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INVARIANT MEASURES FOR THE DEFOCUSING NONLINEAR SCHRÖDINGER EQUATION

by Nikolay TZVETKOV

ABSTRACT. — We prove the existence and the invariance of a Gibbs measure associated to the defocusing sub-quintic Nonlinear Schrödinger equations on the disc of the plane $\mathbb{R}^2$. We also prove an estimate giving some intuition to what may happen in 3 dimensions.

RÉSUMÉ. — On démontre l'existence et l'invariance d'une mesure de Gibbs par le flot de l'équation de Schrödinger non linéaire posée sur le disque du plan $\mathbb{R}^2$. On démontre également une estimée qui donne une idée de ce qui pourrait arriver en dimension 3.

1. Introduction

In [12], we constructed and proved the invariance of a Gibbs measure associated to the sub-cubic, focusing or defocusing Nonlinear Schrödinger equation (NLS) on the disc of the plane $\mathbb{R}^2$. For focusing non-linear interactions the cubic threshold is critical for the argument in [12] because of measure existence obstructions. The main goal of this paper is to show that, in the case of defocusing nonlinearities, one can extend the result of [12] to the case of sub-quintic nonlinearities. Thus we will be able to treat the relevant for the Physics case of cubic defocusing NLS. The argument presented here requires some significant elaborations with respect to [12] both in the measure existence analysis and the Cauchy problem issues. The main facts, proved in [12] which will be used here without proof are some properties of the Bessel functions and their zeros and the bilinear Strichartz estimates of Proposition 4.1 and Proposition 4.2 below.

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1.1. Presentation of the equation

Let $V : \mathbb{C} \longrightarrow \mathbb{R}$ be a $C^\infty(\mathbb{C})$ function. We suppose that V is gauge invariant which means that there exists a smooth function $G : \mathbb{R} \longrightarrow \mathbb{R}$ such that $V(z) = G(|z|^2)$. Set $F = \bar{\partial} V$, *i.e.* $F(z) = G'(|z|^2)z$. Consider the NLS

$$(i\partial_t + \Delta)u - F(u) = 0, \tag{1.1}$$

where $u : \mathbb{R} \times \Theta \to \mathbb{C}$ is a complex valued function defined on the product of the real line (corresponding to the time variable) and Θ, the unit disc of $\mathbb{R}^2$ corresponding to the spatial variable. More precisely

$$\Theta = \big((x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\big).$$

In this paper, we consider (1.1) subject to Dirichlet boundary conditions $u|_{\mathbb{R}\times\partial\Theta} = 0$. It is likely that Neumann boundary conditions are in the scope of applicability of our methods too.

We suppose that

$$\exists \alpha \in]0, 4[: \forall (k_1, k_2), \exists C > 0 : \forall z \in \mathbb{C}, \quad \big|\partial^{k_1}\bar{\partial}^{k_2} V(z)\big| \leqslant C(1 + |z|)^{2+\alpha-k_1-k_2}. \tag{1.2}$$

The number α involved in (1.2) measures the "degree" of the nonlinearity. In this paper we will also suppose the defocusing assumption

$$\exists C > 0, \exists \beta \in [2, 4[: \forall z \in \mathbb{C}, V(z) \geqslant -C(1 + |z|)^\beta. \tag{1.3}$$

A typical example for V is

$$V(z) = \frac{2}{\alpha + 2}(1 + |z|^2)^{\frac{\alpha+2}{2}}$$

with corresponding

$$F(z) = (1 + |z|^2)^{\frac{\alpha}{2}} z.$$

In the case $\alpha = 2$ one can take $V(z) = \frac{1}{2}|z|^4$ which leads to a cubic defocusing nonlinearity $F(u) = |u|^2 u$. Observe that $V(z) = -\frac{1}{2}|z|^4$, which is the potential of the cubic focusing nonlinearity $F(u) = -|u|^2 u$, does not satisfy assumption (1.3).

We restrict our consideration only to radial solutions, *i.e.* we shall suppose that $u = u(t, r)$, where

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad 0 \leqslant r < 1, \quad \phi \in [0, 2\pi].$$

Our goal here is to construct a Gibbs type measure, on a suitable phase space, associated to the radial solutions of (1.1) which is invariant under the (well-defined) global flow of (1.1).

1.2. Bessel expansion and formal Hamiltonian form

Since we deal with radial solutions of (1.1), it is natural to use Bessel function expansions. Denote by $J_0(x)$ the zero order Bessel function. We have that (see [12] and the references therein) $J_0(0) = 1$ and $J(x)$ decays as $x^{-1/2}$ when $x \to \infty$. More precisely

$$J_0(x) = \sqrt{\frac{2}{\pi}} \frac{\cos(x - \pi/4)}{\sqrt{x}} + \mathcal{O}(x^{-3/2}).$$

Let $0 < z_1 < z_2 < \cdots$ be the (simple) zeroes of $J_0(x)$. Then (see e.g. [12]) $z_n \sim n$ as $n \to \infty$. Each L^2 radial function may be expanded with respect to the Dirichlet bases formed by $J_0(z_n r)$, $n = 1, 2, 3, \ldots$. The functions $J_0(z_n r)$ are eigenfunctions of $-\Delta$ with eigenvalues z_n^2. Define $e_n : \Theta \to \mathbb{R}$ by

$$e_n \equiv e_n(r) = \|J_0(z_n \cdot)\|_{L^2(\Theta)}^{-1} J_0(z_n r).$$

We have (see [12]) that $\|J_0(z_n \cdot)\|_{L^2(\Theta)} \sim n^{-1/2}$ as $n \to \infty$. Therefore $\|e_n\|_{L^2(\Theta)} = 1$ but $\|e_n\|_{L^\infty(\Theta)} \sim n^{1/2}$ as $n \to \infty$. Hence we observe a significant difference between the disc and the flat torus $\mathbb{T}^2$, where the sup norm of the eigenfunctions can not grow so fast.

Let us fix from now on a real number s such that

$$\max\Big(\frac{1}{3}, 1 - \frac{2}{\alpha}, 1 - \frac{2}{\beta}\Big) < s < \frac{1}{2} \tag{1.4}$$

(recall that $\alpha, \beta < 4$ and thus a proper choice of the index s is indeed possible). Set $e_{n,s} = z_n^{-s} e_n$ (H^s normalization) and if

$$u(t) = \sum_{n=1}^{\infty} c_n(t) e_{n,s}$$

then we need to analyze the equation

$$i z_n^{-s} \dot{c_n}(t) - z_n^2 z_n^{-s} c_n(t) - \Pi_n\Big(F\Big(\sum_{m=1}^{\infty} c_m(t) e_{m,s}\Big)\Big) = 0, \quad n = 1, 2, \ldots \tag{1.5}$$

where Π_n is the projection on the mode e_n, *i.e.* $\Pi_n(f) = \langle f, e_n \rangle$. Equation (1.5) is a Hamiltonian PDE for $c \equiv (c_n)_{n \geqslant 1}$ with Hamiltonian

$$H(c, \overline{c}) = \sum_{n=1}^{\infty} z_n^{2-2s} |c_n|^2 + 2\pi \int_0^1 V\Big(\sum_{n=1}^{\infty} c_n e_{n,s}(r)\Big) r dr,$$

and a formal Hamiltonian form

$$i c_t = J \frac{\delta H}{\delta \overline{c}}, \quad i \overline{c}_t = -J \frac{\delta H}{\delta c},$$

where $J = \mathrm{diag}(z_n^{2s})_{n\geqslant 1}$ is the map inducing the symplectic form in the coordinates $(c,\overline{c})$. Thus the quantity $H(c,\overline{c})$ is, at least formally, conserved by the flow. In fact we will need to use the energy conservation only for finite dimensional (Hamiltonian) approximations of (1.5). Let us also observe that the L^2 norm of $u(t)$ expressed in terms of c as

$$\|c\|^2 \equiv \sum_{n=1}^{\infty} z_n^{-2s}|c_n|^2$$

is also conserved by the flow. Following Lebowitz-Rose-Speer [10], we will construct a **renormalization** of the formal measure $\chi(\|c\|)\exp(-H(c,\overline{c}))dcd\overline{c}$ (χ being a cut-off) which is invariant under the (well-defined) flow, living on a low regularity phase space (for a finite dimensional Hamiltonian model the invariance would follow from the Liouville theorem for volume preservation by flows induced by divergence free vector fields).

1.3. The free measure

Define the Sobolev spaces $H^\sigma_{\mathrm{rad}}(\Theta)$, $\sigma \geqslant 0$ equipped with the norm

$$\Big\|\sum_{n=1}^{\infty} c_n e_{n,s}\Big\|^2_{H^\sigma_{\mathrm{rad}}(\Theta)} \equiv \sum_{n=1}^{\infty} z_n^{2(\sigma-s)}|c_n|^2.$$

The Sobolev spaces $H^\sigma_{\mathrm{rad}}(\Theta)$ are related to the domains of $\sigma/2$ powers of the Dirichlet Laplacian. In several places in the sequel, we shall denote $\|\cdot\|_{H^\sigma_{\mathrm{rad}}(\Theta)}$ simply by $\|\cdot\|_{H^\sigma(\Theta)}$. We can identify $l^2(\mathbb{N};\mathbb{C})$ with $H^s_{\mathrm{rad}}(\Theta)$ via the map

$$c \equiv (c_n)_{n\geqslant 1} \longmapsto \sum_{n=1}^{\infty} c_n e_{n,s}.$$

Consider the free Hamiltonian

$$H_0(c,\overline{c}) = \sum_{n=1}^{\infty} z_n^{2-2s}|c_n|^2$$

and the measure

$$\frac{\text{“}\exp(-H_0(c,\overline{c}))dcd\overline{c}\text{”}}{\int \exp(-H_0(c,\overline{c}))dcd\overline{c}} = \prod_{n=1}^{\infty} \frac{e^{-z_n^{2-2s}|c_n|^2}dc_n d\overline{c}_n}{\int_{\mathbb{C}} e^{-z_n^{2-2s}|c_n|^2}dc_n d\overline{c}_n} \equiv d\mu(c).$$

Denote by $\mathcal{B}$ the Borel sigma algebra of $H^s_{\mathrm{rad}}(\Theta)$. The measure $d\mu$ is first defined on cylindrical sets (see [12]) in the natural way and since for $s<1/2$,

$$\sum_{n=1}^{\infty} z_n^{2s-2} < \infty$$

we obtain that $d\mu$ is countably additive on the cylindrical sets and thus may be defined as a probability measure on $(H^s_{\rm rad}(\Theta),\mathcal{B})$ via the map considered above. Let us recall that $A\subset H^s_{\rm rad}(\Theta)$ is called cylindrical if there exist an integer N and a Borel set of V of $\mathbb{C}^N$ so that

$$A=\Big\{u\in H^s_{\rm rad}(\Theta):\big((u,e_{1,s}),\dots,(u,e_{N,s})\big)\in V\Big\}.$$

In addition, the minimal sigma algebra on $H^s_{\rm rad}(\Theta)$ containing the cylindrical sets is $\mathcal{B}$.

The measure $d\mu$ may also equivalently be defined as the distribution of the $H^s_{\rm rad}(\Theta)$ valued random variable

$$\varphi(\omega,r)=\sum_{n=1}^{\infty}\frac{g_n(\omega)}{z_n^{1-s}}e_{n,s}(r)=\sum_{n=1}^{\infty}\frac{g_n(\omega)}{z_n}e_n(r),$$

where $g_n(\omega)$ is a sequence of centered, normalized, independent identically distributed (i.i.d.) complex Gaussian random variables, defined in a probability space $(\Omega,\mathcal{F},p)$. By normalized, we mean that

$$g_n(\omega)=\frac{1}{\sqrt{2}}\big(h_n(\omega)+il_n(\omega)\big),$$

where $h_n,l_n\in\mathcal{N}(0,1)$ are standard independent real Gaussian variables. Indeed, if we consider the sequence $(\varphi_N(\omega,r))_{N\in\mathbb{N}}$ defined by

$$\varphi_N(\omega,r)=\sum_{n=1}^{N}\frac{g_n(\omega)}{z_n}e_n(r) \tag{1.6}$$

then using that $s<1/2$ we obtain that $(\varphi_N(\omega,r))_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega;H^s_{\rm rad}(\Theta))$ and $\varphi(\omega,r)$ is, by definition, the limit of this sequence. Thus the map which to $\omega\in\Omega$ associates $\varphi(\omega,r)$ is measurable from $(\Omega,\mathcal{F})$ to $(H^s_{\rm rad}(\Theta),\mathcal{B})$. Therefore $\varphi(\omega,r)$ may be seen as a $H^s_{\rm rad}(\Theta)$ valued random variable and for every Borel set $A\in\mathcal{B}$,

$$\mu(A)=p(\omega:\varphi(\omega,r)\in A).$$

Moreover, if $f:H^s_{\rm rad}(\Theta)\to\mathbb{R}$ is a measurable function then f is integrable if and only if the real random variable $f\circ\varphi:\Omega\to\mathbb{R}$ is integrable and

$$\int_{H^s_{\rm rad}(\Theta)}f(u)d\mu(u)=\int_\Omega f(\varphi(\omega,\cdot))dp(\omega).$$

1.4. Measure existence

Following the basic idea one may expect that the measure (Gibbs measure)

$$(1.7)\qquad d\rho(u) \equiv \chi\big(\|u\|_{L^2(\Theta)}\big)\exp\Big(-\int_\Theta V(u)\Big)d\mu(u)$$

is invariant under the flow of (1.1). In (1.7),

$$\chi : \mathbb{R} \longrightarrow [0,\infty[$$

is a non-negative continuous function with compact support. In (1.7), $\exp\big(-\int_\Theta V(u)\big)$ is the contribution of the nonlinearity of (1.1) to the Hamiltonian, while the free Hamiltonian (coming from the linear part of (1.1)) is incorporated in $d\mu(u)$. One may wish to see $d\rho(u)$ as the image measure on $H^s_{\mathrm{rad}}(\Theta)$ under the map

$$\omega \longmapsto \sum_{n=1}^{\infty} \frac{g_n(\omega)}{z_n} e_n(r)$$

of the measure

$$\chi\big(\|\varphi(\omega,\cdot)\|_{L^2(\Theta)}\big)\exp\Big(-2\pi\int_0^1 V\big(\varphi(\omega,r)\big)rdr\Big)dp(\omega).$$

A first problem (in order to ensure that ρ is not trivial) is whether $\int_\Theta V(u)$ is finite μ almost surely (a.s.). Let us notice that an appeal to the (1.2) and the Sobolev inequality gives

$$(1.8)\qquad \Big|\int_\Theta V(u)\Big| \leqslant C\big(1+\|u\|^{\alpha+2}_{L^{\alpha+2}(\Theta)}\big) \leqslant C\big(1+\|u\|^{\alpha+2}_{H^\sigma_{\mathrm{rad}}(\Theta)}\big),$$

provided $\sigma \geqslant 2(\frac{1}{2}-\frac{1}{2+\alpha}) = \frac{\alpha}{2+\alpha}$. For $\alpha \geqslant 2$ (a case excluded in [12]), inequality (1.8) does not suffice to conclude that $\int_\Theta V(u)$ is finite μ a.s. Indeed, for $\alpha \geqslant 2$ one has $\sigma \geqslant \frac{1}{2}$ and, using for instance the Fernique integrability theorem, one may show that $\|u\|_{H^\sigma_{\mathrm{rad}}(\Theta)} = \infty$, μ a.s. We can however resolve this problem by using a probabilistic argument (which "improves" on the Sobolev inequality). Let us also mention the recent work [2], where one studies L^p properties of Gaussian random series with a particular attention to radial functions. Here is a precise statement.

Theorem 1.1. — *We have that $\int_\Theta V(u) \in L^1(d\mu(u))$ (in particular $\int_\Theta V(u)$ is μ a.s. finite).*

Essentially, the assertion of Theorem 1.1 follows from the considerations in [2]. We will however give below a proof of Theorem 1.1 using an argument slightly different from [2].

1.5. Finite dimensional approximations

Let E_N be the finite dimensional complex vector space spanned by $(e_n)_{n=1}^N$. We consider E_N as a measured space with the measure induced by $\mathbb{C}^n$ under the map from $\mathbb{C}^N$ to E_N defined by

$$(c_1,\dots,c_N)\longmapsto\sum_{n=1}^N c_n e_{n,s}.$$

Following Zhidkov (*cf.* [13] and the references therein), we consider the finite dimensional projection (an ODE) of (1.1)

$$(i\partial_t+\Delta)u-S_N(F(u))=0,\quad u|_{t=0}\in E_N, \tag{1.9}$$

where S_N is the projection on E_N. Notice that $S_N(F(u))$ is well-defined for $u\in E_N$ since $E_N\subset C^\infty(\overline{\Theta})$. The equation (1.9) is a Hamiltonian ODE for $u\in E_N$ with Hamiltonian

$$H_N(u,\bar u)=\int_\Theta|\nabla u|^2+\int_\Theta V(u),\quad u\in E_N.$$

Thus $H_N(u,\bar u)$ is conserved by the flow of (1.9). One may directly check this by multiplying (1.9) with $\bar u_t\in E_N$ and integrating over Θ (observe that the boundary terms in the integration by parts disappear). Multiplying (1.9) by $\bar u$ and integrating over Θ, we see that the $L^2(\Theta)$ norm is also preserved by the flow of (1.9) and thus (1.9) has a well-defined global dynamics. Denote by $\Phi_N(t):E_N\to E_N$, $t\in\mathbb{R}$ the flow of (1.9). Let μ_N be the distribution of the E_N valued random variable $\varphi_N(\omega,r)$ defined by (1.6). Set

$$d\rho_N(u)\equiv\chi\big(\|u\|_{L^2(\Theta)}\big)\exp\big(-\int_\Theta V(u)\big)d\mu_N(u).$$

One may see ρ_N as the image measure on E_N under the map $\omega\mapsto\varphi_N(\omega,r)$ of the measure

$$\chi\big(\|\varphi_N(\omega,\cdot)\|_{L^2(\Theta)}\big)\exp\Big(-2\pi\int_0^1 V\big(\varphi_N(\omega,r)\big)rdr\Big)dp(\omega).$$

From the Liouville theorem for divergence free vector fields, the measure ρ_N is invariant under $\Phi_N(t)$. Indeed, if we write the solution of (1.9) as

$$u(t)=\sum_{n=1}^N c_n(t)e_{n,s},\quad c_n(t)\in\mathbb{C}$$

then in the coordinates c_n, the equation (1.9) can be written as

$$iz_n^{-s}\dot{c_n}(t)-z_n^2z_n^{-s}c_n(t)-\int_\Theta S_N(F(u(t)))\overline{e_n}=0,\quad 1\leqslant n\leqslant N. \tag{1.10}$$

Equation (1.10) in turn can be written in a Hamiltonian format as follows

$$\partial_t c_n = -iz_n^{2s}\frac{\partial H}{\partial \overline{c_n}},\quad \partial_t\overline{c_n} = iz_n^{2s}\frac{\partial H}{\partial c_n},\quad 1\leqslant n\leqslant N,$$

with

$$H(c,\overline{c}) = \sum_{n=1}^{N} z_n^{2-2s}|c_n|^2 + 2\pi\int_0^1 V\Big(\sum_{n=1}^{N} c_n e_{n,s}(r)\Big) r dr,\quad c=(c_1,\dots,c_N).$$

Since

$$\sum_{n=1}^{N}\Big(\frac{\partial}{\partial c_n}\big(-iz_n^{2s}\frac{\partial H}{\partial\overline{c_n}}\big)+\frac{\partial}{\partial\overline{c_n}}\big(iz_n^{2s}\frac{\partial H}{\partial c_n}\big)\Big)=0,$$

we can apply the Liouville theorem for divergence free vector fields to conclude that the measure $dcd\overline{c}$ is invariant under the flow of (1.10). On the other hand the quantities $H(c,\overline{c})$ and

$$\|c\|^2 \equiv \sum_{n=1}^{N} z_n^{-2s}|c_n|^2$$

are conserved under the flow of (1.10). Moreover, by definition if A is a Borel set of E_N defined by

$$A = \Big\{u\in E_N : u = \sum_{n=1}^{N} c_n e_{n,s},\quad (c_1,\dots,c_N)\in A_1\Big\},$$

where A_1 is a Borel set of $\mathbb{C}^N$, then

$$\rho_N(A) = \kappa_N\int_{A_1} e^{-H(c,\overline{c})}\chi(\|c\|)dcd\overline{c},$$

with

$$\kappa_N = \pi^{-N}\Big(\prod_{1\leqslant n\leqslant N} z_n^{2-2s}\Big).$$

Therefore the measure ρ_N is invariant under $\Phi_N(t)$, thanks to the invariance of $dcd\overline{c}$ and the $\Phi_N(t)$ conservations of of $H(c,\overline{c})$ and $\chi(\|c\|)$. Let us also observe that if we write (1.10) in terms of $(\mathrm{Re}(c_n),\mathrm{Im}(c_n))$ then we still obtain a Hamiltonian ODE and one may show the invariance of ρ_N under (1.9) by analyzing that ODE.

One may extend ρ_N to a measure $\tilde{\rho}_N$ on $H^s_{\mathrm{rad}}(\Theta)$. If U is a ρ measurable set then $\tilde{\rho}_N(U)\equiv\rho_N(U\cap E_N)$. A similar definition may be given for μ_N. The measure $\tilde{\rho}_N$ is well-defined since for $U\in\mathcal{B}$ one has that $U\cap E_N$ is a Borel set of E_N. Indeed, this property is clear for U a cylindrical set and then we extend it to $\mathcal{B}$ by the key property of the cylindrical sets. Observe that for U, a ρ measurable set, one has

$$\tilde{\rho}_N(U) = \int_{U_N} \chi\big(\|S_N(u)\|_{L^2(\Theta)}\big) \exp\big(-\int_\Theta V(S_N(u))\big) d\mu(u),$$

where

$$U_N = \big\{u \in H^s_{\rm rad}(\Theta) : S_N(u) \in U\big\}.$$

The following properties relating ρ and ρ_N will be useful in our analysis concerning the long time dynamics of (1.1).

Theorem 1.2. — *One has that for every $p \in [1,\infty[$,*

$$\chi\big(\|u\|_{L^2(\Theta)}\big) \exp\big(-\int_\Theta V(u)\big) \in L^p(d\mu(u)).$$

In addition, if we fix $\sigma \in [s, 1/2[$ then for every U an open set of $H^\sigma_{\rm rad}(\Theta)$ one has

$$\rho(U) \leqslant \liminf_{N\to\infty} \tilde{\rho}_N(U) \ \big(= \liminf_{N\to\infty} \rho_N(U \cap E_N)\big). \tag{1.11}$$

Moreover if F is a closed set of $H^\sigma_{\rm rad}(\Theta)$ then

$$\rho(F) \geqslant \limsup_{N\to\infty} \tilde{\rho}_N(F) \ \big(= \limsup_{N\to\infty} \rho_N(F \cap E_N)\big). \tag{1.12}$$

The proof of Theorem 1.2 is slightly more delicate than an analogous result used in [12]. In contrast with [12] we can not exploit that $\int_\Theta V(S_N u)$ converges μ a.s. to $\int_\Theta V(u)$. In [12] we deal with sub-quartic growth of V and by the Sobolev embedding we can get directly the needed μ a.s. convergence. Here we will need to use a different argument.

1.6. Statement of the main result

With Theorem 1.1 and Theorem 1.2 in hand we can prove our main result.

Theorem 1.3. — *The measure ρ is invariant under the well-defined ρ a.s. global in time flow of the NLS (1.1), posed on the disc. More precisely:*

- *There exists a ρ measurable set Σ of full ρ measure such that for every $u_0 \in \Sigma$ the NLS (1.1), posed on the disc, with initial data data $u|_{t=0} = u_0$ has a unique (in a suitable sense) global in time solution $u \in C(\mathbb{R}; H^s_{\rm rad}(\Theta))$. In addition, for every $t \in \mathbb{R}$, $u(t) \in \Sigma$ and the map $u_0 \mapsto u(t)$ is ρ measurable.*

- *For every $A \subset \Sigma$, a ρ measurable set, for every $t \in \mathbb{R}$, $\rho(A) = \rho(\Phi(t)(A))$, where $\Phi(t)$ denotes the flow defined in the previous point.*

The uniqueness statement of Theorem 1.3 is in the sense of a uniqueness for the integral equation (6.1) in a suitable space continuously embedded in the space of continuous $H^s_{\mathrm{rad}}(\Theta)$ valued functions. Another possibility is to impose zero boundary conditions on $\mathbb{R} \times \partial\Theta$ and then relate the solutions of (1.1) to the solutions of (6.1) (see also Remark 6.1 below).

