & a & & b \\ & & \hat{x} & \\ & c & & d \end{bmatrix} \mid x \in \operatorname{Res}_{K/\mathbf{Q}} \mathrm{GL}_1,\ \ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{U}(1,1) \right\},$$

and (non-abelian) unipotent radical

$$U_Q = \left\{ \begin{bmatrix} 1 & \alpha & \beta & \gamma \\ & 1 & \bar{\gamma} & \\ & & 1 & \\ & & -\bar{\alpha} & 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \operatorname{Res}_{K/\mathbf{Q}} \mathbf{G}_a,\ \ \beta + \gamma\bar{\alpha} \in \mathbf{G}_a \right\}$$

For an associative ring R with identity and an R-module N we write N^n_m to denote the R-module of $n \times m$ matrices with entries in N. We also set $N^n := N^n_1$, and $M_n(N) := N^n_n$. Let $x = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in M_{2n}(N)$ with $A, B, C, D \in M_n(N)$. Define $a_x = A$, $b_x = B$, $c_x = C$, $d_x = D$.

For $M \in \mathbf{Q}$, $N \in \mathbf{Z}$ such that $MN \in \mathbf{Z}$ we will denote by $D(M,N)$ the group $G(\mathbf{R}) \prod_{p \nmid \infty} K_{0,p}(M,N) \subset G(\mathbf{A})$, where

$$\text{(2.1)} \quad K_{0,p}(M,N) = \{x \in G(\mathbf{Q}_p) \mid a_x, d_x \in M_2(\mathcal{O}_{K,p}),\\ b_x \in M_2(M^{-1}\mathcal{O}_{K,p}),\ \ c_x \in M_2(MN\mathcal{O}_{K,p})\}.$$

If $M = 1$, denote $D(M,N)$ simply by $D(N)$ and $K_{0,p}(M,N)$ by $K_{0,p}(N)$. For any finite p, the group $K_{0,p} := K_{0,p}(1) = G(\mathbf{Z}_p)$ is a maximal (open) compact subgroup of $G(\mathbf{Q}_p)$. Note that if $p \nmid N$, then $K_{0,p} = K_{0,p}(N)$. We write $K_{0,\mathrm{f}}(N) := \prod_{p \nmid \infty} K_{0,p}(N)$ and $K_{0,\mathrm{f}} := K_{0,\mathrm{f}}(1)$. Note that $K_{0,\mathrm{f}}$ is a maximal (open) compact subgroup of $G(\mathbf{A}_\mathrm{f})$. Set

$$K_{0,\infty} := \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in G(\mathbf{R}) \mid A, B \in \mathrm{GL}_2(\mathbf{C}), AA^* + BB^* = I_2, AB^* = BA^* \right\}.$$

Then $K_{0,\infty}$ is a maximal compact subgroup of $G(\mathbf{R})$. Let

$$U(m) := \{A \in \mathrm{GL}_m(\mathbf{C}) \mid AA^* = I_m\}.$$

We have

$$K_{0,\infty} = G(\mathbf{R}) \cap U(4) \xrightarrow{\sim} U(2) \times U(2),$$

where the last isomorphism is given by

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \mapsto (A + iB, A - iB) \in U(2) \times U(2).$$

Finally, set $K_0(N) := K_{0,\infty} K_{0,\mathrm{f}}(N)$ and $K_0 := K_0(1)$. The last group is a maximal compact subgroup of $G(\mathbf{A})$. Let $M \in \mathbf{Q}$, $N \in \mathbf{Z}$ be such that $MN \in \mathbf{Z}$. We define the following congruence subgroups of $G(\mathbf{Q})$:

$$\begin{aligned} &\Gamma_0^{\mathrm{h}}(M,N) := G(\mathbf{Q}) \cap D(M,N), \\ &\Gamma_1^{\mathrm{h}}(M,N) := \{\alpha \in \Gamma_0^{\mathrm{h}}(M,N) \mid a_\alpha - 1 \in M_2(N\mathcal{O}_K)\},, \\ &\Gamma^{\mathrm{h}}(M,N) := \{\alpha \in \Gamma_1^{\mathrm{h}}(M,N) \mid b_\alpha \in M_2(M^{-1}N\mathcal{O}_K)\} \end{aligned} \tag{2.2}$$

and set $\Gamma_0^{\mathrm{h}}(N) := \Gamma_0^{\mathrm{h}}(1,N)$, $\Gamma_1^{\mathrm{h}}(N) := \Gamma_1^{\mathrm{h}}(1,N)$ and $\Gamma^{\mathrm{h}}(N) := \Gamma^{\mathrm{h}}(1,N)$. Because we will frequently use the group $\Gamma_0^{\mathrm{h}}(1) = \{A \in \mathrm{GL}_4(\mathcal{O}_K) \mid AJA^* = J\}$, we reserve a special notation for it and denote it by $\Gamma_{\mathbf{Z}}$. Note that the groups $\Gamma_0^{\mathrm{h}}(N)$, $\Gamma_1^{\mathrm{h}}(N)$ and $\Gamma^{\mathrm{h}}(N)$ are $\mathrm{U}(2,2)$-analogues of the standard congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ of $\mathrm{SL}_2(\mathbf{Z})$. In general the superscript 'h' will indicate that an object is in some way related to the group $\mathrm{U}(2,2)$. The letter 'h' stands for 'hermitian', as this is the standard name of modular forms on $\mathrm{U}(2,2)$.

2.3. Modular forms

In this paper we will make use of the theory of modular forms on congruence subgroups of two different groups: $\mathrm{SL}_2(\mathbf{Z})$ and $\Gamma_{\mathbf{Z}}$. We will use both the classical and the adelic formulation of the theories. In the adelic framework one usually speaks of automorphic forms rather than modular forms and in this case SL_2 is usually replaced with GL_2. For more details see e.g. [22], chapter 3. In the classical setting the modular forms on congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$ will be referred to as *elliptic modular forms*, and those on congruence subgroups of $\Gamma_{\mathbf{Z}}$ as *hermitian modular forms.*

2.3.1. Elliptic modular forms

The theory of elliptic modular forms is well-known, so we omit most of the definitions and refer the reader to standard sources, e.g. [42]. Let

$$\mathbf{H} := \{z \in \mathbf{C} \mid \mathrm{Im}\,(z) > 0\}$$

denote the complex upper half-plane. In the case of elliptic modular forms we will denote by $\Gamma_0(N)$ the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ consisting of matrices whose lower-left entries are divisible by N, and by $\Gamma_1(N)$ the subgroup of $\Gamma_0(N)$ consisting of matrices whose upper left entries are congruent to 1 modulo N. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a congruence subgroup. Set $M_m(\Gamma)$ (resp. $S_m(\Gamma)$) to denote the $\mathbf{C}$-space of elliptic modular forms (resp. cusp forms) of weight m and level Γ. We also denote by $M_m(N,\psi)$ (resp. $S_m(N,\psi)$) the space of elliptic modular forms (resp. cusp forms) of weight m, level N and character ψ. For $f,g \in M_m(\Gamma)$ with either f or g a cusp form, and $\Gamma' \subset \Gamma$ a finite index subgroup, we define the Petersson inner product

$$\langle f, g\rangle_{\Gamma'} := \int_{\Gamma'\backslash\mathbf{H}} f(z)\overline{g(z)}(\operatorname{Im} z)^{m-2}\, dx\, dy,$$

and set

$$\langle f, g\rangle := \frac{1}{[\overline{\mathrm{SL}_2(\mathbf{Z})} : \overline{\Gamma}']} \langle f, g\rangle_{\Gamma'},$$

where $\overline{\mathrm{SL}_2(\mathbf{Z})} := \mathrm{SL}_2(\mathbf{Z})/\langle -I_2\rangle$ and $\overline{\Gamma}'$ is the image of Γ' in $\overline{\mathrm{SL}_2(\mathbf{Z})}$. The value $\langle f, g\rangle$ is independent of Γ'.

Every elliptic modular form $f \in M_m(N,\psi)$ possesses a Fourier expansion $f(z) = \sum_{n=0}^{\infty} a(n)q^n$, where throughout this paper in such series q will denote $e(z) := e^{2\pi i z}$. For $\gamma = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbf{R})$, set $j(\gamma, z) = cz + d$.

In this paper we will be particularly interested in the space $S_m\left(4, \left(\frac{-4}{\cdot}\right)\right)$, where $\left(\frac{-4}{\cdot}\right)$ is the non-trivial character of $(\mathbf{Z}/4\mathbf{Z})^{\times}$. Regarded as a function $\mathbf{Z} \to \{1, -1\}$, it assigns the value 1 to all prime numbers p such that (p) splits in K and the value -1 to all prime numbers p such that (p) is inert in K. Note that since the character $\left(\frac{-4}{\cdot}\right)$ is primitive, the space $S_m\left(4, \left(\frac{-4}{\cdot}\right)\right)$ has a basis consisting of primitive normalized eigenforms. We will denote this (unique) basis by $\mathcal{N}$. For $f = \sum_{n=1}^{\infty} a(n)q^n \in \mathcal{N}$, set $f^{\rho} := \sum_{n=1}^{\infty} \overline{a(n)}q^n \in \mathcal{N}$.

Fact 2.1. — ([42], section 4.6) *One has* $a(p) = \left(\frac{-4}{p}\right)\overline{a(p)}$ *for any rational prime* $p \nmid 2$.

This implies that $a(p) = \overline{a(p)}$ if (p) splits in K and $a(p) = -\overline{a(p)}$ if (p) is inert in K.

For $f \in \mathcal{N}$ and E a finite extension of $\mathbf{Q}_\ell$ containing the eigenvalues of T_n, $n = 1, 2, \dots$ we will denote by $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(E)$ the Galois representation attached to f by Deligne (cf. e.g., [11], section 3.1). We will write $\overline{\rho}_f$ for the reduction of ρ_f modulo a uniformizer of E with respect to some lattice Λ in E^2. In general $\overline{\rho}_f$ depends on the lattice Λ, however the isomorphism class of its semisimplification $\overline{\rho}_f^{\mathrm{ss}}$ is independent of Λ. Thus, if $\overline{\rho}_f$ is irreducible (which we will assume), it is well-defined.

2.3.2. Hermitian modular forms

For a systematic treatment of the theory of hermitian modular forms see [6, 7, 8] as well as [23, 37, 36]. We begin by defining the *hermitian upper half-plane*

$$\mathcal{H} = \{Z \in M_2(\mathbf{C}) \mid -\mathbf{i}(Z - Z^*) > 0\},$$

where $\mathbf{i} = \left[\begin{smallmatrix} i & \\ & i \end{smallmatrix}\right]$. Set $\mathrm{Re}\, Z = \frac{1}{2}(Z + Z^*)$ and $\mathrm{Im}\, Z = -\frac{1}{2}\mathbf{i}(Z - Z^*)$. Let

$$G_\mu^+(\mathbf{R}) := \{g \in G_\mu(\mathbf{R}) \mid \mu(g) > 0\}.$$

The group $G_\mu^+(\mathbf{R})$ acts on $\mathcal{H}$ by $\gamma Z = (a_\gamma Z + b_\gamma)(c_\gamma Z + d_\gamma)^{-1}$, with $\gamma \in G_\mu^+(\mathbf{R})$. For a function F on $\mathcal{H}$, an integer m and $\gamma \in G_\mu^+(\mathbf{R})$ put

$$F|_m\gamma = \mu(\gamma)^{2m-4} j(\gamma, Z)^{-m} F(\gamma Z),$$

with the automorphy factor $j(\gamma, Z) = \det(c_\gamma Z + d_\gamma)$.

Let Γ^{h} be a congruence subgroup of $\Gamma_{\mathbf{Z}}$. We say that a holomorphic function F on $\mathcal{H}$ is a *hermitian modular form* of weight m and level Γ^{h} if

$$F|_m\gamma = F \quad \text{for all } \gamma \in \Gamma^{\mathrm{h}}.$$

The group Γ^{h} is called the *level* of F. If $\Gamma^{\mathrm{h}} = \Gamma_0^{\mathrm{h}}(N)$ for some $N \in \mathbf{Z}$, then we say that F is of level N. Forms of level 1 will sometimes be referred to as forms of *full level*. One can also define hermitian modular forms with a character. Let $\Gamma^{\mathrm{h}} = \Gamma_0^{\mathrm{h}}(N)$ and let $\psi : \mathbf{A}_K^\times \to \mathbf{C}^\times$ be a Hecke character such that for all finite p, $\psi_p(a) = 1$ for every $a \in \mathcal{O}_{K,p}^\times$ with $a - 1 \in N\mathcal{O}_{K,p}$. We say that F is *of level N and character ψ* if

$$F|_m\gamma = \psi_N(\det a_\gamma)F \quad \text{for every } \gamma \in \Gamma_0^{\mathrm{h}}(N).$$

A hermitian modular form of level $\Gamma^{\mathrm{h}}(M, N)$ possesses a Fourier expansion

$$F(Z) = \sum_{\tau \in \mathcal{S}(M)} c(\tau) e(\mathrm{tr}\, \tau Z),$$

where $\mathcal{S}(M) = \{x \in S \mid \mathrm{tr}\, xL(M) \subset \mathbf{Z}\}$ with $S = \{h \in M_2(K) \mid h^* = h\}$ and $L(M) = S \cap M_2(M\mathcal{O}_K)$. As we will be particularly interested in the case when $M = 1$, we set

$$\mathcal{S} := \mathcal{S}(1) = \left\{ \begin{bmatrix} t_1 & t_2 \\ \overline{t_2} & t_3 \end{bmatrix} \in M_2(K) \mid t_1, t_3 \in \mathbf{Z}, t_2 \in \frac{1}{2}\mathcal{O}_K \right\}.$$

Denote by $\mathcal{M}_m(\Gamma^{\mathrm{h}})$ the $\mathbf{C}$-space of hermitian modular forms of weight m and level Γ^{h}, and by $\mathcal{M}_m(N, \psi)$ the $\mathbf{C}$-space of hermitian modular forms of weight m, level N and character ψ. For $F \in \mathcal{M}_m(\Gamma^{\mathrm{h}})$ and $\alpha \in G_\mu^+(\mathbf{R})$ one has $F|_m\alpha \in \mathcal{M}_m(\alpha^{-1}\Gamma^{\mathrm{h}}\alpha)$ and there is an expansion

$$F|_m\alpha = \sum_{\tau\in S} c_\alpha(\tau)e(\operatorname{tr}\tau Z).$$

We call F a *cusp form* if for all $\alpha \in G_\mu^+(\mathbf{R})$, $c_\alpha(\tau) = 0$ for every τ such that $\det\tau = 0$. Denote by $\mathcal{S}_m(\Gamma^{\mathrm{h}})$ (resp. $\mathcal{S}_m(N,\psi)$) the subspace of cusp forms inside $\mathcal{M}_m(\Gamma^{\mathrm{h}})$ (resp. $\mathcal{M}_m(N,\psi)$). If $\psi = \mathbf{1}$, set $\mathcal{M}_m(N) := \mathcal{M}_m(N,\mathbf{1})$ and $\mathcal{S}_m(N) := \mathcal{S}_m(N,\mathbf{1})$.

Theorem 2.2 (q-expansion principle, [28], section 8.4). — *Let ℓ be a rational prime and N a positive integer with $\ell \nmid N$. Suppose all Fourier coefficients of $F \in \mathcal{M}_m(N,\psi)$ lie inside the valuation ring $\mathcal{O}$ of a finite extension E of $\mathbf{Q}_\ell$. If $\gamma \in \Gamma_{\mathbf{Z}}$, then all Fourier coefficients of $F|_m\gamma$ also lie in $\mathcal{O}$.*

If F and F' are two hermitian modular forms of weight m, level Γ^{h} and character ψ, and either F or F' is a cusp form, we define for any finite index subgroup Γ_0^{h} of Γ^{h}, the Petersson inner product

$$\langle F,F'\rangle_{\Gamma_0^{\mathrm{h}}} := \int_{\Gamma_0^{\mathrm{h}}\backslash\mathcal{H}} F(Z)\overline{F'(Z)}(\det Y)^{m-4}dXdY,$$

where $X = \operatorname{Re} Z$ and $Y = \operatorname{Im} Z$, and

$$\langle F,F'\rangle = [\overline{\Gamma}_{\mathbf{Z}} : \overline{\Gamma}_0^{\mathrm{h}}]^{-1}\langle F,F'\rangle_{\Gamma_0^{\mathrm{h}}},$$

where $\overline{\Gamma}_{\mathbf{Z}} := \Gamma_{\mathbf{Z}}/\langle \mathbf{i}\rangle$ and $\overline{\Gamma}_0^{\mathrm{h}}$ is the image of Γ_0^{h} in $\overline{\Gamma}_{\mathbf{Z}}$. The value $\langle F,F'\rangle$ is independent of Γ_0^{h}.

There exist adelic analogues of hermitian modular forms. For $F \in \mathcal{M}_m(N,\psi)$, the function $\varphi_F : G(\mathbf{A}) \to \mathbf{C}$ defined by

$$\varphi_F(g) = j(g_\infty,\mathbf{i})^{-m}F(g_\infty,\mathbf{i})\psi^{-1}(\det d_k),$$

where $g = g_{\mathbf{Q}}g_\infty k \in G(\mathbf{Q})G(\mathbf{R})K_{0,\mathrm{f}}(N)$, is an automorphic form on $G(\mathbf{A})$.

3. Eisenstein series

The goal of this section is to compute the residue of the hermitian Klingen Eisenstein series (cf. Definition 3.1 and Theorem 3.10). This computation will be used in the next section.

3.1. Siegel, Klingen and Borel Eisenstein series

Siegel and Klingen Eisenstein series are induced from the maximal parabolic subgroups P and Q of $G = \mathrm{U}(2,2)$ respectively. (For the definitions of P and Q see section 2.2.) Let

$$\delta_P : P(\mathbf{A}) \to \mathbf{R}_+$$

be the modulus character of $P(\mathbf{A})$,

$$\delta_P\left(\begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} u\right) = |\det A\,\overline{\det A}|^2_{\mathbf{A}}, \tag{3.1}$$

with $A \in \mathrm{Res}_{K/\mathbf{Q}}\,\mathrm{GL}_2(\mathbf{A})$, $u \in U_P(\mathbf{A})$, and

$$\delta_Q : Q(\mathbf{A}) \to \mathbf{R}_+$$

the modulus character of $Q(\mathbf{A})$,

$$\delta_Q\left(\begin{bmatrix} x & & & \\ & a & & b \\ & & \hat{x} & \\ & c & & d \end{bmatrix} u\right) = |x\overline{x}|^3_{\mathbf{A}}, \tag{3.2}$$

with $x \in \mathrm{Res}_{K/\mathbf{Q}}\,\mathrm{GL}_1(\mathbf{A})$, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{U}(1,1)(\mathbf{A})$ and $u \in U_Q(\mathbf{A})$. As before, $K_0 = K_{0,\infty}K_{0,\mathrm{f}}$ will denote the maximal compact subgroup of $G(\mathbf{A})$. Using the Iwasawa decomposition $G(\mathbf{A}) = P(\mathbf{A})K_0$ we extend both characters δ_P and δ_Q to functions on $G(\mathbf{A})$ and denote these extensions again by δ_P and δ_Q.

Definition 3.1. — *For $g \in G(\mathbf{A})$, the series*

$$E_P(g,s) := \sum_{P(\mathbf{Q})\backslash G(\mathbf{Q})} \delta_P(\gamma g)^s$$

is called the (hermitian) Siegel Eisenstein series, while the series

$$E_Q(g,s) := \sum_{Q(\mathbf{Q})\backslash G(\mathbf{Q})} \delta_Q(\gamma g)^s$$

is called the (hermitian) Klingen Eisenstein series.

Properties of $E_P(g,s)$ were investigated by Shimura in [50]. We summarize them in the following proposition.

Proposition 3.2 (Shimura). — *The series $E_P(g,s)$ is absolutely convergent for* $\mathrm{Re}\,(s)>1$ *and can be meromorphically continued to the entire s-plane with only a simple pole at $s=1$. One has*

$$\operatorname{res}_{s=1} E_P(g,s) = \frac{45L\left(2,\left(\frac{-4}{\cdot}\right)\right)}{4\pi L\left(3,\left(\frac{-4}{\cdot}\right)\right)}, \tag{3.3}$$

where $L(\cdot,\cdot)$ denotes the Dirichlet L-function.

Properties of the Klingen Eisenstein series were investigated by Raghavan and Sengupta in [44]. The only difference is that instead of $E_Q(g,s)$, [44] uses an Eisenstein series that we will denote by $E_s(Z)$. The connection between $E_Q(g,s)$ and $E_s(Z)$ is provided by Lemma 4.6. After the connection has been established the following proposition follows from Lemma 1 in [44].

Proposition 3.3 (Raghavan-Sengupta). — *The series $E_Q(g,s)$ converges absolutely for* $\mathrm{Re}\,(s)>1$ *and can be meromorphically continued to the entire s-plane. The possible poles of $E_Q(g,s)$ are at most simple and are contained in the set* $\{0,1/3,2/3,1\}$.

In section 3.4 we will show that $E_Q(g,s)$ has a simple pole at $s=1$ and calculate the residue.

Both $E_P(g,s)$ and $E_Q(g,s)$ have their classical analogues, i.e., series in which g is replaced by a variable Z in the hermitian upper half-plane $\mathcal{H}$. Let $g_\infty \in G(\mathbf{R})$ be such that $Z = g_\infty \mathbf{i}$ and set $g=(g_\infty,1)\in G(\mathbf{R})\times G(\mathbf{A}_\mathrm{f})$. Define

$$E_P(Z,s) := E_P(g,s)$$

and

$$E_Q(Z,s) = E_Q(g,s).$$

We will show in Lemma 4.6 that

$$E_Q(Z,s) = \sum_{\gamma\in Q(\mathbf{Z})\backslash\Gamma_\mathbf{Z}} \left(\frac{\det \operatorname{Im}(\gamma Z)}{(\operatorname{Im}(\gamma Z))_{2,2}}\right)^{3s},$$

where for any matrix M we denote its (i,j)-th entry by $M_{i,j}$.

Remark 3.4. — Note that we use the same symbols $E_P(\cdot,s)$ and $E_Q(\cdot,s)$ to denote both the adelic and the classical Eisenstein series. We distinguish them by inserting $g\in G(\mathbf{A})$ or $Z\in\mathcal{H}$ in the place of the dot. We will continue this abuse of notation for other Eisenstein series we study.

We now turn to the Eisenstein series which is induced from the Borel subgroup B of G, which we call the *Borel Eisenstein series*. It is a function of two complex variables s and z, defined by

$$E_B(g,s,z) := \sum_{\gamma \in B(\mathbf{Q})\backslash G(\mathbf{Q})} \delta_Q(\gamma g)^s \delta_P(\gamma g)^z.$$

Note that as the Levi subgroup of B is abelian (it is the torus T), the character $\delta_Q^s \delta_P^z$ is a cuspidal automorphic form on $T(\mathbf{A})$. Thus the following proposition follows from [43], Proposition II.1.5.

Proposition 3.5. — *The series $E_B(g,s,z)$ is absolutely convergent for*

$$(s,z) \in \{(s',z') \in \mathbf{C} \times \mathbf{C} \mid \mathrm{Re}\,(s') > 2/3, \mathrm{Re}\,(z') > 1/2\}.$$

It can be meromorphically continued to all of $\mathbf{C} \times \mathbf{C}$.

Remark 3.6. — It follows from the general theory (cf. [38], chapter 7) that by taking iterated residues of Eisenstein series induced from minimal parabolics one obtains Eisenstein series on other parabolics. These series are usually referred to as *residual Eisenstein series*. In fact E_P and E_Q are residues of E_B taken with respect to the variable s and z respectively. We will prove this fact in section 3.5, but see also [32], Remark 5.6.

3.2. Siegel Eisenstein series with positive weight

In this section we define an Eisenstein series induced from the Siegel parabolic, having positive weight, level and non-trivial character. For notation refer to section 2. Let m, N be integers with $m \geqslant 0$ and $N > 0$. Note that $K_{0,\infty}$ is the stabilizer of $\mathbf{i}$ in $G(\mathbf{R})$. Let $\psi : K^\times \setminus \mathbf{A}_K^\times \to \mathbf{C}^\times$ be a Hecke character of $\mathbf{A}_K^\times$ with local decomposition $\psi = \prod_p \psi_p$, where p runs over all the places of $\mathbf{Q}$. Assume that

$$\psi_\infty(x_\infty) = \left(\frac{x_\infty}{|x_\infty|}\right)^m$$

and

$$\psi_p(x_p) = 1 \quad \text{if } p \neq \infty, x_p \in \mathcal{O}_{K,p}^\times, \text{ and } x_p - 1 \in N\mathcal{O}_{K,p}.$$

As before we set $\psi_N = \prod_{p \mid N} \psi_p$. Let δ_P denote the modulus character of P. We define

$$\mu_P : M_P(\mathbf{Q})U_P(\mathbf{A}) \setminus G(\mathbf{A}) \to \mathbf{C}$$

by setting

$$\mu_P(g) = \begin{cases} 0 & g \notin P(\mathbf{A})K_0(N) \\ \psi(\det d_q)^{-1}\psi_N(\det d_\kappa)^{-1} j(\kappa_\infty, \mathbf{i})^{-m} & g = q\kappa \in P(\mathbf{A})K_0(N). \end{cases}$$

Note that μ_P has a local decomposition $\mu_P = \prod_p \mu_{P,p}$, where

$$\mu_{P,p}(q_p\kappa_p) = \begin{cases} \psi_p(\det d_{q_p})^{-1} & \text{if } p \nmid N\infty, \\ \psi_p(\det d_{q_p})^{-1}\psi_p(\det d_{\kappa_p}) & \text{if } p \mid N, p \neq \infty, \\ \psi_\infty(\det d_{q_\infty})^{-1} j(\kappa_\infty, \mathbf{i})^{-m} & \text{if } p = \infty \end{cases} \tag{3.4}$$

and δ_P has a local decomposition $\delta_P = \prod_p \delta_{P,p}$, where

$$\delta_{P,p}\left(\begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} u\kappa\right) = |\det A \det \overline{A}|_{\mathbf{Q}_p}. \tag{3.5}$$

Definition 3.7. — *The series*

$$E(g,s,N,m,\psi) := \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \mu_P(\gamma g)\delta_P(\gamma g)^{s/2}$$

is called the (hermitian) Siegel Eisenstein series of weight m, level N and character ψ.

The series $E(g,s,N,m,\psi)$ converges for $\mathrm{Re}\,(s)$ sufficiently large, and can be continued to a meromorphic function on all of $\mathbf{C}$ (cf. [50], Proposition 19.1). It also has a complex analogue $E(Z,s,m,\psi,N)$ defined by

$$E(Z,s,m,\psi,N) := j(g_\infty, \mathbf{i})^m E(g,s,N,m,\psi)$$

for $Z = g_\infty \mathbf{i}$, $g = g_{\mathbf{Q}} g_\infty \kappa_{\mathrm{f}} \in G(\mathbf{Q})G(\mathbf{R})K_{0,\mathrm{f}}(N)$. It follows from Lemma 18.7(3) of [50] and formulas (16.40) and (16.48) of [51], together with the fact that K has class number one that

(3.6)

$$\begin{aligned} E(Z,s,m,\psi,N) = & \sum_{\gamma \in (P(\mathbf{Q}) \cap \Gamma_0^{\mathrm{h}}(N))\backslash \Gamma_0^{\mathrm{h}}(N)} \psi_N(\det d_\gamma)^{-1}(\det \operatorname{Im} Z)^{s-m/2}|_m \gamma = \\ = & \sum_{\gamma \in (P(\mathbf{Q}) \cap \Gamma_0^{\mathrm{h}}(N))\backslash \Gamma_0^{\mathrm{h}}(N)} \psi_N(\det d_\gamma)^{-1} \det(c_\gamma Z + d_\gamma)^{-m} \times \\ & \times |\det(c_\gamma Z + d_\gamma)|^{-2s+m}(\det \operatorname{Im} Z)^{s-m/2}. \end{aligned}$$

3.3. The Eisenstein series on $\mathrm{U}(1,1)$

Let B_1 denote the upper-triangular Borel subgroup of $\mathrm{U}(1,1)$ with Levi decomposition $B_1 = T_1 U_1$, where

$$T_1 := \left\{ \begin{bmatrix} a & \\ & \hat{a} \end{bmatrix} \mid a \in \mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_1 \right\}$$

and

$$U_1 = \left\{ \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \mid x \in \mathbf{G}_a \right\}.$$

Let $\delta_1 : B_1(\mathbf{A}) \to \mathbf{R}_+$ be the modulus character given by

$$\delta_1 \left(\begin{bmatrix} a & \\ & \hat{a} \end{bmatrix} u \right) = |a\overline{a}|_{\mathbf{A}}$$

for $u \in U_1(\mathbf{A})$. Let $K_1 = K_{1,\infty} K_{1,\mathrm{f}}$ denote the maximal compact subgroup of $\mathrm{U}(1,1)(\mathbf{A})$ with

$$K_{1,\infty} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \in \mathrm{GL}_2(\mathbf{C}) \mid |\alpha|^2 + |\beta|^2 = 1, \;\; \alpha\overline{\beta} \in \mathbf{R} \right\}$$

being the maximal compact subgroup of $\mathrm{U}(1,1)(\mathbf{R})$ and

$$K_{1,\mathrm{f}} = \prod_{p \neq \infty} \mathrm{U}(1,1)(\mathbf{Z}_p).$$

As usually we extend δ_1 to a map on $\mathrm{U}(1,1)(\mathbf{A})$ using the Iwasawa decomposition. For $g \in \mathrm{U}(1,1)(\mathbf{A})$, set

$$E_{\mathrm{U}(1,1)}(g,s) = \sum_{\gamma \in B_1(\mathbf{Q}) \backslash \mathrm{U}(1,1)(\mathbf{Q})} \delta_1(\gamma g)^s. \tag{3.7}$$

The following proposition follows from [50], Theorem 19.7.

Proposition 3.8. — *The series $E_{\mathrm{U}(1,1)}(g,s)$ converges absolutely for $\mathrm{Re}\,(s) > 1$ and continues meromorphically to all of $\mathbf{C}$. It has a simple pole at $s = 1$ with residue $3/\pi$.*

We now define a complex analogue of $E_{\mathrm{U}(1,1)}(g,s)$. As $\mathrm{SL}_2(\mathbf{R})$ acts transitively on $\mathbf{H}$, so does $\mathrm{U}(1,1)(\mathbf{R}) \supset \mathrm{SL}_2(\mathbf{R})$. Hence for every $z_1 \in \mathbf{H}$ there exists $g_\infty \in \mathrm{U}(1,1)(\mathbf{R})$ such that $z_1 = g_\infty i$. Set $g = (g_\infty, 1) \in \mathrm{U}(1,1)(\mathbf{R}) \times \mathrm{U}(1,1)(\mathbf{A}_\mathrm{f})$. An easy calculation shows that

$$\delta_1(g) = \mathrm{Im}\,(z_1). \tag{3.8}$$

For z_1 and g as above, define the complex Eisenstein series corresponding to $E_{\mathrm{U}(1,1)}(g,s)$ by

$$E_{\mathrm{U}(1,1)}(z_1,s) := E_{\mathrm{U}(1,1)}(g,s). \tag{3.9}$$

It is easy to see that

$$E_{\mathrm{U}(1,1)}(z_1,s) = \sum_{\gamma \in B_1(\mathbf{Z}) \backslash \mathrm{U}(1,1)(\mathbf{Z})} (\mathrm{Im}\,(\gamma z_1))^s. \tag{3.10}$$

The series $E_{\mathrm{U}(1,1)}(z_1, s)$ possesses a Fourier expansion of the form

$$E_{\mathrm{U}(1,1)}(z_1, s) = \sum_{n \in \mathbf{Z}} c_n(y_1, s) e^{2\pi i n x_1},$$

where $x_1 := \mathrm{Re}\,(z_1)$ and $y_1 := \mathrm{Im}\,(z_1)$.

LEMMA 3.9. — *Let z_1 and g be as before, i.e., $z_1 = g_\infty i$. Then*

$$c_0(s, y_1) = y_1^s + \frac{\zeta(2s-1)}{\zeta(2s)} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \sqrt{\pi}\, y_1^{1-s},$$

where $\zeta(s)$ denotes the Riemann zeta function.

Proof. — This is a standard argument. See, e.g., [10], the proof of Theorem 1.6.1. □

3.4. Residue of the Klingen Eisenstein series

Let $E_Q(g, s)$ be the Klingen Eisenstein series defined in section 3.1. This section and section 3.5 are devoted to proving the following theorem.

THEOREM 3.10. — *The series $E_Q(g, s)$ has a simple pole at $s = 1$ and one has*

$$\mathrm{res}_{s=1}\, E_Q(g, s) = \frac{5\pi^2 L\left(2, \left(\frac{-4}{\cdot}\right)\right)}{4\zeta_K(2) L\left(3, \left(\frac{-4}{\cdot}\right)\right)}, \tag{3.11}$$

where $\zeta_K(s)$ denotes the Dedekind zeta function of K.

Theorem 3.10 is a consequence of the following proposition.

PROPOSITION 3.11. — *The following statements hold:*

(i) *For any fixed $s \in \mathbf{C}$ with $\mathrm{Re}\,(s) > 2/3$ the function $E_B(g, s, z)$ has a simple pole at $z = 1/2$ and*

$$\mathrm{res}_{z=1/2} E_B(g, s, z) = \frac{3}{2\pi} E_Q(g, s + 1/3). \tag{3.12}$$

(ii) *For any fixed $z \in \mathbf{C}$ with $\mathrm{Re}\,(z) > 1/2$ the function $E_B(g, s, z)$ has a simple pole at $s = 2/3$ and*

$$\mathrm{res}_{s=\frac{2}{3}}\, E_B(g, s, z) = \frac{\pi^2}{6\zeta_K(2)} E_P\left(g, z + 1/2\right). \tag{3.13}$$

Indeed, using Proposition 3.11 and interchanging the order of taking residues we obtain:

$$\mathrm{res}_{s=\frac{2}{3}} E_Q\left(g, s+\frac{1}{3}\right) = \frac{2\pi}{3}\frac{\pi^2}{6\zeta_K(2)}\mathrm{res}_{z=\frac{1}{2}} E_P\left(g, \frac{1}{2}+z\right).$$

By Proposition 3.2,

$$\mathrm{res}_{z=\frac{1}{2}} E_P\left(g, \frac{1}{2}+z\right) = \frac{45L\left(2, \left(\frac{-4}{\cdot}\right)\right)}{4\pi L\left(3, \left(\frac{-4}{\cdot}\right)\right)},$$

and thus we finally get

$$\mathrm{res}_{s=1} E_Q(g,s) = \frac{5\pi^2 L\left(2, \left(\frac{-4}{\cdot}\right)\right)}{4\zeta_K(2) L\left(3, \left(\frac{-4}{\cdot}\right)\right)},$$

which proves Theorem 3.10.

We now prepare for the proof of Proposition 3.11, which will be completed in section 3.5.

