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A HILBERT LEMNISCATE THEOREM IN $\mathbb{C}^2$

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Abstract. — For a regular, compact, polynomially convex circled set K in $\mathbf{C}^2$, we construct a sequence of pairs $\{P_n, Q_n\}$ of homogeneous polynomials in two variables with $\deg P_n = \deg Q_n = n$ such that the sets $K_n := \{(z,w) \in \mathbf{C}^2 : |P_n(z,w)| \leqslant 1,\ |Q_n(z,w)| \leqslant 1\}$ approximate K and if K is the closure of a strictly pseudoconvex domain the normalized counting measures associated to the finite set $\{P_n = Q_n = 1\}$ converge to the pluripotential-theoretic Monge-Ampère measure for K. The key ingredient is an approximation theorem for subharmonic functions of logarithmic growth in one complex variable.

Résumé. — Pour un compact K dans $\mathbf{C}^2$, regulier, pôlynomiallement convexe et cerclé, on construit une suite de paires $\{P_n, Q_n\}$ avec P_n, Q_n pôlynomes homogènes en deux variables et $\deg P_n = \deg Q_n = n$ tel que les ensembles $K_n := \{(z,w) \in \mathbf{C}^2 : |P_n(z,w)| \leqslant 1,\ |Q_n(z,w)| \leqslant 1\}$ font une approximation de K et quand K est la fermeture d'un domaine strictement pseudoconvexe les mesures de comptage normalisées associées à l'ensemble fini $\{P_n = Q_n = 1\}$ tendent vers la mesure de Monge-Ampère pour K. L'élément principal est un théorème d'approximation pour les fonctions sousharmoniques de croissance logarithmique à une variable.

1. Introduction

Let $K \subset \mathbb{C}$ be a compact set with connected complement. The *Hilbert lemniscate theorem* in one variable says that for such sets, given any $\epsilon > 0$, there exists a polynomial p with

(1.1)
$$K \subset \mathcal{K}_p := \big\{z : |p(z)| \leqslant ||p||_K := \sup_{z\in K} |p(z)|\big\} \subset K^\epsilon := \{z : \mathrm{dist}(z,K) \leqslant \epsilon\}.$$

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The set $\mathcal{K}_p$ is called a *lemniscate*. In general, given $\epsilon > 0$, one can take p to be a *Fekete polynomial* of sufficiently large degree. A Fekete polynomial of degree n for K is a monic polynomial $F_n(z) = \prod_{j=1}^n (z - a_{nj})$ with $a_{nj} \in K$ chosen so that

$$\prod_{j<k}^{n} |a_{nj} - a_{nk}| = \max_{z_1,\ldots,z_n \in K} \prod_{j<k}^{n} |z_j - z_k|.$$

The condition that K have connected complement is equivalent to the *polynomial convexity* of K: this means that $K = \hat{K}$ where

$$\hat{K} := \big\{ z \in \mathbb{C} : |p(z)| \leqslant ||p||_K := \sup_{\zeta \in K} |p(\zeta)| \text{ for all polynomials } p \big\}.$$

(Here and in the entire paper "polynomial" means *holomorphic* polynomial). We call K *regular* if the extremal function

$$V_K(z) := \max \left[0, \sup \left\{ \frac{1}{\deg p} \log |p(z)| : p \text{ polynomial},\ \deg p \geqslant 1, \right.\right.$$
$$\left.\left. ||p||_K \leqslant 1 \right\} \right] \tag{1.2}$$

is continuous on $\mathbb{C}$. For the lemniscate $\mathcal{K}_p$ in (1.1),

$$V_{\mathcal{K}}(z) = \max \left[\frac{1}{\deg p} \log \left[|p(z)| / ||p||_K \right], 0 \right].$$

If K is regular, in choosing, *e.g.*, a sequence of Fekete polynomials $\{F_n\}$, the functions

$$\frac{1}{n} \log \left[|F_n(z)| / ||F_n||_K \right] \to V_K(z) \tag{1.3}$$

locally uniformly outside of K. We also have the normalized counting measure of the zeros

$$\mu_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{a_{nj}} \to \frac{1}{2\pi} \Delta V_K \tag{1.4}$$

weak-* as measures. Here, ΔV_K, the Laplacian of V_K, is to be interpreted as a positive distribution, *i.e.*, a positive measure. Another example of a sequence of polynomials for which (1.3) and (1.4) hold is gotten by taking the interval $K = [-1, 1]$ and the classical Chebyshev polynomials $\{T_n\}$. Here $T_n(x) = \cos n(\arccos x)$ for $x \in \mathbb{R}$; $V_K(z) = \log |z + \sqrt{z^2 - 1}|$ and the normalized counting measure of the zeros approximate the arcsine distribution $\frac{dx}{\sqrt{1-x^2}} = \Delta V_K$.