As a consequence of Theorem 1.3 one may apply the Poincaré recurrence theorem to the flow Φ. For previous works proving the invariance of Gibbs measures under the flow of NLS we refer to [3, 4, 13]. In all these works one considers periodic boundary conditions, *i.e.* the spatial domain is the flat torus. We also refer to [9], for a construction of invariant measures, supported by H^2, for the defocusing NLS.

Let us also remark that the result of Theorem 1.3 implies that the sub-quintic defocusing NLS is almost surely globally well-posed for data $\varphi(\omega, r)$ defined by

$$\varphi(\omega, r) = \sum_{n=1}^{\infty} \frac{g_n(\omega)}{z_n} e_n(r).$$

Because of the low regularity of φ for typical ω's such a result seems to be difficult to achieve by the present deterministic methods for global well-posedness of NLS.

1.7. Structure of the paper and notation

Let us briefly describe the organization of the rest of the paper. In the next section, we prove Theorem 1.1. Section 3, is devoted to the proof of Theorem 1.2. In Section 4, we recall the definition of the Bourgain spaces and we state two bilinear Strichartz estimates which are the main tool in the study of the local Cauchy problem. In Section 5, we prove nonlinear estimates in Bourgain spaces. Section 6 is devoted to the local well-posedness analysis. In Section 7, we establish the crucial control on the dynamics of (1.9). In section 8, we construct the set Σ involved in the statement of Theorem 1.3. In Section 9, we prove the invariance of the measure. In the last section, we prove several bounds for the 3d NLS with random data.

In this paper, we assume that the set of the natural numbers $\mathbb{N}$ is $\{1, 2, 3, \ldots\}$. We call dyadic integers the non-negative powers of 2, *i.e.* $1, 2, 4, 8$ *etc.*

1.8. Acknowledgements.

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2. Proof of Theorem 1.1 (measure existence)

2.1. Large deviation estimates

LEMMA 2.1. — *Let $(g_n(\omega))_{n\in\mathbb{N}}$ be a sequence of normalized i.i.d. complex Gaussian random variables defined in a probability space $(\Omega,\mathcal{F},p)$. There exists $\beta>0$ such that for every $\lambda>0$, every sequence $(c_n)\in l^2(\mathbb{N};\mathbb{C})$ of complex numbers,*

$$p\Big(\omega : \big|\sum_{n=1}^{\infty} c_n g_n(\omega)\big| > \lambda\Big) \leqslant 4e^{-\frac{\beta\lambda^2}{\sum_n |c_n|^2}}$$

(the right hand-side being defined as zero if $(c_n)_{n\in\mathbb{N}}$ is identically zero).

Proof. — By separating the real and the imaginary parts, we can assume that g_n are real valued independent standard Gaussians and c_n are real constants. The bound we need to prove is thus

$$\exists\beta>0 : \forall (c_n)\in l^2(\mathbb{N};\mathbb{R}), \forall\lambda>0,\quad p\Big(\omega : \big|\sum_{n=1}^{\infty} c_n g_n(\omega)\big| > \lambda\Big) \leqslant 2e^{-\frac{\beta\lambda^2}{\sum_n c_n^2}}. \tag{2.1}$$

We may of course assume that the sequence $(c_n)_{n\in\mathbb{N}}$ is not identically zero. For $t>0$ to be determined later, using the independence, we obtain that

$$\int_\Omega \exp\Big(t\sum_{n=1}^{\infty} c_n g_n(\omega)\Big)dp(\omega) = \exp\Big((t^2/2)\sum_{n=1}^{\infty} c_n^2\Big).$$

Therefore

$$\exp\Big((t^2/2)\sum_{n=1}^{\infty} c_n^2\Big) \geqslant \exp(t\lambda)p\Big(\omega : \sum_{n=1}^{\infty} c_n g_n(\omega) > \lambda\Big)$$

and thus

$$p(\omega : \sum_{n=1}^{\infty} c_n g_n(\omega) > \lambda) \leqslant \exp\Big((t^2/2)\sum_{n=1}^{\infty} c_n^2\Big) e^{-t\lambda}.$$

For $a > 0$, $b > 0$ the minimum of $f(t) = at^2 - bt$ is $-b^2/4a$ and this minimum is attained in the positive number $t = b/(2a)$. It is thus natural to choose the positive number t as

$$t \equiv \lambda/\Big(\sum_{n=1}^{\infty} c_n^2\Big)$$

which leads to

$$p\Big(\omega : \sum_{n=1}^{\infty} c_n g_n(\omega) > \lambda\Big) \leqslant \exp\Big(-\frac{\lambda^2}{2\sum_n c_n^2}\Big).$$

In the same way (replacing c_n by $-c_n$), we can show that

$$p\Big(\omega : \sum_{n=1}^{\infty} c_n g_n(\omega) < -\lambda\Big) \leqslant \exp\Big(-\frac{\lambda^2}{2\sum_{nr} c_n^2}\Big)$$

which shows that (2.1) holds with $\beta = 1/2$. This completes the proof of Lemma 2.1. □

We next state the following consequence of Lemma 2.1.

Lemma 2.2. — *Let $(g_n(\omega))_{n\in\mathbb{N}}$ be a sequence of normalized i.i.d. complex Gaussian random variables defined in a probability space $(\Omega, \mathcal{F}, p)$. Then there exist positive numbers c_1, c_2 such that for every non empty finite set of indexes $\Lambda \subset \mathbb{N}$, every $\lambda > 0$,*

$$p\Big(\omega \in \Omega : \sum_{n\in\Lambda} |g_n(\omega)|^2 > \lambda\Big) \leqslant e^{c_1|\Lambda| - c_2\lambda},$$

where $|\Lambda|$ denotes the cardinality of Λ.

Proof. — A proof of this lemma is given in [12, Lemma 3.4]. Here we propose a different proof based on Lemma 2.1. The interest of this proof is that the argument might be useful in more general situations. Again, we can suppose that g_n are real valued standard Gaussians. A simple geometric observation shows that there exists $c_1 > 0$ (independent of $|\Lambda|$) and a set $\mathcal{A}$ of the unit ball of $\mathbb{R}^{|\Lambda|}$ of cardinality bounded by $e^{c_1|\Lambda|}$ such that almost surely in ω,

$$\frac{1}{2}\Big(\sum_{n\in\Lambda} |g_n(\omega)|^2\Big)^{1/2} \leqslant \sup_{c\in\mathcal{A}} \Big|\sum_{n\in\Lambda} c_n g_n(\omega)\Big|$$

($c=(c_n)_{n\in\Lambda}$ with $\sum_n |c_n|^2=1$). Therefore

$$\Big\{\omega : \sum_{n\in\Lambda} |g_n(\omega)|^2 > \lambda\Big\} \subset \bigcup_{c\in\mathcal{A}} \Big\{\omega : |\sum_{n\in\Lambda} c_n g_n(\omega)| \geqslant \frac{\sqrt{\lambda}}{2}\Big\}.$$

Consequently, using Lemma 2.1, we obtain that there exists $c_2>0$, independent of Λ, such that for every $\lambda>0$,

$$p\Big(\omega : \sum_{n\in\Lambda} |g_n(\omega)|^2 > \lambda\Big) \leqslant |\mathcal{A}| 4e^{-c_2\lambda} \leqslant 4e^{c_1|\Lambda|-c_2\lambda} < e^{(c_1+2)|\Lambda|-c_2\lambda}.$$

This completes the proof of Lemma 2.2. □

2.2. Proof of Theorem 1.1

Theorem 1.1 follows from the following statement.

Lemma 2.3. — *The sequence $\int_\Theta V(S_N(u))$ converges to $\int_\Theta V(u)$ in $L^1(d\mu)$.*

Proof. — Let us first show that $(\int_\Theta V(S_N(u)))_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^1(d\mu)$. From the Sobolev embedding, we have that for a fixed N the map from $H^s_{\mathrm{rad}}(\Theta)$ to $\mathbb{R}$ defined by $u\mapsto \int_\Theta V(S_N(u))$ is continuous and thus measurable. Write, for $N<M$, using (1.2)

$$\Big\| \int_\Theta V(S_N(u)) - \int_\Theta V(S_M(u))\Big\|_{L^1(H^s_{\mathrm{rad}};\mathcal{B},d\mu(u))}$$
$$\leqslant C\Big\| \int_\Theta |S_N(u)-S_M(u)|(1+|S_N(u)|^{\alpha+1}+|S_M(u)|^{\alpha+1})\Big\|_{L^1(H^s_{\mathrm{rad}};\mathcal{B},d\mu(u))}.$$

Using the Hölder inequality, we get

$$\Big| \int_\Theta |S_N(u)-S_M(u)|(1+|S_N(u)|^{\alpha+1}+|S_M(u)|^{\alpha+1})\Big|$$
$$\leqslant \|S_N(u)-S_M(u)\|_{L^{\alpha+2}(\Theta)}\big(C+\|S_N(u)\|^{\alpha+1}_{L^{\alpha+2}(\Theta)}+\|S_M(u)\|^{\alpha+1}_{L^{\alpha+2}(\Theta)}\big).$$

Another use of the Hölder inequality, this time with respect to $d\mu$ gives

$$\Big\| \int_\Theta V(S_N(u)) - \int_\Theta V(S_M(u))\Big\|_{L^1(d\mu(u))}$$
$$\leqslant C\Big\| \|S_N(u)-S_M(u)\|_{L^{\alpha+2}(\Theta)}\Big\|_{L^{\alpha+2}(d\mu(u))}$$
$$\times\Big(1+\Big\|\|S_N(u)\|_{L^{\alpha+2}(\Theta)}\Big\|^{\alpha+1}_{L^{\alpha+2}(d\mu(u))}+\Big\|\|S_M(u)\|_{L^{\alpha+2}(\Theta)}\Big\|^{\alpha+1}_{L^{\alpha+2}(d\mu(u))}\Big).$$

Thus

$$\Big\|\int_\Theta V(S_N(u)) - \int_\Theta V(S_M(u))\Big\|_{L^1(d\mu(u))} \leqslant C\|\varphi_N - \varphi_M\|_{L^{\alpha+2}(\Theta\times\Omega)} \times \Big(1 + \big\|\varphi_N\|^{\alpha+1}_{L^{\alpha+2}(\Theta\times\Omega)} + \big\|\varphi_M\|^{\alpha+1}_{L^{\alpha+2}(\Theta\times\Omega)}\Big), \tag{2.2}$$

where φ_N is defined by (1.6). Let us now prove that there exists $C > 0$ such that for every N,

$$\|\varphi_N\|_{L^{\alpha+2}(\Omega\times\Theta)} \leqslant C. \tag{2.3}$$

Using Lemma 2.1 with $c_n = z_n^{-1} e_n(r)$, $1 \leqslant n \leqslant N$ and the definition of the $L^{\alpha+2}$ norms by the aide of the distributional function, we obtain that for a fixed r

$$\begin{aligned}\|\varphi_N(\omega,r)\|^{\alpha+2}_{L^{\alpha+2}(\Omega)} &= (\alpha+2)\int_0^\infty \lambda^{\alpha+1} p\big(\omega : \big|\varphi_N(\omega,r)\big| > \lambda\big) d\lambda \\ &\leqslant C\int_0^\infty \lambda^{\alpha+1}\exp\Big(-(\beta\lambda^2)/\Big(\sum_{n=1}^N z_n^{-2}|e_n(r)|^2\Big)\Big)d\lambda \\ &= C\Big(\int_0^\infty \lambda^{\alpha+1} e^{-\beta\lambda^2} d\lambda\Big)\Big(\sum_{n=1}^N z_n^{-2}|e_n(r)|^2\Big)^{\frac{\alpha+2}{2}}.\end{aligned}$$

Therefore

$$\|\varphi_N(\omega,r)\|_{L^{\alpha+2}(\Omega)} \leqslant C\Big(\sum_{n=1}^N z_n^{-2}|e_n(r)|^2\Big)^{\frac{1}{2}}.$$

Squaring, taking the $L^{\frac{\alpha+2}{2}}(\Theta)$ norm and using the triangle inequality, we get

$$\|\varphi_N\|^2_{L^{\alpha+2}(\Omega\times\Theta)} \leqslant \sum_{n=1}^N z_n^{-2}\|e_n\|^2_{L^{\alpha+2}(\Theta)}.$$

On the other hand, it is shown in [12] that for $\alpha < 2$ one has that $\|e_n\|_{L^{\alpha+2}(\Theta)}$ is uniformly bounded (with respect to n), for $\alpha = 2$, $\|e_n\|_{L^{\alpha+2}(\Theta)} \leqslant C\log(1+z_n)^{1/4}$ and for $\alpha > 2$, $\|e_n\|_{L^{\alpha+2}(\Theta)} \leqslant Cz_n^{1/2-2/(\alpha+2)}$. Since $z_n \sim n$, we obtain that there exists C such that for every $N \in \mathbb{N}$, $\|\varphi_N\|_{L^{\alpha+2}(\Omega\times\Theta)} \leqslant C$. Therefore (2.3) holds.

Similarly, we may obtain that

$$\|\varphi_N - \varphi_M\|^2_{L^{\alpha+2}(\Omega\times\Theta)} \leqslant \sum_{n=N+1}^M z_n^{-2}\|e_n\|^2_{L^{\alpha+2}(\Theta)} \tag{2.4}$$

which tends to zero as $N \to \infty$ thanks to the bounds on the growth of $\|e_n\|_{L^{\alpha+2}(\Theta)}$. Moreover, we have that

$$\lim_{N\to\infty} \varphi_N = \varphi \text{ in } L^{\alpha+2}(\Theta \times \Omega) \tag{2.5}$$

(we can identify the limit thanks to the $L^2(\Theta\times\Omega)$ convergence of φ_N to φ and the fact that $L^{\alpha+2}(\Theta\times\Omega)$ convergence implies $L^2(\Theta\times\Omega)$ convergence).

On the other hand thanks to (1.3), we can write $V(u) = V_1(u) + V_2(u)$, where $V_1 \geqslant 0$ and $|V_2(u))| \leqslant C\big(1+|u|^\beta\big)$. Thanks to the Sobolev embedding and (1.4), we obtain that $\int_\Theta V_2(u)$ is continuous on $H^s_{\text{rad}}(\Theta)$. Therefore the map $u \mapsto \int_\Theta V_2(u)$ is a μ measurable real valued function. Let us next show that the map $u \mapsto \int_\Theta V_1(u)$ is μ measurable. For that purpose, it is sufficient to show that the map

$$c \equiv (c_n)_{n\in\mathbb{N}} \longmapsto \int_\Theta V_1\Big(\sum_{n\in\mathbb{N}} c_n e_{n,s}\Big) \tag{2.6}$$

is measurable from $l^2(\mathbb{N})$ to $\mathbb{R}$. Indeed, we have that the map

$$(c,r) \longmapsto \sum_{n\in\mathbb{N}} c_n e_{n,s}(r)$$

is measurable from from $l^2(\mathbb{N}) \times \Theta$ to $\mathbb{R}$ since we can see $\sum_{n\in\mathbb{N}} c_n e_{n,s}(r)$ as the limit of $\sum_{n=1}^N c_n e_{n,s}(r)$ as $N \to \infty$ in $L^2(l^2(\mathbb{N}) \times \Theta)$ where $l^2(\mathbb{N})$ is equipped with the measure $d\mu(c)$ introduced in the introduction. Therefore $V_1\Big(\sum_{n\in\mathbb{N}} c_n e_{n,s}\Big)$ is a measurable map from $l^2(\mathbb{N}) \times \Theta$ to $\mathbb{R}$. Since $V_1 \geqslant 0$, using for instance the Fubini theorem, we obtain that the map (2.6) is indeed measurable. This in turn implies the measurability of the map $u \mapsto \int_\Theta V(u)$. Next, similarly to the proof of (2.2), we get

$$\begin{aligned}\Big\| \int_\Theta V(S_N(u)) - \int_\Theta V(u)\Big\|_{L^1(d\mu(u))} &\leqslant C\|\varphi - \varphi_N\|_{L^{\alpha+2}(\Theta\times\Omega)} \\ &\quad \times \Big(1 + \|\varphi_N\|^{\alpha+1}_{L^{\alpha+2}(\Theta\times\Omega)} + \|\varphi\|^{\alpha+1}_{L^{\alpha+2}(\Theta\times\Omega)}\Big).\end{aligned}$$

Therefore

$$\lim_{N\to\infty} \Big\| \int_\Theta V(S_N(u)) - \int_\Theta V(u)\Big\|_{L^1(H^s_{\text{rad}};\mathcal{B},d\mu(u))} = 0.$$

This completes the proof of Lemma 2.3. □

Using Lemma 2.3, we have that $\int_\Theta V(u) \in L^1(d\mu(u))$ and thus $\int_\Theta V(u)$ is finite μ a.s. This proves that $d\rho$ is indeed a nontrivial measure. This completes the proof of Theorem 1.1. □

2.3. The necessity of the probabilistic argument

In this section we make a slight digression by showing that for $\alpha \geqslant 2$ an argument based only on the Sobolev embedding may not conclude to the fact that $\int_\Theta V(u)$ is finite μ a.s. More precisely we know that for every $\sigma < 1/2$, $\|u\|_{H^\sigma(\Theta)}$ is finite μ a.s. Therefore the deterministic inequality

$$\exists \sigma < 1/2, \exists C > 0, \forall u \in H^\sigma_{\mathrm{rad}}(\Theta), \quad \|u\|_{L^{\alpha+2}(\Theta)} \leqslant C\|u\|_{H^\sigma_{\mathrm{rad}}(\Theta)} \tag{2.7}$$

would suffice to conclude that $\int_\Theta V(u)$ is finite μ a.s. We have however the following statement.

Lemma 2.4. — *For $\alpha \geqslant 2$, estimate (2.7) fails.*

Proof. — We shall give the proof for $\alpha = 2$. The construction for $\alpha > 2$ is similar. Suppose that (2.7) holds for some $\sigma < 1/2$. Using the Cauchy-Schwarz inequality, we obtain that there exists $\theta \in]0, 1/2]$ such that

$$\exists C > 0 : \forall u \in H^1_{\mathrm{rad}}(\Theta), \quad \|u\|_{H^\sigma(\Theta)} \leqslant C\|u\|^{\frac{1}{2}+\theta}_{L^2(\Theta)}\|u\|^{\frac{1}{2}-\theta}_{H^1_{\mathrm{rad}}(\Theta)}$$

(observe that $H^1_{\mathrm{rad}}(\Theta)$ may be seen as the completion of $C_0^\infty(\Theta)$ radial functions with respect to the $H^1(\Theta)$ norm). Thus by applying (2.7) to $H^1_{\mathrm{rad}}(\Theta)$ functions, we obtain that

$$\exists C > 0 : \forall u \in H^1_{\mathrm{rad}}(\Theta), \quad \|u\|_{L^4(\Theta)} \leqslant C\|u\|^{\frac{1}{2}+\theta}_{L^2(\Theta)}\|u\|^{\frac{1}{2}-\theta}_{H^1(\Theta)}. \tag{2.8}$$

We now show that (2.8) fails. Let $v \in C_0^\infty(\Theta)$ be a radial bump function, not identically zero. We can naturally see v as a $C_0^\infty(\mathbb{R}^2)$ function. For $\lambda \geqslant 1$, we set

$$v_\lambda(x_1, x_2) \equiv v(\lambda x_1, \lambda x_2).$$

Thus $v_\lambda \in C_0^\infty(\Theta)$ and v_λ is still radial. We can therefore substitute v_λ in (2.8) and obtain a contradiction in the limit $\lambda \to \infty$. More precisely, one may directly check that for $\lambda \gg 1$,

$$\|v_\lambda\|_{L^4(\Theta)} \sim \lambda^{-\frac{1}{2}}, \quad \|v_\lambda\|_{L^2(\Theta)} \sim \lambda^{-1}, \quad \|v_\lambda\|_{H^1(\Theta)} \sim 1.$$

This completes the proof of Lemma 2.4. □

3. Proof of Theorem 1.2 (integrability and convergence properties)

3.1. Convergence in measure

Let us define the μ measurable functions f and f_N by

$$f(u) \equiv \chi\big(\|u\|_{L^2(\Theta)}\big) \exp\Big(-\int_\Theta V(u)\Big)$$

and

$$f_N(u) \equiv \chi\big(\|S_N(u)\|_{L^2(\Theta)}\big) \exp\Big(-\int_\Theta V(S_N(u))\Big).$$

We start by the following convergence property.

Lemma 3.1. — *The sequence $(f_N(u))_{N\in\mathbb{N}}$ converges in measure as N tends to infinity, with respect to the measure μ, to $f(u)$.*

Proof. — Since χ and the exponential are continuous functions, it suffices to show that the sequence $\|S_N u\|_{L^2(\Theta)}$ converges in measure as N tends to infinity, with respect to the measure μ, to $\|u\|_{L^2(\Theta)}$ and that the sequence $\int_\Theta V(S_N(u))$ converges in measure as N tends to infinity, with respect to the measure μ, to $\int_\Theta V(u)$. Thanks to the Chebishev inequality, it therefore suffices to prove that $\|S_N u\|_{L^2(\Theta)}$ converges in $L^2(d\mu(u))$ to $\|u\|_{L^2(\Theta)}$ and that $\int_\Theta V(S_N(u))$ converges in $L^1(d\mu(u))$ to $\int_\Theta V(u)$. The first assertion is trivial and the second one follows from Lemma 2.3. This completes the proof of Lemma 3.1. □

3.2. A Gaussian estimate

We now state a property of the measure μ resulting from its Gaussian nature.