Let $\left[\begin{smallmatrix} x &&& \\ & a && b \\ && \hat{x} & \\ & c && d \end{smallmatrix}\right] \in M_Q(\mathbf{A})$. Since $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{U}(1,1)(\mathbf{A})$, we can use the Iwasawa decomposition for $\mathrm{U}(1,1)(\mathbf{A})$ with respect to the upper-triangular Borel to write $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} \alpha & \beta \\ & \hat{\alpha} \end{smallmatrix}\right]\kappa$ with $\kappa \in K_1$, where K_1 is as in section 3.3. Note that if $\kappa = \left[\begin{smallmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & \kappa_4 \end{smallmatrix}\right]$, then $\left[\begin{smallmatrix} 1 &&& \\ & \kappa_1 && \kappa_2 \\ && 1 & \\ & \kappa_3 && \kappa_4 \end{smallmatrix}\right] \in K_0$. Define a character

$$\phi_Q : M_Q(\mathbf{A}) \to \mathbf{R}_+$$

by

$$\phi_Q\left(\begin{bmatrix} x &&& \\ & a && b \\ && \hat{x} & \\ & c && d \end{bmatrix}\right) = \phi_Q\left(\begin{bmatrix} x &&& \\ & 1 && \\ && \hat{x} & \\ &&& 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & \alpha && \beta \\ && 1 & \\ &&& \hat{\alpha} \end{bmatrix}\right) = |\alpha\overline{\alpha}|_{\mathbf{A}},$$

and a character

$$\phi_P : M_P(\mathbf{A}) \to \mathbf{R}_+$$

by:

$$\phi_P\left(\begin{bmatrix} A & \\ & \hat{A} \end{bmatrix}\right) = \tag{3.14}$$

$$\phi_P\left(\begin{bmatrix} x & * && \\ & y && \\ && \hat{x} & \\ && * & \hat{y} \end{bmatrix}\begin{bmatrix} \kappa_1 & \kappa_2 && \\ \kappa_3 & \kappa_4 && \\ && \kappa_1' & \kappa_2' \\ && \kappa_3' & \kappa_4' \end{bmatrix}\right) = |xy^{-1}\overline{(xy^{-1})}|_{\mathbf{A}},$$

where we used the Iwasawa decomposition for $\mathrm{GL}_2(\mathbf{A}_K) = \mathrm{Res}_{K/\mathbf{Q}}\,\mathrm{GL}_2(\mathbf{A})$ with respect to its upper-triangular Borel B_R, and its maximal compact subgroup $K_R = U(2)\prod_{v\nmid\infty}\mathrm{GL}_2(\mathcal{O}_{K_v})$ to write $A \in \mathrm{GL}_2(\mathbf{A}_K)$ as

$$A = \begin{bmatrix} x & * \\ & y \end{bmatrix}\begin{bmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & \kappa_4 \end{bmatrix} \in B_R(\mathbf{A})K_R.$$

We again have $\begin{bmatrix} \kappa_1 & \kappa_2 & & \\ \kappa_3 & \kappa_4 & & \\ & & \kappa_1' & \kappa_2' \\ & & \kappa_3' & \kappa_4' \end{bmatrix} \in K_0$.

Extend ϕ_Q and ϕ_P as well as δ_Q and δ_P to functions on $G(\mathbf{A})$ using the Iwasawa decompositions

$$G(\mathbf{A}) = B(\mathbf{A})\,K_0 = P(\mathbf{A})\,K_0 = Q(\mathbf{A})\,K_0. \tag{3.15}$$

A simple calculation shows that

$$\delta_Q^s\delta_P^z = \delta_Q^{s+\frac{2}{3}z}\phi_Q^{2z} = \delta_P^{\frac{3}{4}s+z}\phi_P^{\frac{3}{2}s} \tag{3.16}$$

for any complex numbers s and z. Let $E_B(g,s,z)$ be the Borel Eisenstein series defined in section 3.1. By Proposition 3.5 the series is absolutely convergent if $\mathrm{Re}\,(s) > 2/3$ and $\mathrm{Re}\,(z) > 1/2$ and admits meromorphic continuation to all of $\mathbf{C}^2$. Using identity (3.16) and rearranging terms we get:

$$E_B(g,s,z) := \sum_{\gamma\in Q(\mathbf{Q})\backslash G(\mathbf{Q})} \delta_Q(\gamma g)^{s+\frac{2}{3}z} \sum_{\alpha\in B(\mathbf{Q})\backslash Q(\mathbf{Q})} \phi_Q(\alpha\gamma g)^{2z} =$$

$$= \sum_{\gamma\in P(\mathbf{Q})\backslash G(\mathbf{Q})} \delta_P(\gamma g)^{\frac{3}{4}s+z} \sum_{\alpha\in B(\mathbf{Q})\backslash P(\mathbf{Q})} \phi_P(\alpha\gamma g)^{\frac{3}{2}s}. \tag{3.17}$$

Let $E_{\mathrm{U}(1,1)}(g,s)$ be the Eisenstein series defined by formula (3.7). We also define an Eisenstein series on $\mathrm{Res}_{K/\mathbf{Q}}\,GL_2(\mathbf{A})$ by:

$$E_{\mathrm{Res}_{K/\mathbf{Q}}\,GL_2}(g,s) = \sum_{\gamma\in B_R(\mathbf{Q})\backslash \mathrm{Res}_{K/\mathbf{Q}}\,GL_2(\mathbf{Q})} \delta_R(\gamma g)^s, \tag{3.18}$$

where δ_R denotes the modulus character on B_R defined by:

$$\delta_R : B_R \to \mathbf{R}_+$$

$$\delta_R\left(\begin{bmatrix} a & * \\ & b \end{bmatrix}\right) = |a\overline{a}b^{-1}\overline{b}^{-1}|_{\mathbf{A}}^{1/2}. \tag{3.19}$$

The following maps

$$\pi_Q : M_Q U_Q \to \mathrm{U}(1,1) \tag{3.20}$$
$$\left(\begin{bmatrix} x & & & \\ & a & & b \\ & & \hat{x} & \\ & c & & d \end{bmatrix}, u\right) \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and

$$\pi_P : P \to \mathrm{Res}_{K/\mathbf{Q}}\, GL_2 \tag{3.21}$$
$$\begin{bmatrix} A & X \\ & \hat{A} \end{bmatrix} \mapsto A$$

give bijections

$$B(\mathbf{Q}) \setminus Q(\mathbf{Q}) \cong B_1(\mathbf{Q}) \setminus \mathrm{U}(1,1)(\mathbf{Q})$$

and

$$B(\mathbf{Q}) \setminus P(\mathbf{Q}) \cong B_R(\mathbf{Q}) \setminus \mathrm{Res}_{K/\mathbf{Q}}\, GL_2(\mathbf{Q}),$$

respectively.
On the $\mathbf{A}$-points we can extend π_Q to a map $G(\mathbf{A}) \to \mathrm{U}(1,1)(\mathbf{A})/K_1$ and π_P to a map $G(\mathbf{A}) \to \mathrm{Res}_{K/\mathbf{Q}}\, \mathrm{GL}_2(\mathbf{A})/K_R$ by declaring them to be trivial on K_0. Hence we can rewrite (3.17) as

$$\begin{aligned} E_B(g,s,z) &:= \sum_{\gamma \in Q(\mathbf{Q})\setminus G(\mathbf{Q})} \delta_Q(\gamma g)^{s+\frac{2}{3}z} E_{\mathrm{U}(1,1)}(\pi_Q(\gamma g), 2z) = \\ &= \sum_{\gamma \in P(\mathbf{Q})\setminus G(\mathbf{Q})} \delta_P(\gamma g)^{\frac{3}{4}s+z} E_{\mathrm{Res}_{K/\mathbf{Q}}\, GL_2}(\pi_P(\gamma g), \frac{3}{2}s). \end{aligned} \tag{3.22}$$

3.5. $E_Q(g,s)$ as a residual Eisenstein series

In this section we complete the proof of Proposition 3.11. We will only present a proof of part (i) of the proposition as the proof of (ii) is completely analogous. (In part (ii) the role of $E_{\mathrm{U}(1,1)}$ (see below) is played by $E_{\mathrm{Res}_{K/\mathbf{Q}}\,\mathrm{GL}_2}$ for which an easy computation shows that $\mathrm{res}_{s=1} E_{\mathrm{Res}_{K/\mathbf{Q}}\,\mathrm{GL}_2}(g,s) = \pi^2/(4\zeta_K(2))$.) In what follows Z will denote a variable in the hermitian upper half-plane $\mathcal{H}$, and z_1 a variable in the complex upper half-plane $\mathbf{H}$. Otherwise we use notation from sections 3.1-3.4. Write $g = g_{\mathbf{Q}} g_\infty \kappa \in G(\mathbf{A})$ with $g_{\mathbf{Q}} \in G(\mathbf{Q})$, $g_\infty \in G(\mathbf{R})$ and $\kappa \in K_{0,\mathrm{f}}$. We have $E_B(g,s,z) = E_B(g_\infty,s,z)$ and $E_Q(g,s) = E_Q(g_\infty,s)$, hence it is enough to prove (3.12) for $g = (g_\infty, 1) \in G(\mathbf{R}) \times G(\mathbf{A}_\mathrm{f})$. Let K_1 denote the maximal compact subgroup of $\mathrm{U}(1,1)(\mathbf{A})$ and let $\pi_Q : G(\mathbf{A}) \to \mathrm{U}(1,1)(\mathbf{A})/K_1$ be as in formula (3.20). Lemmas 3.12 and 3.14 are easy.

Lemma 3.12. — *If $g = (g_\infty, 1) \in G(\mathbf{A})$, then*

$$\mathrm{Im}\,(\pi_Q(g)_\infty i) = \mathrm{Im}\,(g_\infty \mathbf{i})_{2,2}.$$

Remark 3.13. — Note that for any 2×2 matrix M with entries in $\mathbf{C}$ one has $\mathrm{Im}\,(M_{2,2}) = (\mathrm{Im}\,(M))_{2,2}$. Hence the conclusion of Lemma 3.12 can also be written as $\mathrm{Im}\,(\pi_Q(g)_\infty i) = \mathrm{Im}\,((g_\infty \mathbf{i})_{2,2})$.

Lemma 3.14. — *For any $Z \in \mathcal{H}$, there exists $\gamma \in Q(\mathbf{Z})$ such that $(\mathrm{Im}\,\gamma Z)_{2,2} > \frac{1}{2}$.*

The next lemma is just a simple adaptation to the case of hermitian modular forms of the proof of Hilfsatz 2.10 of [21].

Lemma 3.15. — *For every $Z \in \mathcal{H}$, we have*

$$\sup_{\gamma \in \Gamma_\mathbf{Z}} \det \mathrm{Im}\,(\gamma Z) < \infty.$$

Proposition 3.16. — *Let $\delta > 0$ and $g = (g_\infty, 1) \in G(\mathbf{R}) \times G(\mathbf{A}_\mathrm{f})$. For every $s \in \mathbf{C}$ with $\mathrm{Re}\,(s) > 1 + \delta$ and every $z \in \mathbf{C}$ with $|z - \frac{1}{2}| < \delta$, the series*

$$D := |z - 1/2| \sum_{\gamma \in Q(\mathbf{Q}) \backslash G(\mathbf{Q})} \left| \delta_Q(\gamma g)^{s+2z/3} E_{\mathrm{U}(1,1)}(\pi_Q(\gamma g), 2z) \right| \tag{3.23}$$

converges.

Proof. — Using the same arguments as in the proof of Lemma 4.6 (cf. section 4.2) one shows that

$$D = \sum_{\gamma \in Q(\mathbf{Z}) \backslash \Gamma_\mathbf{Z}} \left| \left(\frac{\det \mathrm{Im}\,(\gamma Z)}{(\mathrm{Im}\,(\gamma Z))_{2,2}} \right)^{3s+2z} \right| |z - 1/2|\, |E_{\mathrm{U}(1,1)}(\pi_Q(\gamma g)_\infty i, 2z)|.$$

(Note that $z' := \pi_Q(\gamma g)_\infty i$ is a complex variable.) As $g = (g_\infty, 1)$ and $\gamma \in \Gamma_\mathbf{Z} \subset K_{0,\mathrm{f}}$, we have $\pi_Q(\gamma g)_\infty = \pi_Q((\gamma g_\infty, 1))_\infty$. By Lemmas 3.12 and 3.14 we can find a set S of representatives of $Q(\mathbf{Z}) \setminus \Gamma_\mathbf{Z}$ such that for every $\gamma \in S$ we have

$$\mathrm{Im}\ (\pi_Q(\gamma g)_\infty i) = \mathrm{Im}\ ((\gamma g_\infty \mathbf{i})_{2,2}) > \frac{1}{2}. \tag{3.24}$$

The series $E_{\mathrm{U}(1,1)}(z_1, 2z)$ has a Fourier expansion of the form

$$E_{\mathrm{U}(1,1)}(z_1, 2z) = \sum_{n \in \mathbf{Z}} c_n(2z, \mathrm{Im}\,(z_1)) e^{2\pi i n \mathrm{Re}\,(z_1)},$$

and $E_{\mathrm{U}(1,1)}(z_1,2z)-c_0(2z,\mathrm{Im}\,(z_1))$ for every fixed z_1 continues to a holomorphic function on the entire z-plane and for every fixed z is rapidly decreasing as $\mathrm{Im}\,(z_1)\to\infty$. It follows that for any given $N>0$ there exists a constant $M(N)$ (independent of z_1 and independent of z as long as $|z-1/2|<\delta$) such that $|E_{\mathrm{U}(1,1)}(z_1,2z)-c_0(2z,\mathrm{Im}\,(z_1))|<M(N)$ as long as $\mathrm{Im}\,(z_1)>N$. Set $x_\gamma:=\mathrm{Re}\,(\pi_Q(\gamma g)_\infty i)$ and $y_\gamma:=\mathrm{Im}\,(\pi_Q(\gamma g)_\infty i)=\mathrm{Im}\,((\gamma g_\infty \mathbf{i})_{2,2})$. Taking $N=1/2$, we see by formula (3.24) that there exists a constant M (independent of γ) such that $|E_{\mathrm{U}(1,1)}(x_\gamma+iy_\gamma,2z)|\leqslant M+|c_0(2z,y_\gamma)|$. Using (3.8) and Lemma 3.9 one sees that there exists a positive constant C independent of z and of γ such that

$$|z-1/2||c_0(2z,y_\gamma)|<C+|y_\gamma|^{1+2\delta}.$$

Thus we conclude that there exists a positive constant A (independent of z and γ) such that

$$\begin{aligned}\left|\left(z-\frac{1}{2}\right)E_{\mathrm{U}(1,1)}\left(\pi_Q(\gamma g_\infty)i,2z\right)\right| &\leqslant A(1+\mathrm{Im}\,(\pi_Q(\gamma g_\infty)i)^{1+2\delta})=\\ &=A(1+\mathrm{Im}\,(\gamma g_\infty \mathbf{i})_{2,2}^{1+2\delta}).\end{aligned}\tag{3.25}$$

For $s'\in\mathbf{C}$ lying inside the region of absolute convergence of $E_{s'}(Z)$ let

$$|E|_{s'}(Z):=\sum_{\gamma\in Q(\mathbf{Z})\backslash\Gamma_{\mathbf{Z}}}\left|\left(\frac{\det\mathrm{Im}\,(\gamma Z)}{(\mathrm{Im}\,(\gamma Z))_{2,2}}\right)^{s'}\right|$$

denote the majorant of $E_s(Z)$. By formula (3.25) we have

$$D\leqslant A|E|_{3s+2z}(Z)+A\sum_{\gamma\in S}\left|\left(\frac{\det\mathrm{Im}\,(\gamma Z)}{(\mathrm{Im}\,(\gamma Z))_{2,2}}\right)^{3s+2z}\right|(\mathrm{Im}\,(\gamma Z))_{2,2}^{1+2\delta}.\tag{3.26}$$

Note that $|E|_{3s+2z}(Z)$ is well-defined (i.e., $3s+2z$ is in the region of absolute convergence of $E_{s'}(Z)$) by our assumption on s and z. Denote the second term of the right-hand side of formula (3.26) by D_2. Then

$$D_2=A\sum_{\gamma\in S}\left|\left(\frac{\det\mathrm{Im}\,(\gamma Z)}{(\mathrm{Im}\,(\gamma Z))_{2,2}}\right)^{3s+2z-(1+2\delta)}\right|(\det\mathrm{Im}\,(\gamma Z))^{1+2\delta}.$$

By Lemma 3.15 there exists a constant $M(Z)$ such that $\det\mathrm{Im}\,(\gamma Z)\leqslant M(Z)$ for every $\gamma\in S$ and hence

$$D_2\leqslant AM(Z)^{1+2\delta}|E|_{3s+2z-(1+2\delta)}<\infty$$

as $\mathrm{Re}\,(3s+2z-(1+2\delta))>3$ by our assumptions on z and s. This finishes the proof. □

Proof of Proposition 3.11. — We need to show that for a fixed $s \in \mathbf{C}$ with $\mathrm{Re}\,(s) > 2/3$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that $|z - 1/2| < \delta$ implies

$$D(z) := \left| \left(z - \frac{1}{2}\right) \sum_{\gamma \in Q(\mathbf{Q})\backslash G(\mathbf{Q})} \delta_Q(\gamma g)^{s+2z/3} E_{\mathrm{U}(1,1)}(\pi_Q(\gamma g), 2z) - \right. \\ \left. - \frac{3}{2\pi} \sum_{\gamma \in Q(\mathbf{Q})\backslash G(\mathbf{Q})} \delta_Q(\gamma g)^{s+1/3} \right| < \epsilon. \tag{3.27}$$

As remarked at the beginning of the section we can assume without loss of generality that $g = (g_\infty, 1) \in G(\mathbf{R}) \times G(\mathbf{A}_{\mathrm{f}})$. We first show that (3.27) holds for s with $\mathrm{Re}\,(s) > 1$. Fix $s \in \mathbf{C}$ with $\mathrm{Re}\,(s) > 1$ and $\delta' > 0$ such that $0 < \delta' < \mathrm{Re}\,(s) - 1$. From now on assume $|z - 1/2| < \delta'$. Fix a set S of representatives of $Q(\mathbf{Q}) \setminus G(\mathbf{Q})$. By Proposition 3.16 and the fact that $E_Q(g, s')$ converges absolutely for s' with $\mathrm{Re}\,(s') > 1$, there exists a finite subset S_1 of S such that the following two inequalities:

$$\sum_{\gamma \in S_2} \left| (\delta_Q(\gamma g))^{s+1/3} \right| < \frac{\pi\epsilon}{6}, \tag{3.28}$$

$$\sum_{\gamma \in S_2} \left| z - \frac{1}{2} \right| \left| \delta_Q(\gamma g)^{s+2z/3} E_{\mathrm{U}(1,1)}(\pi_Q(\gamma g), 2z) \right| < \frac{\epsilon}{4} \tag{3.29}$$

are simultaneously satisfied. Here S_2 denotes the complement of S_1 in S. We have $D(z) \leqslant D_1(z) + D_2(z)$, where

$$D_j(z) := \left| \left(z - \frac{1}{2}\right) \sum_{\gamma \in S_j} \delta_Q(\gamma g)^{s+2z/3} E_{\mathrm{U}(1,1)}(\pi_Q(\gamma g), 2z) - \right. \\ \left. - \frac{3}{2\pi} \sum_{\gamma \in S_j} \delta_Q(\gamma g)^{s+1/3} \right|.$$

Note that if we replace δ' with a smaller $\delta'' > 0$, then estimates (3.28) and (3.29) remain true as long as $|z - 1/2| < \delta''$ for the same choice of S_1. Hence we find $\delta > 0$ with $\delta < \delta'$ such that $D_1(z) < \frac{\epsilon}{2}$. This is clearly possible as $D_1(z)$ is a finite sum and it follows from Proposition 3.8 that $3/2\pi$ is the residue of $E_{\mathrm{U}(1,1)}(\pi_Q(\gamma g), 2z)$ at $z = 1/2$. On the other hand $D_2(z) \leqslant D_3(z) + D_4(z)$, where

$$D_3(z) := \sum_{\gamma \in S_2} \left| z - \frac{1}{2} \right| \left| \delta_Q(\gamma g)^{s+2z/3} E_{\mathrm{U}(1,1)}(\pi_Q(\gamma g), 2z) \right|$$

and

$$D_4(z) := \frac{3}{2\pi} \sum_{\gamma \in S_2} \left| (\delta_Q(\gamma g))^{s+1/3} \right|.$$

Formulas (3.28) and (3.29) imply now that $D_3(z) < \epsilon/4$ and $D_4(z) < \epsilon/4$. Hence

$$D(z) \leqslant D_1(z) + D_2(z) \leqslant D_1(z) + D_3(z) + D_4(z) < \epsilon$$

as desired.

We have thus established the equality $\mathrm{res}_{z=1/2} E_B(g,s,z) = \frac{3}{2\pi} E_Q(g, s + 1/3)$ for s with $\mathrm{Re}\,(s) > 1$. However, both sides are meromorphic functions in s and since the right-hand side is holomorphic for $\mathrm{Re}\,(s) > 2/3$, so must be the left-hand side. Hence they agree for $\mathrm{Re}\,(s) > 2/3$. □

4. The Petersson norm of a Maass lift

The goal of this section is to express the denominator of C_{F_f} in formula (1.1) by a special value of the symmetric square L-function of f.

4.1. Maass lifts

Let $\mathbf{H}$, as before, denote the complex upper half-plane. The space $\mathbf{H} \times \mathbf{C} \times \mathbf{C}$ affords an action of the Jacobi modular group $\Gamma^J := \mathrm{SL}_2(\mathbf{Z}) \ltimes \mathcal{O}_K^2$, under which $\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \lambda, \mu\right)$ takes $(\tau, z, w) \in \mathbf{H}\times\mathbf{C}\times\mathbf{C}$ to $\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, \frac{w}{c\tau+d}\right)$.

Definition 4.1. — *A holomorphic function*

$$\phi : \mathbf{H} \times \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

is called a ***Jacobi form of weight k and index m*** *if for every* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$ *and* $\lambda, \mu \in \mathcal{O}_K$,

$$\phi = \phi|_{k,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix} := (c\tau+d)^{-k} e\left(-m \frac{czw}{c\tau+d}\right) \phi_m\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, \frac{w}{c\tau+d}\right)$$

and

$$\phi = \phi|_m[\lambda,\mu] := e(m\lambda\overline{\lambda}t + \overline{\lambda}z + \lambda w)\, \phi_m(\tau, z + \lambda\tau + \mu, w + \overline{\lambda}t + \overline{\mu}).$$

Let k be a positive integer divisible by 4 and F a hermitian cusp form of weight k and full level. By rearranging the Fourier expansion $F(Z) = \sum_{B \in \mathcal{S}} c(B) e(\mathrm{tr}\, BZ)$ of F we obtain

$$F(Z) = \sum_{m \in \mathbf{Z}_{>0}} \phi_m(\tau, z, w) e(m\tau') \tag{4.1}$$

where $Z = \left[\begin{smallmatrix} \tau & z \\ w & \tau' \end{smallmatrix}\right] \in \mathcal{H}$ and

$$\phi_m(\tau, z, w) = \sum_{\substack{l \in \mathbf{Z}_{\geqslant 0}, t \in \frac{1}{2}\mathcal{O}_K \\ t\bar{t} \leqslant lm}} c\left(\begin{bmatrix} l & t \\ \bar{t} & m \end{bmatrix}\right) e(l\tau + \bar{t}z + tw)$$

is a Jacobi form of weight k and index m. The expansion (4.1) is called the *Fourier-Jacobi expansion of F.*

Definition 4.2. — *The Maass space denoted by $\mathcal{S}_k^{\mathrm{M}}(\Gamma_{\mathbf{Z}})$ is the $\mathbf{C}$-linear subspace of $\mathcal{S}_k(\Gamma_{\mathbf{Z}})$ consisting of those $F \in \mathcal{S}_k(\Gamma_{\mathbf{Z}})$ which satisfy the following condition: there exists a function $c_F^* : \mathbf{Z}_{\geqslant 0} \to \mathbf{C}$ such that*

$$c_F(B) = \sum_{d \in \mathbf{Z}_{>0}, d \mid \epsilon(B)} d^{k-1} c_F^*(4 \det B / d^2)$$

for all $B \in \mathcal{S}$, where $\epsilon(B) := \max\left\{q \in \mathbf{Z}_{>0} \mid \frac{1}{q}B \in \mathcal{S}\right\}$. We call $F \in \mathcal{S}_k^{\mathrm{M}}(\Gamma_{\mathbf{Z}})$ a Maass form or a CAP form.

Theorem 4.3 (Raghavan-Sengupta [44]). — *There exists a $\mathbf{C}$-linear isomorphism between the Maass space and the space*

$$S_{k-1}^{+}\left(4, \left(\frac{-4}{\cdot}\right)\right) := \tag{4.2}$$
$$= \left\{\phi \in S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right) \mid \phi = \sum_{n=1}^{\infty} b(n)q^n, \;\; b(n) = 0 \text{ if } \left(\frac{-4}{n}\right) = 1\right\}.$$

We will describe this isomorphism in more detail. Any Jacobi form ψ of weight k and index 1 can be written as a finite linear combination:

$$\psi(\tau, z, w) = \sum_{t \in A} f_t(\tau)\theta_t(\tau, z, w), \tag{4.3}$$

where $A = \left\{0, \frac{1}{2}, \frac{i}{2}, \frac{i+1}{2}\right\}$, $\theta_t(\tau, z, w) := \sum_{\lambda \in t + \mathcal{O}_K} e(\lambda\overline{\lambda}\tau + \lambda\bar{z} + \lambda w)$ and

$$f_t(\tau) = \sum_{l \geqslant 0, l \equiv -4nt \pmod 4} c_F^*(l) e(l\tau/4).$$

The map $\psi(\tau, z, w) \mapsto f_0(\tau)$ gives an injection of $J_{k,1}$, the space of Jacobi forms of weight k and index 1, into $S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right)$. If we put $\psi = \phi_1$ and define ϕ by $\phi|_{k-1}\left[\begin{smallmatrix} & -1 \\ 4 & \end{smallmatrix}\right] = f_0$, the composite $F \mapsto \phi_1(\tau, z, w) \mapsto f_0(\tau) \mapsto \phi$ gives the isomorphism alluded to in Theorem 4.3. Denoting this isomorphism by Ω, we can map any normalized Hecke eigenform $f = \sum_{n \geqslant 1} b(n)q^n \in S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right)$ to the element $F_f := \Omega^{-1}(f - f^\rho) \in \mathcal{S}_k^{\mathrm{M}}(\Gamma_{\mathbf{Z}})$. Here $f^\rho = \sum_{n \geqslant 1} \overline{b(n)}q^n$. This lifting is Hecke equivariant in a sense, which will be explained in section 5.4. Note that $F_f = -F_{f^\rho}$ and $F_f \neq 0$ if and only if $f \neq f^\rho$.

Definition 4.4. — *If $f \neq f^\rho$, then F_f is called the Maass lift of f or the CAP lift of f.*

Proposition 4.5. — *If $f = \sum_{n=1}^{\infty} b(n)q^n \in S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right)$ is a normalized eigenform, then*

$$c^*_{F_f}(n) = \begin{cases} \frac{-2i}{u(n)}(b(n) - \overline{b(n)}) & \text{if } n \not\equiv 1 \pmod 4 \\ 0 & \text{if } n \equiv 1 \pmod 4 \end{cases}, \tag{4.4}$$

where $u(n) := \#\{t \in A \mid 4N(t) \equiv -n \pmod 4\}$.

Proof. — This follows from formula (4) on page 670 in [37]. □

4.2. The Petersson norm of F_f

To express $\langle F_f, F_f\rangle$ by an L-value we will use an identity proved in [44] that involves a variant $E_s(Z)$ (defined below) of the Klingen Eisenstein series $E_Q(g,s)$ (which was defined in section 3.1). For a matrix M, denote by $M_{i,j}$ the (i,j)-th entry of M. Let $\mathcal{C}$ be the subgroup of $\Gamma_{\mathbf{Z}}$ consisting of all matrices whose last row is $\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$. Set

$$E_s(Z) = \sum_{\gamma \in \mathcal{C}\backslash\Gamma_{\mathbf{Z}}} \left(\frac{\det \operatorname{Im} \gamma Z}{(\operatorname{Im} \gamma Z)_{1,1}}\right)^s.$$

The series converges for $\operatorname{Re}(s) > 3$ ([44], Lemma 1).

Lemma 4.6. — *Let $g = (g_\infty, 1) \in G(\mathbf{A})$ and $Z = g_\infty \mathbf{i}$. Then*

$$E_Q(g,s) = \frac{1}{4} E_{3s}(Z). \tag{4.5}$$

Proof. — First note that

$$E_s(Z) = 4 \sum_{\gamma \in \mathcal{C}'\backslash\Gamma_{\mathbf{Z}}} \left(\frac{\det \operatorname{Im} \gamma Z}{(\operatorname{Im} \gamma Z)_{1,1}}\right)^s,$$

where $\mathcal{C}'$ is the subgroup of $\Gamma_{\mathbf{Z}}$ consisting of matrices whose last row is of the form $\begin{bmatrix}0 & 0 & 0 & \alpha\end{bmatrix}$ with $\alpha \in \mathcal{O}_K^\times$. Moreover we have $\mathcal{C}' = wQ(\mathbf{Z})w^{-1}$

with $w = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$. This gives

$$\sum_{\gamma \in \mathcal{C}' \backslash \Gamma_{\mathbf{Z}}} \left(\frac{\det \operatorname{Im} \gamma Z}{(\operatorname{Im} \gamma Z)_{1,1}} \right)^s = \sum_{\gamma \in Q(\mathbf{Z}) \backslash \Gamma_{\mathbf{Z}}} \left(\frac{\det \operatorname{Im} w\gamma w^{-1} Z}{(\operatorname{Im} w\gamma w^{-1} Z)_{1,1}} \right)^s = \sum_{\gamma \in Q(\mathbf{Z}) \backslash \Gamma_{\mathbf{Z}}} \left(\frac{\det \operatorname{Im} \gamma Z}{(\operatorname{Im} \gamma Z)_{2,2}} \right)^s,$$

as $w \in \Gamma_{\mathbf{Z}}$.

Now for $\gamma \in \Gamma_{\mathbf{Z}}$ we have $\delta_Q(\gamma g) = \delta_Q(q)$, where $q = (q_\infty, 1)$ and $\gamma g_\infty = q_\infty \kappa_\infty$ with $q_\infty \in Q(\mathbf{R})$, $\kappa_\infty \in K_{0,\infty}$. If $q_\infty = um$ with $m = \begin{bmatrix} x & & & \\ & a & & b \\ & & \hat{x} & \\ & c & & d \end{bmatrix} \in M_Q(\mathbf{R})$ and $u \in U_Q(\mathbf{R})$, then

$$\delta_Q(\gamma g) = \delta_Q(um) = \delta_Q(m(m^{-1}um)) = \delta_Q(m) = |x\overline{x}|^3_{\mathbf{A}}.$$

Moreover

$$\operatorname{Im} \gamma Z = \operatorname{Im} \gamma g_\infty \mathbf{i} = \operatorname{Im} q_\infty \mathbf{i} = \operatorname{Im} um\, \mathbf{i}.$$

A direct calculation shows that $\det \operatorname{Im} u(m\mathbf{i}) = \det \operatorname{Im} m\, \mathbf{i}$ and that $(\operatorname{Im} u(m\mathbf{i}))_{2,2} = (\operatorname{Im} m\mathbf{i})_{2,2}$. On the other hand

$$\operatorname{Im} m\, \mathbf{i} = \begin{bmatrix} x\overline{x} & \\ & \frac{1}{(ci+d)\overline{(ci+d)}} \end{bmatrix},$$

hence we have

$$\frac{\det \operatorname{Im} \gamma Z}{(\operatorname{Im} \gamma Z)_{2,2}} = \delta_Q(\gamma g)^{1/3}.$$

The lemma now follows from the fact that the natural injection

$$Q(\mathbf{Z}) \setminus \Gamma_{\mathbf{Z}} \to Q(\mathbf{Q}) \setminus G(\mathbf{Q})$$

is a bijection. This is a consequence of the identity

$$Q(\mathbf{A}) = Q(\mathbf{Q})\, Q(\mathbf{R})\, Q(\prod_{p \nmid \infty} \mathbf{Z}_p),$$

which follows from Lemma 8.14 of [50]. □

Set

$$E^*_s(Z) := \pi^{-2s} \Gamma(s) \Gamma(s-1) \zeta(2s-2) \zeta_K(s) E_s(Z). \tag{4.6}$$

In [44] Raghavan and Sengupta prove that $E^*_s(Z)$ can be analytically continued in s to the entire complex plane except for possible simple poles at $s = 0, 1, 2, 3$. Using Lemma 4.6 and Theorem 3.10 we conclude that $E^*_s(Z)$ has a simple pole at $s = 3$ and

$$\operatorname{res}_{s=3} E^*_s(Z) = \frac{2}{\pi^2} \zeta(3). \tag{4.7}$$

Combining results of section 3 of [44] with a formula on page 200 in [loc. cit.] we get

$$\begin{aligned}\left\langle F_f, E^*_{s-k+3}F_f\right\rangle &= 4^{-3s}\pi^{-3s+2k-6}\Gamma(s)\Gamma(s-k+2)\Gamma(s-k+3)\times \\ &\times\left(\prod_{j=1}^{3}\zeta(s-k+j)\right)L(\mathrm{Symm}^2 f,s)\left\langle\phi_1,\phi_1\right\rangle.\end{aligned}\tag{4.8}$$

Here we define $L(\mathrm{Symm}^2 f, s)$ for a normalized eigenform $f = \sum_{n=1}^{\infty} a(n)q^n$ as an Euler product:

(4.9)

$$\begin{aligned}L(\mathrm{Symm}^2 f,s) &= (1-a(2)^2 2^{-s})^{-1}(1-\overline{a(2)}^2 2^{-s})^{-1}\times \\ &\times\prod_{p\neq 2}\left[(1-\alpha_{p,1}^2 p^{-s})(1-\alpha_{p,1}\alpha_{p,2}p^{-s})(1-\alpha_{p,2}^2 p^{-s})\right]^{-1}\end{aligned}$$

where the complex numbers $\alpha_{p,1}$ and $\alpha_{p,2}$ are the p-Satake parameters of f defined by the equation

$$1-a(p)x+\left(\frac{-4}{p}\right)p^{k-2}x^2 = (1-\alpha_{p,1}x)(1-\alpha_{p,2}x).$$

Combining formulas (4.7) and (4.8) we obtain:

$$\langle F_f, F_f\rangle = 2^{-2k-3}\Gamma(k)\cdot\pi^{-k-2}\langle\phi_1,\phi_1\rangle L(\mathrm{Symm}^2 f,k).\tag{4.10}$$

Finally, to relate $\langle\phi_1,\phi_1\rangle$ to $\langle f,f\rangle$, in the next subsection we will prove the following lemma.

LEMMA 4.7. — *The following identity holds:*

$$\langle\phi_1,\phi_1\rangle = 2\langle f,f\rangle_{\Gamma_1(N)} = 24\langle f,f\rangle.\tag{4.11}$$

Combining Lemma 4.7 with formula (4.10) we finally obtain:

THEOREM 4.8. — *The following identity holds:*

$$\langle F_f, F_f\rangle = 2^{-2k+2}\cdot 3\cdot\Gamma(k)\cdot\pi^{-k-2}\langle f,f\rangle L(\mathrm{Symm}^2 f,k).\tag{4.12}$$

4.3. Inner product formula for Jacobi forms

This section is devoted to proving Lemma 4.7.