Lemma 3.2. — *Let $\sigma \in [s, 1/2[$. There exist $C > 0$ and $c > 0$ such that for every integers $M \geqslant N \geqslant 0$ (with the convention that $S_0 \equiv 0$), every real number $\lambda \geqslant 1$,*

$$\mu\Big(u \in H^s_{\mathrm{rad}}(\Theta) : \big\|S_M(u) - S_N(u)\big\|_{H^\sigma(\Theta)} > \lambda\Big) \leqslant Ce^{-c\lambda^2(1+N)^{2(1-\sigma)}}.$$

Proof. — We follow the argument given in [12, Proposition 3.3]. It suffices to prove that $p(A_{N,M}) \leqslant C \exp\big(-c\lambda^2(1+N)^{2(1-\sigma)}\big)$, where

$$A_{N,M} \equiv \Big(\omega \in \Omega : \big\|\varphi_M(\omega,\cdot) - \varphi_N(\omega,\cdot)\big\|_{H^\sigma(\Theta)} > \lambda\Big).$$

Let $\theta > 0$ be such that $2\theta < 1 - 2\sigma$. Notice that a proper choice of θ is possible thanks to the assumption $\sigma < 1/2$. For $0 \leqslant N_1 \leqslant N_2$ two integers and $\kappa > 0$, we consider the set $A_{N_1,N_2,\kappa}$, defined by

$$A_{N_1,N_2,\kappa} \equiv \Big(\omega \in \Omega : \big\|\varphi_{N_2}(\omega,\cdot) - \varphi_{N_1}(\omega,\cdot)\big\|_{H^\sigma(\Theta)} > \kappa\lambda\Big((1+N_2)^{-\theta} + \Big(\frac{1+N}{1+N_2}\Big)^{1-\sigma}\Big)\Big).$$

Let L_1, L_2 be two dyadic integers such that

$$L_1/2 < 1+N \leqslant L_1, \quad L_2 \leqslant M < 2L_2.$$

We will only analyze the case $L_1 \leqslant L_2/2$. If $L_1 > L_2/2$ then the analysis is simpler. Indeed, if $L_1 > L_2/2$ then $L_1 \geqslant L_2$ which implies

$$L_1/2 < 1+N \leqslant 1+M < 1+2L_2 < 4L_1$$

and the analysis of the case $L_1 \leqslant L_2/2$ below (see (3.3), (3.4)) can be performed to this case by writing

$$\varphi_M - \varphi_N = (\varphi_{L_1} - \varphi_N) + (\varphi_M - \varphi_{L_1})$$

(without the summation issue). We thus assume that $L_1 \leqslant L_2/2$. Write

$$\varphi_M - \varphi_N = (\varphi_{L_1} - \varphi_N) + \Big(\sum_{\substack{L_1 \leqslant L \leqslant L_2/2 \\ L-\text{dyadic}}} (\varphi_{2L} - \varphi_L)\Big) + (\varphi_M - \varphi_{L_2}).$$

Using the triangle inequality and summing-up geometric series, we obtain that there exists a sufficiently small $\kappa > 0$ depending on σ but independent of λ, N and M such that

$$A_{N,M} \subset A_{N,L_1,\kappa} \bigcup \Big(\bigcup_{\substack{L_1 \leqslant L \leqslant L_2/2 \\ L-\text{dyadic}}} A_{L,2L,\kappa}\Big) \bigcup A_{L_2,M,\kappa}. \tag{3.1}$$

Since $z_n \sim n$, for $\omega \in A_{L,2L,\kappa}$,

$$\sum_{n=L+1}^{2L} |g_n(\omega)|^2 \geqslant c\lambda^2 L^{2-2\sigma}\big(L^{-2\theta} + (L^{-1}(1+N))^{2-2\sigma}\big).$$

Therefore using Lemma 2.2 and that $2-2\sigma-2\theta > 1$, we obtain that for $\lambda \geqslant 1$,

$$\begin{aligned} p(A_{L,2L,\kappa}) &\leqslant e^{c_1 L - c_2\lambda^2(L^{2-2\sigma-2\theta} + (1+N)^{2-2\sigma})} \\ &\leqslant e^{-c\lambda^2(1+N)^{2-2\sigma}} e^{-c\lambda^2 L^{2-2\sigma-2\theta}}, \end{aligned} \tag{3.2}$$

where the constant $c > 0$ is independent of L, N, M and λ. Similarly

$$p(A_{N,L_1,\kappa}) \leqslant e^{-c\lambda^2(1+N)^{2-2\sigma}} e^{-c\lambda^2 L_1^{2-2\sigma-2\theta}} \leqslant e^{-c\lambda^2(1+N)^{2-2\sigma}} \tag{3.3}$$

and

$$p(A_{L_2,M,\kappa}) \leqslant e^{-c\lambda^2(1+N)^{2-2\sigma}} e^{-c\lambda^2 L_2^{2-2\sigma-2\theta}} \leqslant e^{-c\lambda^2(1+N)^{2-2\sigma}}. \tag{3.4}$$

Collecting estimates (3.2), (3.3), (3.4), coming back to (3.1) and summing an obviously convergent series in L completes the proof of Lemma 3.2. □

3.3. Uniform integrability

We next prove the crucial uniform integrability property of f_N.

LEMMA 3.3. — *Let us fix $p \in [1,\infty[$. Then there exists $C > 0$ such that for every $M \in \mathbb{N}$,*

$$\int_{H^s_{\mathrm{rad}}(\Theta)} |f_M(u)|^p d\mu(u) \leqslant C.$$

Proof. — Using (1.3), we observe that it suffices to prove that

$\exists C > 0,\ \forall M \in \mathbb{N},$

$$\int_\Omega \chi^p\big(\|\varphi_M(\omega,\cdot)\|_{L^2(\Theta)}\big) \exp\big(Cp\|\varphi_M(\omega,\cdot)\|^\beta_{L^\beta(\Theta)}\big) dp(\omega) \leqslant C.$$

Using the Sobolev inequality, we infer that

$$\|\varphi_M(\omega,\cdot)\|_{L^\beta(\Theta)} \leqslant C\|\varphi_M(\omega,\cdot)\|_{H^\sigma(\Theta)},$$

provided

$$\sigma \geqslant 2\Big(\frac{1}{2} - \frac{1}{\beta}\Big). \tag{3.5}$$

Observe that since $\beta < 4$ there exists $\sigma \in [s, 1/2[$ satisfying (3.5). Let us fix such a value of σ for the sequel of the proof. Since χ is with compact support, we need to study the convergence of the integral

$$\int_{\lambda_0}^\infty h_M(\lambda) d\lambda,$$

with

$$h_M(\lambda) \equiv p\Big(\omega \in \Omega : \|\varphi_M(\omega,\cdot)\|_{H^\sigma(\Theta)} \geqslant c(\log(\lambda))^{\frac{1}{\beta}},\ \|\varphi_M(\omega,\cdot)\|_{L^2(\Theta)} \leqslant C\Big),$$

where c and C are independent of λ and M (C is depending on the support of χ) and λ_0 is a large constant, independent of M, to be fixed later.

Since for $N \leqslant M$,

$$\|\varphi_N(\omega,\cdot)\|_{H^\sigma(\Theta)} \leqslant CN^\sigma\|\varphi_N(\omega,\cdot)\|_{L^2(\Theta)} \leqslant CN^\sigma\|\varphi_M(\omega,\cdot)\|_{L^2(\Theta)} \tag{3.6}$$

we obtain that there exists $\alpha > 0$, independent of M and λ such that if M satisfies $M \leqslant \alpha(\log(\lambda))^{\frac{1}{\sigma\beta}}$ then $h_M(\lambda) = 0$ (use (3.6) with $M = N$). We can therefore assume that $M > \alpha(\log(\lambda))^{\frac{1}{\sigma\beta}}$.

Let us fix $\lambda \geqslant \lambda_0$. Define N as the integer part of $\alpha(\log(\lambda))^{\frac{1}{\sigma\beta}-\delta}$, where δ is such that

$$0 < \delta < \frac{2-\sigma\beta}{2\sigma\beta(1-\sigma)}. \tag{3.7}$$

Let us notice that a proper choice of δ is possible since $\beta < 4$ and $\sigma < 1/2$. Observe also that for $\lambda_0 \gg 1$, depending only on α, we have $N \geqslant 1$ and $N \leqslant M$. Using (3.6), we obtain that the event

$$\Big(\omega \in \Omega : \|\varphi_N(\omega,\cdot)\|_{H^\sigma(\Theta)} \geqslant \frac{c}{2}(\log(\lambda))^{\frac{1}{\beta}}, \quad \|\varphi_M(\omega,\cdot)\|_{L^2(\Theta)} \leqslant C\Big)$$

is of probability zero for $\lambda \geqslant \lambda_0$, where λ_0 is a large constant independent of M. At this place we fix the value of λ_0. Using the triangle inequality, we obtain that for $\lambda \geqslant \lambda_0$,

$$h_M(\lambda) \leqslant p\Big(\omega \in \Omega : \|\varphi_M(\omega,\cdot) - \varphi_N(\omega,\cdot)\|_{H^\sigma(\Theta)} \geqslant \frac{c}{2}(\log(\lambda))^{\frac{1}{\beta}}\Big).$$

Using Lemma 3.2, we arrive at

$$\begin{aligned} h_M(\lambda) &\leqslant Ce^{-c(\log(\lambda))^{\frac{2}{\beta}}(1+N)^{2(1-\sigma)}} \\ &\leqslant Ce^{-c(\log(\lambda))^{\frac{2}{\beta}}(\log(\lambda))^{\frac{2(1-\sigma)}{\sigma\beta}-2\delta(1-\sigma)}} \\ &= Ce^{-c(\log(\lambda))^{\frac{2}{\sigma\beta}-2\delta(1-\sigma)}}. \end{aligned}$$

Thanks to (3.7), we have that $\frac{2}{\sigma\beta} - 2\delta(1-\sigma) > 1$ and therefore $h_M(\lambda)$ is integrable on the interval $[\lambda_0, \infty[$. The integrability on $[0,\lambda_0]$ is direct since $0 \leqslant h_M(\lambda) \leqslant 1$. This completes the proof of Lemma 3.3. □

Remark 3.4. — The exponent $\beta = 4$ appears as critical in the above argument, a fact which reflects the critical nature of the cubic non-linearity for the $2d$ NLS. This fact may be related to a blow-up for the cubic focusing NLS for data of positive μ measure. This is however an open problem (see the final section of [12]).

Using Lemma 3.3, we readily arrive at the following statement.

LEMMA 3.5. — *Let $\sigma \in [s, 1/2[$. There exist $C > 0$ and $c > 0$ such that for every integer $M \geqslant 1$, every real number $\lambda \geqslant 1$,*

$$\tilde{\rho}_M\big(u \in H^s_{\rm rad}(\Theta) : \|S_M(u)\|_{H^\sigma(\Theta)} > \lambda\big) \leqslant Ce^{-c\lambda^2}.$$

Proof. — It suffices to use the Cauchy-Schwarz inequality, Lemma 3.2 and Lemma 3.3. □

Another consequence of Lemma 3.3 is the integrability of $f(u)$.

LEMMA 3.6. — *For every $p \in [1, \infty[$, $f(u) \in L^p(H^s_{\rm rad}; \mathcal{B}, d\mu(u))$.*

Proof. — Using Lemma 3.1, we obtain that there is a sub-sequence N_k such that the sequence $(f_{N_k}(u))_{k\in\mathbb{N}}$ converges to $f(u)$, μ almost surely. Thanks to Lemma 3.3 $(f_{N_k}(u))_{k\in\mathbb{N}}$ is uniformly bounded in $L^p(H^s_{\rm rad}(\Theta), \mathcal{B}, d\mu)$. Using Fatou's lemma we deduce that $f(u)$ belongs to $L^p(H^s_{\rm rad}(\Theta), \mathcal{B}, d\mu)$ with a norm bounded by the liminf of the norms of $f_{N_k}(u)$'s. This completes the proof of Lemma 3.6. □

3.4. End of the proof of Theorem 1.2

We have the following convergence property which yields the assertion of Theorem 1.2 in the particular case $U = F = H^s_{\rm rad}(\Theta)$.

LEMMA 3.7. — *Let us fix $p \in [1, \infty[$. The following holds true:*

$$\lim_{N\to\infty} \int_{H^s_{\rm rad}(\Theta)} |f_N(u) - f(u)|^p d\mu(u) = 0.$$

Proof. — Let us fix $\varepsilon > 0$. Consider the set

$$A_{N,\varepsilon} \equiv \big(u \in H^s_{\rm rad}(\Theta) : |f_N(u) - f(u)| \leqslant \varepsilon\big).$$

Denote by $A^c_{N,\varepsilon}$ the complementary set in $H^s_{\rm rad}(\Theta)$ of $A_{N,\varepsilon}$. Observe that f and f_N belong to $L^{2p}(d\mu)$ with norms bounded uniformly in N. Then, using the Hölder inequality, we get

$$\begin{aligned}\Big|\int_{A^c_{N,\varepsilon}} |f_N(u) - f(u)|^p d\mu(u)\Big|^{\frac{1}{p}} &\leqslant \|f_N - f\|_{L^{2p}(d\mu)} [\mu(A^c_{N,\varepsilon})]^{\frac{1}{2p}} \\ &\leqslant C[\mu(A^c_{N,\varepsilon})]^{\frac{1}{2p}}.\end{aligned}$$

On the other hand

$$\int_{A_{N,\varepsilon}} |f_N(u) - f(u)|^p d\mu(u) \leqslant \varepsilon^p$$

and thus we have the needed assertion since the convergence in measure of f_N to f implies that for a fixed ε, $\lim_{N\to\infty} \mu(A^c_{N,\varepsilon}) = 0$. This completes the proof of Lemma 3.7. □

We can now turn to the proof of Theorem 1.2. We follow the arguments of [12, Lemma 3.8]. If we set

$$U_N \equiv \big\{u \in H^s_{\rm rad}(\Theta) : S_N(u) \in U\big\}$$

then

$$U \subset \liminf_N (U_N),$$

where

$$\liminf_N (U_N) \equiv \bigcup_{N=1}^{\infty} \bigcap_{N_1=N}^{\infty} U_{N_1}.$$

Indeed, we have that for every $u \in H^\sigma_{\mathrm{rad}}(\Theta)$, $S_N(u)$ converges to u in $H^\sigma_{\mathrm{rad}}(\Theta)$, as N tends to ∞. Therefore, using that U is an open set, we conclude that for every $u \in U$ there exists $N_0 \geqslant 1$ such that for $N \geqslant N_0$ one has $u \in U_N$. Hence we have $U \subset \liminf_N(U_N)$. If A is a ρ-measurable set, we denote by $\mathbb{1}_A$ the characteristic function of A. Notice that thanks to the property $U \subset \liminf_N(U_N)$,

$$\liminf_{N\to\infty} \mathbb{1}_{U_N} \geqslant \mathbb{1}_U.$$

Recall that

$$\tilde{\rho}_N(U) = \rho_N(U \cap E_N) = \int_{H^s_{\mathrm{rad}}(\Theta)} \mathbb{1}_{U_N}(u) f_N(u) d\mu(u).$$

Using Lemma 3.7, we observe that

$$\lim_{N\to\infty} \Big(\int_{H^s_{\mathrm{rad}}(\Theta)} \mathbb{1}_{U_N}(u) f_N(u) d\mu(u) - \int_{H^s_{\mathrm{rad}}(\Theta)} \mathbb{1}_{U_N}(u) f(u) d\mu(u) \Big) = 0.$$

Next, using the Fatou lemma, we get

$$\begin{aligned} \liminf_{N\to\infty} \rho_N(U \cap E_N) &= \liminf_{N\to\infty} \int_{H^s_{\mathrm{rad}}(\Theta)} \mathbb{1}_{U_N}(u) f(u) d\mu(u) \\ &\geqslant \int_{H^s_{\mathrm{rad}}(\Theta)} \mathbb{1}_U(u) f(u) d\mu(u) \\ &= \int_U f(u) d\mu(u) = \rho(U). \end{aligned}$$

This proves (1.11). Observe that Lemma 3.7 implies that

$$\lim_{N\to\infty} \rho_N(E_N) = \rho(H^s_{\mathrm{rad}}(\Theta)).$$

Therefore to prove (1.12), it suffices to use (1.11) by passing to complementary sets (as in [12], we could give a direct proof of (1.12)). This completes the proof of Theorem 1.2. □

Remark 3.8. — Let us observe that the reasoning in the proof of Theorem 1.2 is of quite general nature. It suffices to know that:

- (f_N) is bounded uniformly with respect to N in $L^p(d\mu)$ for some $p > 1$.
- (f_N) converges to f in measure.

3.5. A corollary of Theorem 1.2

Combining Lemma 3.5 and Theorem 1.2, we arrive at the following statement.

LEMMA 3.9. — *Let $\sigma \in [s, 1/2[$. There exist $C > 0$ and $c > 0$ such that for every real number $\lambda \geqslant 1$,*

$$\rho\Big(u \in H^s_{\mathrm{rad}}(\Theta) : \|u\|_{H^\sigma(\Theta)} \in]\lambda, \infty[\Big) \leqslant Ce^{-c\lambda^2}.$$

Proof. — It suffices to apply Theorem 1.2 to the open set of $H^\sigma_{\mathrm{rad}}(\Theta)$,

$$U = \Big(u \in H^s_{\mathrm{rad}}(\Theta) : \|u\|_{H^\sigma(\Theta)} \in]\lambda, \infty[\Big)$$

and to observe that $\tilde{\rho}_N(U) = \tilde{\rho}_N(U_N)$, where

$$U_N = \Big(u \in H^s_{\mathrm{rad}}(\Theta) : \|S_N(u)\|_{H^\sigma(\Theta)} \in]\lambda, \infty[\Big).$$

Thus by Lemma 3.5, $\tilde{\rho}_N(U) \leqslant C \exp(-c\lambda^2)$ which, combined with Theorem 1.2, completes the proof of Lemma 3.9. □

Remark 3.10. — As a consequence of Lemma 3.9 one obtains that for $\sigma \in [s, 1/2[$ one has $\rho(H^\sigma_{\mathrm{rad}}(\Theta)) = \rho(H^s_{\mathrm{rad}}(\Theta))$. Moreover for every ρ measurable set A,

$$\rho\Big(u \in A : \|u\|_{H^\sigma(\Theta)} \in]\lambda, \infty[\Big) \leqslant Ce^{-c\lambda^2}.$$

and thus A may be approximated by bounded sets of $H^\sigma_{\mathrm{rad}}(\Theta)$ (the intersections of A and the balls of radius $\lambda \gg 1$ centered at the origin of $H^\sigma_{\mathrm{rad}}(\Theta)$).

4. Bourgain spaces and bilinear estimates

The following two statements play a crucial role in the analysis of the local Cauchy problem for (1.1).

PROPOSITION 4.1. — *For every $\varepsilon > 0$, there exists $\beta < 1/2$, there exists $C > 0$ such that for every $N_1, N_2 \geqslant 1$, every $L_1, L_2 \geqslant 1$, every u_1, u_2 two functions on $\mathbb{R} \times \Theta$ of the form*

$$u_j(t, r) = \sum_{N_j \leqslant \langle z_n \rangle < 2N_j} c_j(n, t) e_n(r), \quad j = 1, 2$$

where the Fourier transform of $c_j(n, t)$ with respect to t satisfies

$$\operatorname{supp} \widehat{c_j}(n, \tau) \subset \{\tau \in \mathbb{R} : L_j \leqslant \langle \tau + z_n^2 \rangle \leqslant 2L_j\}, \quad j = 1, 2$$

one has the bound

$$\|u_1u_2\|_{L^2(\mathbb{R}\times\Theta)} \leqslant C(\min(N_1,N_2))^{\varepsilon}(L_1L_2)^{\beta}\|u_1\|_{L^2(\mathbb{R}\times\Theta)}\|u_2\|_{L^2(\mathbb{R}\times\Theta)}.$$

Proposition 4.2. — *For every $\varepsilon > 0$, there exists $\beta < 1/2$, there exists $C > 0$ such that for every $N_1, N_2 \geqslant 1$, every $L_1, L_2 \geqslant 1$, every u_1, u_2 two functions on $\mathbb{R}\times\Theta$ of the form*

$$u_1(t,r) = \sum_{N_1\leqslant\langle z_n\rangle<2N_1} c_1(n,t)e_n(r)$$

and

$$u_2(t,r) = \sum_{N_2\leqslant\langle z_n\rangle<2N_2} c_2(n,t)e_n'(r)$$

where the Fourier transform of $c_j(n,t)$ with respect to t satisfies

$$\operatorname{supp}\widehat{c_j}(n,\tau) \subset \{\tau\in\mathbb{R} : L_j \leqslant \langle\tau+z_n^2\rangle \leqslant 2L_j\}, \quad j=1,2$$

one has the bound

$$\|u_1u_2\|_{L^2(\mathbb{R}\times\Theta)} \leqslant C(\min(N_1,N_2))^{\varepsilon}(L_1L_2)^{\beta}\|u_1\|_{L^2(\mathbb{R}\times\Theta)}\|u_2\|_{L^2(\mathbb{R}\times\Theta)}.$$

For the proof of Propositions 4.1 and 4.2 we refer to [12, Proposition 4.1] and [12, Proposition 4.3] respectively. The results of Propositions 4.1 and 4.2 can be injected in the framework of the Bourgain spaces associated to the Schrödinger equation on the disc, in order to get local existence results for (1.1). Following [12], we define the Bourgain spaces $X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)$ of functions on $\mathbb{R}\times\Theta$ which are radial with respect to the second argument, equipped with the norm

$$\|u\|^2_{X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)} = \sum_{n=1}^{\infty} z_n^{2\sigma}\|\langle\tau+z_n^2\rangle^b\widehat{\langle u(t),e_n\rangle}(\tau)\|^2_{L^2(\mathbb{R}_\tau)},$$

where $\langle\cdot,\cdot\rangle$ stays for the $L^2(\Theta)$ pairing and $\widehat{\cdot}$ denotes the Fourier transform on $\mathbb{R}$. We can express the norm in $X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)$ in terms of the localization operators $\Delta_{N,L}$. More precisely, if for N, L positive integers, we define $\Delta_{N,L}$ by

$$\Delta_{N,L}(u) = \frac{1}{2\pi}\sum_{n:N\leqslant\langle z_n\rangle<2N}\Big(\int_{L\leqslant\langle\tau+z_n^2\rangle\leqslant 2L}\widehat{\langle u(t),e_n\rangle}(\tau)e^{it\tau}d\tau\Big)e_n,$$

then we can write

$$\|u\|^2_{X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)} \approx_{\sigma,b} \sum_{L,N-\text{dyadic}} L^{2b}N^{2\sigma}\|\Delta_{N,L}(u)\|^2_{L^2(\mathbb{R}\times\Theta)},$$

where $\approx_{\sigma,b}$ means that the implicit constant may depend on σ and b. Using that (see [12]),

$$\exists C>0 : \forall n\in\mathbb{N}, \|e_n\|_{L^\infty(\Theta)} \leqslant Cn^{\frac{1}{2}}$$

and the Cauchy-Schwarz inequality in the τ integration and in the n summation, we arrive at the bound

$$\|\Delta_{N,L}(u)\|_{L^\infty(\mathbb{R}\times\Theta)} \leqslant L^{\frac{1}{2}} N \|\Delta_{N,L}(u)\|_{L^2(\mathbb{R}\times\Theta)}. \tag{4.1}$$

Let us next analyze $\partial_r(\Delta_{N,L}(u))$. We can write

$$\Delta_{N,L}(u) = \sum_{N\leqslant\langle z_n\rangle<2N} c(n,t)e_n(r),$$

$$\operatorname{supp}\widehat{c}(n,\tau) \subset \{\tau\in\mathbb{R} : L \leqslant \langle \tau + z_n^2\rangle \leqslant 2L\}$$

and thus

$$\partial_r\big(\Delta_{N,L}(u)\big) = \sum_{N\leqslant\langle z_n\rangle<2N} c(n,t)e_n'(r). \tag{4.2}$$

Recall (see [12]) that for $m\neq n$, e_m' and e_n' are orthogonal in $L^2(\Theta)$ and $\|e_n'\|_{L^2(\Theta)} \approx n$. Therefore

$$\big\|\partial_r\big(\Delta_{N,L}(u)\big)\big\|^2_{L^2(\mathbb{R}\times\Theta)} = c\sum_{N\leqslant\langle z_n\rangle<2N} \|e_n'\|^2_{L^2(\Theta)}\int_{-\infty}^{\infty}|\widehat{c}(n,\tau)|^2 d\tau$$

and thus

$$\big\|\partial_r\big(\Delta_{N,L}(u)\big)\big\|_{L^2(\mathbb{R}\times\Theta)} \approx N\big\|\Delta_{N,L}(u)\big\|_{L^2(\mathbb{R}\times\Theta)}. \tag{4.3}$$

Using [12, Lemma 2.1], $\|\partial_r e_n\|_{L^\infty(\Theta)} \leqslant Cn^{3/2}$ and thus coming back to (4.2), after writing $c(n,t)$ in terms of its Fourier transform and applying the Cauchy-Schwarz inequality in the τ (the dual of t variable) integration, we obtain that

$$\big\|\partial_r\big(\Delta_{N,L}(u)\big)\big\|_{L^\infty(\mathbb{R}\times\Theta)} \leqslant CL^{\frac{1}{2}}N^2\big\|\Delta_{N,L}(u)\big\|_{L^2(\mathbb{R}\times\Theta)}. \tag{4.4}$$

Let us next define two other projectors involved in the well-posedness analysis of (1.1). The projector Δ_N is defined by

$$\Delta_N(u) = \sum_{n:N\leqslant\langle z_n\rangle<2N} \langle u, e_n\rangle e_n.$$

For $N\geqslant 2$ a dyadic integer, we define the projector $\widetilde{S}_N$ by

$$\widetilde{S}_N = \sum_{\substack{N_1\leqslant N/2\\ N_1-\text{dyadic}}} \Delta_{N_1}.$$

For a notational convenience, we assume that $\widetilde{S}_1$ is zero.

5. Nonlinear estimates in Bourgain spaces

In this section, we shall derive nonlinear estimates related to the problems (1.1) and (1.9). We start by a lemma which improves on the Sobolev embedding.

LEMMA 5.1. — *Let us fix $p \geqslant 4$. Then for every $b > \frac{1}{2}$, $\sigma > 1 - \frac{4}{p}$ there exists $C > 0$ such that for all $u \in X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)$ one has*

$$\|u\|_{L^p(\mathbb{R}\times\Theta)} \leqslant C\|u\|_{X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)}. \tag{5.1}$$

Proof. — It suffices to prove the assertion for $p = 4$ and $p = \infty$. Let us first consider the case $p = 4$. Observe that $\Delta_{N,L}(u)$ fits in the scope of applicability of Proposition 4.1. Using Proposition 4.1 with $\varepsilon = \sigma/2 > 0$, we obtain that

$$\|\Delta_{N,L}(u)\|_{L^4(\mathbb{R}\times\Theta)} \leqslant C\|\Delta_{N,L}(u)\|_{X^{\sigma/2,\beta}_{\rm rad}(\mathbb{R}\times\Theta)}.$$

Therefore, by writing $u = \sum_{L,N} \Delta_{N,L}(u)$, where the summation runs over the dyadic values of L, N, by summing geometric series in N and L , we obtain that (5.1) holds true for $p = 4$ (observe that we use Proposition 4.1 with $\varepsilon = \sigma/2$ instead of σ in order to get small negative powers of N and L after applying the triangle inequality to $\sum_{L,N} \Delta_{N,L}(u)$). Let us next consider the case $p = \infty$. In this case, the assertion holds true thanks to (4.1) and another summation of geometric series. This completes the proof of Lemma 5.1. □

The next lemma gives sense of $F(u)$, in the scale of Bourgain's spaces, for u of low regularity.

LEMMA 5.2. — *Let (b,σ) be such that* $\max(1/3, 1-2/\alpha) < \sigma < 1/2$, $b > 1/2$ *and let $u \in X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)$. Then $F(u) \in X^{-\sigma,-b}_{\rm rad}(\mathbb{R}\times\Theta)$. Moreover*

$$\lim_{N\to\infty} \|F(u) - F(\widetilde{S}_N(u))\|_{X^{-\sigma,-b}_{\rm rad}(\mathbb{R}\times\Theta)} = 0.$$

Proof. — For $v \in X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)$, we write

$$\begin{aligned}\int_{\mathbb{R}\times\Theta} |F(u)v| &\leqslant C\Big(\int_{\mathbb{R}\times\Theta} |uv| + \int_{\mathbb{R}\times\Theta} |u|^{\alpha+1}|v|\Big)\\ &\leqslant C\|u\|_{X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)}\|v\|_{X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)}\\ &\qquad + C\|u\|^{\alpha+1}_{L^{\alpha+2}(\mathbb{R}\times\Theta)}\|v\|_{L^{\alpha+2}(\mathbb{R}\times\Theta)}.\end{aligned}$$

Now, using Lemma 5.1, we get

$$\|u\|_{L^{\alpha+2}(\mathbb{R}\times\Theta)} \leqslant C\|u\|_{X^{\sigma_1,b}_{\rm rad}(\mathbb{R}\times\Theta)},$$

where $\sigma_1 > 0$, when $\alpha \leqslant 2$ and $\sigma_1 > 1-4/(\alpha+2)$ when $\alpha \in]2,4[$. Observing that for $\alpha \geqslant 2$, $\max(1/3, 1-2/\alpha) \geqslant 1-4/(\alpha+2)$ shows that

$$\int_{\mathbb{R}\times\Theta} |F(u)v| \leqslant C\|u\|_{X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}\|v\|_{X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}\Big(1+\|u\|^{\alpha}_{X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}\Big)$$

and thus $F(u) \in X^{-\sigma,-b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)$. Similarly one shows that

$$\int_{\mathbb{R}\times\Theta} \big|(F(u)-F(\widetilde{S}_N(u)))v\big| \leqslant C\|u-\widetilde{S}_N(u)\|_{X^{\sigma,b}_{\mathrm{rad}}}\|v\|_{X^{\sigma,b}_{\mathrm{rad}}}\Big(1+\|u\|^{\alpha}_{X^{\sigma,b}_{\mathrm{rad}}}\Big)$$

which yields the needed convergence. This completes the proof of Lemma 5.2. □

One may prove a statement similar to Lemma 5.2 with $\widetilde{S}_N$ replaced by $\widetilde{S}_{N,L}$ where $\widetilde{S}_{N,L}$ is defined similarly to $\widetilde{S}_N$ with Δ_{N_1} replaced by Δ_{N_1,L_1}, $L_1 \leqslant L$. This observation allows to consider only finite sums over dyadic integers in the proof of the next proposition (one can also apply a similar approximation argument to v involved in (5.6)).

In fact a much stronger statement then Lemma 5.2 holds true. It turns out that under the assumptions of Lemma 5.2 one has $F(u) \in X^{\sigma,-b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)$.

Proposition 5.3. — *Let* $\max(1/3, 1-2/\alpha) < \sigma_1 \leqslant \sigma < 1/2$. *Then there exist two positive numbers* b, b' *such that* $b+b' < 1$, $b' < 1/2 < b$, *there exists* $C > 0$ *such that for every* $u, v \in X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)$,

$$\|F(u)\|_{X^{\sigma,-b'}_{\mathrm{rad}}(\mathbb{R}\times\Theta)} \leqslant C\Big(1+\|u\|^{\max(\alpha,2)}_{X^{\sigma_1,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}\Big)\|u\|_{X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)} \tag{5.2}$$

and

$$\|F(u)-F(v)\|_{X^{\sigma,-b'}_{\mathrm{rad}}} \leqslant C\Big(1+\|u\|^{\max(\alpha,2)}_{X^{\sigma,b}_{\mathrm{rad}}}+\|v\|^{\max(\alpha,2)}_{X^{\sigma,b}_{\mathrm{rad}}}\Big)\|u-v\|_{X^{\sigma,b}_{\mathrm{rad}}}. \tag{5.3}$$

Proof. — The proof of this proposition for $\alpha < 2$ is given in [12]. We therefore may assume that $\alpha \geqslant 2$ in the sequel of the proof. The proof will follow closely [12] by incorporating an argument already appeared in [6]. Let us observe that in order to prove (5.2), it suffices to prove that

$$\|F(\widetilde{S}_M(u))\|_{X^{\sigma,-b'}_{\mathrm{rad}}(\mathbb{R}\times\Theta)} \leqslant C\Big(1+\|u\|^{\alpha}_{X^{\sigma_1,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}\Big)\|u\|_{X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}, \tag{5.4}$$

uniformly in $M \in \mathbb{N}$. Indeed, if we can prove (5.4) then $(F(\widetilde{S}_M(u)))_{M\in\mathbb{N}}$ is a bounded sequence in $X^{\sigma,-b'}_{\mathrm{rad}}(\mathbb{R}\times\Theta)$ (and thus also in $X^{-\sigma,-b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)$) and therefore it converges (up to a sub-sequence) weakly to some limit which satisfies the needed bound. In order to identify this limit with $F(u)$ it suffices make appeal to Lemma 5.2. Thanks to the gauge invariance of the nonlinearity $F(u)$, we observe that $F(u)-(\partial F)(0)u$ is vanishing at order 2 at $u=0$ and thus in the proof of (5.4), we may assume that

$$\partial^{k_1}\bar{\partial}^{k_2}(F)(0)=0,\quad \forall k_1+k_2\leqslant 2. \tag{5.5}$$

Observe that (5.4) follows from the estimate

$$\Big|\int_{\mathbb{R}\times\Theta}F(\widetilde{S}_M(u))\bar{v}\Big| \leqslant C\|v\|_{X^{-\sigma,b'}_{\rm rad}(\mathbb{R}\times\Theta)}\Big(1+\|u\|^{\alpha}_{X^{\sigma_1,b}_{\rm rad}(\mathbb{R}\times\Theta)}\Big)\|u\|_{X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)} \tag{5.6}$$

(let us remark that if v contains only very high frequencies with respect to the $\Delta_{N,L}$ decomposition then the right hand-side of (5.6) is small). Using that $\Delta_N=\widetilde{S}_{2N}-\widetilde{S}_N$ and (5.5), we may write

$$\begin{aligned}F(\widetilde{S}_M(u)) &= \sum_{\substack{2\leqslant N_1\leqslant M\\ N_1-\text{dyadic}}}(F(\widetilde{S}_{N_1}(u))-F(\widetilde{S}_{N_1/2}(u))\\ &= \sum_{\substack{N_1\leqslant M/2\\ N_1-\text{dyadic}}}\big(\Delta_{N_1}(u)G_1(\Delta_{N_1}(u),\widetilde{S}_{N_1}(u))\\ &\qquad\qquad +\overline{\Delta_{N_1}(u)}G_2(\Delta_{N_1}(u),\widetilde{S}_{N_1}(u))\big)\\ &\equiv F_1(u)+F_2(u),\end{aligned}$$

where $G_1(z_1,z_2)$ and $G_2(z_1,z_2)$ are smooth functions with a control on their growth at infinity coming from (1.2) (similar bounds to F with α replaced $\alpha-1$). Moreover, thanks to (5.5), $G_1(0,0)=\partial(F)(0)=0$ and $G_2(0,0)=\bar{\partial}(F)(0)=0$. We will only estimate the contribution of $F_1(u)$ to the right hand-side of (5.6), the argument for the contribution of $F_2(u)$ being completely analogous. Next, we set

$$\begin{aligned}I &= \Big|\int_{\mathbb{R}\times\Theta}F_1(u)\bar{v}\Big|,\\ I(N_0,N_1) &= \Big|\int_{\mathbb{R}\times\Theta}\Delta_{N_1}(u)\overline{\Delta_{N_0}(v)}G_1(\Delta_{N_1}(u),\widetilde{S}_{N_1}(u))\Big|.\end{aligned}$$

Then $I\leqslant I_1+I_2$, where

$$I_1=\sum_{\substack{N_0\leqslant N_1\leqslant M/2\\ N_0,N_1-\text{dyadic}}}I(N_0,N_1),\quad I_2=\sum_{\substack{N_1\leqslant\min(N_0,M/2)\\ N_0,N_1-\text{dyadic}}}I(N_0,N_1).$$

We first analyze I_1. Using (5.5) with $(k_1,k_2)=(1,0)$, we decompose $G_1(\Delta_{N_1}(u),\widetilde{S}_{N_1}(u))$ as

$$\sum_{\substack{N_2\leqslant N_1\\ N_2-\text{dyadic}}}\Big(G_1(\widetilde{S}_{2N_2}\Delta_{N_1}(u),\widetilde{S}_{2N_2}\widetilde{S}_{N_1}(u))-G_1(\widetilde{S}_{N_2}\Delta_{N_1}(u),\widetilde{S}_{N_2}\widetilde{S}_{N_1}(u))\Big).$$

Using that $\Delta_{N_1}\Delta_{N_2} = \Delta_{N_1}$, if $N_1 = N_2$ and zero elsewhere, we obtain

$$G_1(\Delta_{N_1}(u), \widetilde{S}_{N_1}(u)) = \sum_{\substack{N_2 \leqslant N_1 \\ N_2-\text{dyadic}}} \Delta_{N_2}(u) G_{11}^{N_2}(\Delta_{N_2}(u), \widetilde{S}_{N_2}(u)) + \sum_{\substack{N_2 \leqslant N_1 \\ N_2-\text{dyadic}}} \overline{\Delta_{N_2}(u)} G_{12}^{N_2}(\Delta_{N_2}(u), \widetilde{S}_{N_2}(u)), \tag{5.7}$$

where $G_{11}^{N_2}(z_1, z_2)$ and $G_{12}^{N_2}(z_1, z_2)$ are smooth functions with a control on their growth at infinity coming from (1.2). Moreover thanks to (5.5), applied with $(k_1, k_2) = (2, 0)$ and $(k_1, k_2) = (1, 1)$, we get $G_{11}^{N_2}(0, 0) = 0$ and $G_{12}^{N_2}(0, 0) = 0$. Therefore, we can expand for $j = 1, 2$,

$$G_{1j}^{N_2}(\Delta_{N_2}(u), \widetilde{S}_{N_2}(u)) = \sum_{\substack{N_3 \leqslant N_2 \\ N_3-\text{dyadic}}} \Delta_{N_3}(u) G_{1j1}^{N_3}(\Delta_{N_3}(u), \widetilde{S}_{N_3}(u)) + \sum_{\substack{N_3 \leqslant N_2 \\ N_3-\text{dyadic}}} \overline{\Delta_{N_3}(u)} G_{1j2}^{N_3}(\Delta_{N_3}(u), \widetilde{S}_{N_3}(u)), \tag{5.8}$$

where, thanks to the growth assumption on the nonlinearity $F(u)$, we have that the functions $G_{1j_1j_2}^{N_3}(z_1, z_2)$, $j_1, j_2 \in \{1, 2\}$ satisfy

$$|G_{1j_1j_2}^{N_3}(z_1, z_2)| \leqslant C(1 + |z_1| + |z_2|)^{\alpha-2}. \tag{5.9}$$

We therefore have the bound

$$I_1 \leqslant C \sum_{\substack{N_0 \leqslant N_1 \\ N_0, N_1-\text{dyadic}}} \sum_{\substack{N_1 \geqslant N_2 \geqslant N_3 \\ N_2, N_3-\text{dyadic}}} \int_{\mathbb{R}\times\Theta} |\Delta_{N_0}(v)\Delta_{N_1}(u)\Delta_{N_2}(u)\Delta_{N_3}(u)|(1 + |\Delta_{N_3}(u)| + |\widetilde{S}_{N_3}(u)|)^{\alpha-2}.$$

By splitting

$$\Delta_N = \sum_{L-\text{dyadic}} \Delta_{N,L},$$

we may write for $b > 1/2$, $0 < \sigma_1 < 1/2$, by using (4.1) and the Cauchy-Schwarz inequality in the L summation

$$\begin{aligned}
\|\Delta_{N_3}(u)\|_{L^\infty(\mathbb{R}\times\Theta)} &\leqslant \sum_{L-\text{dyadic}} \|\Delta_{N_3,L}(u)\|_{L^\infty(\mathbb{R}\times\Theta)} \\
&\leqslant C \sum_{L-\text{dyadic}} N_3 L^{\frac{1}{2}} \|\Delta_{N_3,L}(u)\|_{L^2(\mathbb{R}\times\Theta)} \\
&\leqslant C N_3^{1-\sigma_1} \|u\|_{X^{\sigma_1,b}_{\text{rad}}(\mathbb{R}\times\Theta)},
\end{aligned}$$

where C depends on b and σ_1. Similarly

$$\begin{aligned}
\|\widetilde{S}_{N_3}(u)\|_{L^\infty(\mathbb{R}\times\Theta)} &\leqslant \sum_{\substack{N\leqslant N_3/2\\ N-\text{dyadic}}} \|\Delta_N(u)\|_{L^\infty(\mathbb{R}\times\Theta)} \\
&\leqslant \sum_{\substack{N\leqslant N_3/2\\ L,N-\text{dyadic}}} \|\Delta_{N,L}(u)\|_{L^\infty(\mathbb{R}\times\Theta)} \\
&\leqslant \sum_{\substack{N\leqslant N_3/2\\ L,N-\text{dyadic}}} CNL^{\frac{1}{2}}\|\Delta_{N,L}(u)\|_{L^2(\mathbb{R}\times\Theta)} \\
&\leqslant C\Big(\sum_{\substack{N\leqslant N_3/2\\ L,N-\text{dyadic}}} L^{1-2b}N^{2(1-\sigma_1)}\Big)^{\frac{1}{2}} \\
&\qquad\times\Big(\sum_{\substack{N\leqslant N_3/2\\ L,N-\text{dyadic}}} L^{2b}N^{2\sigma_1}\|\Delta_{N,L}(u)\|_{L^2}^2\Big)^{\frac{1}{2}} \\
&\leqslant CN_3^{1-\sigma_1}\|u\|_{X^{\sigma_1,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
I_1 \leqslant C(1+\|u\|^{\alpha-2}_{X^{\sigma_1,b}(\mathbb{R}\times\Theta)}) &\sum_{\substack{N_0\leqslant N_1\\ N_0,N_1-\text{dyadic}}} \sum_{\substack{N_1\geqslant N_2\geqslant N_3\\ N_2,N_3-\text{dyadic}}} \\
&N_3^{(1-\sigma_1)(\alpha-2)}\Big(\int_{\mathbb{R}\times\Theta}|\Delta_{N_0}(v)\Delta_{N_1}(u)\Delta_{N_2}(u)\Delta_{N_3}(u)|\Big).
\end{aligned}$$

Using Proposition 4.1 and the Cauchy-Schwarz inequality, we obtain that for every $\varepsilon>0$ there exist $\beta<1/2$ and C_ε such that

$$\begin{aligned}
&\int_{\mathbb{R}\times\Theta}\big|\Delta_{N_0,L_0}(v)\Delta_{N_1,L_1}(u)\Delta_{N_2,L_2}(u)\Delta_{N_3,L_3}(u)\big| \\
&\leqslant \big\|\Delta_{N_0,L_0}(v)\Delta_{N_2,L_2}(u)\big\|_{L^2(\mathbb{R}\times\Theta)}\big\|\Delta_{N_1,L_1}(u)\Delta_{N_3,L_3}(u)\big\|_{L^2(\mathbb{R}\times\Theta)} \\
&\leqslant C_\varepsilon(N_2N_3)^\varepsilon(L_0L_1L_2L_3)^\beta\big\|\Delta_{N_0,L_0}(v)\big\|_{L^2(\mathbb{R}\times\Theta)}\prod_{j=1}^{3}\big\|\Delta_{N_j,L_j}(u)r\big\|_{L^2(\mathbb{R}\times\Theta)}.
\end{aligned}$$

Therefore, if we set

$$\begin{aligned}
(5.10)\qquad Q &\equiv Q(N_0,N_1,N_2,N_3,L_0,L_1,L_2,L_3) \\
&= CN_0^{-\sigma}N_1^{\sigma}(N_2N_3)^{\sigma_1}L_0^{b'}(L_1L_2L_3)^b\Big(1+\|u\|^{\alpha-2}_{X^{\sigma_1,b}(\mathbb{R}\times\Theta)}\Big) \\
&\qquad\times\big\|\Delta_{N_0,L_0}(v)\big\|_{L^2(\mathbb{R}\times\Theta)}\prod_{j=1}^{3}\big\|\Delta_{N_j,L_j}(u)\big\|_{L^2(\mathbb{R}\times\Theta)},
\end{aligned}$$

we can write

$$I_1 \leqslant \sum_{L_0,L_1,L_2,L_3-\text{dyadic}} \sum_{\substack{N_1\geqslant N_2\geqslant N_3,\, N_1\geqslant N_0\\ N_0,N_1,N_2,N_3-\text{dyadic}}} L_0^{\beta-b'}(L_1L_2L_3)^{\beta-b}\Big(\frac{N_0}{N_1}\Big)^{\sigma}\frac{N_3^{(1-\sigma_1)(\alpha-2)}}{(N_2N_3)^{\sigma_1-\varepsilon}}Q.$$

Let us take $\varepsilon>0$ such that

$$\varepsilon<1-\frac{\alpha(1-\sigma_1)}{2}.$$

A proper choice of ε is possible thanks to the assumption $\sigma_1>1-2/\alpha$. With this choice of ε we have that $2(\sigma_1-\varepsilon)>(1-\sigma_1)(\alpha-2)$. The choice of ε fixes β via the application of Proposition 4.1. Then we choose b' such that $\beta<b'<1/2$. We finally choose $b>1/2$ such that $b+b'<1$. With this choice of the parameters, coming back to the definition of the projectors $\Delta_{N,L}$ and after summing geometric series in L_0, L_1, L_2, L_3, N_2, N_3, we can write that

$$I_1\leqslant C(1+\|u\|^{\alpha-2}_{X^{\sigma_1,b}(\mathbb{R}\times\Theta)})\|u\|^2_{X^{\sigma_1,b}_{\text{rad}}(\mathbb{R}\times\Theta)}\sum_{\substack{N_0\leqslant N_1\\ N_0,N_1-\text{dyadic}}}\Big(\frac{N_0}{N_1}\Big)^{\sigma}c(N_0)d(N_1),$$

where

$$\begin{aligned}c(N_0)&=N_0^{-\sigma}\|\Delta_{N_0}(v)\|_{X^{0,b'}_{\text{rad}}(\mathbb{R}\times\Theta)},\\ d(N_1)&=N_1^{\sigma}\|\Delta_{N_1}(u)\|_{X^{0,b}_{\text{rad}}(\mathbb{R}\times\Theta)}.\end{aligned}\tag{5.11}$$

Finally, using [12, Lemma 6.2], we arrive at the bound

$$I_1\leqslant C\|v\|_{X^{-\sigma,b'}_{\text{rad}}(\mathbb{R}\times\Theta)}(1+\|u\|^{\alpha-2}_{X^{\sigma_1,b}(\mathbb{R}\times\Theta)})\|u\|^2_{X^{\sigma_1,b}_{\text{rad}}(\mathbb{R}\times\Theta)}\|u\|_{X^{\sigma,b}_{\text{rad}}(\mathbb{R}\times\Theta)}$$

which ends the analysis for I_1.