Proof of Lemma 4.7. — Let ψ_1 and ψ_2 denote two Jacobi forms of weight k and index m. It is easy to show that

(4.13)
$$\langle\psi_1,\psi_2\rangle = \int_{\mathcal{F}} v^k \left(\int_{\mathcal{F}_\tau} \psi_1(\tau,z,w)\overline{\psi_2(\tau,z,w)}e^{\frac{-\pi|z-\overline{w}|^2}{v}} dz_0\, dz_1\, dw_0\, dw_1\right) du\, dv,$$

where $\mathcal{F}$ is the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on the complex upper half-plane and $\mathcal{F}_\tau \subset \{\tau\}\times\mathbf{C}\times\mathbf{C}$ is a fundamental domain for the action of the matrices $\begin{bmatrix}-1 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & \lambda & \mu\end{bmatrix}$ ($\lambda,\mu\in\mathcal{O}_K$) on $\mathbf{C}\times\mathbf{C}$. After performing a change of variables on $\mathbf{C}\times\mathbf{C}$ (keeping τ fixed)

$$z' = z+w \qquad w' = z-w,$$

and denoting by $\mathcal{F}'_\tau$ the fundamental domain $\mathcal{F}_\tau$ in the new variables, the integral over $\mathcal{F}_\tau$ in (4.13) becomes

$$\frac{1}{8}\int_{\mathcal{F}'_\tau} \psi_1(\tau,z',w')\overline{\psi_2(\tau,z',w')}v^k e^{\frac{-\pi|\frac{z'+w'}{2}-\frac{\overline{z'}-\overline{w'}}{2}|^2}{v}}\, dz'_0\, dz'_1\, dw'_0\, dw'_1.$$

Set $\psi_1=\psi_2=\phi_1$, where ϕ_1 is the first Fourier-Jacobi coefficient of the CAP form F_f. Using formula (4.3) we can write:

$$\langle\phi_1,\phi_1\rangle = \frac{1}{8}\sum_{t\in A}\sum_{t'\in A}\int_{\mathcal{F}} f_t(\tau)\overline{f_{t'}(\tau)}v^{k-4}I(t,t',\tau)\, du\, dv \tag{4.14}$$

with

(4.15)
$$I(t,t',\tau) = \int_{\mathcal{F}'_\tau}\Big(\sum_{a\in t+\mathcal{O}_K}\ \sum_{b\in t'+\mathcal{O}_K} e(N(a)\tau+\overline{a}z+aw)\, e(N(b)\overline{\tau}+b\overline{z}+\overline{b}\overline{w})\times$$
$$\times\, e^{-\frac{\pi}{v}((\mathrm{Im}\,(z'))^2+(\mathrm{Re}\,(w'))^2)}\Big)\, dz'_0\, dz'_1\, dw'_0\, dw'_1.$$

Changing variables again we get

$$I(t,t',\tau) = \sum_{a\in t+\mathcal{O}_K}\ \sum_{b\in t'+\mathcal{O}_K} e(N(a)\tau-N(b)\overline{\tau})I_1\, I_2 \tag{4.16}$$

with

$$I_1 = 4\int_{\Omega_1} e(2x'\mathrm{Re}\,(a)-2\overline{x'}\mathrm{Re}\,(b))\, e^{\frac{\pi}{2}(2x'_1)}dx'_0\, dx'_1,$$

where Ω_1 is the parallelogram in $\mathbf{C}$ spanned by the two $\mathbf{R}$-linearly independent complex numbers 1 and τ, and $x' = x'_0 + ix'_1 \in \mathbf{C}$, with $x'_0, x'_1 \in \mathbf{R}$. Before we define I_2 we note that I_1 can be written as

$$I_1 = 4\int_0^{\mathrm{Im}\,(\tau)} e^{-\frac{\pi}{v}\cdot 4(x'_1)^2}\left(\int_0^1 e(2x'\mathrm{Re}\,(a) - 2\overline{x'}\mathrm{Re}\,(b))\,dx'_0\right)dx'_1. \tag{4.17}$$

Now the integral inside the parantheses in (4.17) equals $e^{-8\pi \mathrm{Re}\,(a)\,x'_1}$ if $\mathrm{Re}\,(a) = \mathrm{Re}\,(b)$ and 0 otherwise. Hence

$$I_1 = \begin{cases} 4\displaystyle\int_0^{\mathrm{Im}\,(\tau)} e^{-4\frac{\pi}{v}(x'_1)^2}\, e^{-8\pi\mathrm{Re}\,(a)\,x'_1}\,dx'_1 & \text{if } \mathrm{Re}\,(a) = \mathrm{Re}\,(b) \\ 0 & \text{if } \mathrm{Re}\,(a) \neq \mathrm{Re}\,(b) \end{cases} \tag{4.18}$$

The integral

$$I_2 := 4\int_{\Omega_2} e(-2y'\,\mathrm{Im}\,(a) + 2\overline{y'}\,\mathrm{Im}\,(b))e^{-\frac{\pi}{v}\cdot 4(y'_1)}\,dy'_0\,dy'_1,$$

where Ω_2 denotes the region in the complex plane spanned by the two $\mathbf{R}$-linearly independent complex numbers 1 and $-\tau$ and $y' = y'_0 + iy'_1 \in \mathbf{C}$ with $y'_0, y'_1 \in \mathbf{R}$, can be handled in a similar way. In fact one gets:

$$I_2 = \begin{cases} 4\displaystyle\int_0^{\mathrm{Im}\,(\tau)} e^{-4\frac{\pi}{v}(y'_1)^2}\, e^{8\pi\mathrm{Im}\,(a)\,x'_1}\,dx'_1 & \text{if } \mathrm{Im}\,(a) = \mathrm{Im}\,(b) \\ 0 & \text{if } \mathrm{Im}\,(a) \neq \mathrm{Im}\,(b) \end{cases}. \tag{4.19}$$

Substituting (4.18) and (4.19) into (4.16) one sees that $I(t,t',\tau) = 0$ if $t \neq t'$, and that after rearranging terms

$$\begin{aligned} I(t,t,\tau) &= 16\left(\sum_{\mathrm{Re}\,(a)\in\mathrm{Re}\,(t)+\mathbf{Z}}\int_0^v e^{-\frac{4\pi}{v}((\mathrm{Re}\,(a))v+x'_1)^2}dx'_1\right)\times \\ &\times\left(\sum_{\mathrm{Im}\,(a)\in\mathrm{Im}\,(t)+\mathbf{Z}}\int_0^v e^{-\frac{4\pi}{v}((\mathrm{Im}\,(a))v+y'_1)^2}dy'_1\right) = \\ &= 16\int_{\mathbf{R}} e^{-\frac{4\pi}{v}(\mathrm{Re}\,(t)+x'_1)^2}\,dx'_1\int_{\mathbf{R}} e^{-\frac{4\pi}{v}(\mathrm{Im}\,(t)+y'_1)^2}\,dy'_1 = 4v, \end{aligned} \tag{4.20}$$

where $\tau = u + iv$. Hence

$$\langle\phi_1,\phi_1\rangle = \int_{\mathcal{F}}\sum_{t\in A} f_t(\tau)\overline{f_t(\tau)}v^{k-4}\cdot 4v\,du\,dv. \tag{4.21}$$

From this it follows that $\sum_{t\in A} f_t(\tau)\overline{f_t(\tau)}v^{k-1}$ is "invariant" under $\mathrm{SL}_2(\mathbf{Z})$. We want to relate (4.21) to

$$\langle f,f\rangle' := \int_{\Gamma_1(4)\backslash\mathbf{H}} f(\tau)\overline{f(\tau)}v^{k-3}\,du\,dv.$$

Denote by $\langle f_t, f_t\rangle'$ the integral $\int_{\Gamma_1(4)\backslash\mathbf{H}} f_t(\tau)\overline{f_t(\tau)}v^{k-3}\,du\,dv$. We will use calculations carried out in [36]. In particular one has $f_{1/2} = f_{i/2}$ and $f_{(i+1)/2} = f_0|_{k-1}\left[\begin{smallmatrix}1 & \\ 2 & 1\end{smallmatrix}\right]$, hence we conclude that the quantities $\langle f_t, f_t\rangle'$ are well-defined, since $f_0|_{k-1}\alpha = f_0$ for all $\alpha\in\Gamma_1(4)$. Moreover, we have

(4.22)

$$\begin{aligned}\sum_{t\in A}\langle f_t, f_t\rangle' &= \langle f_0, f_0\rangle' + \left\langle f_{(i+1)/2}, f_{(i+1)/2}\right\rangle' + 2\left\langle f_{1/2}, f_{1/2}\right\rangle' = \\ &= \langle f_0, f_0\rangle' + \left\langle f_{(i+1)/2}|_{k-1}\left[\begin{smallmatrix}1 & \\ 2 & 1\end{smallmatrix}\right], f_{(i+1)/2}|_{k-1}\left[\begin{smallmatrix}1 & \\ 2 & 1\end{smallmatrix}\right]\right\rangle' + 2\left\langle f_{1/2}, f_{1/2}\right\rangle' = \\ &= 2\langle f_0, f_0\rangle' + 2\left\langle f_{1/2}, f_{1/2}\right\rangle'.\end{aligned}$$

We use formula (3.5') from [36], which is erroneously stated there, and should read

$$f_{1/2}(\tau) = -\frac{i}{2}\, f_0|_{k-1}\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right](\tau) - \frac{i}{2} f_0|_{k-1}\left[\begin{smallmatrix} & -1 \\ 1 & -2\end{smallmatrix}\right](\tau),$$

hence

$$\begin{aligned}\left\langle f_{1/2}, f_{1/2}\right\rangle' &= \frac{1}{2}\langle f_0, f_0\rangle' + \frac{i}{2}\left(\langle f_0, f_0|_{k-1}\left[\begin{smallmatrix}1 & \\ 2 & 1\end{smallmatrix}\right]\rangle' - \langle f_0|_{k-1}\left[\begin{smallmatrix}1 & \\ 2 & 1\end{smallmatrix}\right], f_0\rangle'\right) \\ &= \frac{1}{2}\langle f_0, f_0\rangle'\end{aligned}$$

as $f_0|_{k-1}\left[\begin{smallmatrix}1 & \\ 4 & 1\end{smallmatrix}\right] = f_0$. Thus we obtain

$$\sum_{t\in A}\langle f_t, f_t\rangle' = 3\langle f_0, f_0\rangle' = 3\left\langle f_0|_{k-1}\left[\begin{smallmatrix} & -1 \\ 4 & \end{smallmatrix}\right], f_0|_{k-1}\left[\begin{smallmatrix} & -1 \\ 4 & \end{smallmatrix}\right]\right\rangle' = 3\langle\phi,\phi\rangle'.$$

Since $\phi = f - f^\rho$, and $\langle f, f^\rho\rangle' = 0$, we get $\langle\phi,\phi\rangle' = 2\langle f, f\rangle'$, so finally

$$\langle\phi_1,\phi_1\rangle = \frac{4}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_1(4)]}\sum_{t\in A}\langle f_t, f_t\rangle' = \frac{24}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_1(4)]}\langle f, f\rangle = 2\langle f, f\rangle'.$$

□

5. Hecke operators

5.1. Elliptic Hecke algebra

The theory of Hecke operators acting on the space of elliptic modular forms is well-known, so we refer the reader to standard sources (e.g., [42], [13]) for definitions of most of the objects as well as their basic properties used in this subsection.

Definition 5.1. — *Let k be a positive integer divisible by 4, and A a $\mathbf{Z}$-algebra. Denote by $\mathbf{T}_{\mathbf{Z}}$ the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}\left(S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right)\right)$ generated by the Hecke operators T_n, $n = 1, 2, \ldots$. We set*

(1) $\mathbf{T}_A := \mathbf{T}_{\mathbf{Z}} \otimes_{\mathbf{Z}} A$;
(2) $\mathbf{T}'_A$ *to be the* A*-subalgebra of* $\mathbf{T}_A$ *generated by the set*

$$\Sigma' := \{T_p\}_{p \text{ split in } K} \cup \{T_{p^2}\}_{p \text{ inert in } K};$$

(3) $\mathbf{T}^{(2)}_A$ *to be the* A*-subalgebra of* $\operatorname{End}_{\mathbf{C}}\left(S_{k-1}\left(4,\left(\frac{-4}{\cdot}\right)\right)\right)$ *generated by* $\mathbf{T}_A$ *and the (*A*-linear) operator* $\operatorname{Tr} T_2$ *which multiplies any normalized eigenform* $g = \sum a(n)q^n$ *by* $a(2) + \overline{a(2)}$.

Suppose $f = \sum_{n=1}^{\infty} a_f(n)q^n \in S_{k-1}\left(4,\left(\frac{-4}{\cdot}\right)\right)$ is a primitive normalized eigenform. Recall that we denote the set of such forms by $\mathcal{N}$. For $T \in \mathbf{T}_{\mathbf{C}}$, set $\lambda_{f,\mathbf{C}}(T)$ to denote the eigenvalue of T corresponding to f. It is a well-known fact that $\lambda_{f,\mathbf{C}}(T_n) = a_f(n)$ for all $f \in \mathcal{N}$ and that the set $\{a_f(n)\}_{n\in\mathbf{Z}_{>0}}$ is contained in the ring of integers of a finite extension L_f of $\mathbf{Q}$. Let E be a finite extension of $\mathbf{Q}_\ell$ containing the fields L_f for all $f \in \mathcal{N}$. Denote by $\mathcal{O}$ the valuation ring of E and by λ a uniformizer of $\mathcal{O}$. Then $\{a_f(n)\}_{f\in\mathcal{N},n\in\mathbf{Z}_{>0}} \subset \mathcal{O}$. Moreover, one has

$$\mathbf{T}_E = \prod_{f\in\mathcal{N}} E \tag{5.1}$$

and

$$\mathbf{T}_{\mathcal{O}} = \prod_{\mathfrak{m}} \mathbf{T}_{\mathcal{O},\mathfrak{m}}, \tag{5.2}$$

where $\mathbf{T}_{\mathcal{O},\mathfrak{m}}$ denotes the localization of $\mathbf{T}_{\mathcal{O}}$ at $\mathfrak{m}$ and the product runs over all maximal ideals of $\mathbf{T}_{\mathcal{O}}$. Every $f \in \mathcal{N}$ gives rise to an $\mathcal{O}$-algebra homomorphism $\mathbf{T}_{\mathcal{O}} \to \mathcal{O}$ assigning to T the eigenvalue of T corresponding to f. We denote this homomorphism by λ_f and its reduction mod λ by $\overline{\lambda}_f$. If $\mathfrak{m} = \ker\overline{\lambda}_f$, we write $\mathfrak{m}_f$ for $\mathfrak{m}$ or if we want to emphasize the ring $\mathfrak{m}$ lives in, we write $\mathfrak{m}_{\mathbf{T}_{\mathcal{O}},f}$. The algebra $\mathbf{T}'_{\mathbf{Z}}$ is studied in detail in section 8.1.

5.2. Hermitian Hecke algebra

The theory of Hecke operators acting on the space $\mathcal{S}_k(\Gamma_{\mathbf{Z}})$ is discussed in [23] and [37]. We summarize it here to the degree that we need it. For a formulation of the theory which is valid for hermitian modular forms of level higher than one (as well as the non-holomorphic ones) see [34]. See also [35] for a theory of Hecke operators acting on the space of adelic hermitian modular forms.

Set $\Delta := G_\mu^+(\mathbf{Q}) \cap M_4(\mathcal{O}_K)$. For $a \in \Delta$, the double coset space $\Gamma_{\mathbf{Z}} a \Gamma_{\mathbf{Z}}$ decomposes into a finite disjoint union of right cosets

$$\Gamma_{\mathbf{Z}} a \Gamma_{\mathbf{Z}} = \coprod_j \Gamma_{\mathbf{Z}} a_j$$

with $a_j \in \Delta$. For $F \in \mathcal{S}_k(\Gamma_{\mathbf{Z}})$ set $F|_k[\Gamma_{\mathbf{Z}} a \Gamma_{\mathbf{Z}}] := \sum_j F|_k a_j$.

Definition 5.2. — *The hermitian Hecke algebra (over* $\mathbf{C}$*), denoted by* $\mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$ *is the subalgebra of* $\mathrm{End}_{\mathbf{C}}(\mathcal{S}_k(\Gamma_{\mathbf{Z}}))$ *generated by the double cosets of the form* $\Gamma_{\mathbf{Z}} a \Gamma_{\mathbf{Z}}$ *for* $a \in \Delta$*. We call* $F \in \mathcal{S}_k(\Gamma_{\mathbf{Z}})$ *an* eigenform *if it is an eigenfunction for all* $T \in \mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$*. We will denote the eigenvalue of* T *corresponding to* F *by* $\lambda_{F,\mathbf{C}}(T)$*.*

For a rational prime p we define an operator

$$T_p^{\mathrm{h}} := \Gamma_{\mathbf{Z}} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \Gamma_{\mathbf{Z}}, \tag{5.3}$$

if p is inert in K we additionally define

$$T_{1,p}^{\mathrm{h}} := \Gamma_{\mathbf{Z}} \begin{bmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ & & & p \end{bmatrix} \Gamma_{\mathbf{Z}},$$

and if $p = \pi\overline{\pi}$ splits or ramifies in K we define

$$T_\pi^{\mathrm{h}} := \Gamma_{\mathbf{Z}} \begin{bmatrix} 1 & & & \\ & \pi & & \\ & & p & \\ & & & \pi \end{bmatrix} \Gamma_{\mathbf{Z}}. \tag{5.4}$$

We now describe the action of the operators T_p^{h}, $T_{1,p}^{\mathrm{h}}$ and T_π^{h} on the Fourier coefficients of hermitian modular forms. As before let $S := \{h \in M_2(K) \mid h^* = h\}$. To shorten our notation we define the following elements of $\mathrm{GL}_2(K)$:

$$\begin{aligned} \alpha_a &= \begin{bmatrix} 1 & \\ \tilde{a} & p \end{bmatrix}, \quad a \in \mathcal{O}_K/p\mathcal{O}_K, \;\; p \,\text{inert}, \;\; \tilde{a}\,\text{any lift of } a \text{ to } \mathcal{O}_K \\ \alpha_p &= \begin{bmatrix} p & \\ & 1 \end{bmatrix}, \quad p\,\text{inert}; \\ \beta_a &= \begin{bmatrix} 1 & \\ a & \pi \end{bmatrix}, \quad a = 0, 1, \dots, p-1, \;\; p = \pi\overline{\pi}\,\text{split}; \\ \beta_p &= \begin{bmatrix} \pi & \\ & 1 \end{bmatrix}, \quad p = \pi\overline{\pi}\,\text{split}, \end{aligned} \tag{5.5}$$

and for a 2×2 matrix M, we set $\tilde{M} = \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right] M \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]$. Moreover, if $B \in S$, we set

$$s(B) := \begin{cases} p & \mathrm{ord}_p(\det(B)) = 0; \\ -p(p-1) & \mathrm{ord}_p(\det(B)) > 0,\ \mathrm{ord}_p(\epsilon(B)) = 0; \\ p^2(p-1) & \mathrm{ord}_p(\epsilon(B)) > 0, \end{cases}$$

where $\epsilon(B)$ is as in Definition 4.2. Finally, if p is inert we write $\mathbf{P}^1_p$ for the disjoint union of $\mathcal{O}_K/p\mathcal{O}_K$ and p.

Lemma 5.3. — *Let $F \in \mathcal{S}_k(\Gamma_{\mathbf{Z}})$ with Fourier expansion*

$$F(Z) = \sum_{B \in S} c_F(B) e^{2\pi i \mathrm{tr}\,(BZ)},$$

and let $T \in \mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$. Then

$$TF(Z) = \sum_{B \in S} c_{TF}(B) e^{2\pi i \mathrm{tr}\,(BZ)},$$

with

$$(5.6) \quad c_{TF}(B) =$$

$$\begin{cases} p^{2k-4} c_F(p^{-1}B) + c_F(pB) + p^{k-3} \sum\limits_{a \in \mathbf{P}^1_p} c_F(p^{-1}\alpha_a^* B \alpha_a) & T = T^{\mathrm{h}}_p,\ p\,\text{inert}; \\ p^{2k-4} c_F(p^{-1}B) + c_F(pB) + p^{k-3} \sum\limits_{a,b=0}^{p} c_F((\beta_a \hat{\beta}_b)^* B \beta_a \hat{\beta}_b) & T = T^{\mathrm{h}}_p,\ p\,\text{split}; \\ p^{k-2}\pi^{-k} \sum\limits_{a=0}^{p} \left(c_F(\tilde{\beta}_a^* B \tilde{\beta}_a) + p^{k-2} c_F(\hat{\tilde{\beta}}{}_a^* B \hat{\tilde{\beta}}_a) \right) & T = T^{\mathrm{h}}_\pi,\ p\,\text{split}; \\ p^{2k-4} s(B) + p^{k-6} \sum\limits_{a \in \mathbf{P}^1_p} \left(c_F(\tilde{\alpha}_a^* B \tilde{\alpha}_a) + p^{2k-2} c_F(\hat{\alpha}_a^* B \hat{\alpha}_a) \right) & T = T^{\mathrm{h}}_{1,p},\ p\,\text{inert}. \end{cases}$$

Proof. — This follows easily from the right coset decomposition of each of the Hecke operators. The decomposition of T^{h}_p was computed by Krieg in [37], p.677. The decomposition of T^{h}_p for split p and that of T^{h}_π was computed by the author in [35], Lemmas 6.5, 6.8, but see also Lemmas 6.6 and 6.9 in loc. cit. Finally, one can show that $T_{1,p}$ decomposes in the following way:

$$
\begin{aligned}
T^{\mathrm{h}}_{1,p} &:= \Gamma_{\mathbf{Z}} \begin{bmatrix} 1 &&& \\ & p && \\ && p^2 & \\ &&& p \end{bmatrix} \Gamma_{\mathbf{Z}} = \\
&= \Gamma_{\mathbf{Z}} \begin{bmatrix} p^2 &&& \\ & p && \\ && 1 & \\ &&& p \end{bmatrix} \sqcup \coprod_{\alpha \in \mathcal{O}_K/p} \Gamma_{\mathbf{Z}} \begin{bmatrix} p & p\alpha && \\ & p^2 && \\ && p & \\ && -\overline{\alpha} & 1 \end{bmatrix} \\
&\sqcup \coprod_{\substack{\alpha,\gamma \in \mathcal{O}_K/p \\ \beta \in \mathbf{Z}/p^2}} \Gamma_{\mathbf{Z}} \begin{bmatrix} 1 & \alpha & \beta+\alpha\overline{\gamma} & \gamma \\ & p & p\overline{\gamma} & \\ && p^2 & \\ && -\overline{\alpha}p & p \end{bmatrix} \sqcup \coprod_{\substack{\delta \in \mathcal{O}_K/p \\ \phi \in \mathbf{Z}/p}} \Gamma_{\mathbf{Z}} \begin{bmatrix} p &&& p\delta \\ & 1 & \overline{\delta} & \phi \\ && p & \\ &&& p^2 \end{bmatrix} \sqcup \qquad (5.7) \\
&\bigsqcup_{\substack{\beta,\phi \in \mathbf{Z}/p\mathbf{Z} \\ \beta\phi \equiv 0 \pmod{p}}} \Gamma_{\mathbf{Z}} \begin{bmatrix} p && \beta & \\ & p && \phi \\ && p & \\ &&& p \end{bmatrix} \sqcup \bigsqcup_{\substack{\beta \in (\mathbf{Z}/p\mathbf{Z})^\times \\ \gamma \in (\mathcal{O}_K/p\mathcal{O}_K)^\times}} \Gamma_{\mathbf{Z}} \begin{bmatrix} p && \beta & \gamma \\ & p & \overline{\gamma} & |\gamma|^2\beta^{-1} \\ && p & \\ &&& p \end{bmatrix}.
\end{aligned}
$$

This can be deduced from the calculations in [23]. □

Remark 5.4. — Note that in Lemma 5.3, we have $c_F(B) = 0$ unless $B \in \mathcal{S}$.

For any split or ramified prime $p = \pi\overline{\pi}$ set $\Sigma'_p := \{T^{\mathrm{h}}_{\pi}, T^{\mathrm{h}}_{\overline{\pi}}, T_p\}$ and for any inert prime p, set $\Sigma'_p := \{T^{\mathrm{h}}_p, T^{\mathrm{h}}_{1,p}\}$.

Proposition 5.5 (Gritsenko, [23]). — *The Hecke algebra $\mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$ is generated as a $\mathbf{C}$-algebra by the set $\bigcup_p \Sigma'_p$.*

Proposition 5.6. — *The space $\mathcal{S}_k(\Gamma_{\mathbf{Z}})$ has a basis consisting of eigenforms.*

Proof. — This is a standard argument, which uses the fact that $\mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$ is commutative and all $T \in \mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$ are self-adjoint. □

5.3. Integral structure of the hermitian Hecke algebra

For a split or ramified prime $p = \pi\overline{\pi}$ set

$$\Sigma_p = \{T^{\mathrm{h}}_p, \pi^k p^{2-k} T^{\mathrm{h}}_{\pi}, \overline{\pi}^k p^{2-k} T^{\mathrm{h}}_{\overline{\pi}}\}$$

and for an inert prime p set

$$\Sigma_p := \{T^{\mathrm{h}}_p, T^{\mathrm{h}}_{1,p}\}.$$

Definition 5.7. — *Set $\mathbf{T}^{\mathrm{h}}_{\mathbf{Z}}$ (resp. $\mathbf{T}^{\mathrm{h},(2)}_{\mathbf{Z}}$) to be the $\mathbf{Z}$-subalgebra of $\mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$ generated by $\bigcup_p \Sigma_p$ (respectively by $\bigcup_{p\neq 2} \Sigma_p$). For any $\mathbf{Z}$-algebra A, set $\mathbf{T}^{\mathrm{h}}_A := \mathbf{T}^{\mathrm{h}}_{\mathbf{Z}} \otimes_{\mathbf{Z}} A$ and $\mathbf{T}^{\mathrm{h},(2)}_A := \mathbf{T}^{\mathrm{h},(2)}_{\mathbf{Z}} \otimes_{\mathbf{Z}} A$.*

Note that $\mathbf{T}^{\mathrm{h}}_{\mathbf{Z}}$ is a finite free $\mathbf{Z}$-algebra.

Lemma 5.8. — *Let $\ell > 2$ be a rational prime, E a finite extension of $\mathbf{Q}_\ell$ and $\mathcal{O}$ the valuation ring of E. Suppose that $F(Z) = \sum_{B\in\mathcal{S}} c_F(B)e^{2\pi i \operatorname{tr}(BZ)} \in \mathcal{S}_k(\Gamma_{\mathbf{Z}})$ with $c_F(B) \in \mathcal{O}$ for all $B \in \mathcal{S}$. Let $T \in \mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$. Then $TF(Z) = \sum_{B\in\mathcal{S}} c_{TF}(B)e^{2\pi i \operatorname{tr}(BZ)}$ with $c_{TF}(B) \in \mathcal{O}$ for every $B \in \mathcal{S}$.*

Proof. — This follows directly from Lemma 5.3 and the assumption that ℓ be odd. (The latter implies that the operators T^{h}_2 and $(i+1)^k 2^{2-k} T^{\mathrm{h}}_{i+1}$ preserve the $\mathcal{O}$-integrality of the Fourier coefficients of F.) □

From now on $\mathcal{N}^{\mathrm{h}}$ will denote a fixed basis of eigenforms of $\mathcal{S}_k(\Gamma_{\mathbf{Z}})$.

Theorem 5.9. — *Let $F \in \mathcal{N}^{\mathrm{h}}$. There exists a number field L_F with ring of integers $\mathcal{O}_{L_F}$ such that $\lambda_{F,\mathbf{C}}(T) \in \mathcal{O}_{L_F}$ for all $T \in \mathbf{T}^{\mathrm{h}}_{\mathcal{O}_{L_F}}$.*

Proof. — This is similar to the Eichler-Shimura isomorphism in the case of elliptic modular forms. □

Let ℓ be a rational prime and E a finite extension of $\mathbf{Q}_\ell$ containing the fields L_F from Theorem 5.9 for all $F \in \mathcal{N}^{\mathrm{h}}$. Denote by $\mathcal{O}$ the valuation ring of E and by λ a uniformizer of $\mathcal{O}$. As in the case of elliptic modular forms, $F \in \mathcal{N}^{\mathrm{h}}$ gives rise to an $\mathcal{O}$-algebra homomorphism $\mathbf{T}^{\mathrm{h}}_{\mathcal{O}} \to \mathcal{O}$ assigning to T the eigenvalue of T corresponding to the eigenform F. We denote this homomorphism by λ_F and its mod λ reduction by $\overline{\lambda}_F$. Theorem 5.9 implies that we have

$$\mathbf{T}^{\mathrm{h}}_E \cong \prod_{F\in\mathcal{N}^{\mathrm{h}}} E.$$

Moreover, as in the elliptic modular case, we have

$$\mathbf{T}^{\mathrm{h}}_{\mathcal{O}} \cong \prod_{\mathfrak{m}} \mathbf{T}^{\mathrm{h}}_{\mathcal{O},\mathfrak{m}}, \tag{5.8}$$

where the product runs over the maximal ideals of $\mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$ and $\mathbf{T}^{\mathrm{h}}_{\mathcal{O},\mathfrak{m}}$ denotes the localization of $\mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$ at $\mathfrak{m}$. A similar description holds for $\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}}$. As before, if $\mathfrak{m} = \ker \overline{\lambda}_F$, we write $\mathfrak{m}_F$ for $\mathfrak{m}$ or if we want to emphasize what ring $\mathfrak{m}$ lives in, we write $\mathfrak{m}_{\mathbf{T}^{\mathrm{h}}_{\mathcal{O}},F}$ or $\mathfrak{m}_{\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}},F}$ accordingly.

5.4. Action on the Maass space

Theorem 5.10 (Gritsenko, [23], section 2). — *The action of the Hecke algebra $\mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$ respects the decomposition of $\mathcal{S}_k(\Gamma_{\mathbf{Z}})$ into the Maass space and its orthogonal complement.*

Theorem 5.11 (Gritsenko, [23], section 3). — *There exists a* $\mathbf{C}$*-algebra map*

$$\mathrm{Desc} : \mathbf{T}^{\mathrm{h}}_{\mathbf{C}} \to \mathbf{T}^{(2)}_{\mathbf{C}}$$

such that for every $T \in \mathbf{T}^{\mathrm{h}}_{\mathbf{C}}$ *the diagram*

$$\begin{tikzcd} \mathcal{S}_k^{\mathrm{Maass}}(\Gamma_{\mathbf{Z}}) \arrow[r, "T"] & \mathcal{S}_k^{\mathrm{Maass}}(\Gamma_{\mathbf{Z}}) \\ S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right) \arrow[u, "f\mapsto F_f"] \arrow[r, "\mathrm{Desc}(T)"] & S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right) \arrow[u, "f\mapsto F_f"'] \end{tikzcd}$$

commutes. In particular one has

(5.9)

$$\begin{aligned} \mathrm{Desc}(T^{\mathrm{h}}_p) &= p^{k-1} + p^{k-2} + p^{k-3} + T_{p^2} \quad \text{for all } p \neq 2, \\ \mathrm{Desc}(T^{\mathrm{h}}_{1,p}) &= p^{k-4}(1+p^2)T_{p^2} + p^{2k-8}(p^3+p^2+p-1) \text{ if } p \text{ is inert in } K, \\ \mathrm{Desc}(T^{\mathrm{h}}_{\pi}) &= p^{k-2}\pi^{-k}(1+p)T_p \quad \text{if } p = \pi\overline{\pi} \text{ is split in } K, \\ \mathrm{Desc}(T^{\mathrm{h}}_{1+i}) &= 3\cdot 2^{k-4}(1+i)^{-k}\mathrm{Tr}\, T_2 \\ \mathrm{Desc}(T^{\mathrm{h}}_2) &= 2^{k-4}(1+i)^{-k}\big((\mathrm{Tr}\, T_2)^2 - 2^{k-1}\big). \end{aligned}$$

Here T_n *is as in section 5.1, and* $\mathrm{Tr}\, T_2$ *denotes the operator from Definition* 5.1.

Corollary 5.12. — *If* $f \in S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right)$ *is an eigenform, then so is* F_f.

Remark 5.13. — Let $f \in \mathcal{N}$, $f \neq f^\rho$. We will always assume that either F_f or F_{f^ρ} belongs to $\mathcal{N}^{\mathrm{h}}$. Hence we can write $\mathcal{N}^{\mathrm{h}} = \mathcal{N}^{\mathrm{M}} \sqcup \mathcal{N}^{\mathrm{NM}}$, where $\mathcal{N}^{\mathrm{M}}$ consists of Maass lifts F_f with $f \in \mathcal{N}$ and $\mathcal{N}^{\mathrm{NM}}$ consists of eigenforms orthogonal to those in $\mathcal{N}^{\mathrm{M}}$.

5.5. Lifting Hecke operators to the Maass space

Let E and $\mathcal{O}$ be as before. We will now prove a result regarding the map Desc, which will be used in section 7.3. Let $\mathbf{T}_{\mathbf{Z}}$ and $\mathbf{T}'_{\mathbf{Z}}$ be as in Definition 5.1. It is clear from Theorem 5.11 and the definition of $\mathbf{T}^{\mathrm{h},(2)}_{\mathbf{Z}}$ that $\mathrm{Desc}(\mathbf{T}^{\mathrm{h},(2)}_A) = \mathbf{T}'_A$ for any $\mathcal{O}$-algebra A. Moreover, we have the following diagram

$$\begin{tikzcd} \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}} \arrow[r, "\mathrm{Desc}"] \arrow[d, "\wr"] & \mathbf{T}'_{\mathcal{O}} \arrow[d, "\wr"'] \\ \prod_{\mathfrak{m}^{(2)}} \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}} \arrow[r] & \prod_{\mathfrak{m}'} \mathbf{T}'_{\mathcal{O},\mathfrak{m}'} \end{tikzcd} \tag{5.10}$$

with the lower horizontal arrow defined so that the diagram commutes. It is clear that Desc respects the direct product decomposition in diagram (5.10). In particular, for $f \in \mathcal{N}$, $\mathrm{Desc} : \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}} \twoheadrightarrow \mathbf{T}'_{\mathcal{O}}$ factors through $\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}} \twoheadrightarrow \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$. Let $\mathbf{T}^{\mathrm{M}}_{\mathcal{O}}$ be the image of $\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}}$ in $\mathrm{End}_{\mathbf{C}}(\mathcal{S}^{\mathrm{M}}_k(\Gamma_{\mathbf{Z}}))$. The horizontal arrows in diagram (5.10) factor through $\mathbf{T}^{\mathrm{M}}_{\mathcal{O}} \cong \prod_{\mathfrak{m}^{\mathrm{M}}} \mathbf{T}^{\mathrm{M}}_{\mathcal{O},\mathfrak{m}^{\mathrm{M}}}$ and the following diagram

$$\begin{tikzcd} \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}} \arrow[r] \arrow[d, "\wr"] & \mathbf{T}^{\mathrm{M}}_{\mathcal{O}} \arrow[r] \arrow[d, "\wr"] & \mathbf{T}'_{\mathcal{O}} \arrow[d, "\wr"'] \\ \prod_{\mathfrak{m}^{(2)}} \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}} \arrow[r] & \prod_{\tilde{\mathfrak{m}}} \mathbf{T}^{\mathrm{M}}_{\mathcal{O},\mathfrak{m}^{\mathrm{M}}} \arrow[r] & \prod_{\mathfrak{m}'} \mathbf{T}'_{\mathcal{O},\mathfrak{m}'} \end{tikzcd} \tag{5.11}$$

commutes. All the horizontal arrows in diagram (5.11) are surjections and the lower ones are induced from the upper ones, which respect the direct product decompositions. In particular we have

$$\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}} \twoheadrightarrow \mathbf{T}^{\mathrm{M}}_{\mathcal{O},\mathfrak{m}^{\mathrm{M}}_{F_f}} \twoheadrightarrow \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}.$$

Let $\mathcal{N}^{\mathrm{M}}_{F_f} := \{F \in \mathcal{N}^{\mathrm{M}} \mid \mathfrak{m}^{\mathrm{M}}_{F_f} = \mathfrak{m}^{\mathrm{M}}_F\}$. The goal of this section is to prove the following proposition.

Proposition 5.14. — *If $f \in \mathcal{N}$, $f \neq f^{\rho}$ is ordinary at ℓ, and $\ell \nmid (k-1)(k-2)(k-3)$, then for every split prime $p = \pi\overline{\pi}$, $p \nmid \ell$, there exists $T^{\mathrm{M}}(p) \in \mathbf{T}^{\mathrm{M}}_{\mathcal{O},\mathfrak{m}^{\mathrm{M}}_{F_f}}$ such that $\mathrm{Desc}(T^{\mathrm{M}}(p)) \in \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$ equals the image of $T_p \in \mathbf{T}'_{\mathcal{O}}$ under the canonical projection $\mathbf{T}'_{\mathcal{O}} \twoheadrightarrow \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$.*

As will be discussed in section 9.1, to every eigenform $F \in \mathcal{S}_k(\Gamma_{\mathbf{Z}})$ one can attach a 4-dimensional ℓ-adic Galois representation ρ_F. Moreover, if $F = F_g$, for some $g \in \mathcal{N}$, then the Galois representation has a special form

$$\rho_{F_g} = \begin{bmatrix} \rho_g|_{G_K} & \\ & (\rho_g \otimes \epsilon)|_{G_K} \end{bmatrix}, \tag{5.12}$$

where ρ_g is the Galois representation attached to g (cf. section 2.3) and ϵ is the ℓ-adic cyclotomic character. Let f be as in Proposition 5.14. Set

$R' := \prod_{F\in\mathcal{N}^{\mathrm{M}}_{F_f}} \mathcal{O}$ and let R be the $\mathcal{O}$-subalgebra of R' generated by the tuples $(\lambda_F(T))_{F\in\mathcal{N}^{\mathrm{M}}_{F_f}}$ for all $T \in \mathbf{T}^{\mathrm{M}}_{\mathcal{O}}$. Note that the expression $\lambda_F(T)$ makes sense since $\mathcal{S}^{\mathrm{M}}_k(\Gamma_{\mathbf{Z}})$ is Hecke stable. Then R is a complete Noetherian local $\mathcal{O}$-algebra with residue field $\mathbf{F} = \mathcal{O}/\lambda$. It is a standard argument to show that $R \cong \mathbf{T}^{\mathrm{M}}_{\mathcal{O},\mathfrak{m}^{\mathrm{M}}_{F_f}}$.