Let us now turn to the analysis of I_2. The main observation is that after in integration by parts the roles of N_0 and N_1 are exchanged. We have that

$$I_2\leqslant\sum_{\substack{N_1\leqslant\min(N_0,M/2)\\ L_0,N_0,N_1-\text{dyadic}}} I(L_0,N_0,N_1),$$

where

$$I(L_0,N_0,N_1)=\Big|\int_{\mathbb{R}\times\Theta}\Delta_{N_1}(u)\overline{\Delta_{N_0,L_0}(v)}G_1(\Delta_{N_1}(u),\widetilde{S}_{N_1}(u))\Big|.$$

Write

$$\Delta_{N_0,L_0}(v)=\sum_{N_0\leqslant\langle z_{n_0}\rangle<2N_0}c(n_0,t)e_{n_0}(r),\tag{5.12}$$

where

$$\operatorname{supp}\widehat{c}(n_0,\tau)\subset\Big\{\tau\in\mathbb{R}:L_0\leqslant\langle\tau+z_{n_0}^2\rangle\leqslant 2L_0\Big\}.$$

Define $\widetilde{\Delta}_{N_0,L_0}$ as

$$\widetilde{\Delta}_{N_0,L_0}(v)\equiv\sum_{N_0\leqslant\langle z_{n_0}\rangle\leqslant 2N_0}\frac{c(n_0,t)}{z_{n_0}^2}e'_{n_0}(r).$$

Since $\|e'_{n_0}\|_{L^2(\Theta)}\approx n_0$ (see [12]), we have

$$\|\widetilde{\Delta}_{N_0,L_0}(v)\|_{L^2(\mathbb{R}\times\Theta)}\approx N_0^{-1}\|\Delta_{N_0,L_0}(v)\|_{L^2(\mathbb{R}\times\Theta)}. \tag{5.13}$$

Since e_n vanishes on the boundary, using that

$$e_n(r)=-\frac{1}{z_n^2}\frac{1}{r}\partial_r(r\partial_r e_n(r)),$$

an integration by parts gives

$$I(L_0,N_0,N_1)=\Big|\int_{\mathbb{R}\times\Theta}\overline{\widetilde{\Delta}_{N_0,L_0}(v)}\partial_r\Big(\Delta_{N_1}(u)G_1(\Delta_{N_1}(u),\widetilde{S}_{N_1}(u))\Big)\Big|.$$

Recall that equality (5.7) shows that $G_1(\Delta_{N_1}(u),\widetilde{S}_{N_1}(u))$ can be expanded as a sum of two terms and then each term can be expanded according to (5.8). Therefore

$$I(L_0,N_0,N_1)\leqslant I_1(L_0,N_0,N_1)+I_2(L_0,N_0,N_1)\\ +I_3(L_0,N_0,N_1)+I_4(L_0,N_0,N_1),$$

where

$$I_1(L_0,N_0,N_1)=\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{\substack{N_3\leqslant N_2\leqslant N_1\\ N_2,N_3-\text{dyadic}}}\\ \int_{\mathbb{R}\times\Theta}|\widetilde{\Delta}_{N_0,L_0}(v)\partial_r\big(\Delta_{N_1}(u)\big)\Delta_{N_2}(u)\Delta_{N_3}(u)G_{1j_1j_2}^{N_3}(\Delta_{N_3}(u),\widetilde{S}_{N_3}(u))|,$$

$$I_2(L_0,N_0,N_1)=\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{\substack{N_3\leqslant N_2\leqslant N_1\\ N_2,N_3-\text{dyadic}}}\\ \int_{\mathbb{R}\times\Theta}|\widetilde{\Delta}_{N_0,L_0}(v)\Delta_{N_1}(u)\partial_r\big(\Delta_{N_2}(u)\big)\Delta_{N_3}(u)G_{1j_1j_2}^{N_3}(\Delta_{N_3}(u),\widetilde{S}_{N_3}(u))|,$$

$$I_3(L_0,N_0,N_1)=\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{\substack{N_3\leqslant N_2\leqslant N_1\\ N_2,N_3-\text{dyadic}}}$$
$$\int_{\mathbb{R}\times\Theta}|\widetilde{\Delta}_{N_0,L_0}(v)\Delta_{N_1}(u)\Delta_{N_2}(u)\partial_r\big(\Delta_{N_3}(u)\big)G^{N_3}_{1j_1j_2}(\Delta_{N_3}(u),\widetilde{S}_{N_3}(u))|,$$

$$I_4(L_0,N_0,N_1)=\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{\substack{N_3\leqslant N_2\leqslant N_1\\ N_2,N_3-\text{dyadic}}}$$
$$\int_{\mathbb{R}\times\Theta}|\widetilde{\Delta}_{N_0,L_0}(v)\Delta_{N_1}(u)\Delta_{N_2}(u)\Delta_{N_3}(u)\partial_r\big(G^{N_3}_{1j_1j_2}(\Delta_{N_3}(u),\widetilde{S}_{N_3}(u))\big)|,$$

Recall that $G^{N_3}_{1j_1j_2}(z_1,z_2)$ satisfies the bound (5.9). If we define Q by (5.10), expanding with respect to the L localizations, using two times Proposition 4.2 to the products $\widetilde{\Delta}_{N_0,L_0}(v)\Delta_{N_2,L_2}(u)$ and $\partial_r\big(\Delta_{N_1}(u)\big)\Delta_{N_3,L_3}(u)$ and (5.9) (together with (4.1), (4.3) and (5.13)) gives

$$\sum_{\substack{N_1\leqslant\min(N_0,M/2)\\ L_0,N_0,N_1-\text{dyadic}}} I_1(L_0,N_0,N_1)\leqslant\sum_{L_0,L_1,L_2,L_3-\text{dyadic}}\ \sum_{\substack{N_1\geqslant N_2\geqslant N_3,\,N_1\leqslant N_0\\ N_0,N_1,N_2,N_3-\text{dyadic}}}$$
$$L_0^{\beta-b'}(L_1L_2L_3)^{\beta-b}\Big(\frac{N_0}{N_1}\Big)^{\sigma-1}\frac{N_3^{(1-\sigma_1)(\alpha-2)}}{(N_2N_3)^{\sigma_1-\varepsilon}}Q.$$

The last expression may estimated exactly as we did for I_1, by exchanging the roles of N_0 and N_1. Similarly

$$\sum_{\substack{N_1\leqslant\min(N_0,M/2)\\ L_0,N_0,N_1-\text{dyadic}}} I_2(L_0,N_0,N_1)\leqslant\sum_{L_0,L_1,L_2,L_3-\text{dyadic}}\ \sum_{\substack{N_1\geqslant N_2\geqslant N_3,\,N_1\leqslant N_0\\ N_0,N_1,N_2,N_3-\text{dyadic}}}$$
$$L_0^{\beta-b'}(L_1L_2L_3)^{\beta-b}\Big(\frac{N_0}{N_1}\Big)^{\sigma}\Big(\frac{N_2}{N_0}\Big)\frac{N_3^{(1-\sigma_1)(\alpha-2)}}{(N_2N_3)^{\sigma_1-\varepsilon}}Q.$$

On the other hand on the summation region,

$$\Big(\frac{N_0}{N_1}\Big)^{\sigma}\Big(\frac{N_2}{N_0}\Big)\leqslant\Big(\frac{N_0}{N_1}\Big)^{\sigma-1}$$

and thus, again, we may conclude as in the bound for I_1. The sum

$$\sum_{\substack{N_1\leqslant\min(N_0,M/2)\\ L_0,N_0,N_1-\text{dyadic}}} I_3(L_0,N_0,N_1)$$

can be bounded similarly. Let us finally estimate the quantity

$$\sum_{\substack{N_1 \leqslant \min(N_0, M/2) \\ L_0, N_0, N_1 - \text{dyadic}}} I_4(L_0, N_0, N_1).$$

Observe that we can write

$$\Big|\partial_r\big(G^{N_3}_{1j_1j_2}(\Delta_{N_3}(u), \widetilde{S}_{N_3}(u))\big)\Big| \leqslant C\Big(|\partial_r\big(\Delta_{N_3}(u)\big)| + |\partial_r\big(\widetilde{S}_{N_3}(u)\big)|\Big) \times \Big(1 + |\Delta_{N_3}(u)| + |\widetilde{S}_{N_3}(u)|\Big)^{\max(\alpha-3,0)}.$$

Now using (4.4), we can write

$$\begin{aligned}
\|\partial_r(\Delta_{N_3}(u))\|_{L^\infty(\mathbb{R}\times\Theta)} &+ \|\partial_r(\widetilde{S}_{N_3}(u))\|_{L^\infty(\mathbb{R}\times\Theta)} \\
&\leqslant \sum_{\substack{N \leqslant N_3 \\ N - \text{dyadic}}} \|\partial_r(\Delta_N(u))\|_{L^\infty(\mathbb{R}\times\Theta)} \\
&\leqslant \sum_{\substack{L, N \leqslant N_3 \\ L, N - \text{dyadic}}} \|\partial_r(\Delta_{N,L}(u))\|_{L^\infty(\mathbb{R}\times\Theta)} \\
&\leqslant \sum_{\substack{L, N \leqslant N_3 \\ L, N - \text{dyadic}}} CN^2 L^{\frac{1}{2}} \|\Delta_{N,L}(u)\|_{L^2(\mathbb{R}\times\Theta)} \\
&\leqslant CN_3^{2-\sigma_1} \|u\|_{X^{\sigma_1,b}_{\rm rad}(\mathbb{R}\times\Theta)}.
\end{aligned}$$

Similarly

$$\Big(1 + |\Delta_{N_3}(u)| + |\widetilde{S}_{N_3}(u)|\Big)^{\max(\alpha-3,0)} \leqslant C(1 + N_3^{1-\sigma_1}\|u\|_{X^{\sigma_1,b}_{\rm rad}(\mathbb{R}\times\Theta)})^{\max(\alpha-3,0)}.$$

Let us suppose that $\alpha \in [3, 4[$, the analysis for $\alpha \in [2, 3]$ being simpler (one needs to modify slightly the next several lines by invoking the assumption $\sigma_1 > 1/3$). If we define Q by (5.10), expanding with respect to the L localizations, using Proposition 4.1 to the product $\Delta_{N_1,L_1}(u)\Delta_{N_3,L_3}(u)$ and Proposition 4.2 to the product $\widetilde{\Delta}_{N_0,L_0}(v)\Delta_{N_2,L_2}(u)$, we get

$$\sum_{\substack{N_1 \leqslant \min(N_0, M/2) \\ L_0, N_0, N_1 - \text{dyadic}}} I_4(L_0, N_0, N_1) \leqslant \sum_{L_0, L_1, L_2, L_3 - \text{dyadic}} \; \sum_{\substack{N_1 \geqslant N_2 \geqslant N_3, N_1 \leqslant N_0 \\ N_0, N_1, N_2, N_3 - \text{dyadic}}} L_0^{\beta - b'}(L_1L_2L_3)^{\beta-b}\Big(\frac{N_0}{N_1}\Big)^{\sigma}\frac{1}{N_0}\frac{N_3 N_3^{(1-\sigma_1)(\alpha-2)}}{(N_2N_3)^{\sigma_1-\varepsilon}}Q.$$

Since on the region of summation

$$\Big(\frac{N_0}{N_1}\Big)^{\sigma}\frac{1}{N_0}N_3\leqslant\Big(\frac{N_0}{N_1}\Big)^{\sigma-1}$$

we may conclude exactly as we did for I_1. This completes the analysis for I_2 and thus (5.2) is established. Thanks to the multilinear nature of our reasoning (compare with the method of Ginibre-Velo, Kato for treating the Cauchy problem for NLS which is not multilinear), the proof of (5.3) is essentially the same as the proof of (5.2). However one can no longer assume that the frequencies N_1 and N_2 satisfy $N_1\geqslant N_2$ but this fact does not affect the analysis since in contrast with (5.2) all terms in the right hand-side of (5.3) have *the same* spatial regularity σ (this is a standard feature in the analysis of nonlinear PDE's and not related to the spaces $X^{\sigma,b}_{\mathrm{rad}}$ we work with). More precisely, we can write

$$F(u)-F(v)=(u-v)G_1(u,v)+(\overline{u}-\overline{v})G_2(u,v)$$

with $G_j(z_1,z_2)$, $j=1,2$ satisfying the growth assumption

$$\big|\partial^{k_1}_{z_1}\bar{\partial}^{k_2}_{z_1}\partial^{l_1}_{z_2}\bar{\partial}^{l_2}_{z_2}G_j(z_1,z_2)\big|\leqslant C_{k_1,k_2,l_1,l_2}(1+|z_1|+|z_2|)^{\alpha-k_1-k_2-l_1-l_2}. \tag{5.14}$$

Since the analysis is very similar to the proof of (5.2), we shall only outline the estimate for $(u-v)G_1(u,v)$. Again, we can suppose that $F(u)$ is vanishing at order 3 at $u=0$ and $\alpha\geqslant 2$. Let us set

$$w_1=u-v,\quad w_2=u,\quad w_3=v.$$

We thus need to prove that

$$\Big|\int_{\mathbb{R}\times\Theta}w_1G_1(w_2,w_3)\overline{w_4}\Big|\leqslant C\big(1+\|w_2\|_{X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}+\|w_3\|_{X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}\big)^{\alpha}$$
$$\times\|w_1\|_{X^{\sigma,b}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}\|w_4\|_{X^{-\sigma,b'}_{\mathrm{rad}}(\mathbb{R}\times\Theta)}.$$

Next, we expand

$$w_1=\sum_{N_1-\text{dyadic}}\Delta_{N_1}(w_1),\quad w_4=\sum_{N_0-\text{dyadic}}\Delta_{N_0}(w_4)$$

and

$$G_1(w_2,w_3)=\sum_{N_2-\text{dyadic}}\Big(G_1(\widetilde{S}_{2N_2}(w_2),\widetilde{S}_{2N_2}(w_3))-G_1(\widetilde{S}_{N_2}(w_2),\widetilde{S}_{N_2}(w_3))\Big).$$

Thus, modulo complex conjugations irrelevant in this discussion, one has to evaluate quantities of type

$$(5.15)\qquad \sum_{N_0,N_1,N_2-\text{dyadic}} \Big|\int_{\mathbb{R}\times\Theta} \overline{\Delta_{N_0}(w_4)}\Delta_{N_1}(w_1)\Delta_{N_2}(w_j) \times H_j^{N_2}(\Delta_{N_2}(w_2),\widetilde{S}_{N_2}(w_2),\Delta_{N_2}(w_3),\widetilde{S}_{N_2}(w_3))\Big|,\quad j=2,3,$$

where $H_j^{N_2}(z_1,z_2,z_3,z_4)$ are smooth functions satisfying growth restrictions at infinity coming from (1.2). In the analysis of (5.15), we distinguish two cases for N_0, N_1, N_2 in the sum defining (5.15). Since N_1 and N_2 are not ordered, we need to compare N_0 with $\max(N_1,N_2)$ by performing arguments close in the spirit to the proof of (5.2).

Case 1. — The first case is when $N_0\leqslant\max(N_1,N_2)$. In this case, we expand once more $H_j^{N_2}$ which introduces a sum over N_3-dyadic, $N_3\leqslant N_2$ of terms $\Delta_{N_3}(w_j)$ (or complex conjugate) times a function which satisfies a decay property coming form (1.2). As in the analysis of I_1 above, we obtain the bound

$$(5.16)\qquad |(5.15)|\leqslant \sum_{L_0,L_1,L_2,L_3-\text{dyadic}}\ \sum_{\substack{N_1,N_2\geqslant N_3,\max(N_1,N_2)\geqslant N_0\\ N_0,N_1,N_2,N_3-\text{dyadic}}} L_0^{\beta-b'}(L_1L_2L_3)^{\beta-b}\Big(\frac{N_0}{\max(N_1,N_2)}\Big)^{\sigma}\frac{N_3^{(1-\sigma)(\alpha-2)}}{(\min(N_1,N_2)N_3)^{\sigma-\varepsilon}}Q,$$

where Q is defined similarly to (5.10) with the important difference that σ_1 is replaced by σ and the harmless difference that u is replaced by a suitable w_j, $j=1,2,3$ and v is replaced by w_4. If $\max(N_1,N_2)=N_1$ or $N_1\geqslant N_3$ then we conclude exactly as in the proof of (5.2).

We can therefore suppose that $\max(N_1,N_2)=N_2$ and $N_1\leqslant N_3$. Observe that we can also suppose that $F(u)$ is vanishing at order 5 at $u=0$ which allows to expand the non-linearity once more. Indeed, the cubic term in the Taylor expansion of the non-linearity can be dealt with as in (5.16) since in this term $\alpha=2$. Thus in the case $\max(N_1,N_2)=N_2$ and $N_1\leqslant N_3$, we expand once more the non-linearity which introduces a sum over N_4- dyadic, $N_4\leqslant N_3$ of terms $\Delta_{N_4}(w_j)$ (or complex conjugate) times a function which satisfies an appropriate decay property coming form (1.2). We next consider two cases $N_1\geqslant N_4$ and $N_1\leqslant N_4$. Let us suppose first that $N_1\geqslant N_4$. In this case, using the bilinear Strichartz estimates as in the analysis of I_1 above, we obtain the bound

$$(5.17)\quad |(5.15)| \leqslant \sum_{L_0,L_1,L_2,L_3,L_4-\text{dyadic}} \sum_{\substack{N_3\geqslant N_1\geqslant N_4, N_2\geqslant N_3\geqslant N_4, N_2\geqslant N_0 \\ N_0,N_1,N_2,N_3,N_4-\text{dyadic}}} L_0^{\beta-b'}(L_1L_2L_3L_4)^{\beta-b}\Big(\frac{N_0}{N_2}\Big)^{\sigma}\frac{N_4^{(1-\sigma)(\alpha-2)}}{(N_1N_3)^{\sigma-\varepsilon}}Q,$$

where Q is defined similarly to (5.10) with one additional factor in the product, *i.e.* the product runs from 1 to 4 instead of 1 to 3. With (5.17) at our disposal, we can conclude exactly as in the proof of (5.2). Let us suppose finally that $N_1 \leqslant N_4$. In this case we put the term involving Δ_{N_1} in L^∞ and perform the bilinear estimates with the terms involving N_0, N_2, N_3, N_4 to get

$$|(5.15)| \leqslant \sum_{L_0,L_1,L_2,L_3,L_4-\text{dyadic}} \sum_{\substack{N_1\leqslant N_4, N_2\geqslant N_3\geqslant N_4, N_2\geqslant N_0 \\ N_0,N_1,N_2,N_3,N_4-\text{dyadic}}} L_0^{\beta-b'}(L_1L_2L_3L_4)^{\beta-b}\Big(\frac{N_0}{N_2}\Big)^{\sigma}\frac{N_1^{1-\sigma}N_4^{\max(0,(1-\sigma)(\alpha-3))}}{(N_3N_4)^{\sigma-\varepsilon}}Q,$$

where Q is defined similarly to (5.10). Once again we conclude similarly to the proof of (5.2).

Case 2. — If $N_0 \geqslant \max(N_1, N_2)$, then we integrate by parts by the aid of $\Delta_{N_0}(w_4)$ and the analysis is very similar to the bound for I_2 in the proof of (5.2).

This completes the proof of Proposition 5.3. □

Remark 5.4. — We refer to [1], where an analysis similar to the proof of Proposition 5.3 is performed. In [1], one proves bilinear Strichartz estimates for the free evolution and by the transfer principal of [6] these estimates are transformed to estimates involving the projector $\Delta_{N,L}$. This approach is slightly different from the approach used in [12], based on direct bilinear estimates for functions enjoying localization properties similar to $\Delta_{N,L}(u)$.

6. Local analysis for NLS and the approximating ODE

In this section, we state the standard consequence of Proposition 5.3 to the local well-posedness of (1.1) and (1.9). For $T > 0$, we define the restriction spaces $X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)$, equipped with the natural norm

$$\|u\|_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)}=\inf\Big\{\|w\|_{X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)},\qquad w\in X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)\text{ with } w|_{]-T,T[}=u\Big\}.$$

Similarly, for $I\subset\mathbb{R}$ an interval, we can define the restriction spaces $X^{\sigma,b}_{\rm rad}(I\times\Theta)$, equipped with the natural norm. A Sobolev inequality with respect to the time variable yields,

$$\|u\|_{L^\infty([-T,T];H^\sigma_{\rm rad}(\Theta))}\leqslant C_b\|u\|_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)},\quad b>\frac{1}{2}.$$

Thus for $b>1/2$ the space $X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)$ is continuously embedded in $C([-T,T];H^\sigma_{\rm rad}(\Theta))$. We shall solve (1.1) for short times by applying the Banach contraction mapping principle to the "Duhamel formulation" of (1.1)

$$u(t)=e^{it\Delta}u_0-i\int_0^t e^{i(t-\tau)\Delta}F(u(\tau))d\tau, \tag{6.1}$$

where $e^{it\Delta}$ denotes the free propagator.

Remark 6.1. — In (6.1), the operator $e^{it\Delta}$ is defined by the Dirichlet self-adjoint realization of the Laplacian via the functional calculus of self-adjoint operators. As mentioned before the uniqueness statements in the well-posedness results in this paper are understood as uniqueness results for (6.1). On the other hand, despite the low regularity situation in this paper, the solutions of (6.1), we construct here have zero traces on $\mathbb{R}\times\partial\Theta$ (which is a general feature reflecting from the Dirichlet Laplacian, we work with) and thus the uniqueness issue can be studied in the context of the equation (1.1) subject to zero boundary conditions on $\mathbb{R}\times\partial\Theta$. If we set $S(t)=e^{it\Delta}$, then $S(t)e_n=e^{-itz_n^2}e_n$ and the norms in the Bourgain spaces may be expressed as

$$\|u\|_{X^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)}=\|S(-t)u\|_{H^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)},$$

where $H^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)$ is a classical anisotropic Sobolev space equipped with the norm

$$\|v\|^2_{H^{\sigma,b}_{\rm rad}(\mathbb{R}\times\Theta)}=\sum_{n\geqslant 1}z_n^{2\sigma}\big\|\langle\tau\rangle^b\widehat{\langle v(t),e_n\rangle}(\tau)\big\|^2_{L^2(\mathbb{R}_\tau)},$$

where again $\langle\cdot,\cdot\rangle$ stays for the $L^2(\Theta)$ pairing and $\widehat{\cdot}$ denotes the Fourier transform on $\mathbb{R}$. Therefore in the context of (6.1) we are in a situation where the Bourgain approach to well-posedness of dispersive equations may be applied. Let us also observe that the solutions of (6.1) we obtain here solve (1.1) in distributional sense (see e.g. [6, Section 3.2] for details on this point). Let us finally remark that for $\sigma < 1/2$ the spaces $H^{\sigma}_{\rm rad}(\Theta)$ are independent of the choice of the boundary conditions we work with. In particular, the space $H^{s}_{\rm rad}(\Theta)$, on which the invariant measure $d\rho$ is defined, is independent of the boundary conditions. On the other hand both the dynamics and the Gibbs measure $d\rho$ do depend on the choice of the boundary conditions.

Now we state the following standard consequence of Proposition 5.3 (see [8] or [12, Proposition 6.3]).

Proposition 6.2. — *Let* $\max(1/3, 1-2/\alpha) < \sigma_1 \leqslant \sigma < 1/2$. *Then there exist two positive numbers* b, b' *such that* $b+b' < 1$, $b' < 1/2 < b$, *there exists* $C > 0$ *such that for every* $T \in]0,1]$, *every* $u, v \in X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)$, *every* $u_0 \in H^{\sigma}_{\rm rad}(\Theta)$,

$$\big\|e^{it\Delta}u_0\big\|_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)} \leqslant C\|u_0\|_{H^{\sigma}_{\rm rad}(\Theta)},$$

$$\Big\|\int_0^t e^{i(t-\tau)\Delta}F(u(\tau))d\tau\Big\|_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)} \leqslant CT^{1-b-b'}\Big(1+\|u\|^{\max(2,\alpha)}_{X^{\sigma_1,b}_{\rm rad}([-T,T]\times\Theta)}\Big)\|u\|_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)}$$

and

$$\Big\|\int_0^t e^{i(t-\tau)\Delta}(F(u(\tau))-F(v(\tau)))d\tau\Big\|_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)} \leqslant CT^{1-b-b'}\Big(1+\|u\|^{\max(2,\alpha)}_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)} + \|v\|^{\max(2,\alpha)}_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)}\Big)\|u-v\|_{X^{\sigma,b}_{\rm rad}([-T,T]\times\Theta)}.$$

One may also formulate statements in the spirit of Proposition 6.2, where $[-T,T]$ is replaced by an interval $I\subset\mathbb{R}$ of size one and 0 by a point of I. We also remark that the integral terms in Proposition 6.2 are well-defined thanks to a priori estimates in the Bourgain spaces (see e.g. [8]).

Proposition 6.2 implies (see [12, Proposition 7.1]) a local well-posedness result for the Cauchy problem

$$(i\partial_t+\Delta)u - F(u) = 0, \quad u|_{t=0} = u_0. \tag{6.2}$$

Proposition 6.3. — *Let us fix* σ_1 *and* σ *such that* $\max(1/3, 1-2/\alpha) < \sigma_1 \leqslant \sigma < 1/2$. *Then there exist* $b > 1/2$, $\beta > 0$, $C > 0$, $\widetilde{C} > 0$, $c \in]0,1]$ *such that for every* $A > 0$ *if we set* $T = c(1+A)^{-\beta}$ *then for every* $u_0 \in H^{\sigma_1}_{\rm rad}(\Theta)$ *satisfying*

$\|u_0\|_{H^{\sigma_1}} \leqslant A$ *there exists a unique solution* u *of (6.1) in* $X^{\sigma_1,b}_{\mathrm{rad}}([-T,T]\times\Theta)$. *Moreover* u *solves (6.2) and*

$$\|u\|_{L^\infty([-T,T];H^{\sigma_1}(\Theta))} \leqslant C\|u\|_{X^{\sigma_1,b}_{\mathrm{rad}}([-T,T]\times\Theta)} \leqslant \widetilde{C}\|u_0\|_{H^{\sigma_1}(\Theta)}.$$

If in addition $u_0 \in H^\sigma_{\mathrm{rad}}(\Theta)$ *then*

$$\|u\|_{L^\infty([-T,T];H^{\sigma}(\Theta))} \leqslant C\|u\|_{X^{\sigma,b}_{\mathrm{rad}}([-T,T]\times\Theta)} \leqslant \widetilde{C}\|u_0\|_{H^{\sigma}(\Theta)}.$$

Finally if u *and* v *are two solutions with data* u_0, v_0 *respectively, satisfying*

$$\|u_0\|_{H^{\sigma_1}} \leqslant A, \quad \|v_0\|_{H^{\sigma_1}} \leqslant A$$

then

$$\|u-v\|_{L^\infty([-T,T];H^{\sigma_1}(\Theta))} \leqslant C\|u_0-v_0\|_{H^{\sigma_1}(\Theta)}.$$

If in addition $u_0, v_0 \in H^\sigma_{\mathrm{rad}}(\Theta)$ *then*

$$\|u-v\|_{L^\infty([-T,T];H^{\sigma}(\Theta))} \leqslant C\|u_0-v_0\|_{H^{\sigma}(\Theta)}.$$

Since the projector S_N is acting nicely on the Bourgain spaces Proposition 6.2 also implies a well-posedness result (the important point is the independence of N of the constants appearing in the statement) for the ODE (1.9).

Proposition 6.4. — *Let us fix* σ_1 *and* σ *such that* $\max(1/3, 1-2/\alpha) < \sigma_1 \leqslant \sigma < 1/2$. *Then there exist* $b>1/2$, $\beta>0$, $C>0$, $\widetilde{C}>0$, $c\in]0,1]$ *such that for every* $A>0$ *if we set* $T=c(1+A)^{-\beta}$ *then for every* $N\geqslant 1$, *every* $u_0\in E_N$ *satisfying* $\|u_0\|_{H^{\sigma_1}} \leqslant A$ *there exists a unique solution* $u=S_N(u)$ *of (1.9) in* $X^{\sigma_1,b}_{\mathrm{rad}}([-T,T]\times\Theta)$. *Moreover*

$$\|u\|_{L^\infty([-T,T];H^{\sigma_1}(\Theta))} \leqslant C\|u\|_{X^{\sigma_1,b}_{\mathrm{rad}}([-T,T]\times\Theta)} \leqslant \widetilde{C}\|u_0\|_{H^{\sigma_1}(\Theta)}.$$

If in addition $u_0 \in H^\sigma_{\mathrm{rad}}(\Theta)$ *then*

$$\|u\|_{L^\infty([-T,T];H^{\sigma}(\Theta))} \leqslant C\|u\|_{X^{\sigma,b}_{\mathrm{rad}}([-T,T]\times\Theta)} \leqslant \widetilde{C}\|u_0\|_{H^{\sigma}(\Theta)}.$$

Finally if u *and* v *are two solutions with data* u_0, v_0 *respectively, satisfying*

$$\|u_0\|_{H^{\sigma_1}} \leqslant A, \quad \|v_0\|_{H^{\sigma_1}} \leqslant A$$

then

$$\|u-v\|_{L^\infty([-T,T];H^{\sigma_1}(\Theta))} \leqslant C\|u_0-v_0\|_{H^{\sigma_1}(\Theta)}.$$

If in addition $u_0, v_0 \in H^\sigma_{\mathrm{rad}}(\Theta)$ *then*

$$\|u-v\|_{L^\infty([-T,T];H^{\sigma}(\Theta))} \leqslant C\|u_0-v_0\|_{H^{\sigma}(\Theta)}.$$

7. Long time analysis of the approximating ODE

In this section we study the long time dynamics of

$$(i\partial_t + \Delta)u - S_N(F(u)) = 0, \quad u|_{t=0} \in E_N. \tag{7.1}$$

Recall from the introduction that the measure $d\rho_N$ is invariant under the well-defined flow of (7.1). Denote this flow by $\Phi_N(t) : E_N \to E_N$, $t \in \mathbb{R}$. We have the following statement.

PROPOSITION 7.1. — *There exists $\Lambda > 0$ such that for every integer $i \geqslant 1$, every $\sigma \in [s, 1/2[$, every $N \in \mathbb{N}$, there exists a ρ_N measurable set $\Sigma^i_{N,\sigma} \subset E_N$ such that:*

- $\rho_N(E_N \backslash \Sigma^i_{N,\sigma}) \leqslant 2^{-i}$.
- $u \in \Sigma^i_{N,\sigma} \Rightarrow \|u\|_{L^2(\Theta)} \leqslant \Lambda$.
- *There exists C_σ, depending on σ, such that for every $i \in \mathbb{N}$, every $N \in \mathbb{N}$, every $u_0 \in \Sigma^i_{N,\sigma}$, every $t \in \mathbb{R}$, $\|\Phi_N(t)(u_0)\|_{H^\sigma(\Theta)} \leqslant C_\sigma(i + \log(1 + |t|))^{\frac{1}{2}}$.*
- *For every $\sigma \in]s, 1/2[$, every $\sigma_1 \in [s, \sigma[$, every $t \in \mathbb{R}$ there exists i_1 such that for every integer $i \geqslant 1$, every $N \geqslant 1$, if $u_0 \in \Sigma^i_{N,\sigma}$ then one has $\Phi_N(t)(u_0) \in \Sigma^{i+i_1}_{N,\sigma_1}$.*

Remark 7.2. — One may wish to see the invariance property of the sets $\Sigma^i_{N,\sigma}$ displayed by the last assertion as a "weak form of a conservation law".

Proof. — Let $\Lambda > 0$ be such that $\chi(x) = 0$ for $|x| > \Lambda$. For $\sigma \in [s, 1/2[$, i, j integers $\geqslant 1$, we set

$$B^{i,j}_{N,\sigma}(D_\sigma) \equiv \Big\{ u \in E_N : \|u\|_{H^\sigma(\Theta)} \leqslant D_\sigma(i+j)^{\frac{1}{2}}, \quad \|u\|_{L^2(\Theta)} \leqslant \Lambda \Big\},$$

where the number $D_\sigma \gg 1$ (independent of i, j, N) will be fixed later. Thanks to Proposition 6.4, there exist $c > 0$, $C > 0$, $\beta > 0$ only depending on σ such that if we set $\tau \equiv c D_\sigma^{-\beta}(i+j)^{-\beta/2}$ then for every $t \in [-\tau, \tau]$,

$$\Phi_N(t)\big(B^{i,j}_{N,\sigma}(D_\sigma)\big) \subset \Big\{ u \in E_N : \|u\|_{H^\sigma(\Theta)} \leqslant C D_\sigma(i+j)^{\frac{1}{2}}, \quad \|u\|_{L^2(\Theta)} \leqslant \Lambda \Big\}. \tag{7.2}$$

Next, following Bourgain [3], we set

$$\Sigma^{i,j}_{N,\sigma}(D_\sigma) \equiv \bigcap_{k=-[2^j/\tau]}^{[2^j/\tau]} \Phi_N(-k\tau)(B^{i,j}_{N,\sigma}(D_\sigma)),$$

where $[2^j/\tau]$ stays for the integer part of $2^j/\tau$. Using the invariance of the measure ρ_N by the flow Φ_N, we can write

$$\begin{aligned}\rho_N(E_N\backslash\Sigma^{i,j}_{N,\sigma}(D_\sigma)) &= \rho_N\Big(\bigcup_{k=-[2^j/\tau]}^{[2^j/\tau]}\Big(E_N\backslash\Phi_N(-k\tau)\big(B^{i,j}_{N,\sigma}(D_\sigma)\big)\Big)\Big)\\ &\leqslant \big(2[2^j/\tau]+1\big)\rho_N\big(E_N\backslash B^{i,j}_{N,\sigma}(D_\sigma)\big)\\ &\leqslant C2^jD_\sigma^\beta(i+j)^{\beta/2}\rho_N\big(E_N\backslash B^{i,j}_{N,\sigma}(D_\sigma)\big).\end{aligned}$$

Using the support property of χ, we observe that set $(u\in E_N:\|u\|_{L^2(\Theta)}>\Lambda)$ is of zero ρ_N measure and therefore

(7.3)
$$\rho_N(E_N\backslash B^{i,j}_{N,\sigma}(D_\sigma)) = \tilde\rho_N\Big(u\in H^s_{\mathrm{rad}}(\Theta):\|S_N(u)\|_{H^\sigma(\Theta)}>D_\sigma(i+j)^{\frac12}\Big).$$

Therefore, using Lemma 3.5, we can write

$$\rho_N(E_N\backslash\Sigma^{i,j}_{N,\sigma}(D_\sigma))\leqslant C2^jD_\sigma^\beta(i+j)^{\beta/2}e^{-cD_\sigma^2(i+j)}\leqslant 2^{-(i+j)}, \tag{7.4}$$

provided $D_\sigma\gg 1$, depending on σ but independent of i,j,N. Thanks to (7.2), we obtain that for $u_0\in\Sigma^{i,j}_{N,\sigma}(D_\sigma)$, the solution of (7.1) with data u_0 satisfies

$$\|\Phi_N(t)(u_0)\|_{H^\sigma(\Theta)}\leqslant CD_\sigma(i+j)^{\frac12},\quad |t|\leqslant 2^j. \tag{7.5}$$

Indeed, for $|t|\leqslant 2^j$, we may find an integer $k\in[-[2^j/\tau],[2^j/\tau]]$ and $\tau_1\in[-\tau,\tau]$ so that $t=k\tau+\tau_1$ and thus $u(t)=\Phi_N(\tau_1)\big(\Phi_N(k\tau)(u_0)\big)$. Since $u_0\in\Sigma^{i,j}_{N,\sigma}(D_\sigma)$ implies that $\Phi_N(k\tau)(u_0)\in B^{i,j}_{N,\sigma}(D_\sigma)$, we may apply (7.2) and arrive at (7.5). Next, we set

$$\Sigma^i_{N,\sigma}=\bigcap_{j=1}^{\infty}\Sigma^{i,j}_{N,\sigma}(D_\sigma).$$

Thanks to (7.4),

$$\rho_N(E_N\backslash\Sigma^i_{N,\sigma})\leqslant 2^{-i}.$$

In addition, using (7.5), we get that there exists C_σ such that for every i, every N, every $u_0\in\Sigma^i_{N,\sigma}$, every $t\in\mathbb{R}$,

$$\|\Phi_N(t)(u_0)\|_{H^\sigma(\Theta)}\leqslant C_\sigma(i+\log(1+|t|))^{\frac12}.$$

Indeed for $t\in\mathbb{R}$ there exists $j\in\mathbb{N}$ such that $2^{j-1}\leqslant 1+|t|\leqslant 2^j$ and we apply (7.5) with this j.

Let us now turn to the proof of the last assertion. Fix $t\in\mathbb{R}$ and $u_0\in\Sigma^i_{N,\sigma}$. Since $u_0\in\Sigma^i_{N,\sigma}$, for every integer $j\geqslant 1$, we have the bound

$$\|\Phi_N(t_1)(u_0)\|_{H^\sigma(\Theta)}\leqslant C_\sigma(i+j)^{\frac12},\quad |t_1|\leqslant 2^j.$$

Let $i_1 \in \mathbb{N}$ (depending on t) be such that for every $j \geqslant 1$, $2^j + |t| \leqslant 2^{j+i_1}$. Therefore, if we set $u(t) \equiv \Phi_N(t)(u_0)$, we have that

$$\|\Phi_N(t_1)(u(t))\|_{H^\sigma(\Theta)} = \|\Phi_N(t+t_1)(u_0)\|_{H^\sigma(\Theta)} \leqslant C_\sigma(i+j+i_1)^{\frac{1}{2}}, \quad |t_1| \leqslant 2^j.$$

Thanks to the L^2 conservation law, for $u_0 \in \Sigma^i_{N,\sigma}$ one has

$$\|\Phi_N(t_1)(u(t))\|_{L^2(\Theta)} = \|u_0\|_{L^2(\Theta)} \leqslant \Lambda.$$

Therefore

$$\begin{aligned}\big\|\Phi_N(t_1)(u(t))\big\|_{H^{\sigma_1}(\Theta)} &\leqslant \big\|\Phi_N(t_1)(u(t))\big\|^{\frac{\sigma_1}{\sigma}}_{H^\sigma(\Theta)} l \big\|\Phi_N(t_1)(u(t))\big\|^{\frac{\sigma-\sigma_1}{\sigma}}_{L^2(\Theta)} \\ &\leqslant [\Lambda]^{\frac{\sigma-\sigma_1}{\sigma}} \big[C_\sigma(i+j+i_1)\big]^{\frac{\sigma_1}{2\sigma}}.\end{aligned}$$

Let us fix $i_1 \geqslant 1$ such that in addition to the property

$$2^j + |t| \leqslant 2^{j+i_1}, \quad \forall j \geqslant 1,$$

we also have that for every $i, j \geqslant 1$,

$$[\Lambda]^{\frac{1-\sigma_1}{\sigma}} \big[C_\sigma(i+j+i_1)\big]^{\frac{\sigma_1}{2\sigma}} \leqslant D_{\sigma_1}(i+j+i_1)^{\frac{1}{2}}.$$

Thus

$$\|\Phi_N(t_1)(u(t))\|_{H^{\sigma_1}(\Theta)} \leqslant D_{\sigma_1}(i+j+i_1)^{\frac{1}{2}}, \quad |t_1| \leqslant 2^j,$$

i.e. for every $|t_1| \leqslant 2^j$ one has $\Phi_N(t_1)(u(t)) \in B^{i+i_1,j}_{N,\sigma_1}(D_\sigma)$. We can therefore conclude that $u(t) \in \Sigma^{i+i_1,j}_{N,\sigma_1}(D_\sigma)$ for every $j \geqslant 1$. Hence $u(t) \in \Sigma^{i+i_1}_{N,\sigma_1}$ and the restriction on i_1 depends only on σ, σ_1 and t. This completes the proof of Proposition 7.1. □

8. Construction of the statistical ensemble (long time analysis for NLS)

Let us set for integers $i \geqslant 1$, $N \geqslant 1$ and $\sigma \in [s, 1/2[$,

$$\widetilde{\Sigma}^i_{N,\sigma} \equiv \big\{u \in H^\sigma_{\rm rad}(\Theta) : S_N(u) \in \Sigma^i_{N,\sigma}\big\}.$$

Next, for an integer $i \geqslant 1$ and $\sigma \in [s, 1/2[$, we set

$$\begin{aligned}\Sigma^i_\sigma \equiv \big\{u \in H^\sigma_{\rm rad}(\Theta) : \exists N_k \to \infty,\ N_k \in \mathbb{N},\ \exists u_{N_k} \in \Sigma^i_{N_k,\sigma}, \\ u_{N_k} \to u \text{ in } H^\sigma_{\rm rad}(\Theta)\big\}.\end{aligned}$$

We have the following statement.

Lemma 8.1. — *The set Σ^i_σ is a closed set in $H^\sigma_{\rm rad}(\Theta)$ (in particular ρ measurable).*

Proof. — Let $(u_m)_{m\in\mathbb{N}}$ be a sequence of Σ^i_σ which converges to u in $H^\sigma_{\rm rad}(\Theta)$. Our goal is to show that $u\in\Sigma^i_\sigma$. Since $u_m\in\Sigma^i_\sigma$ there exist a sequence of integers $N_{m,k}\to\infty$ as $k\to\infty$ and a sequence $(u_{N_{m,k}})_{k\in\mathbb{N}}$ of $\Sigma^i_{N_{m,k},\sigma}$ such that

$$\lim_{k\to\infty} u_{N_{m,k}} = u_m \text{ in } H^\sigma(\Theta). \tag{8.1}$$

For every $j\in\mathbb{N}$, we can find $m_j\in\mathbb{N}$ such that

$$\|u-u_{m_j}\|_{H^\sigma(\Theta)} < \frac{1}{2j}.$$

Then, thanks to (8.1) (with $m=m_j$), for every $j\in\mathbb{N}$, we can find $N_{m_j,k_j}\in\mathbb{N}$ and $u_{N_{m_j,k_j}}\in\Sigma^i_{N_{m_j,k_j},\sigma}$ such that

$$N_{m_j,k_j} > j,\quad \|u_{m_j}-u_{N_{m_j,k_j}}\|_{H^\sigma(\Theta)} < \frac{1}{2j}.$$

Therefore, if we set $v_j\equiv u_{N_{m_j,k_j}}$ and $M_j\equiv N_{m_j,k_j}$ then $M_j\to\infty$ as $j\to\infty$, $v_j\in\Sigma^i_{M_j,\sigma}$ and $v_j\to u$ as $j\to\infty$ in $H^\sigma_{\rm rad}(\Theta)$. Consequently $u\in\Sigma^i_\sigma$. This completes the proof of Lemma 8.1. □

We have the inclusion

$$\limsup_{N\to\infty}\widetilde{\Sigma}^i_{N,\sigma}\equiv\bigcap_{N=1}^{\infty}\bigcup_{N_1=N}^{\infty}\widetilde{\Sigma}^i_{N_1,\sigma}\subset\Sigma^i_\sigma.$$

Indeed, if $u\in\limsup_{N\to\infty}\widetilde{\Sigma}^i_{N,\sigma}$ then there exists a sequence of integers (N_k) tending to infinity as $k\to\infty$ such that $u\in\widetilde{\Sigma}^i_{N_k,\sigma}$, *i.e.* $S_{N_k}(u)\in\Sigma^i_{N_k,\sigma}$. Thus $u\in\Sigma^i_\sigma$ since $S_{N_k}(u)$ tends to u in $H^\sigma_{\rm rad}(\Theta)$. Therefore

$$\rho(\Sigma^i_\sigma)\geqslant\rho(\limsup_{N\to\infty}\widetilde{\Sigma}^i_{N,\sigma}). \tag{8.2}$$

Let us next show that

$$\rho(\limsup_{N\to\infty}\widetilde{\Sigma}^i_{N,\sigma})\geqslant\limsup_{N\to\infty}\rho(\widetilde{\Sigma}^i_{N,\sigma}). \tag{8.3}$$

Indeed, if we set $A_N\equiv\widetilde{\Sigma}^i_{N,\sigma}$ and $B_N\equiv H^s_{\rm rad}(\Theta)\backslash A_N$ then

$$\begin{aligned}\limsup_{N\to\infty}\rho(A_N) &= \limsup_{N\to\infty}\Big(\rho(H^s_{\rm rad}(\Theta))-\rho(B_N)\Big)\\ &= \rho(H^s_{\rm rad}(\Theta))-\liminf_{N\to\infty}\rho(B_N).\end{aligned} \tag{8.4}$$

Using Fatou's lemma, we can obtain

$$-\liminf_{N\to\infty}\rho(B_N)\leqslant-\rho\Big(\liminf_{N\to\infty}B_N\Big),$$

where

$$\liminf_{N\to\infty} B_N \equiv \bigcup_{N=1}^{\infty} \bigcap_{N_1=N}^{\infty} B_{N_1}.$$

Therefore, coming back to (8.4), we get

$$\limsup_{N\to\infty} \rho(A_N) \leqslant \rho\Big(H^s_{\mathrm{rad}}(\Theta)\backslash \liminf_{N\to\infty} B_N\Big) = \rho\Big(\limsup_{N\to\infty} A_N\Big).$$

Therefore (8.3) holds. Since

$$\rho(\widetilde{\Sigma}^i_{N,\sigma}) = \int_{\widetilde{\Sigma}^i_{N,\sigma}} f(u) d\mu(u)$$

and

$$\rho_N(\Sigma^i_{N,\sigma}) = \int_{\Sigma^i_{N,\sigma}} f_N(u) d\mu_N(u) = \int_{\widetilde{\Sigma}^i_{N,\sigma}} f_N(u) d\mu(u)$$

thanks to Lemma 3.7, we get

$$\lim_{N\to\infty} \big((\rho(\widetilde{\Sigma}^i_{N,\sigma}) - \rho_N(\Sigma^i_{N,\sigma})\big) = 0.$$

Thus, using Proposition 7.1 and Theorem 1.2, we obtain

$$\begin{aligned}\limsup_{N\to\infty} \rho(\widetilde{\Sigma}^i_{N,\sigma}) &= \limsup_{N\to\infty} \rho_N(\Sigma^i_{N,\sigma}) \\ &\geqslant \limsup_{N\to\infty} \big(\rho_N(E_N) - 2^{-i}\big) \\ &= \rho\big(H^s_{\mathrm{rad}}(\Theta)\big) - 2^{-i}.\end{aligned} \tag{8.5}$$

Collecting (8.2), (8.3) and (8.5), we arrive at

$$\rho(\Sigma^i_\sigma) \geqslant \rho\big(H^s_{\mathrm{rad}}(\Theta)\big) - 2^{-i}.$$

Now, we set

$$\Sigma_\sigma \equiv \bigcup_{i\geqslant 1} \Sigma^i_\sigma.$$

Thus Σ_σ is of full ρ measure. It turns out that one has global existence for $u_0 \in \Sigma^i_\sigma$.

Proposition 8.2. — *Let us fix $\sigma \in [s, 1/2[$, $\sigma_1 \in]\max(1/3, 1-2/\alpha), \sigma[$ and $i \in \mathbb{N}$. Then for every $u_0 \in \Sigma^i_\sigma$, the local solution u of (6.2) given by Proposition 6.3 is globally defined. In addition there exists $C > 0$ such that for every $u_0 \in \Sigma^i_\sigma$,*

$$\|u(t)\|_{H^{\sigma_1}(\Theta)} \leqslant C(i + \log(1+|t|))^{\frac{1}{2}}. \tag{8.6}$$

Moreover, if $(u_{0,k})_{k\in\mathbb{N}}$, $u_{0,k} \in \Sigma^i_{\sigma,N_k}$, $N_k \to \infty$ converges to u_0 as $k \to \infty$ in $H^\sigma_{\mathrm{rad}}(\Theta)$ then

$$\lim_{k\to\infty} \|u(t) - \Phi_{N_k}(t)(u_{0,k})\|_{H^{\sigma_1}(\Theta)} = 0. \tag{8.7}$$

Proof. — Let $u_0 \in \Sigma^i_\sigma$ and $(u_{0,k})$ $u_{0,k} \in \Sigma^i_{\sigma,N_k}$, $N_k \to \infty$ a sequence tending to u_0 in $H^\sigma_{\mathrm{rad}}(\Theta)$. Let us fix $T > 0$. Our aim so to extend the solution of (6.2) given by Proposition 6.3 to the interval $[-T, T]$. Using Proposition 7.1, we have that there exists a constant C such that for every $k \in \mathbb{N}$, every $t \in \mathbb{R}$,

$$\|\Phi_{N_k}(t)(u_{0,k})\|_{H^\sigma(\Theta)} \leqslant C(i + \log(1+|t|))^{\frac{1}{2}}. \tag{8.8}$$

Therefore, if we set $u_{N_k}(t) \equiv \Phi_{N_k}(t)(u_{0,k})$ and $\Lambda \equiv C(i + \log(1+T))^{\frac{1}{2}}$, we have the bound

$$\|u_{N_k}(t)\|_{H^\sigma} \leqslant \Lambda, \quad \forall |t| \leqslant T, \quad \forall k \in \mathbb{N}. \tag{8.9}$$

In particular $\|u_0\|_{H^\sigma} \leqslant \Lambda$ (apply (8.9) with $t = 0$ and let $k \to \infty$). Let $\tau > 0$ be the local existence time for (6.2), provided by Proposition 6.3 for σ_1, σ and $A = \Lambda + 1$. Recall that we can assume $\tau = c(2 + \Lambda)^{-\beta}$ for some $c > 0$, $\beta > 0$ depending only on σ and σ_1. We can of course assume that $T > \tau$. Denote by $u(t)$ the solution of (6.2) with data u_0 on the time interval $[-\tau, \tau]$. Then $v_{N_k} \equiv u - u_{N_k}$ solves the equation

$$(i\partial_t + \Delta)v_{N_k} = F(u) - S_{N_k}(F(u_{N_k})), \quad v_{N_k}|_{t=0} = u_0 - u_{0,k}. \tag{8.10}$$

Next, we write

$$F(u) - S_{N_k}(F(u_{N_k})) = S_{N_k}\big(F(u) - F(u_{N_k})\big) + (1 - S_{N_k})F(u).$$

Therefore

$$v_{N_k}(t) = e^{it\Delta}(u_0 - u_{0,k}) - i\int_0^t e^{i(t-\tau)\Delta} S_{N_k}\big(F(u(\tau)) - F(u_{N_k}(\tau))\big)d\tau \\ - i\int_0^t e^{i(t-\tau)\Delta}(1 - S_{N_k})F(u(\tau))d\tau.$$