Proof of Proposition 5.14. — Let I_ℓ denote the inertia group at ℓ. For every $g \in \mathcal{N}$, ordinary at ℓ, we have by (5.12) and Theorem 3.26 (2) in [27] that

$$\rho_{F_g}|_{I_\ell} \cong \begin{bmatrix} \epsilon^{k-2} & * & & \\ & 1 & & \\ & & \epsilon^{k-1} & * \\ & & & \epsilon \end{bmatrix}.$$

If $\ell \nmid (k-1)(k-2)(k-3)$ it is easy to see that there exists $\sigma \in I_\ell$ such that the elements $\beta_1 := \epsilon^{k-2}(\sigma)$, $\beta_2 := 1$, $\beta_3 := \epsilon^{k-1}(\sigma)$, $\beta_4 := \epsilon(\sigma)$ are all distinct mod λ. For every g as above, we choose a basis of the space of ρ_g so that ρ_g is $\mathcal{O}$-valued and $\rho_{F_g}(\sigma) = \mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_4)$. Let S be the set consisting of the places of K lying over ℓ and the place $(i+1)$. Note that we can treat ρ_{F_g} as a representation of $G_{K,S}$, the Galois group of the maximal Galois extension of K unramified away from S. Moreover, $\mathrm{tr}\,\rho_{F_g}(G_{K,S}) \subset R$, since $G_{K,S}$ is generated by conjugates of $\mathrm{Frob}_{\mathfrak{p}}$, $\mathfrak{p} \notin S$ and for such a $\mathfrak{p}$, $\mathrm{tr}\,\rho_{F_g}(\mathrm{Frob}_{\mathfrak{p}}) \in R$ by Theorem 9.2 (i) and the fact that the coefficients of the characteristic polynomial of $\rho_{F_g}(\mathrm{Frob}_{\mathfrak{p}})$ belong to $\mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$. Set

$$e_j = \prod_{l\neq j} \frac{\sigma - \beta_l}{\beta_j - \beta_l} \in \mathcal{O}[G_{K,S}] \hookrightarrow R[G_{K,S}]$$

and $e := e_1 + e_2$. Let

$$\rho := \prod_{F_g\in\mathcal{N}^{\mathrm{M}}_{F_f}} \rho_{F_g} : G_{K,S} \to \prod_{F_g\in\mathcal{N}^{\mathrm{M}}_{F_f}} \mathrm{GL}_4(\mathcal{O}).$$

We extend ρ to an R-algebra map $\rho' : R[G_{K,S}] \to M_4(R')$. Note that

$$\rho'(\mathrm{Frob}_\pi\, e) = \prod_{F_g\in\mathcal{N}^{\mathrm{M}}_{F_f}} \rho_{F_g}(\mathrm{Frob}_\pi)\rho'_{F_g}(e) = \prod_{F_g\in\mathcal{N}^{\mathrm{M}}_{F_f}} \rho_g(\mathrm{Frob}_\pi)$$

and thus

$$\mathrm{tr}\,\rho'(\mathrm{Frob}_\pi\, e) = (a_g(p))_{F_g\in\mathcal{N}^{\mathrm{M}}_{F_f}} \in R,$$

where $g = \sum_{n=1}^{\infty} a_g(n)q^n$. Define $T^{\mathrm{M}}(p)$ to be the image of $\mathrm{tr}\,\rho'(\mathrm{Frob}_\pi\, e)$ under the $\mathcal{O}$-algebra isomorphism $R \xrightarrow{\sim} \mathbf{T}^{\mathrm{M}}_{\mathcal{O},\mathfrak{m}^{\mathrm{M}}_{F_f}}$. □

Corollary 5.15. — *If* $f \in \mathcal{N}$, $f \neq f^\rho$ *is ordinary at* ℓ, *and* $\ell \nmid (k-1)(k-2)(k-3)$, *then for every split prime* $p = \pi\overline{\pi}$, $p \nmid \ell$, *there exists* $T^{\mathrm{h}}(p) \in \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}}$ *such that* $\mathrm{Desc}(T^{\mathrm{h}}(p)) \in \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$ *equals the image of* $T_p \in \mathbf{T}'_{\mathcal{O}}$ *under the canonical projection* $\mathbf{T}'_{\mathcal{O}} \twoheadrightarrow \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$.

6. The standard L-function of a Maass lift

Let F_f be the Maass lift of $f \in \mathcal{N}$. The goal of this section is to study the numerator of the coefficient C_{F_f} in formula (1.1). To do so we need to define the cusp form Ξ in (1.1). This will be done in subsection 7.3 (formula (7.15)). In this section we define an Eisenstein series $E(Z,s,m,\Gamma^{\mathrm{h}})$ and a theta series θ_χ such that their product is closely related to Ξ. We then express the inner product $\langle F_f, E(Z,s,m,\Gamma^{\mathrm{h}})\theta_\chi \rangle$ by an L-function associated to f.

We begin by defining the appropriate theta series which will be used in the inner product. Let $\mathfrak{f}$ be an ideal of $\mathcal{O}_K$ and χ a Hecke character of K with conductor $\mathfrak{f}$. We assume that the infinity component of χ has the form

$$\chi_\infty(x_\infty) = \frac{|x_\infty|^t}{x_\infty^t},$$

for some integer $-k \leqslant t < -6$. Following [50] we fix a Hecke character ϕ of K such that

$$\phi_\infty(y_\infty) = \frac{|y_\infty|}{y_\infty} \quad \text{and} \quad \phi|_{\mathbf{A}^\times} = \left(\frac{-4}{\cdot}\right).$$

Such a character always exists, but is not unique (cf. [51], lemma A.5.1). Put $\psi' = \chi^{-1}\phi^{-2}$. Let $l = t+k+2$ and $\mu = l-2$. Let $\tau \in \mathcal{S}$ be such that the Fourier coefficient $c_{F_f}(\tau)$ is non-zero. Let $b \in \mathbf{Q}$ be such that $g^*\,\tau\,g \in b\mathbf{Z}$ for all $g \in \mathcal{O}_K^2$, and let $c' \in \mathbf{Z}$ be such that $g^*\,\tau^{-1}\,g \in (c')^{-1}\mathbf{Z}$ for all $g \in \mathcal{O}_K^2$. Let $c \in \mathbf{Z}$ be such that bc generates the $\mathbf{Z}$-fractional ideal $(4c')N_{K/\mathbf{Q}}(\mathfrak{f}) \cap (b)\mathfrak{f}$. Note that when $b=1$, $(c) = (4c'N_{K/\mathbf{Q}}(\mathfrak{f}))$.

Define a Schwartz function $\lambda : M_2(\mathbf{A}_{K,\mathrm{f}}) \to \mathbf{C}$ by setting $\lambda(x) = \chi_{\mathfrak{f}}(\det x)$ if $x \in \prod_{\mathfrak{p}\nmid\infty} M_2(\mathcal{O}_{K,\mathfrak{p}})$ and $\lambda(x) = 0$ otherwise. Then the theta series of our interest is defined by:

$$\theta_\chi(Z) = \sum_{\xi \in M_2(K)} \lambda(\xi)\,(\overline{\det \xi})^\mu\, e(\mathrm{tr}\,(\xi^*\,\tau\xi Z)).$$

We have $\theta_\chi \in \mathcal{M}_l(\Gamma_0^{\mathrm{h}}(b,c),\psi')$ by [50], appendix, Proposition 7.16 and [51], page 278. In fact, since $\mu \neq 0$, θ_χ is cusp form ([50], appendix, page 277). In this section we will denote by Γ_1^{h} a congruence subgroup of $\Gamma_{\mathbf{Z}}$ such that

$\theta_\chi \in \mathcal{M}_l(\Gamma_1^{\mathrm{h}})$ and $\Gamma_1^{\mathrm{h}} \cap K^\times = \{1\}$. We set $\Gamma^{\mathrm{h}} := \Gamma_1^{\mathrm{h}} \cap G_1(\mathbf{Q})$. Note that we have $F_f \in \mathcal{M}_k(\Gamma^{\mathrm{h}})$. We also define an Eisenstein series of weight $m = k - l$ and level Γ^{h} by putting:

$$E(Z, s, m, \Gamma^{\mathrm{h}}) = \sum_{\gamma \in \Gamma^{\mathrm{h}} \cap P(\mathbf{Q}) \backslash \Gamma^{\mathrm{h}}} (\det \operatorname{Im} Z)^{s - \frac{m}{2}} |_m \gamma.$$

The Petersson inner product of F_f against $E(\cdot, s, m, \Gamma^{\mathrm{h}})\, \theta_\chi$ has the form

$$\left\langle F_f, E(\cdot, s, m, \Gamma^{\mathrm{h}})\, \theta_\chi) \right\rangle_{\Gamma^{\mathrm{h}}} = \\ = \int_{\Gamma^{\mathrm{h}} \backslash \mathcal{H}} F_f(Z)\, \overline{E(Z, s, m, \Gamma^{\mathrm{h}})}\, \overline{\theta_\chi(Z)} (\det \operatorname{Im} Z)^{k-4} dX dY.$$

Note that we use a volume form, which is 4 times the volume form used in [51]. By combining formulas (22.9), (22.18b) and (20.19) from [51] we arrive at the following:

$$\begin{aligned} &\left\langle F_f, E(\cdot, \overline{s}, m, \Gamma^{\mathrm{h}})\, \theta_\chi) \right\rangle_{\Gamma^{\mathrm{h}}} = \\ &= 64[\Gamma_0^{\mathrm{h}}(c) : \Gamma_1^{\mathrm{h}}(c)] b^{-4} \Gamma((s-2))(\det \tau)^{-s - \frac{1}{2}(k+l)+2} \times \\ &\times \frac{c_{F_f}(\tau) L_{\mathrm{st}}(F_f, s+1, \chi)}{B(s)\, L_c(2s, \chi_{\mathbf{Q}})\, L_c\left(2s-1, \chi_{\mathbf{Q}} \left(\frac{-4}{\cdot}\right)\right)}. \end{aligned} \tag{6.1}$$

The meaning of the various factors in the product is explained below. We start with the L-function

$$L_{\mathrm{st}}(F_f, s, \chi) = \prod_{p \nmid \infty} L_{\mathrm{st}}(F_f, s, \chi)_p.$$

This is the standard L-function of F_f twisted by the Hecke character χ:

$$L_{\mathrm{st}}(F_f, s, \chi)_p = \tag{6.2}$$

$$= \begin{cases} \prod_{j=1}^{4} \left\{(1 - N(\mathfrak{p})^4 \lambda_{p,j}^{-1} \chi^*(\mathfrak{p}) N(\mathfrak{p})^{-s})(1 - N(\overline{\mathfrak{p}})^4 \lambda_{p,j} \chi^*(\overline{\mathfrak{p}}) N(\overline{\mathfrak{p}})^{-s})\right\}^{-1} \\ \prod_{j=1}^{2} \left\{(1 - N(\mathfrak{p})^2 \lambda_{p,j}^{-1} \chi^*(\mathfrak{p}) N(\mathfrak{p})^{-s})(1 - N(\mathfrak{p}) \lambda_{p,j} \chi^*(\mathfrak{p}) N(\mathfrak{p})^{-s})\right\}^{-1} \end{cases},$$

for $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ and $(p) = \mathfrak{p}^e$, respectively. Here $\lambda_{p,i}$ denote the p-Satake parameters of F_f. (For the definition of p-Satake parameters when p inerts or ramifies in K, see [29], and for the case when p splits in K, see [24].) The L-function in the denominator of (6.1) is the Dirichlet L-function with Euler factors at all $p \mid c$ removed (cf. Definition 7.2). Furthermore,

$$\Gamma((s)) = (4\pi)^{-2s-k-l+1}\, \Gamma\left(s + \frac{1}{2}(k+l)\right) \Gamma\left(s + \frac{1}{2}(k+l) - 1\right),$$

and $B(s) = \prod_{v \in \mathbf{b}} g_p(\chi^*(p\mathcal{O}_K)p^{-2s})$, where $\mathbf{b}$ denotes the set of primes at which $b^{-1}\tau$ is not regular in the sense of ([51], 16.1) and g_p is a polynomial with coefficients in $\mathbf{Z}$ and constant term 1.

For a prime $\mathfrak{p}$ of $\mathcal{O}_K$ of residue characteristic p, with p odd, set $\alpha_{\mathfrak{p},j} := \alpha_{p,j}^d$, where $\alpha_{p,j}$, $j = 1, 2$ denote the p-Satake parameters of f (cf. section 4.2), and d is the degree over $\mathbf{F}_p$ of the field $\mathcal{O}_K/\mathfrak{p}$. For the prime $\mathfrak{p} = (i+1)$ of $\mathcal{O}_K$, set $\alpha_{\mathfrak{p},1} := a(2)$ and $\alpha_{\mathfrak{p},2} := \overline{a(2)}$.

Definition 6.1. — *For a Hecke character ψ of K, set*

$$L(\mathrm{BC}(f), s, \psi) := \prod_{\mathfrak{p}\nmid\infty} \prod_{j=1}^{2} (1 - \psi^*(\mathfrak{p})\alpha_{\mathfrak{p},j}(N\mathfrak{p})^{-s})^{-1}.$$

Remark 6.2. — If π_f denotes the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated with f, then $L(\mathrm{BC}(f), s, \psi)$ is the classical analogue of the L-function attached to the base change of π_f to K twisted by ψ.

Remark 6.3. — Let g_ψ be the modular form associated with the character ψ (cf. [31], section 12.3) and suppose that $\psi_\infty(x_\infty) = \left(\frac{x_\infty}{|x_\infty|}\right)^u$. Then

$$L(\mathrm{BC}(f), s, \psi) = (1 - \psi^*(\mathfrak{p})\overline{a(2)}2^{-s})^{-1} D(s + u/2, f^\rho, g_\psi),$$

where $D(s, \cdot, \cdot)$ denotes the convolution L-function defined in [26], where it is denoted by $L(\lambda_{f^\rho} \otimes \lambda_{g_\psi}, s)$. Here $\mathfrak{p}$ denotes the prime of $\mathcal{O}_K$ lying over (2).

Proposition 6.4. — *Let χ be as before. The following identity holds*

$$L_{\mathrm{st}}(F_f, s, \chi) = L(\mathrm{BC}(f), s - 2 + k/2, \omega\chi) L(\mathrm{BC}(f), s - 3 + k/2, \omega\chi). \tag{6.3}$$

Here ω is the unique Hecke character of K unramified at all finite places with infinity type $\omega_\infty(z) = \left(\frac{z}{\bar{z}}\right)^{-k/2}$.

Proof. — This is a straightforward calculation on the Satake parameters of f and of F_f. □

7. Congruence

In this section we define a hermitian modular form Ξ as in (1.1) and formulate the main congruence result (Theorem 7.12). The form Ξ will be constructed (in section 7.3) as a combination of a certain Eisenstein series and a theta series, whose arithmetic properties are studied below.

7.1. Fourier coefficients of Eisenstein series

We keep the notation from section 6 and assume $b = 1$. Consider the set $X_{m,c}$ of Hecke characters χ' of K, such that

$$\chi'_\infty(x_\infty) = \frac{x_\infty^m}{|x_\infty|^m}, \tag{7.1}$$

$$\chi'_p(x_p) = 1 \text{ if } p \nmid \infty,\ x_p \in \mathcal{O}_{K,p}^\times \text{ and } x_p - 1 \in c\mathcal{O}_{K,p}. \tag{7.2}$$

Here $m = k - l = -t - 2 > 0$ (since $t < -6$) denotes the weight of the Eisenstein series $E(Z, s, m, \Gamma^{\mathrm{h}})$ defined in section 6. For $g \in G(\mathbf{A})$, let $E(g, s, c, m, \chi')$ denote the Siegel Eisenstein series defined in section 3.2. We put, as before,

$$E(Z, s, m, \chi', c) = j(g_\infty, \mathbf{i})^m E(g, s, c, m, \chi'),$$

where $Z = g_\infty \mathbf{i}$ and $g = (g_\infty, 1)$. Recall that in section 6 we made use of a congruence subgroup Γ_1^{h} of $G(\mathbf{Q})$ such that $\theta_\chi \in \mathcal{M}_l(\Gamma_1^{\mathrm{h}})$ and $\Gamma_1^{\mathrm{h}} \cap K^\times = \{1\}$. In this section we fix a particular choice of Γ_1^{h}, namely, we set $\Gamma_1^{\mathrm{h}} := \Gamma_1^{\mathrm{h}}(c)$. Note that as long as $c \nmid 2$, we have $\Gamma_1^{\mathrm{h}}(c) \cap K^\times = \{1\}$ and since $(\operatorname{cond}\psi') \mid c$, where ψ' is the character of θ_χ, we have $\theta_\chi \in \mathcal{M}_l(\Gamma_1^{\mathrm{h}}(c))$. The following lemma provides a connection between $E(Z, s, m, \chi', c)$ and $E(Z, s, m, \Gamma_1^{\mathrm{h}}(c))$. Here $E(Z, s, m, \Gamma_1^{\mathrm{h}}(c))$ is defined in the same way as $E(Z, s, m, \Gamma^{\mathrm{h}})$ in section 6. Recall that $\Gamma^{\mathrm{h}} := \Gamma_1^{\mathrm{h}} \cap G_1(\mathbf{Q})$.

Lemma 7.1. — *The set $X_{m,c}$ is non-empty and*

$$(\# X_{m,c})\, E(Z, s, m, \Gamma_1^{\mathrm{h}}(c)) = \sum_{\chi' \in X} E(Z, s, m, \chi', c).$$

Proof. — This is identical to the proof of Lemma 17.2 in [51]. Note that $\Gamma_1^{\mathrm{h}}(c) \supset \Gamma^{\mathrm{h}}(c)$. ☐

Definition 7.2. — *Let M be a non-zero integer. For a Hecke character ψ of $\mathbf{Q}$ set*

$$L_M(s, \psi) := L(s, \psi) \prod_{p \mid M} (1 - \psi^*(p) p^{-s}),$$

where $L(s, \psi)$ denotes the Dirichlet L-function.

Recall that for any Hecke character $\psi : K^\times \setminus \mathbf{A}_K^\times \to \mathbf{C}^\times$ we denote by $\psi_{\mathbf{Q}}$ its restriction to $\mathbf{A}^\times$. Moreover, if ψ satisfies (7.1) and (7.2) for $c \in \mathbf{Z}$, set $\psi^{\mathrm{c}}(x) = \psi(\overline{x})$. Let

$$D(Z, s, m, \chi', c) = L_c(2s, \chi'_{\mathbf{Q}}) L_c\left(2s - 1, \chi'_{\mathbf{Q}}\left(\frac{-4}{\cdot}\right)\right) E(Z, s, m, \chi', c). \tag{7.3}$$

It has been shown in [51] (Theorem 17.12(iii)) that $D(Z,s,m,\chi',c)$ is holomorphic in the variable Z for $s = 2 - \frac{m}{2}$ as long as $m \geqslant 2$. In our case $m = -t-2 > 4$ as $t < -6$. It follows from formula (18.6.2) in [50] that

$$(7.4) \quad D(Z,s,m,\chi',c)|_m\gamma = (\chi'_c)(\det d_\gamma)D(Z,s,m,\chi',c) = \\ = ((\chi')^{\mathrm{c}})_c^{-1}(\det a_\gamma)D(Z,s,m,\chi',c).$$

Instead of looking at $D(Z,s,m,\chi',c)$ we will study the Fourier expansion of a transform $D^*(Z,s,m,\chi',c)$ defined by

$$(7.5) \qquad D^*(Z,s,m,\chi',c) = D(Z,s,m,\chi',c)|_m J.$$

First note that since D is holomorphic at $s = 2 - \frac{m}{2}$, so is D^*. Write

$$D^*(Z,2-m/2,m,\chi',c) = \sum_{h\in S} c_h^{\chi'} e(\operatorname{tr} hZ)$$

for the Fourier expansion of D^*. Here $S := \{h \in M_2(K) \mid h^* = h\}$.

Lemma 7.3. —

$$(7.6) \quad c_h^{\chi'} = i^{-2m}2^{2m+1}\pi^3c^2 \times \\ \times \prod_{j=0}^{1-\operatorname{rank}(h)} L_c\left(2-m-j, \chi'\left(\frac{-4}{\cdot}\right)^{j-1}\right) \prod_{p\in\mathbf{c}} f_{h,Y^{1/2},p}(\chi'(p)p^{m-4}),$$

where $f_{h,Y^{1/2},p}$ is a polynomial with coefficients in $\mathbf{Z}$ and constant term 1, and $\mathbf{c}$ is a certain finite set of primes. If $n < 1$ we set $\prod_{j=0}^n = 1$.

Proof. — The lemma follows from Propositions 18.14 and 19.2 in [50], combined with Lemma 18.7 of [50] and formulas (4.34K) and (4.35K) in [49]. It is a straightforward calculation. □

Proposition 7.4. — *Fix a prime $\ell \nmid 2c$, and assume that $-k \leqslant t < -6$. Set $\chi_{\mathbf{Q},c} := \prod_{p|c}\chi_{\mathbf{Q},p}$. Let E' be a finite extension of $\mathbf{Q}_\ell$ containing $K(\chi_{\mathbf{Q},c})$, the finite extension of K generated by the values of $\chi_{\mathbf{Q},c}$. Denote by $\mathcal{O}'$ the valuation ring of E'. For every $h \in S$, we have $\pi^{-3}c_h^{\chi_{\mathbf{Q}}} \in \mathcal{O}'$.*

Proof. — The proposition follows from Lemma 7.3 upon noting that for every Dirichlet character ψ of conductor dividing c and every $n \in \mathbf{Z}_{<0}$, one has $L(n,\psi) \in \mathbf{Z}_\ell[\psi]$ (by a simple argument using [59], Corollary 5.13) and $(1-\psi(p)p^{-n}) \in \mathbf{Z}_\ell[\psi]$ for every $p \mid c$. □

Let θ_χ be as above. Set $\theta_\chi^* := \theta_\chi|_l J$.

Corollary 7.5. — *Fix a prime $\ell \nmid 2c$, and assume that $-k \leqslant t < -6$. Let E' be a finite extension of $\mathbf{Q}_\ell$ containing $K(\chi_{\mathbf{Q},c},\mu_c)$, where μ_c denotes* *the set of c-th roots of* 1*. Denote by $\mathcal{O}'$ the valuation ring of E'. Then the Fourier coefficients of*

$$\pi^{-3} D^*(Z, 2-m/2, m, (\psi')^{\mathrm{c}}, c)\, \theta^*_\chi(Z)$$

all lie in $\mathcal{O}'$.

Proof. — Note that it follows from the definition of θ_χ and Theorem 2.2 that the Fourier coefficients of $\theta^*_\chi(Z)$ lie in $\mathcal{O}$. Thus the Corollary is a consequence of Proposition 7.4. □

7.2. Some formulae

We keep notation from the previous section. Note that since $\theta_\chi \in \mathcal{M}_l(c, \psi')$, we have $D(Z, 2-m/2, m, (\psi')^{\mathrm{c}}, c)\, \theta_\chi(Z) \in \mathcal{M}_k(c)$ by (7.4). For $f \in \mathcal{N}$ we can write

$$(7.7)\quad D(Z, 2-m/2, m, (\psi')^{\mathrm{c}}, c)\, \theta_\chi(Z) = \\ = \frac{\langle D(\cdot, 2-m/2, m, (\psi')^{\mathrm{c}}, c)\, \theta_\chi, F_f\rangle_{\Gamma_0^{\mathrm{h}}(c)}}{\langle F_f, F_f\rangle_{\Gamma_0^{\mathrm{h}}(c)}} F_f + F',$$

where $F' \in \mathcal{M}_k(c)$ and $\langle F_f, F'\rangle = 0$. Our goal now is to express

$$(7.8)\qquad \langle D(\cdot, 2-m/2, m, (\psi')^{\mathrm{c}}, c)\, \theta_\chi, F_f\rangle_{\Gamma_0^{\mathrm{h}}(c)}$$

in terms of L-functions of f. In section 6 we already carried out this task for the inner product $\left\langle F_f, E(\cdot, s, m, \Gamma^{\mathrm{h}})\, \theta_\chi\right\rangle_{\Gamma^{\mathrm{h}}}$ with $\Gamma^{\mathrm{h}} = \Gamma_1^{\mathrm{h}}(c) \cap G_1(\mathbf{Q})$, so we will now relate the two inner products to each other. We first relate the inner product (7.8) to $\left\langle F_f, E(\cdot, s, m, \Gamma_1^{\mathrm{h}}(c))\, \theta_\chi\right\rangle_{\Gamma_1^{\mathrm{h}}(c)}$.

We have

$$(7.9)\quad \left\langle E(\cdot, s, m, \Gamma_1^{\mathrm{h}}(c))\, \theta_\chi, F_f\right\rangle_{\Gamma_1^{\mathrm{h}}(c)} = \\ \int_{\Gamma_1^{\mathrm{h}}(c)\backslash\mathcal{H}} E(Z, s, m, \Gamma_1^{\mathrm{h}}(c))\, \theta_\chi(Z)\, \overline{F_f(Z)}\, (\det Y)^{k-4}\, dX\, dY = \\ = \int_{\Gamma_0^{\mathrm{h}}(c)\backslash\mathcal{H}} \theta_\chi(Z) \left(\sum_{\gamma \in \Gamma_1^{\mathrm{h}}(c)\backslash\Gamma_0^{\mathrm{h}}(c)} \psi'_c(\det a_\gamma)\, E(Z, s, m, \Gamma_1^{\mathrm{h}}(c))|_m \gamma \right) \overline{F_f(Z)}\, dX\, dY.$$

Using Lemma 7.1 (note that $(\psi')^{\mathrm{c}} \in X_{m,c}$) we get

$$\begin{aligned}
(7.10)\qquad & \sum_{\gamma\in\Gamma_1^{\mathrm{h}}(c)\backslash\Gamma_0^{\mathrm{h}}(c)} \psi'_c(\det a_\gamma)\, E(Z,s,m,\Gamma_1^{\mathrm{h}}(c))|_m\gamma = \\
&= (\#X_{m,c})^{-1} \sum_{\chi'\in X} \sum_{\gamma\in\Gamma_1^{\mathrm{h}}(c)\backslash\Gamma_0^{\mathrm{h}}(c)} \psi'_c(\det a_\gamma)\, E(Z,s,m,\chi',c)|_m\gamma = \\
&= (\#X_{m,c})^{-1} \sum_{\chi'\in X} E(Z,s,m,\chi',c) \sum_{\gamma\in\Gamma_1^{\mathrm{h}}(c)\backslash\Gamma_0^{\mathrm{h}}(c)} (\psi'((\chi')^{\mathrm{c}})^{-1})_c(\det a_\gamma) = \\
&= (\#X_{m,c})^{-1}[\Gamma_0^{\mathrm{h}}(c):\Gamma_1^{\mathrm{h}}(c)]E(Z,s,m,(\psi')^{\mathrm{c}},c),
\end{aligned}$$

where the last equality follows from the orthogonality relation for characters upon noting that both ψ' and χ' are trivial on $\Gamma_1^{\mathrm{h}}(c)$. Thus (7.3), (7.9) and (7.10) imply that

$$\begin{aligned}
(7.11)\qquad & \\
\langle D(\cdot,s,m,(\psi')^{\mathrm{c}},c)\theta_\chi, F_f\rangle_{\Gamma_0^{\mathrm{h}}(c)} &= [\Gamma_0^{\mathrm{h}}(c):\Gamma_1^{\mathrm{h}}(c)]^{-1}\,\#X_{m,c}\cdot L_c(2s,\psi'_{\mathbf{Q}})\times \\
&\times L_c\left(2s-1,\psi'_{\mathbf{Q}}\left(\frac{-4}{\cdot}\right)\right)\left\langle E(\cdot,s,m,\Gamma_1^{\mathrm{h}}(c))\theta_\chi, F_f\right\rangle_{\Gamma_1^{\mathrm{h}}(c)}.
\end{aligned}$$

Moreover, by [51], formula (17.5) and Remark 17.12(2), we have

$$E(Z,s,m,\Gamma_1^{\mathrm{h}}(c)) = \frac{1}{[\Gamma_1^{\mathrm{h}}(c):\Gamma^{\mathrm{h}}]} \sum_{\alpha\in\Gamma^{\mathrm{h}}\backslash\Gamma_1^{\mathrm{h}}(c)} E(Z,s,m,\Gamma^{\mathrm{h}})|_m\alpha.$$

Hence we get

$$\begin{aligned}
(7.12)\qquad & \\
\langle D(\cdot,s,m,(\psi')^{\mathrm{c}},c)\theta_\chi, F_f\rangle_{\Gamma_0^{\mathrm{h}}(c)} &= [\Gamma_0^{\mathrm{h}}(c):\Gamma^{\mathrm{h}}]^{-1}\,\#X_{m,c}\cdot L_c(2s,\psi'_{\mathbf{Q}})\times \\
&\times L_c\left(2s-1,\psi'_{\mathbf{Q}}\left(\frac{-4}{\cdot}\right)\right)\left\langle E(\cdot,s,m,\Gamma^{\mathrm{h}})\theta_\chi, F_f\right\rangle_{\Gamma^{\mathrm{h}}}.
\end{aligned}$$

Using (6.1) we obtain

$$\begin{aligned}
& \langle D(\cdot,s,m,(\psi')^{\mathrm{c}},c)\theta_\chi, F_f\rangle_{\Gamma_0^{\mathrm{h}}(c)} \\
&\quad = 16\pi\overline{((4\pi)^{-2s'})}(\det\tau)^{-s'}B(\overline{s})^{-1}\cdot\#X_{m,c}\times \\
(7.13)\qquad &\qquad \times L_c(2s,\psi'_{\mathbf{Q}})L_c\left(2s-1,\psi'_{\mathbf{Q}}\left(\frac{-4}{\cdot}\right)\right)\times \\
&\qquad \times\overline{\Gamma(s')}\,\overline{\Gamma(s'-1)}\,\frac{\overline{c_{F_f}(\tau)L_{\mathrm{st}}(F_f,\overline{s}+1,\chi)}}{\overline{L_c(2\overline{s},\chi_{\mathbf{Q}})L_c\left(2\overline{s}-1,\chi_{\mathbf{Q}}\left(\frac{-4}{\cdot}\right)\right)}},
\end{aligned}$$

where $s' := \overline{s} + k - 1 + t/2$, and finally

$$(7.14)\quad \langle D(\cdot, 2-m/2, m, (\psi')^{\mathrm{c}}, c)\theta_\chi, F_f\rangle_{\Gamma_0^{\mathrm{h}}(c)} = \\ = R\pi^{-2t-2k-3}\Gamma(t+k+2)\Gamma(t+k+1) \times \overline{L_{\mathrm{st}}(F, 3-m/2, \chi)},$$

where $R := \# X_{m,c} \cdot 2^{-4(t+k+1)}\overline{c_{F_f}(\tau)}B(2-m/2)^{-1}\,(\det\tau)^{-t-k-2}$.

7.3. Main congruence result

We will now prove the first main result of this paper. We will show that λ^n-divisibility of the algebraic part of $L(\mathrm{Symm}^2 f, k)$ implies the existence of a non-Maass cusp form congruent to F_f modulo λ^n. We keep notation from previous sections.

7.3.1. Algebraicity of C_{F_f}

Let $\ell \nmid 2c$ be a rational prime, and let E be a finite extension of $\mathbf{Q}_\ell$ with valuation ring $\mathcal{O}$. We will always assume that E is "sufficiently large" in the sense that it contains certain algebraic numbers/number fields, which will be specified later. In particular, we assume that E contains the field E' of Corollary 7.5. Fix a uniformizer $\lambda \in \mathcal{O}$. We denote the λ-adic valuation by ord_λ. To shorten notation in this section we set $D(Z) := D(Z, 2 - m/2, m, (\psi')^{\mathrm{c}}, c)$, and $D^*(Z) = D^*(Z, 2-m/2, m, (\psi')^{\mathrm{c}}, c)$. Applying the operator $|_k J$ to both sides of (7.7), we get

$$D^*\theta^*_\chi = \frac{\langle D\theta_\chi, F_f\rangle}{\langle F_f, F_f\rangle}F_f + G' \in \mathcal{M}_k(J^{-1}\Gamma_0^{\mathrm{h}}(c)J)$$

where $G' := F'|_k J$ and we have $\langle F_f, G'\rangle = 0$. By Corollary 7.5, the Fourier coefficients of $\pi^{-3}D^*\theta^*_\chi$ lie in $\mathcal{O}$. Define a trace operator

$$\mathrm{tr}\,:\mathcal{M}_k(J^{-1}\Gamma_0^{\mathrm{h}}(c)J) \to \mathcal{M}_k(\Gamma_{\mathbf{Z}})$$

by

$$F' \mapsto \sum_{\gamma \in J^{-1}\Gamma_0^{\mathrm{h}}(c)J\backslash\Gamma_{\mathbf{Z}}} F'|_k\gamma$$

and set

$$(7.15)\qquad \Xi := \pi^{-3}\mathrm{tr}\,(D^*\theta^*_\chi) = [\Gamma_{\mathbf{Z}} : \Gamma_0^{\mathrm{h}}(c)]\pi^{-3}\frac{\langle D\theta_\chi, F_f\rangle}{\langle F_f, F_f\rangle}\,F_f + G'',$$

where $G'' = \pi^{-3}\mathrm{tr}\, G' \in \mathcal{M}_k(\Gamma_{\mathbf{Z}})$ and we have $\langle F_f, G''\rangle = 0$. By the q-expansion principle (Theorem 2.2), the Fourier coefficients of Ξ lie in $\mathcal{O}$. Set

$$C_{F_f} := [\Gamma_{\mathbf{Z}} : \Gamma_0^{\mathrm{h}}(c)]\pi^{-3}\frac{\langle D\theta_\chi, F_f\rangle}{\langle F_f, F_f\rangle}.$$

By Proposition 4.5, the Fourier coefficients of F_f lie in the ring of integers $\overline{\mathbf{Z}}_\ell$ of $\overline{\mathbf{Q}}_\ell$ and generate a finite extension of $\mathbf{Q}_\ell$. We assume E contains all the Fourier coefficients of F_f. The numerator and denominator of $\frac{\langle D\theta_\chi, F_f\rangle}{\langle F_f, F_f\rangle}$ were studied in sections 7.2 and 4.2 respectively.

Lemma 7.6. —

$$\frac{\langle D\theta_\chi, F_f\rangle}{\langle F_f, F_f\rangle} = (*)\,\alpha\,\frac{L^{\mathrm{alg}}(\mathrm{BC}(f), 1+\frac{t+k}{2}, \overline{\chi\omega})L^{\mathrm{alg}}(\mathrm{BC}(f), 2+\frac{t+k}{2}, \overline{\chi\omega})}{L^{\mathrm{alg}}(\mathrm{Symm}^2 f, k)},$$

where

$$\alpha := \#X_{m,c}\cdot B(2-m/2)^{-1}(\det\tau)^{-k-t-2}\pi^3\overline{c_{F_f}(\tau)},$$

$$L^{\mathrm{alg}}(\mathrm{BC}(f), j+(t+k)/2, \overline{\chi\omega}) := \frac{\Gamma(t+k+j)L(\mathrm{BC}(f), j+\frac{t+k}{2}, \overline{\chi\omega})}{\pi^{t+k+2j}\langle f, f\rangle},$$

$$L^{\mathrm{alg}}(\mathrm{Symm}^2 f, n) := \frac{\Gamma(n)L(\mathrm{Symm}^2 f, n)}{\pi^{n+2}\langle f, f\rangle}$$

for any integer n, and $(*) \in \overline{\mathbf{Q}}\cap E$ is a λ-adic unit.