Let us observe that for $\sigma_1 < \sigma$ the map $1 - S_N$ sends $H^\sigma_{\mathrm{rad}}(\Theta)$ to $H^{\sigma_1}_{\mathrm{rad}}(\Theta)$ with norm $\leqslant CN^{\sigma_1-\sigma}$. Similarly, for $I \subset \mathbb{R}$ an interval, the map $1 - S_N$ sends $X^{\sigma,b}_{\mathrm{rad}}(I \times \Theta)$ to $X^{\sigma_1,b}_{\mathrm{rad}}(I \times \Theta)$ with norm $\leqslant CN^{\sigma_1-\sigma}$. Moreover S_N acts as a bounded operator (with norm $\leqslant 1$) on the Bourgain spaces $X^{\sigma,b}_{\mathrm{rad}}$. Therefore, using Proposition 6.2, we obtain that there exist $C > 0$, $b > 1/2$ and $\theta > 0$ (depending only on σ, σ_1) such that one has the bound

$$
\begin{aligned}
\|v_{N_k}\|_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)} \leqslant C\Big(&\|u_0-u_{0,k}\|_{H^{\sigma_1}(\Theta)} \\
&+\tau^{\theta}\|v_{N_k}\|_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)}\big(1+\|u\|^{\alpha_1}_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)} \\
&+\|u_{N_k}\|^{\alpha_1}_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)}\big) \\
&+\tau^{\theta}N_k^{\sigma_1-\sigma}\|u\|_{X^{\sigma,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)} \\
&\qquad\qquad\times\big(1+\|u\|^{\alpha_1}_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)}\big)\Big),
\end{aligned}
$$

where $\alpha_1\equiv\max(2,\alpha)$. A use of Proposition 6.3 and Proposition 6.4 yields

$$
\begin{aligned}
\|v_{N_k}\|_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)} &\leqslant C\|u_0-u_{0,k}\|_{H^{\sigma_1}(\Theta)} \\
&\quad+C\tau^{\theta}\|v_{N_k}\|_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)}(1+C\|u_0\|^{\alpha_1}_{H^{\sigma_1}(\Theta)} \\
&\quad+C\|u_{0,k}\|^{\alpha_1}_{H^{\sigma_1}(\Theta)}) \\
&\quad+C\tau^{\theta}N_k^{\sigma_1-\sigma}\|u_0\|_{H^{\sigma}(\Theta)}(1+C\|u_0\|^{\alpha_1}_{H^{\sigma_1}(\Theta)}) \\
&\leqslant C\|u_0-u_{0,k}\|_{H^{\sigma_1}(\Theta)} \\
&\quad+C\tau^{\theta}(1+\Lambda)^{\alpha_1}\|v_{N_k}\|_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)} \\
&\quad+C\tau^{\theta}(1+\Lambda)^{\alpha_1}N_k^{\sigma_1-\sigma}\|u_0\|_{H^{\sigma}(\Theta)}.
\end{aligned}
$$

Recall that $\tau=c(2+\Lambda)^{-\beta}$, where $c>0$ and $\beta>0$ are depending only on σ and σ_1. In the last estimate the constants C and θ also depend only on σ_1 and σ. Therefore, if we assume that $\beta>\alpha_1/\theta$ then the restriction on β remains to depend only on σ_1 and σ. Similarly, if we assume that c is so small that

$$
C\tau^{\theta}(1+\Lambda)^{\alpha_1}\leqslant Cc^{\theta}(2+\Lambda)^{-\beta\theta}(1+\Lambda)^{\alpha_1}\leqslant Cc^{\theta}<1/2
$$

then the smallness restriction on c remains to depend only on σ_1 and σ. Therefore, we have that after possibly slightly modifying the values of c and β (keeping c and β only depending on σ and σ_1 and independent of N_k) in the definition of τ that

$$
\begin{aligned}
\|v_{N_k}\|_{X^{\sigma_1,b}_{\mathrm{rad}}([-\tau,\tau]\times\Theta)} &\leqslant C\|u_0-u_{0,k}\|_{H^{\sigma_1}(\Theta)}+\frac{1}{2}N_k^{\sigma_1-\sigma}\|u_0\|_{H^{\sigma}(\Theta)} \\
&=C\|v_{N_k}(0)\|_{H^{\sigma_1}(\Theta)}+\frac{1}{2}N_k^{\sigma_1-\sigma}\|u(0)\|_{H^{\sigma}(\Theta)}.
\end{aligned}
$$

Since $b>1/2$, the last inequality implies

$$
\|v_{N_k}(t)\|_{H^{\sigma_1}(\Theta)}\leqslant C\|v_{N_k}(0)\|_{H^{\sigma_1}(\Theta)}+CN_k^{\sigma_1-\sigma}\|u(0)\|_{H^{\sigma}(\Theta)},\quad |t|\leqslant\tau=c(1+\Lambda)^{-\beta}, \tag{8.11}
$$

where the constants c, C and β depend only σ_1 and σ. Therefore, using that

$$\lim_{k\to\infty} \|v_{N_k}(0)\|_{H^{\sigma_1}(\Theta)} = 0,$$

we obtain that

$$\lim_{k\to\infty} \|v_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} = 0, \quad |t| \leqslant \tau. \tag{8.12}$$

Thus by taking N_k large enough in (8.11) one has via a use of the triangle inequality,

$$\|u(t)\|_{H^{\sigma_1}(\Theta)} \leqslant \|u_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} + \|v_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} \leqslant \Lambda + 1, \quad |t| \leqslant \tau. \tag{8.13}$$

Let us define the function $g_k(t)$ by

$$g_k(t) \equiv \|v_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} + N_k^{\sigma_1-\sigma}\|u(t)\|_{H^{\sigma}(\Theta)}.$$

The function $g_k(t)$ is a priori defined only on $[-\tau,\tau]$. Our goal is to extend it on $[-T,T]$. Using (8.11) and the bound

$$\|u(t)\|_{H^{\sigma}(\Theta)} \leqslant C\|u(0)\|_{H^{\sigma}(\Theta)}, \quad |t| \leqslant \tau,$$

provided from Proposition 6.3, we obtain that there exists a constant $C(\sigma,\sigma_1)$ depending only on σ_1 and σ such that

$$g_k(t) \leqslant C(\sigma,\sigma_1) g_k(0), \quad \forall t \in [-\tau,\tau].$$

We now repeat the argument for obtaining (8.11) on $[\tau, 2\tau]$ and thanks to the bounds (8.9) and (8.13), we obtain that $v_{N_k}(t)$ and u exist on $[\tau, 2\tau]$ and one has the bound

$$\|v_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} \leqslant C\|v_{N_k}(\tau)\|_{H^{\sigma_1}(\Theta)} + CN_k^{\sigma_1-\sigma}\|u(\tau)\|_{H^{\sigma}(\Theta)}, \quad t \in [\tau, 2\tau].$$

Therefore, thanks to (8.12) (with $t=\tau$)

$$\lim_{k\to\infty} \|v_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} = 0, \quad \tau \leqslant t \leqslant 2\tau.$$

By taking $N_k \gg 1$, we get via a use of the triangle inequality

$$\|u(t)\|_{H^{\sigma_1}(\Theta)} \leqslant \|u_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} + \|v_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} \leqslant \Lambda + 1, \quad \tau \leqslant t \leqslant 2\tau.$$

Using (8.11) and the bound

$$\|u(t)\|_{H^{\sigma}(\Theta)} \leqslant C\|u(\tau)\|_{H^{\sigma}(\Theta)}, \quad \tau \leqslant t \leqslant 2\tau,$$

provided from Proposition 6.3, we obtain that

$$g_k(t) \leqslant C(\sigma,\sigma_1) g_k(\tau), \quad \forall t \in [\tau, 2\tau].$$

Then, we can continue by covering the interval $[-T,T]$ with intervals of size τ, which yields the existence of $u(t)$ on $[-T,T]$ (the point is that at each step the H^σ norm of u remains bounded by $\Lambda+1$ and the limit as $k\to\infty$ of the H^σ norm of v_{N_k} is zero). Since $T>0$ was chosen arbitrary, we obtain that for every $u_0 \in \Sigma_\sigma^i$ the local solution of (6.2) is globally defined. Moreover

$$\|u(t)\|_{H^{\sigma_1}(\Theta)} \leqslant \Lambda + 1, \quad |t| \leqslant T$$

which by recalling the definition of Λ implies the bound (8.6). In addition, by iterating the bounds on g_k we get at each step, we obtain the existence of a constant C depending only on σ and σ_1 such that

$$g_k(t) \leqslant e^{C(1+|t|)} g_k(0)$$

which implies that there exists a constant C depending only on σ_1 and σ such that v_{N_k} enjoys the bound

$$\|v_{N_k}(t)\|_{H^{\sigma_1}(\Theta)} \leqslant C^{1+T}\Big(N_k^{\sigma_1-\sigma}\|u_0\|_{H^\sigma(\Theta)} + \|u_0 - u_{0,k}\|_{H^{\sigma_1}(\Theta)}\Big), \quad |t| \leqslant T.$$

Therefore for every $\varepsilon > 0$ there exists $N^\star$ such that for $N_k \geqslant N^\star$ one has the inequality

$$\sup_{|t|\leqslant T} \|u(t) - \Phi_{N_k}(t)(u_{0,k})\|_{H^{\sigma_1}(\Theta)} < \varepsilon.$$

Hence we have (8.7). This completes the proof of Proposition 8.2. □

By the Proposition 8.2, we can define a flow Φ acting on Σ_σ, $\sigma \in [s, 1/2[$ and defining the global dynamics of (6.2) for $u_0 \in \Sigma_\sigma$. Let us now turn to the construction of a set invariant under Φ. Let $l = (l_j)_{j\in\mathbb{N}}$ be a increasing sequence of real numbers such that $l_0 = s$, $l_j < 1/2$ and $\lim_{j\to\infty} l_j = 1/2$. Then, we set

$$\Sigma = \bigcap_{\sigma\in l} \Sigma_\sigma.$$

The set Σ is of full ρ measure. It is the one involved in the statement of Theorem 1.3. Using the invariance property of $\Sigma_{N,\sigma}^i$, we now obtain that the set Σ is invariant under Φ.

Proposition 8.3. — *For every $t \in \mathbb{R}$, $\Phi(t)(\Sigma) = \Sigma$. In addition for every $\sigma \in l$, $\Phi(t)$ is continuous with respect to the induced by $H^\sigma_{\mathrm{rad}}(\Theta)$ on Σ topology. In particular, the map $\Phi(t) : \Sigma \to \Sigma$ is a measurable map with respect to ρ.*

Proof. — Since the flow is time reversible, it suffices to show that

$$\Phi(t)(\Sigma) \subset \Sigma, \quad \forall t \in \mathbb{R}. \tag{8.14}$$

Indeed, if we suppose that (8.14) holds true then for $u \in \Sigma$ and $t \in \mathbb{R}$, we have that thanks to (8.14) $u_0 \equiv \Phi(-t)u \in \Sigma$ (recall that Φ is well-defined on Σ by Proposition 8.2) and thus $u = \Phi(t)u_0$, *i.e.* $\Sigma \subset \Phi(t)(\Sigma)$. Hence $\Phi(t)(\Sigma) = \Sigma$ is a consequence of (8.14).

Let us now prove (8.14). Fix $u_0 \in \Sigma$ and $t \in \mathbb{R}$. It suffices to show that for every $\sigma_1 \in l$, we have

$$\Phi(t)(u_0) \in \Sigma_{\sigma_1}.$$

Let us take $\sigma \in]\sigma_1, 1/2[$, $\sigma \in l$. Since $u_0 \in \Sigma$, we have that $u_0 \in \Sigma_\sigma$. Therefore there exists i such that $u_0 \in \Sigma_\sigma^i$. Let $u_{0,k} \in \Sigma^i_{N_k,\sigma}$, $N_k \to \infty$ be a sequence which tends to u_0 in $H^\sigma(\Theta)$. Thanks to Proposition 7.1 there exists i_1 such that

$$\Phi_{N_k}(t)(u_{0,k}) \in \Sigma^{i+i_1}_{N_k,\sigma_1}, \quad \forall k \in \mathbb{N}.$$

Therefore using (8.7) of Proposition 8.2, we obtain that

$$\Phi(t)(u_0) \in \Sigma^{i+i_1}_{\sigma_1}.$$

Thus $\Phi(t)(u_0) \in \Sigma_{\sigma_1}$ which proves (8.14).

Let us finally prove the continuity of $\Phi(t)$ on Σ with respect to the $H^\sigma_{\rm rad}(\Theta)$ topology. Let $u \in \Sigma$ and $u_n \in \Sigma$ be a sequence such that $u_n \to u$ in $H^\sigma_{\rm rad}(\Theta)$. We need to prove that for every $t \in \mathbb{R}$, $\Phi(t)(u_n) \to \Phi(t)(u)$ in $H^\sigma_{\rm rad}(\Theta)$. Let us fix $t \in \mathbb{R}$. Since $u \in \Sigma$ (and thus in all Σ_σ, $\sigma \in l$), using Proposition 8.2, we obtain that there exists $C > 0$ such that

$$\sup_{|\tau|\leqslant|t|} \|\Phi(\tau)(u)\|_{H^\sigma(\Theta)} \leqslant C(\log(2+|t|))^{\frac{1}{2}} \equiv \Lambda. \tag{8.15}$$

Let us denote by τ_0 the local existence time in Proposition 6.3, associated to σ and $A = 2\Lambda$. Then, by the continuity of the flow given by Proposition 6.3, we have $\Phi(\tau_0)(u_n) \to \Phi(\tau_0)(u)$ in $H^\sigma_{\rm rad}(\Theta)$. Next, we cover the interval $[0,t]$ by intervals of size τ_0 and we apply the continuity of the flow established in Proposition 6.3 at each step. The applicability of Proposition 6.3 is possible thanks to the bound (8.15). Therefore, we obtain that $\Phi(t)(u_n) \to \Phi(t)(u)$ in $H^\sigma_{\rm rad}(\Theta)$. This completes the proof of Proposition 8.3. □

9. Proof of the measure invariance

Fix $\sigma \in]s, 1/2[$, $\sigma \in l$. Thanks to the invariance by Φ of the set Σ, using the regularity of the measure μ (which is a finite Borel measure) and Remark 3.10, we deduce that it suffices to prove the measure invariance for subsets K of Σ which are compacts of $H^s_{\rm rad}(\Theta)$ and which are bounded in $H^\sigma_{\rm rad}(\Theta)$. Let us fix $t \in \mathbb{R}$ and a compact K of $H^s_{\rm rad}(\Theta)$ which is a bounded set in $H^\sigma_{\rm rad}(\Theta)$. Our aim is to show that $\rho(\Phi(t)(K)) = \rho(K)$. By the time reversibility of the flow, we may suppose that $t > 0$. Since K is bounded in $H^\sigma_{\rm rad}(\Theta)$ and a compact in $H^s_{\rm rad}(\Theta)$, using the continuity property displayed by Proposition 8.3 and Proposition 6.3, we infer that there exists $R > 0$ such that

$$\{\Phi(\tau)(K), 0 \leqslant \tau \leqslant t\} \subset \{u \in H^\sigma_{\mathrm{rad}}(\Theta) : \|u\|_{H^\sigma(\Theta)} \leqslant R\} \equiv B_R. \tag{9.1}$$

Indeed, the left hand-side of (9.1) is included in a sufficiently large $H^s_{\mathrm{rad}}(\Theta)$ ball thanks to the continuity property of the flow on $H^s_{\mathrm{rad}}(\Theta)$ shown in Proposition 8.3 and the compactness of K. Then, by iterating the propagation of regularity statement of Proposition 6.3, applied with A such that the $H^s_{\mathrm{rad}}(\Theta)$ ball centered at the origin of radius A contains the left hand-side of (9.1), we arrive at (9.1) (observe that we only have the poor bound $R \sim e^{Ct}$).

Let c and β (depending only on s and σ) be fixed by an application of Proposition 6.3 with $s = \sigma_1$ and $\sigma = \sigma$. Next, we set

$$\tau_0 \equiv c_0(1+R)^{-\beta_0},$$

where $0 < c_0 \leqslant c$, $\beta_0 \geqslant \beta$, depending only on s and σ, are to be fixed in the next lemma which allows to compare Φ and Φ_N for data in B_R.

Lemma 9.1. — *There exist c_0 and β_0 depending only on s and σ such that for every $\varepsilon > 0$ there exists $N_0 \geqslant 1$ such that for every $N \geqslant N_0$, every $u_0 \in B_R$, every $\tau \in [0, \tau_0]$,*

$$\|\Phi(\tau)(u_0) - \Phi_N(\tau)(S_N(u_0))\|_{H^s(\Theta)} < \varepsilon.$$

Proof. — For $u_0 \in B_R$, we denote by u the solution of (6.2) with data u_0 and by u_N the solution of (7.1) with data $S_N(u_0)$, defined on $[0, \tau_0]$. Next, we set $v_N \equiv u - u_N$. Then v_N solves

$$(i\partial_t + \Delta)v_N = F(u) - S_N(F(u_N)), \quad v_N(0) = (1 - S_N)u_0. \tag{9.2}$$

By writing

$$F(u) - S_N(F(u_N)) = S_N\big(F(u) - F(u_N)\big) + (1 - S_N)F(u)$$

and using Proposition 6.2, we obtain that there exists $b > 1/2$ and $\theta > 0$ depending only on s and σ such that one has

$$\begin{aligned}\|v_N\|_{X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)} \leqslant{}& CN^{s-\sigma}\|u_0\|_{H^\sigma(\Theta)} \\ &+ C\tau_0^\theta \|v_N\|_{X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)}\Big(1 + \|u\|^{\max(2,\alpha)}_{X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)} \\ &+ \|u_N\|^{\max(2,\alpha)}_{X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)}\Big) \\ &+ C\tau_0^\theta N^{s-\sigma}\|u\|_{X^{\sigma,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)}\Big(1 + \|u\|^{\max(2,\alpha)}_{X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)}\Big).\end{aligned}$$

Using Proposition 6.3 and Proposition 6.4, we get

$$\begin{aligned}\|v_N\|_{X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)} \leqslant CN^{s-\sigma}\|u_0\|_{H^\sigma(\Theta)} \\ + C\tau_0^\theta\|v_N\|_{X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)}\Big(1+C\|u_0\|^{\max(2,\alpha)}_{H^s(\Theta)}\Big) \\ + C\tau_0^\theta N^{s-\sigma}\|u_0\|_{H^\sigma(\Theta)}\Big(1+C\|u_0\|^{\max(2,\alpha)}_{H^s(\Theta)}\Big).\end{aligned}$$

Coming back to the definition of τ_0 we can choose c_0 small enough and β_0 large enough, but keeping their dependence only on s and σ, to infer that

$$\|v_N\|_{X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)} \leqslant CN^{s-\sigma}\|u_0\|_{H^\sigma(\Theta)}.$$

Since $b > 1/2$, by the Sobolev embedding, the space $X^{s,b}_{\mathrm{rad}}([0,\tau_0]\times\Theta)$ is continuously embedded in $L^\infty([0,\tau_0];H^s_{\mathrm{rad}}(\Theta))$ and thus there exists C depending only on s, σ such that

$$\|v_N(t)\|_{H^s(\Theta)} \leqslant CRN^{s-\sigma}, \quad t\in[0,\tau_0].$$

This completes the proof of Lemma 9.1. □

It suffices to prove that

$$\rho(\Phi(\tau)(K)) = \rho(K), \quad \tau\in[0,\tau_0]. \tag{9.3}$$

Indeed, it suffices to cover $[0,t]$ by intervals of size τ_0 and apply (9.3) at each step. Such an iteration is possible since by the continuity property of $\Phi(t)$ at each step the image remains a compact of $H^s_{\mathrm{rad}}(\Theta)$ included in the ball B_R. Let us now prove (9.3). Let B_ε be the open ball in $H^s_{\mathrm{rad}}(\Theta)$ centered at the origin and of radius ε. We have that $\Phi(\tau)(K)$ is a closed set of $H^s_{\mathrm{rad}}(\Theta)$ contained in Σ. Therefore, by Theorem 1.2, we can write

$$\rho\Big(\Phi(\tau)(K)+\overline{B_{2\varepsilon}}\Big) \geqslant \limsup_{N\to\infty}\rho_N\Big(\big(\Phi(\tau)(K)+\overline{B_{2\varepsilon}}\big)\cap E_N\Big),$$

where $\overline{B_{2\varepsilon}}$ is the closed ball in $H^s_{\mathrm{rad}}(\Theta)$, centered at the origin and of radius 2ε. Using Lemma 9.1, we obtain that for every $\varepsilon>0$, if we take N large enough, we have

$$\big(\Phi_N(\tau)(S_N(K))+B_\varepsilon\big)\cap E_N \subset \big(\Phi(\tau)(K)+\overline{B_{2\varepsilon}}\big)\cap E_N$$

and therefore

$$\begin{aligned}\limsup_{N\to\infty}\rho_N\Big(\big(\Phi(\tau)(K)+\overline{B_{2\varepsilon}}\big)\cap E_N\Big) \\ \geqslant \limsup_{N\to\infty}\rho_N\Big(\big(\Phi_N(\tau)(S_N(K))+B_\varepsilon\big)\cap E_N\Big).\end{aligned}$$

Next, using the Lipschitz continuity of the flow Φ_N (see Proposition 6.4), we obtain that there exists $c \in]0,1[$, independent of ε such that for N large enough, we have

$$\Phi_N(\tau)\big((K+B_{c\varepsilon})\cap E_N\big) \subset \big(\Phi_N(\tau)(S_N(K))+B_\varepsilon\big)\cap E_N,$$

where $B_{c\varepsilon}$ is the open ball in $H^s_{\text{rad}}(\Theta)$ centered at the origin and of radius $c\varepsilon$. Therefore

$$\limsup_{N\to\infty}\rho_N\Big(\big(\Phi_N(\tau)(S_N(K))+B_\varepsilon\big)\cap E_N\Big) \geqslant \limsup_{N\to\infty}\rho_N\Big(\Phi_N(\tau)\big((K+B_{c\varepsilon})\cap E_N\big)\Big).$$

Further, using the invariance of ρ_N under Φ_N, we obtain that

$$\rho_N\Big(\Phi_N(\tau)\big((K+B_{c\varepsilon})\cap E_N\big)\Big) = \rho_N\Big((K+B_{c\varepsilon})\cap E_N\Big)$$

and thus

$$\limsup_{N\to\infty}\rho_N\Big(\Phi_N(\tau)\big((K+B_{c\varepsilon})\cap E_N\big)\Big) \geqslant \liminf_{N\to\infty}\rho_N\Big((K+B_{c\varepsilon})\cap E_N\Big).$$

Finally, invoking once again Theorem 1.2, we can write

$$\liminf_{N\to\infty}\rho_N\Big((K+B_{c\varepsilon})\cap E_N\Big) \geqslant \rho(K+B_{c\varepsilon}) \geqslant \rho(K).$$

Therefore, we have the inequality

$$\rho\Big(\Phi(\tau)(K)+\overline{B_{2\varepsilon}}\Big) \geqslant \rho(K).$$

By letting $\varepsilon \to 0$, thanks to the dominated convergence, we obtain that

$$\rho(\Phi(\tau)(K)) \geqslant \rho(K).$$

By the time reversibility of the flow we get $\rho(\Phi(\tau)(K)) = \rho(K)$ and thus the measure invariance.