Proof. — This is a straightforward calculation using (7.14), Proposition 6.4 and Theorem 4.8. $\square$

It follows from Remark 6.3 and from Theorem 1 on page 325 in [26] that

$$L^{\mathrm{alg}}(\mathrm{BC}(f), 1+(t+k)/2, \overline{\chi\omega}) \in \overline{\mathbf{Q}} \tag{7.16}$$

and

$$L^{\mathrm{alg}}(\mathrm{BC}(f), 2+(t+k)/2, \overline{\chi\omega}) \in \overline{\mathbf{Q}} \tag{7.17}$$

and from a result of Sturm [53] that

$$L^{\mathrm{alg}}(\mathrm{Symm}^2 f, k) \in \overline{\mathbf{Q}}. \tag{7.18}$$

We note here that [53] uses a definition of the Petersson norm of f which differs from ours by a factor of $\frac{3}{\pi}$, the volume of the fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on the complex upper half-plane. We assume that E contains values (7.16), (7.17), and (7.18).

Corollary 7.7. — $C_{F_f} \in \overline{\mathbf{Q}}\cap E$.

As we are ultimately interested in (mod λ) congruences between hermitian modular forms, we will use "integral periods" Ω_f^+, Ω_f^- instead of $\langle f,f\rangle$ (cf. section 8.3). It follows from Proposition 8.15 in section 8.3 that we have:

$$\langle f,f\rangle = (*)\,\eta\,\Omega_f^+\Omega_f^-, \tag{7.19}$$

where $\eta \in \overline{\mathbf{Z}}_\ell$ is defined in section 8.3 and $(*)$ is a λ-adic unit as long as f is ordinary at ℓ and $\ell > k$, which we assume in what follows. We also assume that E contains η and that $\ell \nmid \# X_{m,c}$.

Corollary 7.8. —

$$\begin{aligned} C_{F_f} &= (*)\,\overline{c_{F_f}(\tau)}\,\eta^{-1}\times \\ &\times \frac{L^{\mathrm{int}}(\mathrm{BC}(f),2+(t+k)/2,\overline{\chi\omega})L^{\mathrm{int}}(\mathrm{BC}(f),1+(t+k)/2,\overline{\chi\omega})}{L^{\mathrm{int}}(\mathrm{Symm}^2 f,k)}, \end{aligned} \tag{7.20}$$

where

$$L^{\mathrm{int}}(\mathrm{BC}(f),j+(t+k)/2,\overline{\chi\omega}) := \frac{\Gamma(t+k+j)L(\mathrm{BC}(f),j+(t+k)/2,\overline{\chi\omega})}{\pi^{t+k+2j}\,\Omega_f^+\Omega_f^-},$$

$$L^{\mathrm{int}}(\mathrm{Symm}^2 f,k) := \frac{\Gamma(k)L(\mathrm{Symm}^2 f,k)}{\pi^{k+2}\,\Omega_f^+\Omega_f^-},$$

and $(*) \in E$ *with* $\mathrm{ord}_\lambda((*)) \leqslant 0$.

Proof. — This follows directly from Lemma 7.6 upon noting that $\mathrm{ord}_\lambda(B(2-m/2)) \geqslant 0$ and $\mathrm{ord}_\lambda(\det\tau) \geqslant 0$. ☐

We are now going to show that we can choose τ to make $\overline{c_{F_f}(\tau)}$ in (7.20) a λ-adic unit. Since we have derived our formulas with the assumption $b=1$, where b is defined in section 6, we need to choose τ appropriately so this assumption remains valid. If τ is ℓ-ordinary in the sense of the following definition, then we can take $b=1$.

Definition 7.9. — *For a rational prime ℓ, we will say that $\tau \in \mathcal{S}$ is ℓ-ordinary if the following two conditions are simultaneously satisfied:*

- $\{g^*\tau g\}_{g\in\mathcal{O}_K^2} = \mathbf{Z}$
- *there exists $c' \in \mathbf{Z}$ with $(c',\ell)=1$ such that $\{g^*\tau^{-1}g\}_{g\in\mathcal{O}_K^2} \subset (c')^{-1}\mathbf{Z}$.*

Lemma 7.10. — *If $f \in \mathcal{N}$ is such that the Galois representation $\overline{\rho}_f|_{G_K}$ is absolutely irreducible, then there exists an ℓ-ordinary $\tau \in \mathcal{S}$ such that* $\mathrm{ord}_\lambda(\overline{c_{F_f}(\tau)}) = 0$.

Proof. — Write the Fourier expansion of f as $f = \sum_{n=1}^{\infty} b(n)q^n$. Since F_f is a Maass form, we have $c_{F_f}(\tau) = \sum_{d\in\mathbf{Z}_{>0},\, d\mid\epsilon(\tau)} d^{k-1} c^*_{F_f}(4\det\tau/d^2)$, where $c^*_{F_f}$ and $\epsilon(\cdot)$ were defined in Definition 4.2. Note that $\tau = \left[\begin{smallmatrix} n & \\ & 1\end{smallmatrix}\right]$ is ℓ-ordinary (with $c' = n$) for any positive integer n with $(n,\ell) = 1$. Using Proposition 4.5 and Fact 2.1 we get

$$\overline{c_{F_f}\left(\left[\begin{smallmatrix} 1 & \\ & 1\end{smallmatrix}\right]\right)} = -2i(b(2)^2 - \overline{b(2)}^2)$$

and

$$\overline{c_{F_f}\left(\left[\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right]\right)} = 2ib(p)(b(2)^2 + \overline{b(2)}^2)$$

if $p \neq \ell$ is inert in K. As will be shown in Proposition 8.13, absolute irreducibility of $\overline{\rho}_f|_{G_K}$ implies that there exists an inert prime p_0, distinct from ℓ, such that $b(p_0)$ is a λ-adic unit. Suppose now that both $\operatorname{ord}_\lambda\left(\overline{c_{F_f}\left(\left[\begin{smallmatrix} 1 & \\ & 1\end{smallmatrix}\right]\right)}\right) > 0$ and $\operatorname{ord}_\lambda\left(\overline{c_{F_f}\left(\left[\begin{smallmatrix} p_0 & \\ & 1\end{smallmatrix}\right]\right)}\right) > 0$. Then we must have $\operatorname{ord}_\lambda(b(2)) > 0$, which is impossible as ℓ is odd and $|b(2)| = 2^{(k-2)/2}$ (cf. [31], formula (6.90)). □

Definition 7.11. — *For $f \in \mathcal{N}$ such that the Galois representation $\overline{\rho}_f|_{G_K}$ is absolutely irreducible, let $S_{f,\ell}$ denote the set of positive integer n with $(n,\ell) = 1$ such that* $\operatorname{ord}_\lambda\left(\overline{c_{F_f}\left(\left[\begin{smallmatrix} n & \\ & 1\end{smallmatrix}\right]\right)}\right) = 0$. *By the proof of Lemma 7.10 the set $S_{f,\ell}$ is non-empty.*

7.3.2. Congruence between F_f and a non-Maass form

Our goal is to prove that F_f is congruent to a non-Maass form. Note that if $C_{F_f} = a\lambda^{-n}$, with $a \in \mathcal{O}^\times$ and $n > 0$, then F_f is congruent to $-a^{-1}\lambda^n G''$ mod λ^n. However, G'' need not a priori be orthogonal to the Maass space. We overcome this obstacle by introducing a certain Hecke operator $\tilde{T}^{\mathrm{h}}$ which will kill the "Maass part" of G''. For $g \in \mathcal{N}$ and $F \in \mathcal{N}^{\mathrm{h}}$, the set

$$\Sigma := \{\lambda_{g,\mathbf{C}}(T) \mid g \in \mathcal{N}, T \in \mathbf{T}_{\mathbf{Z}}\} \cup \{\lambda_{F,\mathbf{C}}(T) \mid F \in \mathcal{N}^{\mathrm{h}}, T \in \mathbf{T}^{\mathrm{h}}_{\mathbf{Z}}\}$$

is contained in the ring of integers of a finite extension of $\mathbf{Q}$ (cf. section 5.1 and Theorem 5.9). We assume that E contains Σ. From now on assume that the Galois representation $\overline{\rho}_f|_{G_K}$ is absolutely irreducible. Without loss of generality we also assume that $F_f \in \mathcal{N}^{\mathrm{h}}$ (cf. Remark 5.13). For any $F \in \mathcal{N}^{\mathrm{h}}$, let $\mathfrak{m}_F \subset \mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$ be the maximal ideal corresponding to F. It follows from (5.8) that there exists $T^{\mathrm{h}} \in \mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$ such that $T^{\mathrm{h}}F_f = F_f$ and $T^{\mathrm{h}}F = 0$ for all $F \in \mathcal{N}^{\mathrm{h}}$ such that $\mathfrak{m}_F \neq \mathfrak{m}_{F_f}$. We apply T^{h} to both sides of

$$\Xi = C_{F_f}F_f + G''.$$

As the Fourier coefficients of F_f and Ξ lie in $\mathcal{O}$, so do the Fourier coefficients of $T^{\mathrm{h}}\Xi$ by Lemma 5.8. Moreover, since θ_χ is a cusp form, so are Ξ and $T^{\mathrm{h}}\Xi$. Let $\mathcal{S}^{(2)}_{k,F_f} \subset \mathcal{S}_k(\Gamma_{\mathbf{Z}})$ denote the subspace spanned by

$$\mathcal{N}^{\mathrm{h},(2)}_{F_f} := \{F \in \mathcal{N}^{\mathrm{h}} \mid \mathfrak{m}^{(2)}_F = \mathfrak{m}^{(2)}_{F_f}\},$$

where $\mathfrak{m}^{(2)}_F$ and $\mathfrak{m}^{(2)}_{F_f}$ are the maximal ideals of $\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}}$ corresponding to F and F_f, respectively (cf. section 5.5). Then $T^{\mathrm{h}}\Xi, T^{\mathrm{h}}F_f = F_f, T^{\mathrm{h}}G'' \in \mathcal{S}^{(2)}_{k,F_f}$. The image of $\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}}$ inside $\mathrm{End}_{\mathbf{C}}(\mathcal{S}^{(2)}_{k,F_f})$ can be naturally identified with $\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}}$. By the commutativity of diagram (5.10) and the discussion following the diagram, the $\mathcal{O}$-algebra map $\mathrm{Desc} : \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}} \twoheadrightarrow \mathbf{T}'_{\mathcal{O}}$ factors through $\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}} \twoheadrightarrow \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$. The algebra $\mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$ can be identified with the image of $\mathbf{T}'_{\mathcal{O}}$ inside $\mathrm{End}_{\mathbf{C}}(S_{k-1,f})$, where $S_{k-1,f} \subset S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right)$ is the subspace spanned by $\mathcal{N}'_f := \{g \in \mathcal{N} \mid \mathfrak{m}'_f = \mathfrak{m}'_g\}$. Here $\mathfrak{m}'_f$ and $\mathfrak{m}'_g$ denote the maximal ideals of $\mathbf{T}'_{\mathcal{O}}$ corresponding to f and g, respectively. Denote by ϕ_f the natural projection $\mathbf{T}'_{\mathcal{O}} \twoheadrightarrow \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$, and by Φ_f the natural projection $\mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O}} \twoheadrightarrow \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}}$. Assume $\ell > k$, hence in particular $\ell \nmid (k-1)(k-2)(k-3)$. By Corollary 5.15, for every split prime $p = \pi\overline{\pi}$, $p \nmid \ell$, there exists $T^{\mathrm{h}}(p) \in \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}}$ such that $\mathrm{Desc}(T^{\mathrm{h}}(p)) = \phi_f(T_p) \in \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$. As will be proven in section 8 (cf. Proposition 8.14) there exists a Hecke operator $T \in \mathbf{T}'_{\mathcal{O},\mathfrak{m}'_f}$ such that $Tf = \eta f$, $Tf^\rho = \eta f^\rho$, and $Tg = 0$ for all $g \in \mathcal{N}'_f$, $g \neq f, f^\rho$. The operator T is a polynomial P_T in the elements of $\phi_f(\Sigma')$ with coefficients in $\mathcal{O}$ (here Σ' is as in Definition 5.1). Let $\tilde{T}^{\mathrm{h}} \in \mathbf{T}^{\mathrm{h},(2)}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}}$ be the Hecke operator given by the polynomial $P_{\tilde{T}^{\mathrm{h}}}$ obtained from P_T by substituting

- $\Phi_f(T^{\mathrm{h}}_p - p^{k-1} - p^{k-2} - p^{k-3})$ for $\phi_f(T_{p^2})$ if p inert in K,
- $T^{\mathrm{h}}(p)$ for $\phi_f(T_p)$ if $p \nmid \ell$ splits in K,
- $\Phi_f(\lambda_0^k \ell^{2-k}(\ell+1)^{-1} T^{\mathrm{h}}_{\lambda_0})$ for $\phi(T_\ell)$ if $\ell = \lambda_0\overline{\lambda}_0$ splits in K.

Note that $\lambda_0^k \ell^{2-k}(\ell+1)^{-1} T^{\mathrm{h}}_{\lambda_0}$ is indeed an element of $\mathbf{T}^{\mathrm{h}}_{\mathcal{O},\mathfrak{m}^{(2)}_{F_f}}$ as $\ell + 1$ is invertible in $\mathcal{O}$. It follows from (5.9) that $\mathrm{Desc}(\tilde{T}^{\mathrm{h}}) = T$. Apply $\tilde{T}^{\mathrm{h}}$ to both sides of

$$T^{\mathrm{h}}\Xi = C_{F_f} F_f + T^{\mathrm{h}} G''.$$

Note that $\tilde{T}^{\mathrm{h}} T^{\mathrm{h}}\Xi$ is again a cusp form. The operator $\tilde{T}^{\mathrm{h}}$ preserves the Maass space and its orthogonal complement by Theorem 5.10. Another application of Lemma 5.8 shows that the Fourier coefficients of $\tilde{T}^{\mathrm{h}} T^{\mathrm{h}}\Xi$ lie in $\mathcal{O}$. Moreover, since Desc is a $\mathbf{C}$-algebra map, it is clear from the definition of $\tilde{T}^{\mathrm{h}}$ that $\tilde{T}^{\mathrm{h}}F_f = \eta F_f$ and $\tilde{T}^{\mathrm{h}}F = 0$ for any F inside the Maass space of $\mathcal{S}_k(\Gamma_{\mathbf{Z}})$ which is orthogonal to F_f. We thus get

$$\tilde{T}^{\mathrm{h}}T^{\mathrm{h}}\Xi = \eta C_{F_f}F_f + \tilde{T}^{\mathrm{h}}T^{\mathrm{h}}G'' \tag{7.21}$$

with $\tilde{T}^{\mathrm{h}}T^{\mathrm{h}}G''$ orthogonal to the Maass space.

As $C_{F_f} \in \overline{\mathbf{Q}} \cap E \subset \mathbf{C}$ by Corollary 7.7, it makes sense to talk about its λ-adic valuation. Suppose $\mathrm{ord}_\lambda(\eta\, C_{F_f}) = -n \in \mathbf{Z}_{<0}$. We write $F \equiv F'$ (mod λ^n) to mean that $\mathrm{ord}_\lambda(c_F(h) - c_{F'}(h)) \geqslant n$ for every $h \in \mathcal{S}$. Note that since the Fourier coefficients of $\tilde{T}^{\mathrm{h}}T^{\mathrm{h}}\Xi$ and of F_f lie in $\mathcal{O}$, but $\eta\, C_{F_f} \notin \mathcal{O}$, we must have that either $\tilde{T}^{\mathrm{h}}T^{\mathrm{h}}G'' \neq 0$ or $F_f \equiv 0 \bmod \lambda$. However, by Proposition 4.5, the latter is only possible if $f \equiv f^\rho \bmod \lambda$ and this contradicts absolute irreducibility of $\overline{\rho}_f|_{G_K}$ by Proposition 8.13 in section 8.2. Hence we must have $\tilde{T}^{\mathrm{h}}T^{\mathrm{h}}G'' \neq 0$. Write $\eta\, C_{F_f} = a\lambda^{-n}$ with $a \in \mathcal{O}^\times$. Then the Fourier coefficients of $\lambda^n\tilde{T}^{\mathrm{h}}T^{\mathrm{h}}G''$ lie in $\mathcal{O}$ and one has

$$F_f \equiv -a^{-1}\lambda^n\tilde{T}^{\mathrm{h}}T^{\mathrm{h}}G'' \pmod{\lambda^n}.$$

As explained above, $-a^{-1}\lambda^n\tilde{T}^{\mathrm{h}}T^{\mathrm{h}}G''$ is a hermitian modular form orthogonal to the Maass space.

We have proven the following theorem:

Theorem 7.12. — *Let k a positive integer divisible by 4 and $\ell > k$ a rational prime. Let $f \in \mathcal{N}$ be ordinary at ℓ and such that $\overline{\rho}_f|_{G_K}$ is absolutely irreducible. Fix a positive integer $m \in S_{f,\ell}$ and a Hecke character χ of K such that $\mathrm{ord}_\ell(\mathrm{cond}\,\chi) = 0$, $\chi_\infty(z) = \left(\frac{z}{|z|}\right)^{-t}$ with $-k \leqslant t < -6$, and $\ell \nmid \#X_{-t-2,4mN_{K/\mathbf{Q}}(\mathrm{cond}\,\chi)}$. Let E be a sufficiently large finite extension of $\mathbf{Q}_\ell$ with uniformizer λ. If*

$$-n := \mathrm{ord}_\lambda\left(\prod_{j=1}^{2} L^{\mathrm{int}}(\mathrm{BC}(f), j + (t+k)/2, \overline{\chi\omega})\right) - \mathrm{ord}_\lambda(L^{\mathrm{int}}(\mathrm{Symm}^2 f, k)) < 0$$

where ω is the unique Hecke character of K which is unramified at all finite places and such that $\omega_\infty(z) = \left(\frac{z}{|z|}\right)^{-k}$, then there exists $F' \in \mathcal{S}_k(\Gamma_{\mathbf{Z}})$, orthogonal to the Maass space, such that $F' \equiv F_f$ (mod λ^n).

Remark 7.13. — For χ and m as in Theorem 7.12, set

$$c = 4mN_{K/\mathbf{Q}}(\mathrm{cond}\,\chi).$$

In Theorem 7.12, we say that E is sufficiently large if it contains the field $K(\chi_{\mathbf{Q},c}, \mu_c)$, the set Σ, the elements (7.16), (7.17), (7.18), the Fourier coefficients of F_f and the number η.

Corollary 7.14. — *Suppose that χ in Theorem 7.12 can be chosen so that*

$$\mathrm{ord}_{\lambda}\left(\prod_{j=1}^{2} L^{\mathrm{int}}(\mathrm{BC}(f), j+(t+k)/2, \overline{\chi\omega})\right)=0,$$

then n in Theorem 7.12 can be taken to be $\mathrm{ord}_{\lambda}(L^{\mathrm{int}}(\mathrm{Symm}^2 f, k))$.

Remark 7.15. — The existence of character χ as in Corollary 7.14 is not known in general. Some results in this direction (although not applicable to the case considered here) have been obtained by Vatsal in [57]. The problem in our case is that one would need to control the λ-adic valuation of two L-values at the same time.

Remark 7.16. — The ordinarity assumption on f in Theorem 7.12 is crucial to our method and is used in section 8 to construct the Hecke operator T annihilating the Maass part of G'' as above as well as to ensure that $(*)$ in (7.19) is a λ-adic unit. One expects that the set of primes ℓ of $\mathbf{Q}$ such that a given (non-CM) form f is ordinary at ℓ has Dirichlet density one, but for now no proof of this fact is known. An analogous statement for elliptic curves was proved by Serre [48].

7.4. Congruence between F_f and a non-CAP eigenform

Corollary 7.17. — *Under the assumptions of Theorem 7.12 there exists a non-CAP cuspidal Hecke eigenform F such that* $\mathrm{ord}_{\lambda}(\lambda_{F_f}(T^{\mathrm{h}})-\lambda_F(T^{\mathrm{h}}))>0$ *for all Hecke operators $T^{\mathrm{h}}\in\mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$.*

Proof. — Let F' be as in Theorem 7.12. Using the decomposition (5.8), we see that there exists a Hecke operator $T^{\mathrm{h}}_0\in\mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$ such that $T^{\mathrm{h}}_0F_f=F_f$ and $T^{\mathrm{h}}_0F=0$ for each $F\in\mathcal{S}_k(\Gamma_{\mathbf{Z}})$ which is orthogonal to all Hecke eigenforms whose eigenvalues are congruent to those of F_f (mod λ). Suppose all the elements of $\mathcal{N}^{\mathrm{h}}$ whose eigenvalues are congruent to those of F_f (mod λ) are CAP forms. Then applying T^{h}_0 to the congruence $F_f\equiv F'$, we get $F_f\equiv 0$ (mod λ). By Proposition 4.5 this is only possible if $f\equiv f^{\rho}$ (mod λ). This however leads to a contradiction by Proposition 8.13. □

7.5. The CAP ideal

Recall that we have a Hecke-stable decomposition

$$\mathcal{S}_k(\Gamma_{\mathbf{Z}})=\mathcal{S}^{\mathrm{M}}_k(\Gamma_{\mathbf{Z}})\oplus\mathcal{S}^{\mathrm{NM}}_k(\Gamma_{\mathbf{Z}}),$$

where $\mathcal{S}_k^{\rm NM}(\Gamma_{\mathbf{Z}})$ denotes the orthogonal complement of $\mathcal{S}_k^{\rm M}(\Gamma_{\mathbf{Z}})$ inside $\mathcal{S}_k(\Gamma_{\mathbf{Z}})$. Denote by $\mathbf{T}_{\mathcal{O}}^{\rm NM}$ the image of $\mathbf{T}_{\mathcal{O}}^{\rm h}$ inside $\operatorname{End}_{\mathbf{C}}(\mathcal{S}_k^{\rm NM}(\Gamma_{\mathbf{Z}}))$ and let $\phi : \mathbf{T}_{\mathcal{O}}^{\rm h} \twoheadrightarrow \mathbf{T}_{\mathcal{O}}^{\rm NM}$ be the canonical $\mathcal{O}$-algebra epimorphism. Let $\operatorname{Ann}(F_f) \subset \mathbf{T}_{\mathcal{O}}^{\rm h}$ denote the annihilator of F_f. It is a prime ideal of $\mathbf{T}_{\mathcal{O}}^{\rm h}$ and $\lambda_{F_f} : \mathbf{T}_{\mathcal{O}}^{\rm h} \twoheadrightarrow \mathcal{O}$ induces an $\mathcal{O}$-algebra isomorphism $\mathbf{T}_{\mathcal{O}}^{\rm h}/\operatorname{Ann}(F_f) \xrightarrow{\sim} \mathcal{O}$.

Definition 7.18. — *As ϕ is surjective, $\phi(\operatorname{Ann}(F_f))$ is an ideal of $\mathbf{T}_{\mathcal{O}}^{\rm NM}$. We call it the CAP ideal associated to F_f.*

There exists a non-negative integer r for which the diagram

$$\begin{tikzcd} \mathbf{T}_{\mathcal{O}}^{\rm h} \arrow[r, "\phi"] \arrow[d] & \mathbf{T}_{\mathcal{O}}^{\rm NM} \arrow[d] \\ \mathbf{T}_{\mathcal{O}}^{\rm h}/\operatorname{Ann}(F_f) \arrow[r, "\phi"] \arrow[d, "\lambda_{F_f}"', "\wr"] & \mathbf{T}_{\mathcal{O}}^{\rm NM}/\phi(\operatorname{Ann}(F_f)) \arrow[d, "\wr"] \\ \mathcal{O} \arrow[r] & \mathcal{O}/\lambda^r\mathcal{O} \end{tikzcd} \tag{7.22}$$

all of whose arrows are $\mathcal{O}$-algebra epimorphisms, commutes.

Corollary 7.19. — *If r is the integer from diagram (7.22), and n is as in Theorem 7.12, then $r \geqslant n$.*

Proof. — Set $\mathcal{N}^{\rm NM} := \{F \in \mathcal{N}^{\rm h} \mid F \in \mathcal{S}_k^{\rm NM}(\Gamma_{\mathbf{Z}})\}$. Choose any $T^{\rm h} \in \phi^{-1}(\lambda^r) \subset \mathbf{T}_{\mathcal{O}}^{\rm h}$. Suppose that $r < n$, and let F' be as in Theorem 7.12. We have

$$F_f \equiv F' \pmod{\lambda^n}. \tag{7.23}$$

and $T^{\rm h}F' = \lambda^r F'$. Hence applying $T^{\rm h}$ to both sides of (7.23), we obtain $0 \equiv \lambda^r F' \pmod{\lambda^n}$, which leads to

$$F' \equiv 0 \pmod{\lambda^{n-r}}. \tag{7.24}$$

Since $r < n$, (7.23) and (7.24) imply that $F_f \equiv 0 \pmod{\lambda}$, which is impossible as shown in the proof of Corollary 7.17. □

Remark 7.20. — The CAP ideal can be regarded as an analogue of the Eisenstein ideal in the case of classical modular forms (see e.g. [39]). It measures congruences between F_f and non-CAP modular forms. We will show in section 9.2 that $\operatorname{ord}_\ell(\#\mathbf{T}_{\mathcal{O}}^{\rm NM}/\phi(\operatorname{Ann}(F_f)))$ provides a lower bound for the ℓ-adic valuation of the order of the Selmer group we study in section 9.1.

8. Hecke algebras and deformation rings

The goal of this section is to prove Proposition 8.14 which was used in section 7.3 to prove Theorem 7.12, as well as some auxiliary results.

8.1. Congruences and weak congruences

Let E denote a finite extension of $\mathbf{Q}_\ell$ containing all Hecke eigenvalues of all the elements of $\mathcal{N}$. Let $\mathcal{O}$ be the valuation ring of E with uniformizer λ, and put $\mathbf{F} = \mathcal{O}/\lambda$. Whenever we refer to a prime p being split or inert we will always mean split in K or inert in K. Let $\mathbf{T}_\mathbf{Z}$, $\mathbf{T}'_\mathbf{Z}$ be as in Definition 5.1. To ease notation in this section we set $\mathbf{T} := \mathbf{T}_\mathcal{O}$ and $\mathbf{T}' := \mathbf{T}'_\mathcal{O}$. Moreover, if $a, b \in \mathcal{O}$, we write $a \equiv b$ if $\lambda \mid (a - b)$.

Let $\lambda_f : \mathbf{T} \to \mathcal{O}$ be as in section 5.1 and as before set $\mathfrak{m}_f = \ker \overline{\lambda}_f$. Moreover, set $\lambda'_f := \lambda_f|_{\mathbf{T}'}$ and denote by $\overline{\lambda}'_f$ the reduction of λ'_f modulo λ. Put $\mathfrak{m}'_f := \ker \overline{\lambda}'_f$.

From now on let $f = \sum_{n=1}^\infty a(n)q^n$ and $g = \sum_{n=1}^\infty b(n)q^n$ denote two elements of $\mathcal{N}$. We denote by $\rho_f, \rho_g : G_\mathbf{Q} \to \mathrm{GL}_2(E)$ the ℓ-adic Galois representations attached to f and g, respectively and by $\overline{\rho}_f$ and $\overline{\rho}_g$ their mod λ reductions with respect to some lattice in E^2. We write $\overline{\rho}_f^{\mathrm{ss}}$ for the semi-simplification of $\overline{\rho}_f$. The isomorphism class of $\overline{\rho}_f^{\mathrm{ss}}$ is independent of the choice of the lattice. (cf. section 2.3).

Definition 8.1. — *We will say that f and g are congruent (resp. weakly congruent), denoted by $f \equiv g$ (resp. $f \equiv_w g$) if $\mathfrak{m}_f = \mathfrak{m}_g$ (resp. $\mathfrak{m}'_f = \mathfrak{m}'_g$). We will say that f and g are congruent at p if $a(p) \equiv b(p)$. Let A be a set of finite primes of $\mathbf{Z}$ of density zero. We will say that f and g are A-congruent, denoted by $f \equiv_A g$ if f and g are congruent at p for all primes $p \notin A$.*

We note that decompositions analogous to (5.1) and (5.2) hold for $\mathbf{T}'$ and that the localizations $\mathbf{T}_\mathfrak{m}$ and $\mathbf{T}'_{\mathfrak{m}'}$ are Noetherian, local, complete $\mathcal{O}$-algebras. For a maximal ideal $\mathfrak{m}' \subset \mathbf{T}'$, we denote by $\mathcal{M}(\mathfrak{m}')$ the set of maximal ideals of $\mathbf{T}$ which contract to $\mathfrak{m}'$. Note that the inclusion $\mathbf{T}' \hookrightarrow \mathbf{T}$ factors into a direct product (over all maximal ideals $\mathfrak{m}'$ of $\mathbf{T}'$) of injections $\mathbf{T}'_{\mathfrak{m}'} \hookrightarrow \prod_{\mathfrak{m} \in \mathcal{M}(\mathfrak{m}')} \mathbf{T}_\mathfrak{m}$. We will now examine the sets $\mathcal{M}(\mathfrak{m}')$ a little closer.

Lemma 8.2. — *Let $f, g \in \mathcal{N}$ and let A be a density zero set of finite primes of $\mathbf{Z}$ not containing ℓ. Then $f \equiv g$ if and only if $f \equiv_A g$.*

Proof. — One direction is a tautology, so assume $f \equiv_A g$. We have $\mathrm{tr}\,\overline{\rho}_f(\mathrm{Frob}_p) = a(p) \pmod{\lambda}$, $\mathrm{tr}\,\overline{\rho}_g(\mathrm{Frob}_p) = b(p) \pmod{\lambda}$ and

$\det \overline{\rho}_f(\mathrm{Frob}_p) = \left(\frac{-4}{p}\right) p^{k-2} = \det \overline{\rho}_g(\mathrm{Frob}_p)$ for $p \neq 2, \ell$. Hence by the Tchebotarev Density Theorem together with the Brauer-Nesbitt Theorem we get $\overline{\rho}_f^{\mathrm{ss}} \cong \overline{\rho}_g^{\mathrm{ss}}$, and thus, $a(p) \equiv b(p)$ for all $p \in A, p \neq 2$. Moreover, we have $\rho_f|_{D_2} \cong \begin{bmatrix} \mu_f^1 \chi & \\ & \mu_f^2 \end{bmatrix}$, where D_2 denotes the decomposition group at 2, μ_f^1 and μ_f^2 are unramified characters, with $\mu_f^2(\mathrm{Frob}_2) = a(2)$, and χ is the Galois character associated with the Dirichlet character $\left(\frac{-4}{\cdot}\right)$ (cf. [27], Theorem 3.26 (3)). An analogous result holds for ρ_g. Let $\sigma \in D_2$ be any lift of Frob_2, and let $\tau \in I_2$ be such that $\chi(\tau) = -1$, where I_2 denotes the inertia group at 2. We want to show that $\mu_f^2(\sigma) \equiv \mu_g^2(\sigma) \pmod{\lambda}$. We have $\mathrm{tr}\, \rho_f(\sigma) = \mu_f^1(\sigma)\chi(\sigma) + \mu_f^2(\sigma)$ and $\mathrm{tr}\, \rho_f(\tau\sigma) = \mu_f^1(\sigma)\chi(\tau)\chi(\sigma) + \mu_f^2(\sigma)$. Then as $\chi(\tau) = -1$, we get $\mu_f^2(\sigma) = \frac{1}{2}(\mathrm{tr}\, \rho_f(\sigma) + \mathrm{tr}\, \rho_f(\tau\sigma))$. Similarly we get $\mu_g^2(\sigma) = \frac{1}{2}(\mathrm{tr}\, \rho_g(\sigma) + \mathrm{tr}\, \rho_g(\tau\sigma))$. Since $\overline{\rho}_f^{\mathrm{ss}} \cong \overline{\rho}_g^{\mathrm{ss}}$ implies the equality of traces of $\overline{\rho}_f$ and $\overline{\rho}_g$, $\mu_f^2(\sigma) \equiv \mu_g^2(\sigma)$ and the lemma is proved. □

Proposition 8.3. — *If $f \equiv_w g$, then either $f \equiv g$ or $f \equiv g^\rho$.*

Proof. — Assume $f \equiv_w g$. Using the Tchebotarev density Theorem and the Brauer-Nesbitt Theorem, we see that $\overline{\rho}_f^{\mathrm{ss}}|_{G_K} \cong \overline{\rho}_g^{\mathrm{ss}}|_{G_K}$. By possibly changing a basis of, say, $\overline{\rho}_g$, we may assume that $\overline{\rho}_f^{\mathrm{ss}}|_{G_K} = \overline{\rho}_g^{\mathrm{ss}}|_{G_K}$. This implies that $\overline{\rho}_f^{\mathrm{ss}} = \chi \overline{\rho}_g^{\mathrm{ss}}$, where χ is either as in the proof of Lemma 8.2 or trivial. Hence $a(p) \equiv \left(\frac{-4}{\cdot}\right)^i b(p)$ for some i and all $p \neq 2, \ell$. Thus by Lemma 8.2 we are done if we show that $a(\ell) \equiv \left(\frac{-4}{\ell}\right)^i b(\ell)$. If ℓ is split, then (since $f \equiv_w g$) we have $a(\ell) \equiv b(\ell)$, so assume ℓ is inert. In that case, $a(\ell)^2 \equiv b(\ell)^2$ hence if $a(\ell) \equiv 0$, we are done. Otherwise, f and g are ℓ-ordinary, and in such case $\rho_f|_{D_\ell} \cong \begin{bmatrix} \mu_f^1 & * \\ & \mu_f^2 \end{bmatrix}$ with μ_f^2 unramified and $\mu_f^2(\mathrm{Frob}_\ell)$ is the unit root α_f of $X^2 - a(\ell)X + \left(\frac{-4}{\ell}\right)\ell^{k-2}$ (cf. [27], Theorem 3.26 (2)). Analogous statements hold for ρ_g. Now, since $\overline{\rho}_f \cong \overline{\rho}_g \otimes \chi^i$, we must have $\alpha_f \equiv \left(\frac{-4}{\ell}\right)^i \alpha_g$. As α_f is the unique unit root of the polynomial $X^2 - a(\ell)X + \left(\frac{-4}{\ell}\right)\ell^{k-2}$, we must have $a(\ell) \equiv \alpha_f$, and similarly $b(\ell) \equiv \alpha_g$, hence the proposition is proved. □

Corollary 8.4. — *If $f \equiv f^\rho$, then $\mathcal{M}(\mathfrak{m}'_f) = \{\mathfrak{m}_f\}$. If $f \not\equiv f^\rho$, then $\mathcal{M}(\mathfrak{m}'_f) = \{\mathfrak{m}_f, \mathfrak{m}_{f^\rho}\}$. Hence, if $f \equiv f^\rho$, we have an injection $\mathbf{T}'_{\mathfrak{m}'_f} \hookrightarrow \mathbf{T}_{\mathfrak{m}_f}$, while if $f \not\equiv f^\rho$, we have $\mathbf{T}'_{\mathfrak{m}'_f} \hookrightarrow \mathbf{T}_{\mathfrak{m}_f} \times \mathbf{T}_{\mathfrak{m}_{f^\rho}}$.*

Proposition 8.5. — *If $f \in \mathcal{N}$, then the canonical $\mathcal{O}$-algebra map $\phi_0 : \mathbf{T}'_{\mathfrak{m}'_f} \to \mathbf{T}_{\mathfrak{m}_f}$ is injective.*

Proof. — If $f \equiv f^\rho$, then $\mathbf{T}'_{\mathfrak{m}'_f}$ injects into $\mathbf{T}_{\mathfrak{m}_f}$ by Corollary 8.4. Assume that $f \not\equiv f^\rho$. Note that in that case $g \equiv_w f$ implies $g \not\equiv g^\rho$. By Proposition 8.3, $g \equiv f$ or $g \equiv f^\rho$. Without loss of generality assume that $f \equiv g$. Consider $\mathbf{T}_{\mathfrak{m}_f}$ as a subalgebra of $\prod_{g \in \mathcal{N}, g \equiv f} \mathcal{O}$ via $T \mapsto (\lambda_g(T))_g$, and $\mathbf{T}_{\mathfrak{m}_{f^\rho}}$ as a subalgebra of $\prod_{g \in \mathcal{N}, g \equiv f^\rho} \mathcal{O}$ via $T \mapsto (\lambda_{g^\rho}(T))_g$. By Corollary 8.4 we have $\mathbf{T}'_{\mathfrak{m}'_f} \hookrightarrow \mathbf{T}_{\mathfrak{m}_f} \times \mathbf{T}_{\mathfrak{m}_{f^\rho}}$, so we just need to prove that the composite $\mathbf{T}'_{\mathfrak{m}'_f} \hookrightarrow \mathbf{T}_{\mathfrak{m}_f} \times \mathbf{T}_{\mathfrak{m}_{f^\rho}} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}_f}$ is injective, where the last arrow is projection. Identifying $\mathbf{T}_{\mathfrak{m}_f} \times \mathbf{T}_{\mathfrak{m}_{f^\rho}}$ with a subalgebra of $R := \prod_{g \in \mathcal{N}, g \equiv f} \mathcal{O} \times \prod_{g \in \mathcal{N}, g \equiv f^\rho} \mathcal{O}$ by the embeddings specified above, we see that $T \in \mathbf{T}'_{\mathfrak{m}'_f}$ maps to an element of R, whose g-entry in the first product is the same as the corresponding g^ρ-entry in the second product for every $g \in \mathcal{N}$, $g \equiv f$ (this is so, because $Tg = ag$ implies $Tg^\rho = ag^\rho$ for $T \in \mathbf{T}'_{\mathfrak{m}'_f}$). Hence if T maps to zero under the composite $\mathbf{T}'_{\mathfrak{m}'_f} \hookrightarrow \mathbf{T}_{\mathfrak{m}_f} \times \mathbf{T}_{\mathfrak{m}_{f^\rho}} \to \mathbf{T}_{\mathfrak{m}_f}$, it must be zero in $\mathbf{T}_{\mathfrak{m}_f} \times \mathbf{T}_{\mathfrak{m}_{f^\rho}}$. $\square$

8.2. Deformations of Galois representations

The goal of this section is to prove surjectivity of $\phi_0 : \mathbf{T}'_{\mathfrak{m}'_f} \to \mathbf{T}_{\mathfrak{m}_f}$. We will use the theory of deformations of Galois representations. For an introduction to the subject see e.g. [40].

8.2.1. Universal deformation ring

Let $\mathcal{C}$ denote the category of local, complete $\mathcal{O}$-algebras with residue field $\mathbf{F}$. A morphism between two objects in $\mathcal{C}$ is a continuous $\mathcal{O}$-algebra homomorphism which induces the identity on the residue fields. For an object R of $\mathcal{C}$ we denote by $\mathfrak{m}_R$ its maximal ideal. Let $\mathcal{G}$ be a profinite group. Two continuous representations $\rho : \mathcal{G} \to \mathrm{GL}_2(R)$ and $\rho' : \mathcal{G} \to \mathrm{GL}_2(R)$ are called *strictly equivalent* if $\rho(g) = x\rho'(g)x^{-1}$ for every $g \in \mathcal{G}$ with $x \in 1 + M_2(\mathfrak{m}_R)$ independent of g. We will write $\rho \approx \rho'$ if ρ and ρ' are strictly equivalent. Consider a continuous representation $\overline{\rho} : \mathcal{G} \to \mathrm{GL}_2(\mathbf{F})$. If R is an object of $\mathcal{C}$, a continuous representation $\rho : \mathcal{G} \to \mathrm{GL}_2(R)$ or, more precisely, a strict equivalence of such, is called a *deformation of* $\overline{\rho}$ if $\overline{\rho} = \rho \bmod \mathfrak{m}_R$. A pair $(R^{\mathrm{univ}}, \rho^{\mathrm{univ}})$ consisting of an object R^{univ} of $\mathcal{C}$ and a deformation $\rho^{\mathrm{univ}} : \mathcal{G} \to \mathrm{GL}_2(R^{\mathrm{univ}})$ is called a *universal couple* if for every deformation $\rho : \mathcal{G} \to \mathrm{GL}_2(R)$, where R is an object in $\mathcal{C}$, there exists a unique $\mathcal{O}$-algebra homomorphism $\phi : R^{\mathrm{univ}} \to R$ such that $\phi \circ \rho^{\mathrm{univ}} \approx \rho$ in $\mathrm{GL}_2(R)$. The ring R^{univ} is called *the universal deformation ring* of $\overline{\rho}$.

By the universal property stated above, it is unique if it exists. Note that any $\mathcal{O}$-algebra homomorphism between objects in $\mathcal{C}$ is automatically local, since all objects of $\mathcal{C}$ have the same residue fields.

THEOREM 8.6 (Mazur). — *Suppose that* $\overline{\rho}: \mathcal{G} \to GL_2(\mathbf{F})$ *is absolutely irreducible. Then there exists a universal deformation ring* R^{univ} *in* $\mathcal{C}$ *and a universal deformation* $\rho^{\mathrm{univ}}: \mathcal{G} \to GL_2(R^{\mathrm{univ}})$.

Proof. — [27], Theorem 2.26. □

8.2.2. Hecke algebras as quotients of deformation rings

Consider $f \in \mathcal{N}$ and let $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be the associated Galois representation (after fixing a lattice in E^2). Let $\overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F})$ be its reduction modulo λ. Since ρ_f is unramified away from $S = \{2, \ell\}$, it factors through $G_{\mathbf{Q},S}$, the Galois group of the maximal Galois extension of $\mathbf{Q}$ unramified away from S. Let $G_{K,S}$ be the image of G_K under the map $G_K \hookrightarrow G_{\mathbf{Q}} \twoheadrightarrow G_{\mathbf{Q},S}$. We will be considering deformations of the representation $\overline{\rho}_f : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{F})$ and of $\overline{\rho}_{f,K} := \overline{\rho}_f|_{G_{K,S}}$. From now on we assume that $\overline{\rho}_{f,K}$ is absolutely irreducible. Let (R_Q, ρ_Q) and (R_K, ρ_K) denote the universal couples of $\overline{\rho}_f$ and $\overline{\rho}_{f,K}$, respectively, which exist by Theorem 8.6. We will denote $\mathfrak{m}_{R_Q}$ and $\mathfrak{m}_{R_K}$ by $\mathfrak{m}_Q$ and $\mathfrak{m}_K$, respectively. Let A be a density zero set of primes of $\mathbf{Q}$ and $g \in \mathcal{N}$, $g \equiv_A f$. Then after possibly changing the basis of ρ_g we may assume (by the Tchebotarev Density Theorem together with the Brauer-Nesbitt Theorem) that $\overline{\rho}_f = \overline{\rho}_g$. Hence $\rho_g : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathcal{O})$ is a deformation of $\overline{\rho}_f$, and $\rho_g|_{G_K}$ is a deformation of $\overline{\rho}_{f,K}$. As in the proof of Proposition 8.5 we identify $\mathbf{T}_{\mathfrak{m}_f}$ and $\mathbf{T}'_{\mathfrak{m}'_f}$ with appropriate subalgebras of $\prod_{g \in \mathcal{N},\, g \equiv f} \mathcal{O}$ and of $\prod_{g \in \mathcal{N},\, g \equiv_w f} \mathcal{O}$, respectively. Let $\tilde{\mathbf{T}}$ denote the $\mathcal{O}$-subalgebra of $\mathbf{T}$ generated by the operators T_p for $p \neq 2, \ell$ and let $\tilde{\mathbf{T}}'$ denote the $\mathcal{O}$-subalgebra of $\mathbf{T}'$ generated by the set $\Sigma' \setminus \{T_\ell\}$, where Σ' is as in Definition 5.1. We put $\tilde{\mathfrak{m}}_f := \tilde{\mathbf{T}} \cap \mathfrak{m}_f$ and $\tilde{\mathfrak{m}}'_f := \tilde{\mathbf{T}}' \cap \mathfrak{m}_f$. Let Σ_f denote the subset of $\mathcal{N}$ consisting of those eigenforms which are congruent to f except possibly at 2 or ℓ. Similarly let Σ'_f be the subset of $\mathcal{N}$ consisting of those eigenforms which are weakly congruent to f except possibly at 2 or ℓ. We have $\Sigma_f \subset \Sigma'_f$. We again identify $\tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f}$ (resp. $\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f}$) with a subalgebra of $\prod_{g \in \Sigma_f} \mathcal{O}$ (resp. $\prod_{g \in \Sigma'_f} \mathcal{O}$) in an obvious way. Consider the representations $\rho := \prod_{g \in \Sigma_f} \rho_g : G_{\mathbf{Q},S} \to \mathrm{GL}_2\left(\prod_{g \in \Sigma_f} \mathcal{O}\right)$, and $\rho' := \rho|_{G_{K,S}}$. Choose bases for each ρ_g so that $\overline{\rho}_g = \overline{\rho}_{g'}$ for all $g, g' \in \Sigma_f$, and so that $\rho_g(c) = \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$ for all $g \in \Sigma_f$, where c is the complex conjugation. We allow ourselves to enlarge E, $\mathcal{O}$ and $\mathbf{F}$ if necessary.

Lemma 8.7. — *The image of the representation ρ is contained in* $\mathrm{GL}_2(\tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f})$.

Proof. — [11], Lemma 3.27. $\square$

We claim that $\rho'(G_{K,S})$ is contained in the image of $\mathrm{GL}_2(\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f})$ inside $\mathrm{GL}_2(\tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f})$. To prove it, let ϕ denote the map $\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f} \to \tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f}$ induced by $\tilde{\mathbf{T}}' \hookrightarrow \tilde{\mathbf{T}}$. It is easy to see that $\phi(\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f})$ is an object of $\mathcal{C}$. Consider $\tilde{\rho}' : G_{K,S} \to \mathrm{GL}_2\left(\prod_{g\in\Sigma'_f}\mathcal{O}\right)$, $\tilde{\rho}'(\sigma) = (\rho_g(\sigma))_{g\in\Sigma'_f}$. We have $\phi\circ\tilde{\rho}' = \rho'$. For $\tau \in G_{K,S}$ we denote by $[\tau]$ the conjugacy class of τ in $G_{K,S}$. Note that $G_{K,S}$ is topologically generated by the set $\bigcup_{\mathfrak{p}\in\mathrm{Spec}\,\mathcal{O}_K,\mathfrak{p}\cap\mathbf{Z}\notin S}[\mathrm{Frob}_{\mathfrak{p}}]$. For a split $p = \mathfrak{p}\overline{\mathfrak{p}}$, we have $\mathrm{tr}\,\rho'(\mathrm{Frob}_{\mathfrak{p}}) = \mathrm{tr}\,\rho'(\mathrm{Frob}_{\overline{\mathfrak{p}}}) = \mathrm{tr}\,\rho'(\mathrm{Frob}_p) = T_p \in \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f}$ while for p inert, $\mathrm{tr}\,\rho'(\mathrm{Frob}_p^2) = T_p^2 - p^{k-2} \in \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f}$. Thus $\mathrm{tr}\,\tilde{\rho}'(G_{K,S}) \subset \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f}$, and hence $\mathrm{tr}\,\rho'(G_{K,S}) \subset \phi(\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f})$. Since we know that $\rho'(G_{K,S}) \subset \mathrm{GL}_2(\tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f})$, a theorem of Mazur ([40], Corollary 6, page 256)) implies that (after possibly changing the basis of ρ'), we have $\rho'(G_{K,S}) \subset \mathrm{GL}_2(\phi(\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f}))$. Then ρ is a deformation of $\overline{\rho}_f$ and $\rho' : G_{K,S} \to \mathrm{GL}_2(\phi(\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f}))$ is a deformation of $\overline{\rho}_{f,K}$. Hence there are unique $\mathcal{O}$-algebra homomorphisms $\phi_Q : R_Q \to \tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f}$ and $\phi_K : R_K \to \phi(\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f})$, such that $\phi_Q \circ \rho_Q \approx \rho$, and $\phi_K \circ \rho_K \approx \rho'$. In fact as $\rho_Q|_{G_K}$ is a deformation of $\overline{\rho}_{f,K}$, there is a unique $\mathcal{O}$-algebra homomorphism $\psi : R_K \to R_Q$, such that $\psi \circ \rho_K \approx \rho_Q|_{G_K}$. Hence we get the following diagram

$$\begin{tikzcd} R_K \arrow[r, "\psi"] \arrow[d, "\phi_K"'] & R_Q \arrow[d, "\phi_Q"] \\ \phi(\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f}) \arrow[r, "\iota"] & \tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f} \end{tikzcd} \tag{8.1}$$

where ι denotes the embedding $\phi(\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f}) \subset \tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f}$. Note that diagram (8.1) commutes. [Indeed, as $\iota \circ \rho'$ is a deformation of $\overline{\rho}_{f,K}$, there is a unique $\mathcal{O}$-algebra homomorphism $\alpha : R_K \to \tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f}$, such that $\alpha \circ \rho_K \approx \iota \circ \rho'$. Since $\phi_K \circ \rho_K \approx \rho'$ we get $\iota\circ\phi_K\circ\rho_K \approx \iota\circ\rho'$, and hence $\iota\circ\phi_K = \alpha$ by uniqueness of α. On the other hand as stated in the paragraph before diagram (8.1), $\psi \circ \rho_K \approx \rho_Q|_{G_K}$, thus $\phi_Q \circ \psi \circ \rho_K \approx \phi_Q \circ \rho_Q|_{G_K}$. Since $\phi_Q \circ \rho_Q \approx \rho$, we have $\phi_Q \circ \rho_Q|_{G_K} \approx \rho|_{G_K} = \iota \circ \rho'$. Hence $\phi_Q \circ \psi \circ \rho_K \approx \iota \circ \rho'$, which implies as before that $\phi_Q \circ \psi = \alpha$. So, $\iota \circ \phi_K = \phi_Q \circ \psi$.] Furthermore, note that ϕ_Q and ϕ_K are surjective. Our goal is to prove surjectivity of ψ which will imply surjectivity of ι. From this we will deduce surjectivity of ϕ_0.

The map $\psi : R_K \to R_Q$ is local, hence induces an $\mathbf{F}$-linear homomorphism on the cotangent spaces $\mathfrak{m}_K/(\mathfrak{m}_K^2, \lambda R_K) \to \mathfrak{m}_Q/(\mathfrak{m}_Q^2, \lambda R_Q)$, which we will call ψ_{ct}. We will show that $\psi_{ct}^* : \alpha \mapsto \alpha \circ \psi_{ct}$ in the exact sequence of dual maps

$$0 \to \mathrm{Hom}_{\mathbf{F}}(C, \mathbf{F}) \to \mathrm{Hom}_{\mathbf{F}}(\mathfrak{m}_Q/(\mathfrak{m}_Q^2, \lambda R_Q), \mathbf{F}) \xrightarrow{\psi_{ct}^*} \mathrm{Hom}_{\mathbf{F}}(\mathfrak{m}_K/(\mathfrak{m}_K^2, \lambda R_K), \mathbf{F})$$

is injective, which will imply $C := \operatorname{coker} \psi_{ct} = 0$.

Let $\mathcal{G}$ be a profinite group and $(R^{\mathrm{univ}}, \rho^{\mathrm{univ}})$ the universal couple of an absolutely irreducible representation $\overline{\rho} : \mathcal{G} \to \mathrm{GL}_2(\mathbf{F})$.

Lemma 8.8. — *One has*

$$\mathrm{Hom}_{\mathbf{F}}(\mathfrak{m}_{R^{\mathrm{univ}}}/(\mathfrak{m}_{R^{\mathrm{univ}}}^2, \lambda R^{\mathrm{univ}}), \mathbf{F}) \cong H^1(\mathcal{G}, \mathrm{ad}(\overline{\rho})),$$

where H^1 stands for continuous group cohomology and $\mathrm{ad}(\overline{\rho})$ denotes the discrete $\mathcal{G}$-module $M_2(\mathbf{F})$ with the $\mathcal{G}$-action given by $g \cdot M := \overline{\rho}(g) M \overline{\rho}(g)^{-1}$.

Proof. — [27], Lemma 2.29. □

When $\mathcal{G} = G_{\mathbf{Q},S}$ (or $\mathcal{G} = G_{K,S}$) and $R^{\mathrm{univ}} = R_Q$ (or $R^{\mathrm{univ}} = R_K$), we will denote the isomorphism from Lemma 8.8 by t_Q (or t_K, respectively).

Proposition 8.9. — *The following diagram is commutative:*

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{F}}(\mathfrak{m}_Q/(\mathfrak{m}_Q^2, \lambda R_Q), \mathbf{F}) \arrow[r, "\psi_{ct}^*"] \arrow[d, "t_Q"', "\wr"] & \mathrm{Hom}_{\mathbf{F}}(\mathfrak{m}_K/(\mathfrak{m}_K^2, \lambda R_K), \mathbf{F}) \arrow[d, "t_K", "\wr"'] \\ H^1(G_{\mathbf{Q},S}, \mathrm{ad}(\overline{\rho})) \arrow[r, "\mathrm{res}"] & H^1(G_{K,S}, \mathrm{ad}(\overline{\rho})) \end{tikzcd} \tag{8.2}$$

Proof. — This follows from unraveling the definitions of the maps in diagram (8.2). We omit the details. □

8.2.3. Isomorphism between $\mathbf{T}'_{\mathfrak{m}'_f}$ and $\mathbf{T}_{\mathfrak{m}_f}$

Note that since $\# \, \mathrm{ad}(\overline{\rho}_f)$ is a power of ℓ, and $\mathrm{Gal}(K/\mathbf{Q})$ has order 2, the first cohomology group in the inflation-restriction exact sequence

$$0 \to H^1(\mathrm{Gal}(K/\mathbf{Q}), \mathrm{ad}(\overline{\rho}_f)^{G_{K,S}}) \to H^1(G_{\mathbf{Q},S}, \mathrm{ad}(\overline{\rho}_f)) \to H^1(G_{K,S}, \mathrm{ad}(\overline{\rho}_f))$$

is zero, hence the restriction map in diagram (8.2) is injective, and thus so is ψ_{ct}^*. Hence $C = 0$ and thus ψ_{ct} is surjective. An application of the complete version of Nakayama's Lemma (cf. [18], exercise 7.2) now implies that ψ is surjective.

Corollary 8.10. — *Let $f \in \mathcal{N}$ and suppose that $\overline{\rho}_f|_{G_K}$ is absolutely irreducible. Then $\phi : \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f} \to \tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f}$ is surjective.*

Proof. — This is essentially a summary of the arguments we have carried out so far. □

Proposition 8.11. — *Assume that $f \in \mathcal{N}$ is ordinary at ℓ and that $\overline{\rho}_f|_{G_K}$ is absolutely irreducible. Then $\phi_0 : \mathbf{T}'_{\mathfrak{m}'_f} \to \mathbf{T}_{\mathfrak{m}_f}$ is surjective.*

Proof. — Consider the commutative diagram

$$\begin{tikzcd} \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_f} \arrow[r, "\phi"] \arrow[d] & \tilde{\mathbf{T}}_{\tilde{\mathfrak{m}}_f} \arrow[d] \\ \mathbf{T}'_{\mathfrak{m}'_f} \arrow[r, "\phi_0"] & \mathbf{T}_{\mathfrak{m}_f} \end{tikzcd} \tag{8.3}$$

where $\tilde{\mathfrak{m}}_f$, $\mathfrak{m}'_f$ and $\tilde{\mathfrak{m}}'_f$ are contractions of $\mathfrak{m}_f$ to $\tilde{\mathbf{T}}$, $\mathbf{T}'$ and $\tilde{\mathbf{T}}'$ respectively. Since f satisfies the assumptions of Corollary 8.10, ϕ is surjective. For $p \neq 2, \ell$, it is clear that $T_p \in \mathbf{T}_{\mathfrak{m}_f}$ is inside the image of ϕ_0. If ℓ is split, then $\mathbf{T}_{\mathfrak{m}_f}$ contains T_ℓ by definition, so assume ℓ is inert. Then $T_\ell^2 \in \mathbf{T}_{\mathfrak{m}_f}$. Since $f = \sum_{n=1}^\infty a(n)q^n$ is ordinary at ℓ, we must have $a(\ell) \notin \lambda$, hence the image of T_ℓ in $\mathbf{F}$ is not zero, i.e., $T_\ell \notin \mathfrak{m}_f$. Thus the equation $X^2 - T_\ell^2$ splits in $\mathbf{T}_{\mathfrak{m}_f}/\mathfrak{m}_f$ into relatively prime factors $X - T_\ell$ and $X + T_\ell$. Since $\mathbf{T}'_{\mathfrak{m}'_f}/\mathfrak{m}'_f \cong \mathbf{T}_{\mathfrak{m}_f}/\mathfrak{m}_f$, $X^2 - T_\ell^2$ splits in $\mathbf{T}'_{\mathfrak{m}'_f}/\mathfrak{m}'_f$, and then by Hensel's lemma it splits in $\mathbf{T}'_{\mathfrak{m}'_f}$. This shows that T_ℓ is in the image of ϕ_0. It remains to show that T_2 is in the image of ϕ_0.

Let $\rho_g : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathcal{O})$ denote the Galois representation associated to $g = \sum_{n=1}^\infty b(n)q^n$, $g \equiv f$. Arguing as in Lemma 8.2, we get $\rho_g|_{D_2} \cong \begin{bmatrix} \mu_g^1\chi & \\ & \mu_g^2 \end{bmatrix}$ with $\mu_g^2(\sigma) = \frac{1}{2}(\mathrm{tr}\,\rho_g(\sigma) + \mathrm{tr}\,\rho_g(\tau\sigma))$ (for notation see the proof of Lemma 8.2). Let L be the fixed field of $G_{\mathbf{Q},S}$, and $L' \subset L$ always denote a finite Galois extension of $\mathbf{Q}$. Using the Tchebotarev Density Theorem we can write

$$\sigma = \varprojlim_{\mathbf{Q} \subset L' \subset L} \xi(L')\,\mathrm{Frob}_{p(L')}\,\xi(L')^{-1},$$

where $p(L')$ is a choice of $p \in S$ and $\xi(L') \in G_{\mathbf{Q},S}$ is such that

$$\sigma|_{L'} = \xi(L')|_{L'}\,\mathrm{Frob}_{p(L')}\,|_{L'}\xi(L')^{-1}|_{L'}.$$

Hence $(\mathrm{tr}\,\rho_g(\sigma))_g = \varprojlim_{\mathbf{Q} \subset L' \subset L} (\mathrm{tr}\,(\mathrm{Frob}_{p(L')}))_g = \varprojlim_{\mathbf{Q} \subset L' \subset L} T_{p(L')}$, where each T_p is considered as an element of $\prod_{g \in \mathcal{N}, g \equiv f} \mathcal{O}$. Since every $T_{p(L')} \in \mathrm{Im}(\phi_0)$, and $\mathrm{Im}(\phi_0)$ being the image of $\mathbf{T}'_{\mathfrak{m}'_f}$ is complete, $(\mathrm{tr}\,\rho_g(\sigma))_g \in$

$\mathrm{Im}(\phi_0)$. Similarly one shows that $(\mathrm{tr}\,\rho_g(\tau\sigma))_g \in \mathrm{Im}(\phi_0)$, and hence $T_2 \in \mathrm{Im}(\phi_0)$. $\square$

Corollary 8.12. — *Assume $f \in \mathcal{N}$ is ordinary at ℓ. If $\overline{\rho}_f|_{G_K}$ is absolutely irreducible then the canonical $\mathcal{O}$-algebra map $\mathbf{T}'_{\mathfrak{m}'_f} \to \mathbf{T}_{\mathfrak{m}_f}$ is an isomorphism.*

Proposition 8.13. — *If $\overline{\rho}_f|_{G_K}$ is absolutely irreducible, then $f \not\equiv f^\rho$.*

Proof. — Assume that $\overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F})$ is absolutely irreducible when restricted to G_K. Suppose $f = \sum_{n=1}^\infty a(n)q^n \equiv f^\rho = \sum_{n=1}^\infty \overline{a(n)}q^n$. Let p be a prime inert in K. By Fact 2.1, $a(p) = -\overline{a(p)}$, hence $a(p) \equiv -a(p)$, and thus $\mathrm{tr}\,\overline{\rho}_f(\mathrm{Frob}_p) \equiv a(p) \equiv 0$. Let L be the splitting field of $\overline{\rho}_f$ and denote by $c \in \mathrm{Gal}(L/\mathbf{Q})$ the complex conjugation. By possibly replacing $\mathbf{F}$ with a finite extension, we can choose a basis of the space of $\overline{\rho}_f$ such that with respect to that basis $\overline{\rho}_f(c) = \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$. Let $\sigma \in \mathrm{Gal}(L/K)$, and suppose that $\overline{\rho}_f(\sigma) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. By Tchebotarev Density Theorem there exists a prime p and an element $\tau \in \mathrm{Gal}(L/\mathbf{Q})$ such that $c\sigma = \tau\,\mathrm{Frob}_p\,\tau^{-1}$. Since $\sigma \in \mathrm{Gal}(L/K)$, we must have $\mathrm{Frob}_p \notin \mathrm{Gal}(L/K)$, and thus p is inert in K. Hence $\mathrm{tr}\,\overline{\rho}_f(\mathrm{Frob}_p) = a - d = 0$. Let $\sigma' \in \mathrm{Gal}(L/K)$ and write $\overline{\rho}_f(\sigma') = \left[\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right]$. Then $\overline{\rho}_f(\sigma\sigma') = \left[\begin{smallmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{smallmatrix}\right]$. Since the argument carried out for σ may also be applied to σ' and $\sigma\sigma' \in \mathrm{Gal}(L/K)$, we have $a' = d'$ and $bc' = cb'$, and this condition implies that $\sigma\sigma' = \sigma'\sigma$. Hence $\mathrm{Gal}(L/K)$ is abelian, which contradicts the absolute irreducibility of $\overline{\rho}_f|_{G_K}$. The proposition follows. $\square$

8.3. Hida's congruence modules

Fix $f \in \mathcal{N}$ and set $\mathcal{N}_f := \{g \in \mathcal{N} \mid \mathfrak{m}_g = \mathfrak{m}_f\}$. Write $\mathbf{T}_{\mathfrak{m}_f} \otimes E = E \times B_E$, where $B_E = \prod_{g \in \mathcal{N}_f \setminus \{f\}} E$ and let B denote the image of $\mathbf{T}_{\mathfrak{m}}$ under the composite $\mathbf{T}_{\mathfrak{m}} \hookrightarrow \mathbf{T}_{\mathfrak{m}} \otimes E \xrightarrow{\pi_f} B_E$, where π_f is projection. Denote by $\delta : \mathbf{T}_{\mathfrak{m}_f} \hookrightarrow \mathcal{O} \times B$ the map $T \mapsto (\lambda_f(T), \pi_f(T))$. If E is sufficiently large, there exists $\eta \in \mathcal{O}$ such that $\mathrm{coker}\,\delta \cong \mathcal{O}/\eta\mathcal{O}$. This cokernel is usually called the *congruence module of f*. Set $\mathcal{N}'_f := \{g \in \mathcal{N} \mid \mathfrak{m}'_g = \mathfrak{m}'_f\}$.

Proposition 8.14. — *Assume $f \in \mathcal{N}$ is ordinary at ℓ and the associated Galois representation ρ_f is such that $\overline{\rho}_f|_{G_K}$ is absolutely irreducible. Then there exists $T \in \mathbf{T}'_{\mathfrak{m}'_f}$ such that $Tf = \eta f$, $Tf^\rho = \eta f^\rho$ and $Tg = 0$ for all $g \in \mathcal{N}'_f \setminus \{f, f^\rho\}$.*

Proof. — First note that $\mathbf{T}'_{\mathfrak{m}'_f}$ can be identified with the image of $\mathbf{T}'$ inside $\mathrm{End}_{\mathbf{C}}(S_{k-1,f})$, where $S_{k-1,f} \subset S_{k-1}\left(4, \left(\frac{-4}{\cdot}\right)\right)$ is the subspace spanned by $\mathcal{N}'_f$. By Corollary 8.12, the natural $\mathcal{O}$-algebra map $\mathbf{T}'_{\mathfrak{m}'_f} \to \mathbf{T}_{\mathfrak{m}_f}$ is an isomorphism. So, it is enough to find $T \in \mathbf{T}_{\mathfrak{m}_f}$ such that $Tf = \eta f$ and $Tg = 0$ for every $g \in \mathcal{N}_f \setminus \{f\}$. (Note that by Proposition 8.13, $f^\rho \notin \mathcal{N}_f$.) It follows from the exactness of the sequence $0 \to \mathbf{T}_{\mathfrak{m}_f} \xrightarrow{\delta} \mathcal{O} \times B \to \mathcal{O}/\eta\mathcal{O} \to 0$, that $(\eta, 0) \in \mathcal{O} \times B$ is in the image of $\mathbf{T}_{\mathfrak{m}_f} \hookrightarrow \mathcal{O} \times B$. Let T be the preimage of $(\eta, 0)$ under this injection. Then T has the desired property. □

Proposition 8.15 ([25], Theorem 2.5). — *Suppose $\ell > k$. If $f \in \mathcal{N}$ is ordinary at ℓ, then*

$$\eta = (*)\frac{\langle f, f\rangle}{\Omega_f^+\Omega_f^-},$$

where Ω_f^+, Ω_f^- denote the "integral" periods defined in [56] *and* $(*)$ *is a λ-adic unit.*

9. Galois representations and Selmer groups

In this section we will give a lower bound on the order of (the Pontryagin dual of) the Selmer group $\mathrm{ad}^0\,\rho_f|_{G_K}(-1)$ in terms of the CAP ideal (Theorem 9.10) as well as in terms of the special L-value $L^{\mathrm{int}}(\mathrm{Symm}^2 f, k)$ (Corollary 9.11). We will also discuss the relationship between Corollary 9.11 and the Bloch-Kato conjecture for the "motives" $\mathrm{ad}^0\,M_0(-1)$ and $\mathrm{ad}^0\,M_0(2)$, where M_0 is the motif (over $\mathbf{Q}$) associated to f (section 9.3).

9.1. Galois representations

It is well-known that one can attach an ℓ-adic Galois representation to every $f \in \mathcal{N}$ (cf. section 2.3). In this section we gather some basic facts concerning Galois representations attached to hermitian modular forms.

Let $F \in \mathcal{S}_k(\Gamma_{\mathbf{Z}})$ be an eigenform. For every rational prime p, let $\lambda_{p,j}(F)$, $j = 1, \dots, 4$, denote the p-Satake parameters of F. (For the definition of p-Satake parameters when p inerts or ramifies in K, see [29], and for the case when p splits in K, see [24].) Let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ lying over p. Set

$$\tilde{\lambda}_{\mathfrak{p},j}(F) := (N\mathfrak{p})^{-2+k/2}\omega^*(\mathfrak{p})\lambda_{p,j}(F),$$

where ω is the unique Hecke character of K unramified at all finite places with infinity type $\omega_\infty(x_\infty) = \left(\frac{z}{\bar{z}}\right)^{-k/2}$.

DEFINITION 9.1. — *The elements $\tilde{\lambda}_{\mathfrak{p},j}(F)$ will be called the Galois-Satake parameters of F at $\mathfrak{p}$.*

By Theorem 5.9 there exists a finite extension L_F of $\mathbf{Q}$ containing the Hecke eigenvalues of F. In what follows for a number field L and a prime $\mathfrak{p}$ of L we denote by $\mathrm{Frob}_{\mathfrak{p}}$ the arithmetic Frobenius at $\mathfrak{p}$.

THEOREM 9.2. — *There exists a finite extension E_F of $\mathbf{Q}_\ell$ containing L_F and a 4-dimensional semisimple Galois representation $\rho_F : G_K \to \mathrm{GL}_{E_F}(V)$ unramified away from the primes of K dividing 2ℓ and such that*

(i) *For any prime $\mathfrak{p}$ of K such that $\mathfrak{p} \nmid 2\ell$, the set of eigenvalues of $\rho_F(\mathrm{Frob}_{\mathfrak{p}})$ coincides with the set of the Galois-Satake parameters of F at $\mathfrak{p}$ (cf. Definition 9.1);*

(ii) *If $\mathfrak{p}$ is a place of K over ℓ, the representation $\rho_F|_{D_{\mathfrak{p}}}$ is crystalline (cf. section 9.2).*

(iii) *If $\ell > m$, and $\mathfrak{p}$ is a place of K over ℓ, the representation $\rho_F|_{D_{\mathfrak{p}}}$ is short. (For a definition of short we refer the reader to* [14]*, section 1.1.2.)*

Remark 9.3. — We know of no reference in the existing literature for the proof of this theorem, although it is widely regarded as a known result. For some discussion regarding Galois representations attached to hermitian modular forms, see [4] or [1]. We assume Theorem 9.2 in what follows.

Remark 9.4. — It is not known if the representation ρ_F is also unramified at the prime $i+1$. See [2] for a discussion of this issue.

As before, we assume that E is a sufficiently large finite extension of $\mathbf{Q}_\ell$ with valuation ring $\mathcal{O}$, uniformizer λ and residue field $\mathbf{F} = \mathcal{O}/\lambda$. Let $f = \sum_{n=1}^\infty a(n)q^n \in \mathcal{N}$ be such that $\overline{\rho}_f|_{G_K}$ is absolutely irreducible. Then by Proposition 8.13, $F_f \neq 0$. From now on we also assume that $\mathrm{ad}^0\,\overline{\rho}_f|_{G_K}$, the trace-0-endomorphisms of the representation space of $\overline{\rho}_f|_{G_K}$ with the usual G_K-action, is absolutely irreducible. Let ϵ denote the ℓ-adic cyclotomic character. It follows from Proposition 6.4 that the Galois representation $\rho_{F_f} \cong \rho_{f,K} \oplus (\rho_{f,K} \otimes \epsilon)$. From now on we assume in addition that $2^k \not\equiv a(2) \not\equiv 2^{k-4} \pmod{\lambda}$.

9.2. Selmer group

Set $\mathcal{N}^{\mathrm{NM}} := \{F \in \mathcal{N}^{\mathrm{h}} \mid F \in \mathcal{S}_k^{\mathrm{NM}}(\Gamma_{\mathbf{Z}})\}$. Let $\mathcal{M}^{\mathrm{h}}$ denote the set of maximal ideals of $\mathbf{T}^{\mathrm{h}}_{\mathcal{O}}$ and $\mathcal{M}^{\mathrm{NM}}$ the set of maximal ideals of $\mathbf{T}^{\mathrm{NM}}_{\mathcal{O}}$. We have

$\mathbf{T}_{\mathcal{O}}^{\mathrm{NM}} = \prod_{\mathfrak{m}\in\mathcal{M}^{\mathrm{NM}}} \mathbf{T}_{\mathfrak{m}}^{\mathrm{NM}}$, where $\mathbf{T}_{\mathfrak{m}}^{\mathrm{NM}}$ denotes the localization of $\mathbf{T}_{\mathcal{O}}^{\mathrm{NM}}$ at $\mathfrak{m}$. Let $\phi : \mathbf{T}_{\mathcal{O}}^{\mathrm{h}} \to \mathbf{T}_{\mathcal{O}}^{\mathrm{NM}}$ be the natural projection. We have $\mathcal{M}^{\mathrm{h}} = \mathcal{M}^{c} \sqcup \mathcal{M}^{nc}$, where $\mathcal{M}^{c}$ consists of those $\mathfrak{m} \in \mathcal{M}^{\mathrm{h}}$ which are preimages (under ϕ) of elements of $\mathcal{M}^{\mathrm{NM}}$ and $\mathcal{M}^{nc} := \mathcal{M}^{\mathrm{h}} \backslash \mathcal{M}^{c}$. Note that ϕ factors into a product $\phi = \prod_{\mathfrak{m}\in\mathcal{M}^{c}} \phi_{\mathfrak{m}} \times \prod_{\mathfrak{m}\in\mathcal{M}^{nc}} 0_{\mathfrak{m}}$, where $\phi_{\mathfrak{m}} : \mathbf{T}_{\mathfrak{m}}^{\mathrm{h}} \to \mathbf{T}_{\mathfrak{m}'}^{\mathrm{h}}$ is the projection, with $\mathfrak{m}' \in \mathcal{M}^{\mathrm{NM}}$ being the unique maximal ideal such that $\phi^{-1}(\mathfrak{m}') = \mathfrak{m}$ and $0_{\mathfrak{m}}$ is the zero map. For $F \in \mathcal{N}^{\mathrm{h}}$ we denote by $\mathfrak{m}_F$ (respectively $\mathfrak{m}_F^{\mathrm{NM}}$) the element of $\mathcal{M}^{\mathrm{h}}$ (resp. of $\mathcal{M}^{\mathrm{NM}}$) corresponding to F. In particular, $\mathfrak{m}_{F_f}^{\mathrm{NM}} \in \mathcal{M}^{\mathrm{NM}}$ is such that $\phi^{-1}(\mathfrak{m}_{F_f}^{\mathrm{NM}}) = \mathfrak{m}_{F_f}$.