This completes the proof of Theorem 1.3. □

10. Concerning the three dimensional case

10.1. General discussion

The extension of the result to the 3d case is an interesting problem. In this case one can still prove the measure existence. The Cauchy problem issue is much more challenging. Despite the fact that the Cauchy problem for H^σ, $\sigma < 1/2$ data is ill-posed, in the sense of failure of continuity of the flow map (see the work of Christ-Colliander-Tao [7], or the appendix of [6]), we may hope that estimates on Wiener chaos can help us to resolve globally (with uniqueness) the Cauchy problem a.s. on a suitable statistical ensemble Σ (which is included in the intersection of H^σ, $\sigma < 1/2$ and misses $H^{1/2}$). This would be an example showing the possibility to get strong solutions of a dispersive equation, a.s. with respect to a measure, beyond the Hadamard well-posedness threshold. In this section, we prove an estimate which shows that one has a control on the second Picard iteration, in all H^σ, $\sigma < 1/2$, a.s. with respect to the measure. We will consider zonal solutions of the cubic defocusing NLS on the sphere S^3. The analysis of this model has a lot of similarities with the analysis on the ball of $\mathbb{R}^3$ (which is the three dimensional analogue of (1.1)). There are however some simplifications because of the absence of boundary on S^3 and a nice formula for the products of zonal eigenfunctions. In this section, we will benefit from some computations of the unpublished manuscript [5].

10.2. Zonal functions on S^3

Let S^3 be the unit sphere in $\mathbb{R}^4$. If we consider functions on S^3 depending only on the geodesic distance to the north pole, we obtain the zonal functions on S^3. The zonal functions can be expressed in terms of zonal spherical harmonics which in their turn can be expressed in terms the classical Jacobi polynomials. Let $\theta \in [0, \pi]$ be a local parameter measuring the geodesic distance to the north pole of S^3. Define the space $L^2_{\mathrm{rad}}(S^3)$ to be equipped with the following norm

$$\|f\|_{L^2_{\mathrm{rad}}(S^3)} = \Big(\int_0^\pi |f(\theta)|^2 (\sin\theta)^2 d\theta\Big)^{\frac{1}{2}},$$

where f is a zonal function on S^3 and $(\sin\theta)^2 d\theta$ is the surface measure on S^3. One can define similarly other functional spaces of zonal functions, for example $L^p_{\mathrm{rad}}(S^3)$, $H^s_{\mathrm{rad}}(S^3)$ *etc.* The Laplace-Beltrami operator on $L^2(S^3)$ can be restricted to $L^2_{\mathrm{rad}}(S^3)$ and in the coordinate θ it reads

$$\frac{\partial^2}{\partial\theta^2} + \frac{2}{\mathrm{tg}\theta}\frac{\partial}{\partial\theta}$$

since using the parametrization of S^3 in terms of θ and S^2, one can write,

$$\Delta_{S^3} = \frac{\partial^2}{\partial\theta^2} + \frac{2}{\mathrm{tg}\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{S^2}.$$

It follows from the Sturm-Liouville theory (see also e.g. [11]) that an orthonormal basis of $L^2_{\mathrm{rad}}(S^3)$ can be build by the functions

$$P_n(\theta) = \sqrt{\frac{2}{\pi}}\frac{\sin n\theta}{\sin\theta}, \quad \theta \in [0,\pi], \quad n \geqslant 1,$$

where θ connotes the geodesic distance to the north pole of S^3. The functions P_n are eigenfunctions of $-\Delta_{S^3}$ with corresponding eigenvalue $\lambda_n = n^2 - 1$. We next define the function $\gamma : \mathbb{N}^4 \longrightarrow \mathbb{R}$ by

$$\gamma(n, n_1, n_2, n_3) \equiv \int_{S^3} P_n P_{n_1} P_{n_2} P_{n_3}.$$

Then clearly

$$P_{n_1} P_{n_2} P_{n_3} = \sum_{n=1}^{\infty} \gamma(n, n_1, n_2, n_3) P_n$$

and thus the behavior of γ would be of importance when analyzing cubic expressions on S^3. In the next lemma we give a bound for $\gamma(n, n_1, n_2, n_3)$.

Lemma 10.1. — *One has the bound*

$$0 \leqslant \gamma(n, n_1, n_2, n_3) \leqslant (2/\pi) \min(n, n_1, n_2, n_3).$$

Proof. — Using the explicit formula for P_n and some trigonometric considerations, we obtain the relation

$$P_k P_l = \sqrt{\frac{2}{\pi}} \sum_{j=1}^{\min(k,l)} P_{|k-l|+2j-1}, \quad k \geqslant 1, l \geqslant 1. \tag{10.1}$$

By symmetry we can suppose that $n_1 \geqslant n_2 \geqslant n_3 \geqslant n$. Then due to (10.1) we obtain that $P_n P_{n_3}$ can be expressed as a sum of n terms while the sum corresponding to $P_{n_1} P_{n_2}$ contains n_2 terms. Since for $k \neq l$ one has $\int_{S^3} P_k P_l = 0$, we obtain that the contribution to $\gamma(n, n_1, n_2, n_3)$ of any of the term of the sum for $P_n P_{n_3}$ is not more than $2/\pi$ and therefore $\gamma(n, n_1, n_2, n_3) \leqslant (2/\pi)n$. This completes the proof of Lemma 10.1. □

We shall also make use of the following property of $\gamma(n, n_1, n_2, n_3)$.

Lemma 10.2. — *Let $n > n_1 + n_2 + n_3$. Then $\gamma(n, n_1, n_2, n_3) = 0$.*

Proof. — One needs simply to observe that in the spectral decomposition of $P_{n_1} P_{n_2} P_{n_3}$ there are only spherical harmonics of degree $\leqslant n_1 + n_2 + n_3$ and therefore $P_{n_1} P_{n_2} P_{n_3}$ is orthogonal to P_n. This completes the proof of Lemma 10.2. □

Remark 10.3. — Let us observe that $(\pi/2)\gamma(n, n_1, n_2, n_3) \in \mathbb{Z}$. This fact is however not of importance for the sequel.

10.3. The cubic defocusing NLS on S^3

Consider the cubic defocusing nonlinear Schrödinger equation, posed on S^3,

$$(i\partial_t + \Delta_{S^3})u - |u|^2 u = 0, \tag{10.2}$$

where $u : \mathbb{R} \times S^3 \longrightarrow \mathbb{C}$. By the variable change $u \to e^{it}u$, we can reduce (10.2) to

$$(i\partial_t + \Delta_{S^3} - 1)u - |u|^2 u = 0. \tag{10.3}$$

We will perform our analysis to the equation (10.3). The Hamiltonian associated to (10.3) is

$$H(u, \bar{u}) = \int_{S^3} |\nabla u|^2 + \int_{S^3} |u|^2 + \frac{1}{2}\int_{S^3} |u|^4,$$

where ∇ denotes the Riemannian gradient on S^3. We will study zonal solutions of (10.3), *i.e.* solutions such that $u(t, \cdot\,)$ is a zonal function on S^3. Let us fix $s < 1/2$. The free measure, denoted by μ, associated to (10.3) is the distribution of the $H^s_{\mathrm{rad}}(S^3)$ random variable

$$\varphi(\omega, \theta) = \sum_{n=1}^{\infty} \frac{g_n(\omega)}{n} P_n(\theta),$$

where $g_n(\omega)$ is a sequence of centered, normalized, independent identically distributed (i.i.d.) complex Gaussian random variables, defined in a probability space $(\Omega, \mathcal{F}, p)$. Using Lemma 10.1, we obtain that

$$\|P_n\|_{L^4(S^3)} \leqslant n^{\frac{1}{4}}.$$

and therefore using Lemma 2.1, as in the proof of Theorem 1.1, we get

$$\|\varphi(\omega, \theta)\|^2_{L^4(\Omega \times S^3)} \leqslant \sum_{n=1}^{\infty} \frac{C}{n^2}\|P_n\|^2_{L^4(S^3)} \leqslant C \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{n^2} < \infty.$$

Hence the image measure on $H^s_{\mathrm{rad}}(S^3)$ under the map

$$\omega \longmapsto \sum_{n=1}^{\infty} \frac{g_n(\omega)}{n} P_n(\theta),$$

of

$$\exp\Big(-\frac{1}{2}\|\varphi(\omega, \cdot\,)\|^4_{L^4(S^3)}\Big)\, dp(\omega)$$

is a nontrivial measure which could be expected to be invariant under a flow of (10.3). For that purpose one should define global dynamics of (10.3) on a set of full μ measure, *i.e.* solutions of (10.3) with data $\varphi(\omega, \theta)$ for typical ω's. Using for instance the Fernique integrability theorem one has that

$\|\varphi(\omega,\cdot)\|_{H^{1/2}(S^3)} = \infty$ μ a.s. Thus one needs to establish a well-define (and stable in a suitable sense) dynamics for data of Sobolev regularity $< 1/2$. There is a major problem if one tries to solve this problem for individual ω's since the result of [7] (see also the appendix of [6]) shows that (10.3) is in fact ill-posed for data of Sobolev regularity $< 1/2$ and the data giving the counterexample can be chosen to be a zonal function since the analysis uses only point concentrations. Therefore, it is possible that solving (10.3) with data $\varphi(\omega,\theta)$, for typical ω's, would require a probabilistic argument in the spirit of the definition of the stochastic integration. Below, we present an estimate which gives a control on the second Picard iteration with data $\varphi(\omega,\theta)$.

Let us consider the integral equation (Duhamel form) corresponding to (10.3) with data $\varphi(\omega,\theta)$

$$u(t) = S(t)(\varphi(\omega,\cdot)) - i\int_0^t S(t-\tau)(|u(\tau)|^2 u(\tau))d\tau, \tag{10.4}$$

where $S(t) = \exp(it(\Delta_{S^3}-1))$ is the unitary group generated by the free evolution. The operator $S(t)$ acts as an isometry on $H^s(S^3)$ which can be easily seen by expressing $S(t)$ in terms of the spectral decomposition. One can show (see [6]) that for $s > 1/2$, the Picard iteration applied in the context of (10.4) converges, if we replace $\varphi(\omega,\cdot)$ in (10.4) by data in $u_0 \in H^s(S^3)$, in the Bourgain spaces $X^{s,b}([-T,T]\times S^3)$, where $b > 1/2$ is close to $1/2$, $T \sim (1+\|u_0\|_{H^s(S^3)})^{-\beta}$ (for some $\beta > 0$ depending on b and s). For the definition the Bourgain spaces $X^{s,b}([-T,T]\times S^3)$ associated to Δ_{S^3}, we refer to [6] (see also (10.5) below). The modification for $\Delta_{S^3}-1$ is then direct. Let us set (the first Picard iteration)

$$u_1(\omega,t,\theta) \equiv S(t)(\varphi(\omega,\cdot)) = \sum_{n=1}^{\infty} \frac{g_n(\omega)}{n} P_n(\theta) e^{-itn^2}.$$

The random variable u_1 represents the free evolution. Notice that again

$$\|u_1(\omega,t,\cdot)\|_{H^{1/2}(S^3)} = \infty, \quad \text{a.s.}$$

but for every $\sigma < 1/2$,

$$\|u_1(\omega,t,\cdot)\|_{H^{\sigma}(S^3)} < \infty, \quad \text{a.s.}$$

Let us consider the second Picard iteration

$$u_2(\omega,t,\theta) \equiv S(t)(\varphi(\omega,\cdot)) - i\int_0^t S(t-\tau)(|u_1(\omega,\tau)|^2 u_1(\omega,\tau))d\tau.$$

Set

$$v_2(\omega,t,\theta) \equiv \int_0^t S(t-\tau)(|u_1(\omega,\tau)|^2 u_1(\omega,\tau))d\tau.$$

Thanks to the "dispersive effect", v_2 is again a.s. in all $H^\sigma(S^3)$ for $\sigma < 1/2$.

Proposition 10.4. — *Let us fix $\sigma < 1/2$. Then for $b > 1/2$ close to $1/2$ and every $T > 0$,*

$$\|v_2(\omega,t,\theta)\|_{L^2(\Omega;X^{\sigma,b}([-T,T]\times S^3))} < \infty.$$

In particular

$$\|v_2(\omega,t,\theta)\|_{L^2(\Omega;L^\infty([-T,T];H^\sigma(S^3)))} < \infty$$

and thus $\|v_2(\omega,\cdot,\cdot)\|_{L^\infty([-T,T];H^\sigma(S^3))}$ is a.s. finite which implies that the second Picard iteration for (10.4) is a.s. in H^σ.

Remark 10.5. — Using estimates on the third order Wiener chaos, we might show that higher moments and Orlitch norms with respect to ω are finite.

Proof of Proposition 10.4. — Let $\psi \in C_0^\infty(\mathbb{R};\mathbb{R})$ be a bump function localizing in $[-T,T]$. Let $\psi_1 \in C_0^\infty(\mathbb{R};\mathbb{R})$ be a bump function which equals one on the support of ψ. Set

$$w_1(t) \equiv \psi_1(t)u_1(t).$$

Then using [8], for $b > 1/2$ (close to $1/2$),

$$\begin{aligned}\|v_2(\omega,\cdot)\|_{X^{\sigma,b}([-T,T]\times S^3)} &\leqslant \|\psi v_2(\omega,\cdot)\|_{X^{\sigma,b}(\mathbb{R}\times S^3)}\\ &\leqslant C\||w_1(\omega,\cdot)|^2 w_1(\omega,\cdot)\|_{X^{\sigma,b-1}(\mathbb{R}\times S^3)}.\end{aligned}$$

Set

$$w(\omega,t,\theta) \equiv |w_1(\omega,t,\theta)|^2 w_1(\omega,t,\theta).$$

We need to show that the $L^2(\Omega)$ of $\|w(\omega,\cdot)\|_{X^{\sigma,b-1}(\mathbb{R}\times S^3)}$ is finite. If

$$w(\omega,t,\theta) = \sum_{n=1}^{\infty} c(\omega,n,t)P_n(\theta)$$

then we have

$$\|w(\omega,\cdot)\|^2_{X^{\sigma,b-1}(\mathbb{R}\times S^3)} = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\langle\tau+n^2\rangle^{2(b-1)}n^{2\sigma}|\widehat{c}(\omega,n,\tau)|^2 d\tau, \tag{10.5}$$

where $\widehat{c}(\omega,n,\tau)$ denotes the Fourier transform with respect to t of $c(\omega,n,t)$. Let us next compute $c(\omega,n,t)$. This will of course make appeal to the function γ introduced in the previous section. We have that

$$\begin{aligned}w(\omega,t,\theta) = \psi_1^3(t)\sum_{(n_1,n_2,n_3)\in\mathbb{N}^3}&\frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}g_{n_3}(\omega)}{n_1n_2n_3}\\ &\times P_{n_1}(\theta)P_{n_2}(\theta)P_{n_3}(\theta)e^{-it(n_1^2-n_2^2+n_3^2)}\end{aligned}$$

and therefore

$$c(\omega,n,t)=\psi_1^3(t)\sum_{(n_1,n_2,n_3)\in\mathbb{N}^3}\gamma(n,n_1,n_2,n_3)\frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}g_{n_3}(\omega)}{n_1n_2n_3}\times e^{-it(n_1^2-n_2^2+n_3^2)}.$$

If we denote $\psi_2=\psi_1^3$ then

$$\widehat{c}(\omega,n,\tau)=\sum_{(n_1,n_2,n_3)\in\mathbb{N}^3}\gamma(n,n_1,n_2,n_3)\frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}g_{n_3}(\omega)}{n_1n_2n_3}\times\widehat{\psi_2}\big(\tau+n_1^2-n_2^2+n_3^2\big).$$

Let us observe that thanks to the independence of $(g_n)_{n\in\mathbb{N}}$ we have that there are essentially two different situations when the expression

$$\int_\Omega g_{n_1}(\omega)\overline{g_{n_2}(\omega)}g_{n_3}(\omega)\overline{g_{m_1}(\omega)}g_{m_2}(\omega)\overline{g_{m_3}(\omega)}dp(\omega)\tag{10.6}$$

is different from zero. Namely

- $n_1=m_1$, $n_2=m_2$, $n_3=m_3$,
- $n_1=n_2$, $n_3=m_1$, $m_2=m_3$.

Indeed, the complex Gaussians g_n satisfy

$$\begin{aligned}\int_\Omega g_n(\omega)dp(\omega)=\int_\Omega g_n^2(\omega)dp(\omega)&=\int_\Omega g_n^3(\omega)dp(\omega)\\&=\int_\Omega|g_n(\omega)|^2g_n(\omega)dp(\omega)=0\end{aligned}$$

and thus in order to have a nonzero contribution of (10.6) each Gaussian without a bar in the integral (10.6) should be coupled with another Gaussian having a bar and the same index.

Therefore coming back to (10.5), we get

$$\int_\Omega\|w(\omega,\cdot)\|^2_{X^{\sigma,b-1}(\mathbb{R}\times S^3)}dp(\omega)\leqslant C(I_1+I_2),$$

where

$$I_1=\sum_{(n,n_1,n_2,n_3)\in\mathbb{N}^4}\int_{-\infty}^{\infty}\frac{n^{2\sigma}}{\langle\tau+n^2\rangle^\beta}\frac{\gamma^2(n,n_1,n_2,n_3)}{(n_1n_2n_3)^2}\times|\widehat{\psi_2}|^2\big(\tau+n_1^2-n_2^2+n_3^2\big)d\tau,\tag{10.7}$$

with $\beta \equiv 2(1-b)$ and

$$(10.8)\quad I_2 = \sum_{(n,n_1,n_2,n_3)\in\mathbb{N}^4} \int_{-\infty}^{\infty} \frac{n^{2\sigma}}{\langle \tau + n^2\rangle^{\beta}} \frac{\gamma(n,n_1,n_1,n_2)\gamma(n,n_2,n_3,n_3)}{(n_1n_1n_2)(n_2n_3n_3)} \times |\widehat{\psi_2}|^2\big(\tau + n_2^2\big)d\tau.$$

Notice that $\beta < 1$ is close to 1 when $b > 1/2$ is close to $1/2$. Thus our goal is to show the convergence of (10.7) and (10.8). For that purpose we make appeal to the following lemma.

LEMMA 10.6. — *For every $\sigma \in]0,1/2[$ there exist $\beta < 1$ and $C > 0$ such that for every $\alpha \in \mathbb{R}$,*

$$(10.9)\qquad \sum_{n=1}^{\infty} \frac{n^{2\sigma}}{(1+|n^2-\alpha|)^{\beta}} \leqslant C(1+|\alpha|)^{\sigma}.$$

Proof. — Let $\beta < 1$ be such that $2\beta - 2\sigma > 1$, *i.e.* $1/2 + \sigma < \beta < 1$. We prove (10.9) for such values of β. The contribution of the region $\frac{1}{4}n^2 \geqslant |\alpha|$ to the left hand-side of (10.9) can be bounded by

$$\sum_{n=1}^{\infty} \frac{n^{2\sigma}}{(1+\frac{3}{4}n^2)^{\beta}} \leqslant C_{\sigma} \leqslant C_{\sigma}(1+|\alpha|)^{\sigma}$$

thanks to the assumption $2\beta - 2\sigma > 1$ and since for $\frac{1}{4}n^2 \geqslant |\alpha|$ one has $|n^2-\alpha| \geqslant \frac{3}{4}n^2$. We next estimate the contribution of the region $\frac{1}{4}n^2 \leqslant |\alpha|$ (if it is not empty) by

$$(4|\alpha|)^{\sigma} \sum_{n=1}^{\infty} \frac{1}{(1+|n^2-\alpha|)^{\beta}} \leqslant C_{\sigma}|\alpha|^{\sigma}.$$

This completes the proof of Lemma 10.6. □

Let us now show the convergence of (10.7). Using the rapid decay of $|\widehat{\psi_2}|^2$, we can eliminate the τ integration and arrive at

$$(10.10)\quad (10.7) \leqslant C \sum_{(n,n_1,n_2,n_3)\in\mathbb{N}^4} \frac{n^{2\sigma}\gamma^2(n,n_1,n_2,n_3)}{(1+|n^2-n_1^2+n_2^2-n_3^2|)^{\beta}(n_1n_2n_3)^2}.$$

Using Lemma 10.1 and Lemma 10.6, we obtain that with a suitable choice of $\beta < 1$ one has

$$
\begin{aligned}
(10.7) &\leqslant C \sum_{(n_1,n_2,n_3)\in\mathbb{N}^3} \frac{(1+|n_1^2-n_2^2+n_3^2|)^{\sigma}(\min(n_1,n_2,n_3))^2}{(n_1n_2n_3)^2} \\
&\leqslant C \sum_{(n_1,n_2,n_3)\in\mathbb{N}^3} \frac{(\max(n_1,n_2,n_3))^{2\sigma}(\min(n_1,n_2,n_3))^2}{(n_1n_2n_3)^2} \\
&\leqslant C \sum_{n_3\leqslant n_2\leqslant n_1} \frac{n_1^{2\sigma}n_2n_3}{(n_1n_2n_3)^2} \leqslant C \sum_{n_1=1}^{\infty} \frac{n_1^{2\sigma}(\log(1+n_1))^2}{n_1^2} < \infty.
\end{aligned}
$$

Let us next analyze (10.8). Using the rapid decay of $|\widehat{\psi_2}|^2$, we can eliminate the τ integration and arrive at

$$
(10.8) \leqslant C \sum_{(n,n_1,n_2,n_3)\in\mathbb{N}^4} \frac{n^{2\sigma}}{(1+|n_2^2-n^2|)^{\beta}} \frac{\gamma(n,n_1,n_1,n_2)\gamma(n,n_2,n_3,n_3)}{(n_1n_1n_2)(n_2n_3n_3)}.
$$

Using Lemma 10.1 and Lemma 10.6, we obtain that with a suitable choice of $\beta < 1$ one has

$$
(10.8) \leqslant C \sum_{(n_1,n_2,n_3)\in\mathbb{N}^3} \frac{n_2^{2\sigma}\min(n_1,n_2)\min(n_3,n_2)}{(n_1n_2n_3)^2}.
$$

Let us fix $\varepsilon > 0$ such that $\sigma + \varepsilon < 1/2$. Therefore, we can write

$$
(10.8) \leqslant C \sum_{(n_1,n_2,n_3)\in\mathbb{N}^3} \frac{n_2^{2\sigma+2\varepsilon}(n_1n_3)^{1-\varepsilon}}{(n_1n_2n_3)^2} < \infty.
$$

This completes the proof of Proposition 10.4. □

Note addeded in proof. — In a recent joint work by Nicolas Burq and the author, it is shown that the phenomenon discussed in Section 10.1 is indeed possible in the context of the nonlinear wave equation.

BIBLIOGRAPHY

[1] R. ANTON, "Cubic nonlinear Schrödinger equation on three dimensional balls with radial data", Preprint, 2006.

[2] A. AYACHE & N. TZVETKOV, "L^p properties of Gaussian random series", to appear in Trans. AMS.

[3] J. BOURGAIN, "Periodic nonlinear Schrödinger equation and invariant measures", *Comm. Math. Phys.* **166** (1994), p. 1-26.

[4] ———, "Invariant measures for the 2D-defocusing nonlinear Schrödinger equation", *Comm. Math. Phys.* **176** (1996), p. 421-445.

[5] N. BURQ, P. GÉRARD & N. TZVETKOV, "Zonal low regularity solutions of the nonlinear Schrödinger equation on S^d", Unpublished manuscript, Summer 2002.

[6] ———, "Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations", *Ann. ENS* **38** (2005), p. 255-301.

[7] M. Christ, J. Colliander & T. Tao, "Ill-posedness for nonlinear Schrödinger and wave equations", Preprint, 2003.

[8] J. Ginibre, "Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain)", *Séminaire Bourbaki, Exp. 796, Astérisque* **237** (1996), p. 163-187.

[9] S. Kuksin & A. Shirikyan, "Randomly forced CGL equation : stationary measures and the inviscid limit", *J. Phys A* **37** (2004), p. 1-18.

[10] J. Lebowitz, R. Rose & E. Speer, "Statistical dynamics of the nonlinear Schrödinger equation", *J. Stat. Physics V* **50** (1988), p. 657-687.

[11] E. Stein & G. Weiss, "Introduction to Fourier analysis on euclidean spaces" (P. N. Princeton University Press, ed.), Princeton Mathematical Series, vol. 32, 1971.

[12] N. Tzvetkov, "Invariant measures for the nonlinear Schrödinger equation on the disc", *Dynamics of PDE* **3** (2006), p. 111-160.

[13] P. Zhidkov, *Korteweg de Vries and nonlinear Schrödinger equations : qualitative theory*, Lecture Notes in Mathematics, vol. 1756, Springer-Verlag, Berlin, 2001, vi+147 pages.

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