We now define the Selmer group relevant for our purposes. For a profinite group $\mathcal{G}$ and a $\mathcal{G}$-module M (where we assume the action of $\mathcal{G}$ on M to be continuous) we will consider the group $H^1_{\mathrm{cont}}(\mathcal{G}, M)$ of cohomology classes of continuous cocycles $\mathcal{G} \to M$. To shorten notation we will suppress the subscript 'cont' and simply write $H^1(\mathcal{G}, M)$. For a field L, and a $\mathrm{Gal}(\overline{L}/L)$-module M (with a continuous action of $\mathrm{Gal}(\overline{L}/L)$) we sometimes write $H^1(L, M)$ instead of $H^1_{\mathrm{cont}}(\mathrm{Gal}(\overline{L}/L), M)$. We also write $H^0(L, M)$ for the submodule $M^{\mathrm{Gal}(\overline{L}/L)}$ consisting of the elements of M fixed by $\mathrm{Gal}(\overline{L}/L)$.

Let L be a number field. For a rational prime p denote by Σ_p the set of primes of L lying over p. Let $\Sigma \supset \Sigma_\ell$ be a finite set of primes of L and denote by G_Σ the Galois group of the maximal Galois extension L_Σ of L unramified outside of Σ. Let V be a finite dimensional E-vector space with a continuous G_Σ-action. Let $T \subset V$ be a G_Σ-stable $\mathcal{O}$-lattice. Set $W := V/T$.

We begin by defining local Selmer groups. For every $\mathfrak{p} \in \Sigma$ set

$$H^1_{\mathrm{un}}(L_\mathfrak{p}, M) := \ker\{H^1(L_\mathfrak{p}, M) \xrightarrow{\mathrm{res}} H^1(I_\mathfrak{p}, M)\}.$$

Define the local $\mathfrak{p}$-Selmer group (for V) by

$$H^1_{\mathrm{f}}(L_\mathfrak{p}, V) := \begin{cases} H^1_{\mathrm{un}}(L_\mathfrak{p}, V) & \mathfrak{p} \in \Sigma \setminus \Sigma_\ell \\ \ker\{H^1(L_\mathfrak{p}, V) \to H^1(L_\mathfrak{p}, V \otimes B_{\mathrm{crys}})\} & \mathfrak{p} \in \Sigma_\ell. \end{cases}$$

Here B_{crys} denotes Fontaine's ring of ℓ-adic periods (cf. [19]).

For $\mathfrak{p} \in \Sigma_\ell$, we call the $D_\mathfrak{p}$-module V *crystalline* (or the G_L-module V *crystalline at* $\mathfrak{p}$) if $\dim_{\mathbf{Q}_\ell} V = \dim_{\mathbf{Q}_\ell} H^0(L_\mathfrak{p}, V \otimes B_{\mathrm{crys}})$. When we refer to a Galois representation $\rho : G_L \to GL(V)$ as being crystalline at $\mathfrak{p}$, we mean that V with the G_L-module structure defined by ρ is crystalline at $\mathfrak{p}$.

For every $\mathfrak{p}$, define $H^1_{\mathrm{f}}(L_\mathfrak{p}, W)$ to be the image of $H^1_{\mathrm{f}}(L_\mathfrak{p}, V)$ under the natural map $H^1(L_\mathfrak{p}, V) \to H^1(L_\mathfrak{p}, W)$. Using the fact that $\mathrm{Gal}(\overline{\kappa}_\mathfrak{p} : \kappa_\mathfrak{p}) = \hat{\mathbf{Z}}$ has cohomological dimension 1, one easily sees that if W is unramified at $\mathfrak{p}$ and $\mathfrak{p} \notin \Sigma_\ell$, then $H^1_{\mathrm{f}}(L_\mathfrak{p}, W) = H^1_{\mathrm{un}}(L_\mathfrak{p}, W)$. Here $\kappa_\mathfrak{p}$ denotes the residue field of $L_\mathfrak{p}$.

For a $\mathbf{Z}_\ell$-module M, we write $M^\vee$ for its Pontryagin dual defined as

$$M^\vee = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbf{Q}_\ell/\mathbf{Z}_\ell).$$

Moreover, if M is a Galois module, we denote by $M(n) := M \otimes \epsilon^n$ its n-th Tate twist.

Definition 9.5. — *For each finite set $\Sigma' \subset \Sigma \setminus \Sigma_\ell$, the group*

$$\mathrm{Sel}_\Sigma(\Sigma', W) := \ker \left\{ H^1(G_\Sigma, W) \xrightarrow{\mathrm{res}} \bigoplus_{\mathfrak{p} \in \Sigma' \cup \Sigma_\ell} \frac{H^1(L_\mathfrak{p}, W)}{H^1_{\mathrm{f}}(L_\mathfrak{p}, W)} \right\}$$

is called the (global) Selmer group of the triple (Σ, Σ', W). We also set $S_\Sigma(\Sigma', W) := \mathrm{Sel}_\Sigma(\Sigma', W)^\vee$, $\mathrm{Sel}_\Sigma(W) := \mathrm{Sel}_\Sigma(\emptyset, W)$ and $S_\Sigma(W) = S_\Sigma(\emptyset, W)$. Define $\mathrm{Sel}_\Sigma(\Sigma', V)$ in the same way with V instead of W.

For $L = \mathbf{Q}$, the group $\mathrm{Sel}_\Sigma(\Sigma \setminus \Sigma_\ell, W)$ is the standard Selmer group $H^1_{\mathrm{f}}(\mathbf{Q}, W)$ defined by Bloch and Kato [5], section 5.

Let Σ, Σ' be as above. Let $\rho : G_\Sigma \to \mathrm{GL}_E(V)$ denote the representation giving the action of G_Σ on V. The following two lemmas are easy (cf. [46], Lemma 1.5.7 and [52]).

Lemma 9.6. — *$S_\Sigma(\Sigma', W)$ is a finitely generated $\mathcal{O}$-module.*

Lemma 9.7. — *If the mod λ reduction $\overline{\rho}$ of ρ is absolutely irreducible, then the length of $S_\Sigma(\Sigma', W)$ as an $\mathcal{O}$-module is independent of the choice of the lattice T.*

Remark 9.8. — For an $\mathcal{O}$-module M,

$$\mathrm{ord}_\ell(\# M) = [\mathcal{O}/\lambda : \mathbf{F}_\ell] \, \mathrm{length}_{\mathcal{O}}(M).$$

Example 9.9. — Let $L = K$, $\Sigma = \Sigma_\ell$, $\rho_{f,K} := \rho_f|_{G_K}$ and let V denote the representation space of

$$\mathrm{ad}^0 \rho_{f,K}(-1) = \mathrm{ad}^0 \rho_{f,K} \otimes \epsilon^{-1} \subset \mathrm{Hom}_E(\rho_{f,K} \otimes \epsilon, \rho_{f,K})$$

of G_K. Let $T \subset V$ be some choice of a G_K-stable lattice. Set $W = V/T$. Note that the action of G_K on V factors through G_Σ. Since the mod λ reduction of $\mathrm{ad}^0 \rho_{f,K} \otimes \epsilon^{-1}$ is absolutely irreducible by assumption, $\mathrm{ord}_\ell(S_\Sigma(W))$ is independent of the choice of T.

Our goal is to prove the following theorem.

Theorem 9.10. — *Let L, Σ and W be as in Example 9.9. Suppose that for each $F \in \mathcal{N}^{\mathrm{NM}}_{F_f}$, the representation $\rho_F : G_K \to \mathrm{GL}_4(E)$ is absolutely irreducible. Then*

$$\mathrm{ord}_\ell(\# S_\Sigma(W)) \geqslant \mathrm{ord}_\ell(\# \mathbf{T}^{\mathrm{NM}}_{\mathfrak{m}_{F_f}} / \phi_{\mathfrak{m}_{F_f}}(\mathrm{Ann}(F_f))).$$

Corollary 9.11. — *With the same assumptions and notation as in Theorem 7.12 and Theorem 9.10 we have*

$$\mathrm{ord}_\ell(\#S_\Sigma(W)) \geqslant n.$$

If in addition the character χ in Theorem 7.12 can be taken as in Corollary 7.14, then

$$\mathrm{ord}_\ell(\#S_\Sigma(W)) \geqslant \mathrm{ord}_\ell(\#\mathcal{O}/L^{\mathrm{int}}(\mathrm{Symm}^2 f, k)).$$

Proof. — The corollary follows immediately from Theorem 9.10 and Corollary 7.19. □

9.3. Relations to the Bloch-Kato Conjecture

In this section we discuss how our results (Theorem 9.10 and Corollary 9.11) are related to the Bloch-Kato conjecture. We begin by recalling the statement of the conjecture in our particular case. We follow closely the exposition in [14]. For more details as well as precise definitions the reader is encouraged to consult [14], section 2.4 and [20].

Let E_0 be a number field, which we will assume to be "sufficiently large" (in particular we assume that E_0 contains all the Hecke eigenvalues of f) and write E for its completion at a prime λ lying over ℓ determined by our choice of the embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$. This is consistent with our previous definition of E as a sufficiently large finite extension of $\mathbf{Q}_\ell$. Let M_0 be the "premotivic structure" attached to f over $\mathbf{Q}$ with coefficients in E_0. Write

$$M := \{M_B, M_{\mathrm{dR}}, \{M_v\}_v, I^\infty, \{I_B^v\}_v, \{I^v\}_v, \{W^i\}_i\},$$

for the premotivic structure $\mathrm{ad}^0 M_0(-1)$. Here v runs over the set of finite places of E_0, M_B (resp. M_{dR}; M_v) is a finite dimensional vector space over E_0 (resp. E_0; $E_{0,v}$), with an action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ (resp. with a decreasing filtration Fil^i; with a pseudo-geometric action of $G_{\mathbf{Q}}$), $I^\infty : \mathbf{C}\otimes M_{\mathrm{dR}} \to \mathbf{C}\otimes M_{\mathrm{B}}$ (resp. $I_{\mathrm{B}}^v : E_{0,v}\otimes_{E_0} M_{\mathrm{B}} \to M_v$; $I^v : B_{\mathrm{dR},p}\otimes_{\mathbf{Q}_p} E_{0,v}\otimes_{E_0} M_{\mathrm{dR}} \to B_{\mathrm{dR},p}\otimes_{\mathbf{Q}_p} M_v$) is a $\mathbf{C}\otimes E_0$-linear (resp. $E_{0,v}$-linear; $B_{\mathrm{dR},p}\otimes_{\mathbf{Q}_p} E_{0,v}$-linear with $v \mid p$) isomorphism respecting the $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-action (resp. the $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-action; the $G_{\mathbf{Q}_p}$-action and filtrations), where $B_{\mathrm{dR},p}$ is the ring defined by Fontaine, and W^i are the so called weight filtrations, whose definition we omit. Similarly one defines the premotivic structure $\mathrm{ad}^0 M_0(2)$, which we denote by M^*. We have $M_\lambda = V_{\mathbf{Q}}$ and $M_\lambda^* = V_{\mathbf{Q}}^* := V_{\mathbf{Q}}(3)$, where $V_{\mathbf{Q}}$ is the $E[G_{\mathbf{Q}}]$-module $\mathrm{ad}^0 \rho_f(-1)$.

From now on let $\mathcal{M} \in \{M, M^*\}$. We adopt similar notation for other symbols, e.g., $\mathcal{V}_{\mathbf{Q}} \in \{V_{\mathbf{Q}}, V_{\mathbf{Q}}^*\}$, where the choice of $\mathcal{M}$ determines the choice of $\mathcal{V}$ and other symbols related to $\mathcal{M}$ in an obvious way.

Let

$$\Delta_{\mathrm{f}}(\mathcal{M}) = \mathrm{Hom}_{E_0}(\det_{E_0} \mathcal{M}_B^+, \det_{E_0}(\mathcal{M}_{\mathrm{dR}}/\mathrm{Fil}^0 \mathcal{M}_{\mathrm{dR}}))$$

be the *fundamental line* for $\mathcal{M}$ and write $\delta_{\mathrm{f}}(\mathcal{M})$ for the $\mathcal{O}$-lattice in $E \otimes_{E_0} \Delta_{\mathrm{f}}(\mathcal{M})$ defined by Fontaine and Perrin-Riou (see [20], section II.4 or [14], p. 700 or [33] for details). Here $^+$ indicates the subspace fixed by $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$.

The Bloch-Kato conjecture relates the lattice $\delta_{\mathrm{f}}(\mathcal{M})$ to the value at 0 of an L-function of $\mathcal{M}$ normalized by a certain period, both of which we now define.

The isomorphism I^∞ gives rise to an $\mathbf{R} \otimes E_0$-linear isomorphism

$$\mathbf{R} \otimes \mathcal{M}_{\mathrm{B}}^+ \to (\mathbf{C} \otimes \mathcal{M}_{\mathrm{B}})^+ \xrightarrow{(I^\infty)^{-1}} \mathbf{R} \otimes \mathcal{M}_{\mathrm{dR}} \to \mathbf{R} \otimes \mathcal{M}_{\mathrm{dR}}/\mathrm{Fil}^0 \mathcal{M}_{\mathrm{dR}},$$

whose determinant over $\mathbf{R} \otimes E_0$ is $c^+(\mathcal{M}) \in \mathbf{R} \otimes \Delta_{\mathrm{f}}(\mathcal{M})$ which we will refer to as the *Deligne period* of $\mathcal{M}$. Note that $c^+(\mathcal{M})$ is canonically defined, i.e., not just up to a multiplication by an element of $E_0^\times$ as in [12].

Let $\alpha_{p,1}$ and $\alpha_{p,2}$ be the p-Satake parameters of f as defined in section 4.2. For an odd prime p, set

$$L_p(\mathrm{ad}^0 M_0, s) := (1 - \alpha_{p,1}\alpha_{p,2}^{-1} p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \alpha_{p,1}^{-1}\alpha_{p,2} p^{-s})^{-1}$$

and put

$$L_2(\mathrm{ad}^0 M_0, s) := (1 - 2^{-s})^{-1}.$$

Then the L-function of $\mathrm{ad}^0 M_0(n)$ is defined as

$$L(\mathrm{ad}^0 M_0(n), s) := \prod_p L_p(\mathrm{ad}^0 M_0, s + n). \tag{9.1}$$

In particular we have

$$L(M, s) = L(\mathrm{ad}^0 M_0, s - 1)$$

and

$$L(M^*, s) = L(\mathrm{ad}^0 M_0, s + 2).$$

The properties of $L(\mathrm{ad}^0 M_0, s)$ are summarized in [14], p. 686. In particular $L(\mathrm{ad}^0 M_0, s)$ is entire as a function of s and satisfies a functional equation with respect to $s \mapsto 1 - s$.

Remark 9.12 (Geometric vs. arithmetic Frobenius). — In general one defines the L-function of a motive $\mathcal{N}$ as $\prod_p \det(1 - f_p p^{-s})^{-1}$, where f_p denotes the action of the Frobenius element at p on the inertia invariants of the space of $\mathcal{N}_\lambda$ (or on $\mathrm{Crys}(\mathcal{N}_\lambda)$ if $p = \ell$) - see [5], p. 361 for more details. However, this definition depends in general on whether one uses the geometric or arithmetic Frobenius. In [5] Bloch and Kato use the geometric Frobenius and then their conjecture relates $L(\mathcal{N},0)$ to the Selmer group of $\mathcal{N}$. Moreover, in that case one has $L(\mathcal{N}(n),s) = L(\mathcal{N},s+n)$. Also note that $L(\mathrm{ad}^0 M_0, s)$ is independent of the choice of geometric or arithmetic Frobenius, since the set of eigenvalues of $\mathrm{ad}^0 \rho_f$ is of the form $\{\alpha, 1, \alpha^{-1}\}$ hence is invariant under taking the inverse. To be able to keep with the spirit of the original paper of Bloch and Kato, we defined our L-function in (9.1) so that it agrees with the L-function defined in [5], i.e., $L(\mathrm{ad}^0 M_0(n), s) := \prod_p (1 - f_p p^{-s})$, where f_p denotes the action of Frob_p^{-1} on the appropriate space (see above) and Frob_p is the arithmetic Frobenius at p as before.

It follows from a result of Sturm [53] that there exists a basis $b(\mathcal{M})$ of $\Delta_{\mathrm{f}}(\mathcal{M})$ such that

$$L(\mathcal{M},0)(1 \otimes b(\mathcal{M})) = c^+(\mathcal{M}).$$

The first version of (the λ-part of) the Bloch-Kato conjecture can be formulated as follows.

Conjecture 9.13 (Bloch-Kato, cf. [14], Conjecture 2.14). — *One has*

$$\delta_{\mathrm{f}}(\mathcal{M}) = (1 \otimes b(\mathcal{M}))\mathcal{O} \tag{9.2}$$

as lattices in $E \otimes \Delta_{\mathrm{f}}(\mathcal{M})$.

In [20], Diamond, Flach and Guo give an alternative description of the lattice $\delta_{\mathrm{f}}(\mathcal{M})$ in terms of Tate-Shafarevich groups. This description will allow us to state a different version of Conjecture 9.13 and relate it to Theorem 9.10 and Corollary 9.11.

Set $\Sigma := \Sigma_\ell \cup \Sigma_2$. Note that the representation $\mathcal{V}_{\mathbf{Q}}$ is unramified outside Σ. To formulate the second version of the Bloch-Kato conjecture one needs to choose "integral structures" on the one-dimensional E-vector spaces $E \otimes \det_{E_0} \mathcal{M}_{\mathrm{B}}^+$ and $E \otimes \det_{E_0}(\mathcal{M}_{\mathrm{dR}}/\,\mathrm{Fil}^0 \mathcal{M}_{\mathrm{dR}})$. One does this by choosing a free rank one $\mathcal{O}$-module $\omega(\mathcal{M}) \subset E \otimes \det_{E_0}(\mathcal{M}_{\mathrm{dR}}/\mathrm{Fil}^0\mathcal{M}_{\mathrm{dR}}))$, which in the following we abbreviate as ω, and a Galois stable $\mathcal{O}$-lattice $\mathcal{T}_{\mathbf{Q}} \subset \mathcal{V}_{\mathbf{Q}}$ which gives rise to a free rank one $\mathcal{O}$-module in $E \otimes \det_{E_0} \mathcal{M}_{\mathrm{B}}^+$ via the isomorphism I_{B}^λ. Set $\mathcal{W}_{\mathbf{Q}} := \mathcal{V}_{\mathbf{Q}}/\mathcal{T}_{\mathbf{Q}}$. Let

$$Ш(\mathcal{T}_{\mathbf{Q}}) := \frac{\mathrm{Sel}_\Sigma(\Sigma_2, \mathcal{W}_{\mathbf{Q}})}{\mathrm{Sel}_\Sigma(\Sigma_2, \mathcal{V}_{\mathbf{Q}}) \otimes (E/\mathcal{O})}$$

be the Tate-Shafarevich group of $\mathcal{T}_{\mathbf{Q}}$. The group $Ш(\mathcal{T}_{\mathbf{Q}})$ is finite ([20], Proposition II.5.3.5). Put $\mathcal{T}_{\mathbf{Q}}^D := \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}_{\mathbf{Q}}, \mathcal{O}(1))$ and set $\mathcal{V}_{\mathbf{Q}}^D := \mathcal{T}_{\mathbf{Q}}^D \otimes_{\mathcal{O}} E$ and $\mathcal{W}_{\mathbf{Q}}^D := \mathcal{V}_{\mathbf{Q}}^D/\mathcal{T}_{\mathbf{Q}}^D$. Note that $V_{\mathbf{Q}}^D \cong V_{\mathbf{Q}}^*$ and $(V_{\mathbf{Q}}^*)^D \cong V_{\mathbf{Q}}$.

Assume that

$$\mathrm{Sel}_\Sigma(\Sigma_2, V_{\mathbf{Q}}) = \mathrm{Sel}_\Sigma(\Sigma_2, V_{\mathbf{Q}}^D) = 0. \tag{9.3}$$

This follows from a conjecture on the order of vanishing of $L(\mathcal{M},0)$ (cf. [20], section III.4.2.2) and the fact that in our case

$$H^0(\mathbf{Q}, V_{\mathbf{Q}}) = H^0(\mathbf{Q}, V_{\mathbf{Q}}^D) = 0. \tag{9.4}$$

For a commutative ring R and a finitely generated R-module N, denote by $\mathrm{Fitt}_R(N)$ the Fitting ideal of N in R. For the definition and basic properties of Fitting ideals see for example the Appendix of [41]. Using Theorem II.5.3.6 in [20], Diamond, Flach and Guo show that

$$\delta_{\mathrm{f}}(\mathcal{M}) = \frac{\mathrm{Fitt}_{\mathcal{O}} H^0(\mathbf{Q}, \mathcal{W}_{\mathbf{Q}}) \cdot \mathrm{Fitt}_{\mathcal{O}} H^0(\mathbf{Q}, \mathcal{W}_{\mathbf{Q}}^D)}{\mathrm{Fitt}_{\mathcal{O}} Ш(\mathcal{T}_{\mathbf{Q}}^D) \cdot \mathrm{Tam}^0_\omega(\mathcal{T}_{\mathbf{Q}})} L_\omega(\mathcal{T}_{\mathbf{Q}}), \tag{9.5}$$

where $L_\omega(\mathcal{T}_{\mathbf{Q}})$ is a lattice in $E \otimes \Delta_{\mathrm{f}}(\mathcal{M})$ depending on the choice of the "integral structures" $\mathcal{T}_{\mathbf{Q}}$ and $\omega = \omega(\mathcal{M})$, and

$$\mathrm{Tam}^0_\omega(\mathcal{T}_{\mathbf{Q}}) = \mathrm{Tam}^0_{\ell,\omega}(\mathcal{T}_{\mathbf{Q}}) \cdot \mathrm{Tam}^0_\infty(\mathcal{T}_{\mathbf{Q}}) \cdot \prod_{p \neq \ell, \infty} \mathrm{Tam}^0_p(\mathcal{T}_{\mathbf{Q}})$$

is the Tamagawa ideal of $\mathcal{T}_{\mathbf{Q}}$ relative to ω (cf. [20], section II.5.3). It follows from Proposition II.4.2.2 in [20] that $\mathrm{Tam}^0_p(\mathcal{T}_{\mathbf{Q}}) = \mathrm{Tam}^0_\infty(\mathcal{T}_{\mathbf{Q}}) = \mathcal{O}$ for all $p \neq \ell$.

The integral structures $\mathcal{T}_{\mathbf{Q}}$ and ω give an identification of $E \otimes \Delta_{\mathrm{f}}(\mathcal{M})$ with E. Then the quotient $L_\omega(\mathcal{T}_{\mathbf{Q}})/\mathrm{Tam}^0_{\ell,\omega}(\mathcal{T}_{\mathbf{Q}})$ is identified with a fractional ideal of E whose inverse we denote by $\mathrm{Tam}_\omega(\mathcal{T}_{\mathbf{Q}})$. Similarly, $(1 \otimes b(\mathcal{M}))\mathcal{O}$ is identified with a fractional ideal $(\Omega_\omega(\mathcal{T}_{\mathbf{Q}})/L(\mathcal{M},0)) \cdot \mathcal{O}$ of E for some $\Omega_\omega(\mathcal{T}_{\mathbf{Q}}) \in E/\mathcal{O}^\times$. Using our assumption (9.3), we get $Ш(\mathcal{T}_{\mathbf{Q}}) = \mathrm{Sel}_\Sigma(\Sigma_2, \mathcal{W}_{\mathbf{Q}})$. Moreover, as explained in [14], p. 708, one also has an $\mathcal{O}$-linear isomorphism $Ш(\mathcal{T}_{\mathbf{Q}}^D) \cong \mathrm{Hom}_{\mathbf{Z}_\ell}(Ш(\mathcal{T}_{\mathbf{Q}}), \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ and the latter group is just $S_\Sigma(\Sigma_2, \mathcal{W}_{\mathbf{Q}})$.

Using the above arguments and (9.4), Conjecture 9.13 can be rephrased in the following way.

Conjecture 9.14 (Bloch-Kato conjecture, second version). — *For $\mathcal{M} \in \{M, M^*\}$ one has*

$$\# S_\Sigma(\Sigma_2, \mathcal{W}_{\mathbf{Q}}) \cdot \mathrm{Tam}_\omega(\mathcal{T}_{\mathbf{Q}}) = \frac{L(\mathcal{M},0)}{\Omega_\omega(\mathcal{T}_{\mathbf{Q}})} \mathcal{O} \tag{9.6}$$

as fractional ideals of E.

Let $\mathcal{T}_K$ be $\mathcal{T}_{\mathbf{Q}}$ considered as an $\mathcal{O}[G_K]$-module. Set $\mathcal{V}_K := \mathcal{T}_K \otimes E$ and $\mathcal{W}_K := \mathcal{V}_K/\mathcal{T}_K$. Then $\mathcal{V}_K$ is unramified away from primes of K lying over ℓ.

The following is just a restatement of Corollary 9.11.

Theorem 9.15. — *With the same assumptions as in Theorem 7.12 and in Corollary 9.11 we have the following containment of fractional ideals of E:*

$$\#S_{\Sigma_\ell}(W_K)\cdot\mathcal{O}\subset L^{\mathrm{int}}(\mathrm{Symm}^2 f,k)\cdot\mathcal{O}. \tag{9.7}$$

We will now discuss the relation between Theorem 9.15 and Conjecture 9.14. As before, let $\mathcal{M}\in\{M,M^*\}$. Write $s(M)=k-3$ and $s(M^*)=k$.

Theorem 9.15 falls short of proving that the left-hand side of (9.6) is contained in the right-hand side of (9.6), but gives some evidence for this containment. First note that

$$L(\mathcal{M},0)=L(\mathrm{ad}^0 M_0,s(\mathcal{M})-(k-2))=\alpha(\mathcal{M})L(\mathrm{Symm}^2 f,s(\mathcal{M})),$$

where $\alpha(\mathcal{M})$ comes from the discrepancy in the definitions of the Euler factors at 2 of $L(\mathcal{M},s)$ and $L(\mathrm{Symm}^2 f,s)$ and $\alpha(\mathcal{M})\in\mathcal{O}$ since $\lambda\nmid 2$. Moreover, using the functional equation for $L(\mathrm{Symm}^2 f,s)$ (cf. [47]) one concludes that

$$\mathrm{ord}_\ell(L^{\mathrm{int}}(\mathrm{Symm}^2 f,k-3))=\mathrm{ord}_\ell(L^{\mathrm{int}}(\mathrm{Symm}^2 f,k)).$$

Recall that the order of the Selmer group is independent of the choice of the lattice $\mathcal{T}_{\mathbf{Q}}$, hence we can fix $\mathcal{T}_{\mathbf{Q}}$ as in [14], section 1.6.2 and $\omega(\mathcal{M})$ as in [14], p. 709. It has been shown by Dummigan [17], p.11 using Proposition 7.7 in [12] and some arguments in [14] that with these choices of $\mathcal{T}_{\mathbf{Q}}$ and $\omega(\mathcal{M})$, one has $\Omega_\omega(\mathcal{T}_{\mathbf{Q}})=u\pi^{-s(\mathcal{M})-2}\Omega_f^+\Omega_f^-$ for u an ℓ-adic unit. Thus the right-hand side of (9.6) is the same for M and M^* and is contained in the right-hand side of (9.7). This containment is an equality if $\alpha(\mathcal{M})$ is a λ-adic unit.

Similarly, using the fact that $Ш(\mathcal{T}_{\mathbf{Q}}^D)\cong Ш(\mathcal{T}_{\mathbf{Q}})^\vee$ one sees that $\#S_\Sigma(\Sigma_2,W_{\mathbf{Q}})=\#S_\Sigma(\Sigma_2,W_{\mathbf{Q}}^*)$. On the other hand Theorem 9.15 concerns the group $S_{\Sigma_\ell}(W_K)=S_\Sigma(\Sigma_2,W_K)$, which can be potentially larger than $S_\Sigma(\Sigma_2,W_{\mathbf{Q}})$ (this follows from the inflation-restriction sequence), so (9.7) does not imply an analogous containment for $S_\Sigma(W_{\mathbf{Q}})$ or $S_\Sigma(W_{\mathbf{Q}}^*)$ - see also Remark 9.16 below. Finally, we are not able to show that with the choice of $\mathcal{T}_{\mathbf{Q}}$ and ω as above, one has $\mathrm{Tam}_\omega(\mathcal{T}_{\mathbf{Q}})\supset\mathcal{O}$. Dummigan in [15], section 7 and [16], section 6 showed that $\mathrm{Tam}_\omega(\mathcal{T}_{\mathbf{Q}})=\mathcal{O}$ if f is a modular form of level 1. See [15], section 10 for a discussion of the difficulties involved, when the level of f is larger than one. Diamond, Flach and Guo in [14] have computed the Tamagawa ideal for the motives $\mathrm{ad}^0 M_0$ and $\mathrm{ad}^0 M_0(1)$ (cf. the proof of Theorem 2.15 and Proposition 2.16 in [14]). However, their calculations cannot be extended to our case.

To summarize, if f is such that $\#S_\Sigma(\Sigma_2, \mathcal{W}_K) = \#S_\Sigma(\Sigma_2, \mathcal{W}_{\mathbf{Q}})$ and $\mathrm{Tam}_\omega(\mathcal{T}_{\mathbf{Q}}) = \mathcal{O}$, Theorem 9.15 implies that the right-hand side of (9.6) contains the left-hand side of (9.6).

Remark 9.16. — One can state a conjecture similar to Conjecture 9.14 for the restriction $M|_K$ of the premotivic structure M to G_K. Then the L-function $L(M|_K, 0)$ factors as

$$L(M,0)L\left(M,0,\left(\frac{-4}{\cdot}\right)\right) = \alpha(M)L(\mathrm{Symm}^2 f, k-3)L\left(\mathrm{Symm}^2 f, k-3, \left(\frac{-4}{\cdot}\right)\right)$$

and the Selmer group on the left-hand-side of (9.6) is replaced by $S_{\Sigma_\ell}(W_K)$. So, this version of the Bloch-Kato conjecture gives us the same Selmer group as the one in Theorem 9.15, but an extra L-value $L\Big(\mathrm{Symm}^2 f, k - 3, \left(\frac{-4}{\cdot}\right)\Big)$. Unfortunately we are unaware of any rationality results for that special value. One needs a statement that would involve the period $c^+(M|_K)$ used in the formulation of the Bloch-Kato conjecture. For the value $L(\mathrm{Symm}^2 f, k)$ we have used a rationality result due to Sturm [53], who uses a period related to the Petersson inner product $\langle f, f\rangle$, but his theorem (cf. [53], p. 220) specifically excludes the value $L\left(\mathrm{Symm}^2 f, k-3, \left(\frac{-4}{\cdot}\right)\right)$.

Remark 9.17. — In [14] Diamond, Flach and Guo proved the λ-part of the Bloch-Kato conjecture for the motives $\mathrm{ad}^0 M_0$ and $\mathrm{ad}^0 M_0(1)$ using an extension of the methods of Taylor and Wiles [61, 54]. The latter two motives are in duality and the two L-values $L(\mathrm{ad}^0 M_0, 0)$ and $L(\mathrm{ad}^0 M_0(1), 0)$ are related by the functional equation. Hence our result provides evidence for an extension of their theorem to the motives $\mathrm{ad}^0 M_0(-1)$ and $\mathrm{ad}^0 M_0(2)$.

9.4. Degree n Selmer groups

In this section we collect some technical results regarding Selmer groups which will be used in the proof of Theorem 9.10. Let $\mathcal{G}$ be a group, R a commutative ring with identity, M a finitely generated R-module with an R-linear action of $\mathcal{G}$ given by a homomorphism $\rho : \mathcal{G} \to \mathrm{Aut}_R(M)$. For any two such pairs (M', ρ'), (M'', ρ''), the R-module $\mathrm{Hom}_R(M'', M')$ is naturally a $\mathcal{G}$-module with the $\mathcal{G}$-action given by

$$(g \cdot \phi)(m'') = \rho'(g)\phi(\rho''(g^{-1})m'').$$

Suppose there exists (M,ρ) which fits into an exact sequence of $R[\mathcal{G}]$-modules

$$X: \quad 0 \to M' \to M \to M'' \to 0,$$

that splits as a sequence of R-modules. Choose $s_X : M'' \to M$, an R-section of X. Define $\phi_X : \mathcal{G} \to \mathrm{Hom}_R(M'', M')$ to be the map sending g to the homomorphism $m'' \mapsto \rho(g)s_X(\rho''(g)^{-1}m'') - s_X(m'')$.

Lemma 9.18. — *Let* $\mathrm{Ext}_{R[\mathcal{G}]}(M'', M')$ *denote the set of equivalence classes of* $R[\mathcal{G}]$*-extensions of* M'' *by* M' *which split as extensions of* R*-modules. The map* $X \mapsto \phi_X$ *defines a bijection between* $\mathrm{Ext}_{R[\mathcal{G}]}(M'', M')$ *and* $H^1(\mathcal{G}, \mathrm{Hom}_R(M'', M'))$.

Proof. — The proof is a simple modification of the proof of Proposition 4 in [58]. □

Let E, $\mathcal{O}$ and λ be as before. Let L be a number field and Σ a finite set of places of L containing Σ_ℓ. Let $\rho' : G_\Sigma \to \mathrm{GL}_E(V')$, $\rho'' : G_\Sigma \to \mathrm{GL}_E(V'')$ be two Galois representations. Choose G_Σ-stable $\mathcal{O}$-lattices $T' \subset V'$, $T'' \subset V''$, and denote the corresponding representations by $(T', \rho'_{T'})$ and $(T'', \rho''_{T''})$ respectively. Define $W' := V'/T'$, and $W'' := V''/T''$. Set $V = \mathrm{Hom}_E(V'', V')$. Let $T \subset V$ be a G_Σ-stable $\mathcal{O}$-lattice, and set $W = V/T$. For an $\mathcal{O}$-module M, let $M[n]$ denote the submodule consisting of elements killed by λ^n. For $\mathfrak{p} \in \Sigma$, Lemma 9.18 provides a natural bijection between $H^1(L_\mathfrak{p}, W[n])$ and $\mathrm{Ext}_{\mathcal{O}/\lambda^n[D_\mathfrak{p}]}(W''[n], W'[n])$. We now define *degree n local Selmer groups.* If $\mathfrak{p} \in \Sigma \setminus \Sigma_\ell$, set

$$H^1_{\mathrm{f}}(L_\mathfrak{p}, W[n]) := H^1_{\mathrm{un}}(L_\mathfrak{p}, W[n]), \quad \text{where } W \text{ is as above.}$$

If $\mathfrak{p} \in \Sigma_\ell$, define $H^1_{\mathrm{f}}(L_\mathfrak{p}, W[n]) \subset H^1(L_\mathfrak{p}, W[n])$ to be the subset consisting of those cohomology classes which correspond to extensions

$$0 \to W'[n] \to \tilde{W}[n] \to W''[n] \to 0 \quad \in \mathrm{Ext}_{\mathcal{O}/\lambda^n[D_\mathfrak{p}]}(W''[n], W'[n])$$

such that $\tilde{W}[n]$ is in the essential image of the functor $\mathbf{V}$ defined in [14], section 1.1.2. We will not need the precise definition of $\mathbf{V}$. It is shown in [14] that $H^1_{\mathrm{f}}(L_\mathfrak{p}, W[n])$ is an $\mathcal{O}$-submodule of $H^1(L_\mathfrak{p}, W[n])$ and that $H^1_{\mathrm{f}}(L_\mathfrak{p}, W[n])$ is the preimage of $H^1_{\mathrm{f}}(L_\mathfrak{p}, W[n+1])$ under the natural map $H^1(L_\mathfrak{p}, W[n]) \to H^1(L_\mathfrak{p}, W[n+1])$ (cf. Section 2.1, loc. cit.).

Lemma 9.19. — *Fix* $\mathfrak{p} \in \Sigma_\ell$. *Let* $\tilde{\rho} : G_L \to \mathrm{GL}_E(\tilde{V})$ *be a Galois representation short at* $\mathfrak{p}$, $\tilde{T} \subset \tilde{V}$ *an* $\mathcal{O}[D_\mathfrak{p}]$*-stable lattice and* $\tilde{W} := \tilde{V}/\tilde{T}$. *If* $\tilde{W}[n]$ *fits into an exact sequence*

$$0 \to W'[n] \to \tilde{W}[n] \to W''[n] \to 0 \quad \in \mathrm{Ext}_{\mathcal{O}/\lambda^n[D_\mathfrak{p}]}(W''[n], W'[n]),$$

then such an extension gives rise to an element of $H^1_{\mathrm{f}}(L_\mathfrak{p}, W[n])$.

Proof. — See [14], Section 1.1.2. □

PROPOSITION 9.20. — *The natural isomorphism*

$$\varinjlim_n H^1(L_{\mathfrak{p}}, W[n]) \cong H^1(L_{\mathfrak{p}}, W)$$

induces a natural isomorphism

$$\varinjlim_n H^1_{\mathrm{f}}(L_{\mathfrak{p}}, W[n]) \cong H^1_{\mathrm{f}}(L_{\mathfrak{p}}, W).$$

Proof. — See [14], Proposition 2.2. □

9.5. Proof of Theorem 9.10

The key ingredient in the proof of Theorem 9.10 is Lemma 9.21 below. Before we state it, we need some notation. Let L be any number field, $\Sigma \supset \Sigma_\ell$ a finite set of primes of L. Let $n', n'' \in \mathbf{Z}_{\geqslant 0}$ and $n := n'+n''$. Let V' (respectively V'') be an E-vector space of dimension n' (resp. n''), affording a continuous absolutely irreducible representation $\rho' : G_\Sigma \to \mathrm{Aut}_E(V')$ (resp. $\rho'' : G_\Sigma \to \mathrm{Aut}_E(V'')$). Assume that the residual representations $\overline{\rho}'$ and $\overline{\rho}''$ are also absolutely irreducible (hence well-defined) and non-isomorphic. Let $V_1, \dots, V_m$ be n-dimensional E-vector spaces each of them affording an absolutely irreducible continuous representation $\rho_i : G_\Sigma \to \mathrm{Aut}_E(V_i)$, $i = 1, \dots, m$. Moreover assume that the mod λ reductions $\overline{\rho}_i$ (with respect to some G_Σ-stable lattice in V_i and hence with respect to all such lattices) satisfy

$$\overline{\rho}_i^{\mathrm{ss}} \cong \overline{\rho}' \oplus \overline{\rho}''.$$

For $\sigma \in G_\Sigma$, let $\sum_{j=0}^n a_j(\sigma) X^j \in \mathcal{O}[X]$ be the characteristic polynomial of $(\rho' \oplus \rho'')(\sigma)$ and let $\sum_{j=0}^n c_j(i,\sigma) X^j \in \mathcal{O}[X]$ be the characteristic polynomial of $\rho_i(\sigma)$. Put $c_j(\sigma) := \begin{bmatrix} c_j(1,\sigma) \\ \dots \\ c_j(m,\sigma) \end{bmatrix} \in \mathcal{O}^m$ for $j = 0, 1, \dots, n-1$. Let $\mathbf{T} \subset \mathcal{O}^m$ be the $\mathcal{O}$-subalgebra generated by the set $\{c_j(\sigma) \mid 0 \leqslant j \leqslant n-1, \sigma \in G_\Sigma\}$. By continuity of the ρ_i this is the same as the $\mathcal{O}$-subalgebra of $\mathcal{O}^m$ generated by $\{c_j(\mathrm{Frob}_{\mathfrak{p}}) \mid 0 \leqslant j \leqslant n-1, \mathfrak{p} \notin \Sigma\}$. Note that $\mathbf{T}$ is a finite $\mathcal{O}$-algebra. Let $I \subset \mathbf{T}$ be the ideal generated by the set $\{c_j(\mathrm{Frob}_{\mathfrak{p}}) - a_j(\mathrm{Frob}_{\mathfrak{p}}) \mid 0 \leqslant j \leqslant n-1, \mathfrak{p} \notin \Sigma\}$. From the definition of I it follows that the $\mathcal{O}$-algebra structure map $\mathcal{O} \to \mathbf{T}/I$ is surjective. Let J be the kernel of this map, so we have $\mathcal{O}/J = \mathbf{T}/I$. The following lemma is due to Urban.

Lemma 9.21. — *Suppose $\mathbf{F}^\times$ contains n distinct elements. Then there exists a G_Σ-stable $\mathbf{T}$-submodule $\mathcal{L} \subset \bigoplus_{i=1}^m V_i$, $\mathbf{T}$-submodules $\mathcal{L}', \mathcal{L}'' \subset \mathcal{L}$ (not necessarily G_Σ-stable) and a finitely generated $\mathbf{T}$-module $\mathcal{T}$ such that*

(1) *as $\mathbf{T}$-modules we have $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}''$ and $\mathcal{L}'' \cong \mathbf{T}^{n''}$;*
(2) *$\mathcal{L}$ has no $\mathbf{T}[G_\Sigma]$-quotient isomorphic to $\overline{\rho}'$;*
(3) *$\mathcal{L}'/I\mathcal{L}'$ is G_Σ-stable and there exists a $\mathbf{T}[G_\Sigma]$-isomorphism*

$$\mathcal{L}/(I\mathcal{L} + \mathcal{L}') \cong M'' \otimes_{\mathcal{O}} \mathbf{T}/I$$

for any G_Σ-stable $\mathcal{O}$-lattice $M'' \subset V''$.

(4) $\mathrm{Fitt}_{\mathbf{T}}(\mathcal{T}) = 0$ *and there exists a $\mathbf{T}[G_\Sigma]$-isomorphism*

$$\mathcal{L}'/I\mathcal{L}' \cong M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}$$

for any G_Σ-stable $\mathcal{O}$-lattice $M' \subset V'$.

Proof. — Lemma 9.21 follows from Theorem 1.1 of [55]. We only indicate how one proves that $\mathrm{Fitt}_{\mathbf{T}}(\mathcal{T}) = 0$, which is not directly stated in [55]. By Lemma 1.5 (i) in [loc. cit.], $\mathcal{L}' \cong \mathcal{T}^{n'}$, hence it is enough to show that $\mathfrak{a} := \mathrm{Fitt}_{\mathbf{T}}(\mathcal{L}') = 0$. Since $\mathfrak{a} \subset \mathrm{Ann}_{\mathbf{T}}(\mathcal{L}')$, if $\mathfrak{a} \neq 0$, there exists a non-zero $t \in \mathbf{T}$ such that $t\mathcal{L}' = 0$. Let $1 \leqslant i \leqslant m$ be such that the projection t_i of t onto the ith component of $\mathbf{T} \subset \mathcal{O}^m$ is non-zero. Then t_i annihilates the image of $\mathcal{L}'$ under the projection of $\bigoplus_{j=1}^m V_j \twoheadrightarrow V_i$. Since $0 \neq t_i \in \mathcal{O}$ and $\mathcal{O}$ is a domain, we must have that the image of $\mathcal{L}'$ in V_i is zero. Thus the composite $\mathcal{L} \hookrightarrow \bigoplus_{j=1}^m V_j \twoheadrightarrow V_i$ factors through $\mathcal{L}/\mathcal{L}' \cong \mathcal{L}'' \cong \mathbf{T}^{n''}$ by part (1) of the Lemma. Hence the image of $\mathcal{L}$ in V_i is a G_Σ-stable, rank n'' $\mathcal{O}$-module which contradicts the assumption that ρ_i is absolutely irreducible. We conclude that $\mathrm{Fitt}_{\mathbf{T}}(\mathcal{L}') = 0$. □

We will now show how Lemma 9.21 implies Theorem 9.10. For this we set

- $n' = n'' = 2$;
- $L = K$, $\Sigma = \Sigma_\ell \cup \{(i+1)\}$, $\Sigma' := \{(i+1)\}$;
- $\rho' = \rho_{f,K}$, $\rho'' = \rho_{f,K} \otimes \epsilon$, V', V'' = representation spaces of ρ', ρ'' respectively;
- $\mathbf{T} = \mathbf{T}^{\mathrm{NM}}_{\mathfrak{m}_{F_f}}$;
- $\mathcal{N}^{\mathrm{NM}}_{F_f} = \{F \in \mathcal{N}^{\mathrm{NM}} \mid \phi^{-1}(\mathfrak{m}^{\mathrm{NM}}_F) = \mathfrak{m}_{F_f}\}$ (we denote the elements of $\mathcal{N}^{\mathrm{NM}}_{F_f}$ by $F_1, \dots, F_m$);
- I = the ideal of $\mathbf{T}$ generated by $\phi_{\mathfrak{m}_{F_f}}(\mathrm{Ann}\, F_f)$
- (V_i, ρ_i) = the representation ρ_{F_i}, $i = 1, \dots, m$.

Remark 9.22. — As mentioned in section 9.2, ρ' and ρ'' factor not only through G_Σ, but also through G_{Σ_ℓ}, however, the ρ_i do not necessarily factor through G_{Σ_ℓ} (cf. Theorem 9.2), and hence we have to work with Σ as defined above. Nevertheless, for any G_Σ module M which is unramified at $(i+1)$ we have an exact sequence (cf. [58], Proposition 6)

$$0 \to H^1(G_{\Sigma_\ell}, M) \to H^1(G_\Sigma, M) \to H^1(I_{(i+1)}, M).$$

Hence in particular the group $S_\Sigma(W)$ from Theorem 9.10 is isomorphic to $S_\Sigma(\{(i+1)\}, W)$, which we study below.

Lemma 9.21 guarantees the existence of $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$ and $\mathcal{T}$ with properties (1)-(4) as in the statement of the lemma. Let M' (resp. M'') be a G_Σ-stable $\mathcal{O}$-lattice inside V' (resp. V''). The split short exact sequence of $\mathbf{T}$-modules (cf. Lemma 9.21, (1))

$$0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}/\mathcal{L}' \to 0 \tag{9.8}$$

gives rise to a short exact sequence of $(\mathbf{T}/I)[G_\Sigma]$-modules, which splits as a sequence of $\mathbf{T}/I$-modules (cf. Lemma 9.21, (3) and (4))

$$0 \to M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T} \to \mathcal{L}/I\mathcal{L} \to M'' \otimes_{\mathcal{O}} \mathbf{T}/I \to 0. \tag{9.9}$$

(Note that $\mathcal{L}/I\mathcal{L} \cong \mathcal{L}\otimes_{\mathbf{T}}\mathbf{T}/I \cong \mathcal{L}\otimes_{\mathcal{O}}\mathbf{T}/I$, hence (9.9) recovers the sequence from Theorem 1.1 of [55].) Let $s : M''\otimes_{\mathcal{O}}\mathbf{T}/I \to \mathcal{L}/I\mathcal{L}$ be a section of $\mathbf{T}/I$-modules. Define a class $c \in H^1(G_\Sigma, \mathrm{Hom}_{\mathbf{T}/I}(M'' \otimes_{\mathcal{O}} \mathbf{T}/I, M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}))$ by

$$g \mapsto (m'' \otimes t \mapsto s(m'' \otimes t) - g \cdot s(g^{-1} \cdot m'' \otimes t)).$$

The following lemma will be used in the proof of Lemma 9.25.

Lemma 9.23. — *Let $I_{(i+1)}$ denote the inertia group of the prime ideal $(i+1)$. We have $c|_{I_{(i+1)}} = 0$.*

Proof. — For simplicity set $J := I_{(i+1)}$ and $D := D_{(i+1)}$. We identify D with $\mathrm{Gal}(\overline{K}_{(i+1)}/K_{(i+1)})$. It is enough to show that J acts trivially on $\mathcal{L}/I\mathcal{L}$. Let $\phi : D \to \mathrm{Aut}_{\mathbf{T}/I}(\mathcal{L}/I\mathcal{L})$ be the homomorphism giving the action of D on $\mathcal{L}/I\mathcal{L}$ and denote by K_s the splitting field of ϕ. Set $\mathcal{G} := \mathrm{Gal}(K_\mathrm{s}/K_{(i+1)})$. Note that for $g \in D$ we can write $\phi(g) = \begin{bmatrix} \phi_{11}(g) & \phi_{12}(g) \\ & \phi_{22}(g) \end{bmatrix}$, where $\phi_{11}(g) \in \mathrm{Aut}_{\mathbf{T}/I}(M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T})$, $\phi_{22}(g) \in \mathrm{Aut}_{\mathbf{T}/I}(M'' \otimes_{\mathcal{O}} \mathbf{T}/I)$ and $\phi_{12}(g) \in \mathrm{Hom}_{\mathbf{T}/I}(M'' \otimes_{\mathcal{O}} \mathbf{T}/I, M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T})$. Also note that ϕ_{11} and ϕ_{22} are group homomorphisms from $\mathcal{G}$ into the appropriate groups of automorphisms. Let $K^{\mathrm{un}}/K_{(i+1)}$ be the maximal unramified subextension of $K_\mathrm{s}/K_{(i+1)}$ and let $\sigma \in \mathrm{Gal}(K^{\mathrm{un}}/K_{(i+1)})$ be the Frobenius generator. Let τ be a topological generator of the totally tamely ramified extension $K_\mathrm{s}/K^{\mathrm{un}}$.

On the one hand $\phi(\tau) = \left[\begin{smallmatrix} 1 & \phi_{12}(\tau) \\ & 1 \end{smallmatrix}\right]$ since M' and M'' are unramified at $(i+1)$, and on the other hand, $\phi(\sigma\tau\sigma^{-1}) = \epsilon(\sigma)\phi(\tau)$. This implies that

$$\phi_{11}(\sigma)\phi_{12}(\tau)\phi_{22}(\sigma)^{-1} = \epsilon(\sigma)\phi_{12}(\tau). \tag{9.10}$$

Writing $f = \sum_{n=1}^{\infty} a(n)q^n$ and using Theorem 3.26(ii) from [27], we get $\rho_{f,K}|_D \cong \left[\begin{smallmatrix} \mu_1 & \\ & \mu_2 \end{smallmatrix}\right]$, where μ_j are unramified characters with $\mu_2(\sigma) = a(2)$. Assume $M' = M''$ as $\mathcal{O}$-submodules of V' ($= V''$ as an E-vector space) and choose an $\mathcal{O}$-basis $\{e_1, e_2\}$ of M' so that in that basis $\rho_{f,K}|_D = \left[\begin{smallmatrix} \mu_1 & \\ & \mu_2 \end{smallmatrix}\right]$. Since $M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T} \cong (\mathcal{T}/I\mathcal{T})^2$, it follows that every element $x \in M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}$ can be written as $e_1 \otimes t_1 + e_2 \otimes t_2$, where $t_1, t_2 \in \mathcal{T}/I\mathcal{T}$ are uniquely determined by x. Hence $\rho'(\sigma)(e_j) = \mu_j(\sigma)e_j$ and $\rho''(\sigma)(e_j) = \mu_j(\sigma)\epsilon(\sigma)e_j$. Write $\phi_{12}(\tau)(e_j \otimes 1) = e_1 \otimes t_{j1} + e_2 \otimes t_{j2}$. Then (9.10) implies that $t_{11} = t_{22} = 0$. Moreover, if $t_{12} \neq 0$, we must have $\mu_1(\sigma)\mu_2(\sigma)^{-1} \equiv \epsilon(\sigma)^{-2} \pmod{\lambda}$, while if $t_{21} \neq 0$, we must have $\mu_1(\sigma)\mu_2(\sigma)^{-1} \equiv \epsilon(\sigma)^2 \pmod{\lambda}$. Since $\det \rho'(\sigma) \equiv \mu_1(\sigma)\mu_2(\sigma) \equiv \epsilon^{k-2}(\sigma) \pmod{\lambda}$ by the Tchebotarev Density Theorem, we get $\mu_2(\sigma) \equiv \epsilon(\sigma)^k \equiv 2^k \pmod{\lambda}$ if $t_{12} \neq 0$ and $\mu_2(\sigma) \equiv \epsilon(\sigma)^{k-4} \equiv 2^{k-4} \pmod{\lambda}$ if $t_{21} \neq 0$. Since none of these congruences can hold due to our assumption on f, we get $\phi_{12}(\tau) = 0$ and the lemma follows. □

Note that $\mathrm{Hom}_{\mathbf{T}/I}(M'' \otimes_{\mathcal{O}} \mathbf{T}/I, M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}) \cong \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}$, so c can be regarded as an element of

$$H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}).$$

Define a map

$$\begin{aligned} \iota : \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O}) &\to H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} E/\mathcal{O}) \\ f &\mapsto (1 \otimes f)(c). \end{aligned} \tag{9.11}$$

Note that $\tilde{T} := \mathrm{Hom}_{\mathcal{O}}(M'', M')$ is a G_Σ-stable $\mathcal{O}$-lattice inside $\tilde{V} = \mathrm{ad}\,\rho_{f,K} \otimes \epsilon^{-1} = \mathrm{Hom}_E(V'', V')$. Then $\tilde{W} = \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} E/\mathcal{O} = W \oplus E/\mathcal{O}(-1)$, where W is as in Theorem 9.10.

Lemma 9.24. — *We have $S_\Sigma(W) = S_\Sigma(\tilde{W})$.*

Proof. — Let $\mathcal{O}_\chi$ be a free rank-one $\mathcal{O}$-module on which G_Σ operates by a (non-trivial) character χ, and set $W_\chi = E/\mathcal{O} \otimes \mathcal{O}_\chi$. Since every element in $\mathrm{Sel}_\Sigma(W_\chi)$ is killed by a power of ℓ, we have $\mathrm{Sel}_\Sigma(W_\chi) = 0$ if and only if the λ-torsion part $\mathrm{Sel}_\Sigma(W_\chi)[1]$ of $\mathrm{Sel}_\Sigma(W_\chi)$ is zero. Hence it is enough to show that $\mathrm{Sel}_\Sigma(W_{\epsilon^{-1}})[1] = 0$. Note that the natural map $H^1(G_\Sigma, W_\chi[1]) \to H^1(G_\Sigma, W_\chi)$ is an injection since $H^0(G_\Sigma, W_\chi) = 0$ for a non-trivial χ. Hence $\mathrm{Sel}_\Sigma(W_\chi)[1] = \mathrm{Sel}_\Sigma(W_\chi) \cap H^1(G_\Sigma, W_\chi[1])$. Thus, we have $\mathrm{Sel}_\Sigma(W_{\epsilon^{-1}})[1] = \mathrm{Sel}_\Sigma(W_{\epsilon^{-1}}) \cap H^1(G_\Sigma, W_{\epsilon^{-1}}[1])$. Since $W_{\epsilon^{-1}}[1] =$

$W_{\omega^{-1}}[1]$, where $\omega : G_\Sigma \to \mathbf{Z}_\ell^\times$ is the Teichmuller lift of the mod ℓ cyclotomic character, we conclude that $\mathrm{Sel}_\Sigma(W_{\epsilon^{-1}})[1] = \mathrm{Sel}_\Sigma(W_{\omega^{-1}})[1]$. So it suffices to show that $\mathrm{Sel}_\Sigma(W_{\omega^{-1}})[1] = 0$. Its Pontryagin dual $S_\Sigma(W_{\omega^{-1}})$ is isomorphic to $\mathrm{Cl}_{K(\zeta_\ell)}^{\omega^{-1}}$, the ω^{-1}-isotypical part of the ℓ-primary part of the class group of $K(\zeta_\ell)$. This in turn is isomorphic to $\mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)}^{\omega^{-1}}$, since ℓ is odd ([41], Remark (3), p. 216). By [41], Theorem 2, p. 216, the ℓ-adic valuation of the order of $\mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)}^{\omega^{-1}}$ is equal to the ℓ-adic valuation of $B_1(\omega)^{[E:\mathbf{Q}_\ell]}$, where $B_1(\chi)$ is the first generalized Bernoulli number of χ. Since $B_1(\omega) \equiv \frac{1}{6} \pmod{\ell}$, and $\ell > 3$, we obtain our claim. $\square$

By Lemma 9.24 it is enough to work with $S_\Sigma(\tilde{W})$ instead of $S_\Sigma(W)$. Since the mod λ reduction of the representation $\mathrm{ad}^0(\rho_{f,K}) \otimes \epsilon^{-1}$ is absolutely irreducible, Lemma 9.7 implies that our conclusion is independent of the choice of T. Hence we can work with $\tilde{T}$ chosen as above.

Lemma 9.25. — *The image of ι is contained inside* $\mathrm{Sel}_\S(\{(i+1)\}, W)$.

Lemma 9.26. — $\ker(\iota)^\vee = 0$.

We first prove that Lemma 9.25 and Lemma 9.26 imply Theorem 9.10.

Proof of Theorem 9.10. — By Remark 9.22, $S_{\Sigma_\ell}(\tilde{W}) \cong S_\Sigma(\{(i+1)\}, \tilde{W})$, so it is enough to bound the size of the latter group. It follows from Lemma 9.25 that

$$\mathrm{ord}_\ell(\# S_\Sigma(\tilde{W})) \geqslant \mathrm{ord}_\ell(\# \mathrm{Im}(\iota)^\vee),$$

and from Lemma 9.26 that

$$\mathrm{ord}_\ell(\# \mathrm{Im}(\iota)^\vee) = \mathrm{ord}_\ell(\# \mathrm{Hom}_\mathcal{O}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O})^\vee). \tag{9.12}$$

Since $\mathrm{Hom}_\mathcal{O}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O})^\vee \cong (\mathcal{T}/I\mathcal{T})^{\vee\vee} = \mathcal{T}/I\mathcal{T}$ (cf. [27], page 98), we have

$$\mathrm{ord}_\ell(\# \mathrm{Im}(\iota)^\vee) = \mathrm{ord}_\ell(\# \mathcal{T}/I\mathcal{T}).$$

So, it remains to show that $\mathrm{ord}_\ell(\#\mathcal{T}/I\mathcal{T}) \geqslant \mathrm{ord}_\ell(\#\mathbf{T}/I)$. Since $\mathrm{Fitt}_\mathbf{T}(\mathcal{T}) = 0$ (Lemma 9.21 (4)), we have $\mathrm{Fitt}_\mathbf{T}(\mathcal{T} \otimes_\mathbf{T} \mathbf{T}/I) \subset I$ and thus $\mathrm{ord}_\ell(\#(\mathcal{T} \otimes_\mathbf{T} \mathbf{T}/I)) \geqslant \mathrm{ord}_\ell(\#\mathbf{T}/I)$. As $\mathrm{ord}_\ell(\#\mathcal{T}/I\mathcal{T}) = \mathrm{ord}_\ell(\#(\mathcal{T} \otimes_\mathbf{T} \mathbf{T}/I))$, the claim follows. $\square$

Proof of Lemma 9.25. — Consider $f \in \mathrm{Hom}_\mathcal{O}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O})$. Since $c|_{I_{(i+1)}} = 0$ by Lemma 9.23, we only need to show that $(1 \otimes f)(c)|_{D_\mathfrak{p}} \in H^1_\mathrm{f}(L_\mathfrak{p}, \tilde{W})$ for $\mathfrak{p} \in \Sigma_\ell$. Fix such a $\mathfrak{p}$. Note that since $\mathcal{T}/I\mathcal{T}$ is a finitely generated $\mathbf{T}$-module, it is also a finitely generated $\mathcal{O}$-module (since $\mathbf{T}/I = \mathcal{O}/J$). In fact it is even of finite cardinality for the same reason. In any case, there exists a positive integer n such that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O}[n])$. Thus

$$\mathrm{Im}(\iota) \subset H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} E/\mathcal{O}[n]) = H^1(G_\Sigma, \tilde{W}[n]).$$

By Lemma 9.20, we have $\varinjlim_j H^1_{\mathrm{f}}(L_{\mathfrak{p}}, \tilde{W}[j]) \cong H^1_{\mathrm{f}}(L_{\mathfrak{p}}, \tilde{W})$, hence it is enough to show that $\mathrm{Im}(\iota) \subset H^1_{\mathrm{f}}(L_{\mathfrak{p}}, \tilde{W}[n])$. However, this is clear by Lemma 9.19 since by Theorem 9.2, each ρ_i is short at $\mathfrak{p}$ (note that we are assuming that $\ell > k$). □

Proof of Lemma 9.26. — We follow [52], but see also [55], Fact 1 on page 520. First note that if $f \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O})$, then $\ker f$ has finite index in $\mathcal{T}/I\mathcal{T}$. Suppose that $f \in \ker \iota$. We will show that the image of c under the map

$$\phi : H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}) \to H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} K_f)$$

is zero. Here $K_f := (\mathcal{T}/I\mathcal{T})/\ker f$. Assuming $f \neq 0$, we will use this fact to produce a $\mathbf{T}[G_\Sigma]$-quotient of $\mathcal{L}$ isomorphic to $\overline{\rho}'$ and thus arrive at a contradiction. Set $I_f := (E/\mathcal{O})/\mathrm{Im}\, f$ and $\tilde{T} := \mathrm{Hom}_{\mathcal{O}}(M'', M')$. Tensoring the short exact sequence of $\mathcal{O}[G_\Sigma]$-modules

$$0 \to K_f \xrightarrow{f} E/\mathcal{O} \to I_f \to 0,$$

with $\otimes_{\mathcal{O}} \tilde{T}$ and considering a piece of the long exact sequence in cohomology together with the map ϕ we obtain commutative diagram with the bottom row being exact

(9.13)

$$\begin{tikzcd}
 & H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}) \arrow[d, "\phi"] \arrow[dr, "H^1(1\otimes f)"] & \\
H^0(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} I_f) \arrow[r] & H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} K_f) \arrow[r, "H^1(1\otimes f)"] & H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} E/\mathcal{O}).
\end{tikzcd}$$

Since $f \in \ker \iota$, we get $H^1(1 \otimes f) \circ \phi(c) = 0$. As the action of G_Σ on M' and M'' respectively gives rise to absolutely irreducible non-isomorphic representations, $H^0(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} I_f) = 0$. So, exactness of the bottom row of (9.13) implies that $\phi(c) = 0$. From now on assume that $0 \neq f \in \ker \iota$. Since $\ker f \neq 0$, there exists an $\mathcal{O}$-module A with $\ker f \subset A \subset \mathcal{T}/I\mathcal{T}$ such that $(\mathcal{T}/I\mathcal{T})/A \cong \mathcal{O}/\lambda = \mathbf{F}$. Since the image of c in $H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} ((\mathcal{T}/I\mathcal{T})/A))$ under the composite

$$(9.14)\quad H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}) \xrightarrow{\phi} H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} ((\mathcal{T}/I\mathcal{T})/\ker f)) \to$$
$$\to H^1(G_\Sigma, \tilde{T} \otimes_{\mathcal{O}} ((\mathcal{T}/I\mathcal{T})/A)).$$

is zero, the sequence

$$0 \to M' \otimes_{\mathcal{O}} \mathbf{F} \to (\mathcal{L}/I\mathcal{L})/(\lambda\mathcal{L} + M' \otimes_{\mathcal{O}} A) \to M'' \otimes_{\mathcal{O}} \mathbf{F} \to 0 \tag{9.15}$$

splits a sequence of $\mathbf{T}[G_\Sigma]$-modules. As G_Σ acts on $M' \otimes_{\mathcal{O}} \mathbf{F}$ via $\overline{\rho}'$, this contradicts the fact that $\mathcal{L}$ has no quotient isomorphic $\overline{\rho}'$. Hence $\ker \iota = 0$ and thus $(\ker \iota)^\vee = 0$ as well. □

BIBLIOGRAPHY

[1] J. Bellaïche & G. Chenevier, "p-adic families of Galois representations and higher rank Selmer groups", Astérisque.

[2] ———, "Formes non tempérées pour U(3) et conjectures de Bloch-Kato", *Ann. Sci. École Norm. Sup. (4)* **37** (2004), no. 4, p. 611-662.

[3] T. Berger, "An Eisenstein ideal for imaginary quadratic fields and the Bloch-Kato conjecture for Hecke characters", arXiv:math.NT/0701177, 2007.

[4] D. Blasius & J. D. Rogawski, "Zeta functions of Shimura varieties", in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, p. 525-571.

[5] S. Bloch & K. Kato, "L-functions and Tamagawa numbers of motives", in *The Grothendieck Festschrift, Vol. I*, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, p. 333-400.

[6] H. Braun, "Hermitian modular functions", *Ann. of Math. (2)* **50** (1949), p. 827-855.

[7] ———, "Hermitian modular functions. II. Genus invariants of Hermitian forms", *Ann. of Math. (2)* **51** (1950), p. 92-104.

[8] ———, "Hermitian modular functions. III", *Ann. of Math. (2)* **53** (1951), p. 143-160.

[9] J. Brown, "Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture", *Compos. Math.* **143** (2007), no. 2, p. 290-322.

[10] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997, xiv+574 pages.

[11] H. Darmon, F. Diamond & R. Taylor, "Fermat's last theorem", in *Elliptic curves, modular forms & Fermat's last theorem (Hong Kong, 1993)*, Internat. Press, Cambridge, MA, 1997, p. 2-140.

[12] P. Deligne, "Valeurs de fonctions L et périodes d'intégrales", in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, With an appendix by N. Koblitz and A. Ogus, p. 313-346.

[13] F. Diamond & J. Im, "Modular forms and modular curves", in *Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994)*, CMS Conf. Proc., vol. 17, Amer. Math. Soc., Providence, RI, 1995, p. 39-133.

[14] F. Diamond, M. Flach & L. Guo, "The Tamagawa number conjecture of adjoint motives of modular forms", *Ann. Sci. École Norm. Sup. (4)* **37** (2004), no. 5, p. 663-727.

[15] N. Dummigan, "Period ratios of modular forms", *Math. Ann.* **318** (2000), no. 3, p. 621-636.

[16] ———, "Symmetric square L-functions and Shafarevich-Tate groups", *Experiment. Math.* **10** (2001), no. 3, p. 383-400.

[17] ———, “Symmetric square L-functions and Shafarevich-Tate groups, II”, Preprint, 2008.

[18] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry, xvi+785 pages.

[19] J.-M. Fontaine, “Sur certains types de représentations p-adiques du groupe de Galois d’un corps local; construction d’un anneau de Barsotti-Tate”, *Ann. of Math. (2)* **115** (1982), no. 3, p. 529-577.

[20] J.-M. Fontaine & B. Perrin-Riou, “Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L”, in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, p. 599-706.

[21] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 254, Springer-Verlag, Berlin, 1983, x+341 pages.

[22] S. Gelbart, *Automorphic forms on adèle groups*, Princeton University Press, Princeton, N.J., 1975, Annals of Mathematics Studies, No. 83, x+267 pages.

[23] V. A. Gritsenko, “The Maass space for $\mathrm{SU}(2,2)$. The Hecke ring, and zeta functions”, *Trudy Mat. Inst. Steklov.* **183** (1990), p. 68-78, 223-225, Translated in Proc. Steklov Inst. Math. **1991**, no. 4, 75–86, Galois theory, rings, algebraic groups and their applications (Russian).

[24] ———, “Parabolic extensions of the Hecke ring of the general linear group. II”, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **183** (1990), no. Modul. Funktsii i Kvadrat. Formy. 1, p. 56-76, 165, 167.

[25] H. Hida, “Galois representations and the theory of p-adic Hecke algebras”, *Sūgaku* **39** (1987), no. 2, p. 124-139, Sugaku Expositions **2** (1989), no. 1, 75–102.

[26] ———, *Elementary theory of L-functions and Eisenstein series*, London Mathematical Society Student Texts, vol. 26, Cambridge University Press, Cambridge, 1993, xii+386 pages.

[27] ———, *Modular forms and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 69, Cambridge University Press, Cambridge, 2000, x+343 pages.

[28] ———, *p-adic automorphic forms on Shimura varieties*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2004, xii+390 pages.

[29] T. Hina & T. Sugano, “On the local Hecke series of some classical groups over $\mathfrak{p}$-adic fields”, *J. Math. Soc. Japan* **35** (1983), no. 1, p. 133-152.

[30] T. Ikeda, “On the lifting of hermitian modular forms”, Preprint, 2007.

[31] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997, xii+259 pages.

[32] H. Kim, “Automorphic L-functions”, in *Lectures on automorphic L-functions*, Fields Inst. Monogr., vol. 20, Amer. Math. Soc., Providence, RI, 2004, p. 97-201.

[33] G. Kings, “The Bloch-Kato conjecture on special values of L-functions. A survey of known results”, *J. Théor. Nombres Bordeaux* **15** (2003), no. 1, p. 179-198, Les XXIIèmes Journées Arithmetiques (Lille, 2001).

[34] K. Klosin, *Congruences among automorphic forms on the unitary group* $\mathrm{U}(2,2)$, Thesis, University of Michigan, Ann Arbor, 2006, v+179 pages.

[35] ———, “Adelic Maass spaces on $\mathrm{U}(2,2)$”, arXiv:math.NT/0706.2828, 2007.

[36] H. Kojima, “An arithmetic of Hermitian modular forms of degree two”, *Invent. Math.* **69** (1982), no. 2, p. 217-227.

[37] A. Krieg, “The Maaß spaces on the Hermitian half-space of degree 2”, *Math. Ann.* **289** (1991), no. 4, p. 663-681.

[38] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Springer-Verlag, Berlin, 1976, Lecture Notes in Mathematics, Vol. 544.

[39] B. Mazur, "Modular curves and the Eisenstein ideal", *Inst. Hautes Études Sci. Publ. Math.* (1977), no. 47, p. 33-186 (1978).

[40] ———, "An introduction to the deformation theory of Galois representations", in *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, Springer, New York, 1997, p. 243-311.

[41] B. Mazur & A. Wiles, "Class fields of abelian extensions of **Q**", *Invent. Math.* **76** (1984), no. 2, p. 179-330.

[42] T. Miyake, *Modular forms*, Springer-Verlag, Berlin, 1989, Translated from the Japanese by Yoshitaka Maeda, x+335 pages.

[43] C. Mœglin & J.-L. Waldspurger, "Le spectre résiduel de GL(n)", *Ann. Sci. École Norm. Sup. (4)* **22** (1989), no. 4, p. 605-674.

[44] S. Raghavan & J. Sengupta, "A Dirichlet series for Hermitian modular forms of degree 2", *Acta Arith.* **58** (1991), no. 2, p. 181-201.

[45] K. A. Ribet, "A modular construction of unramified p-extensions of $Q(\mu_p)$", *Invent. Math.* **34** (1976), no. 3, p. 151-162.

[46] K. Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000, Hermann Weyl Lectures. The Institute for Advanced Study, xii+227 pages.

[47] C.-G. Schmidt, "p-adic measures attached to automorphic representations of GL(3)", *Invent. Math.* **92** (1988), no. 3, p. 597-631.

[48] J.-P. Serre, "Groupes de Lie l-adiques attachés aux courbes elliptiques", in *Les Tendances Géom. en Algébre et Théorie des Nombres*, Éditions du Centre National de la Recherche Scientifique, Paris, 1966, p. 239-256.

[49] G. Shimura, "Confluent hypergeometric functions on tube domains", *Math. Ann.* **260** (1982), no. 3, p. 269-302.

[50] ———, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics, vol. 93, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997, xx+259 pages.

[51] ———, *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs, vol. 82, American Mathematical Society, Providence, RI, 2000, x+302 pages.

[52] C. M. Skinner, "Selmer groups", Preprint, 2004.

[53] J. Sturm, "Special values of zeta functions, and Eisenstein series of half integral weight", *Amer. J. Math.* **102** (1980), no. 2, p. 219-240.

[54] R. Taylor & A. Wiles, "Ring-theoretic properties of certain Hecke algebras", *Ann. of Math. (2)* **141** (1995), no. 3, p. 553-572.

[55] E. Urban, "Selmer groups and the Eisenstein-Klingen ideal", *Duke Math. J.* **106** (2001), no. 3, p. 485-525.

[56] V. Vatsal, "Canonical periods and congruence formulae", *Duke Math. J.* **98** (1999), no. 2, p. 397-419.

[57] ———, "Special values of anticyclotomic L-functions", *Duke Math. J.* **116** (2003), no. 2, p. 219-261.

[58] L. C. Washington, "Galois cohomology", in *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, Springer, New York, 1997, p. 101-120.

[59] ———, *Introduction to cyclotomic fields*, second ed., Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1997, xiv+487 pages.

[60] A. Wiles, "The Iwasawa conjecture for totally real fields", *Ann. of Math. (2)* **131** (1990), no. 3, p. 493-540.

[61] ———, “Modular elliptic curves and Fermat’s last theorem”, *Ann. of Math. (2)* **141** (1995), no. 3, p. 443-551.

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