In several complex variables, given a compact set $K \subset \mathbb{C}^N$, $N > 1$, we can define the extremal function V_K as in (1.2) where $p(z) = p(z_1, \ldots, z_N)$ is a polynomial of the complex variables $z_1, \ldots, z_N$. The definitions of regularity and polynomial convexity are defined as in the one-variable case; however this latter definition is no longer equivalent to the complement of K being connected. It follows from the definition of V_K and $\hat{K}$ that $V_K = V_{\hat{K}}$ and that $\hat{K} = \{z : V_K(z) = 0\}$ so that an assumption of polynomial convexity is a natural one. In this paper, we will prove a version of Hilbert's lemniscate theorem for circled compact sets in $\mathbb{C}^2$, including a convergence of measures result in the spirit of (1.4).

To motivate this result, we note that in several complex variables, sublevel sets $\{z : |p(z)| \leqslant M\}$ for a polynomial p are unbounded; in general, one needs at least N polynomials $p_1, ..., p_N$ to have hopes of a sublevel set $\{z \in \mathbb{C}^N : |p_1(z)| \leqslant M_1, ..., |p_N(z)| \leqslant M_N\}$ being compact. Moreover, the topology of such sublevel sets can be complicated. A *polynomial polyhedron* is a set P which is the closure of the union of a finite number of connected components of

$$\mathcal{P} := \{z \in \mathbb{C}^N : |p_1(z)| < 1, ..., |p_m(z)| < 1\}$$

where $p_1, ..., p_m$ are polynomials. It is an easy exercise to see that *given any polynomially convex compact set $K \subset \mathbb{C}^N$, and any open neighborhood Ω of K, there exists a set of the form $\mathcal{P}$ with $K \subset \mathcal{P} \subset \Omega$* (*cf.* [11]). What is not at all obvious is a deep result of Bishop [4]: *there exists a* **special** *polynomial polyhedron P with the same property.* We call a polynomial polyhedron $P \subset \mathbb{C}^N$ *special* if it can be defined by *exactly* N polynomials. We emphasize that not all components of $\mathcal{P}$ need be included in P. It is known (*cf.* [13], Theorem 5.3.1) that if the set

$$\mathcal{P} := \{z \in \mathbb{C}^N : |p_1(z)| < 1, ..., |p_N(z)| < 1\},$$

consisting of the union of *all* components of a special polynomial polyhedron defined by $p_1, ..., p_N$ with $\deg p_1 = ... \deg p_N =: n$ is compact, and if $(p_1, ..., p_N) : \mathbb{C}^N \to \mathbb{C}^N$ is proper, then we have

$$V_{\mathcal{P}}(z) = \max\Big[\frac{1}{n}\log|p_1(z)|, ..., \frac{1}{n}\log|p_N(z)|, 0\Big].$$

Thus, it will be helpful to know when a compact set K can be approximated not just by a special polynomial polyhedron P, but by the full component set $\mathcal{P}$ of such an object. It turns out that if we work in $\mathbb{C}^2$ with variables (z, w) and we assume, in addition to $K = \hat{K}$, that $K \subset \mathbb{C}^2$ is *circled*; *i.e.*, $z \in K$ if and only if $e^{it}z \in K$, then such an approximation is possible. Moreover, in this case, utilizing one-variable techniques, we can construct Bishop-type approximants which satisfy an analogue of (1.3) and (1.4).

THEOREM 1.1. — *Let $K \subset \mathbb{C}^2$ be a regular, circled, polynomially convex compact set. Then there exists a sequence of pairs of homogeneous polynomials $\{P_n, Q_n\}$, $\deg P_n = \deg Q_n = n$ with no common linear factors such that*

$$\tilde{u}_n(z,w) := \max\left[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|, 0\right]$$

uniformly approximates V_K on $\mathbb{C}^2$;

$$U_n(z,w) := \max\left[\frac{1}{n}\log|P_n(z,w)-1|, \frac{1}{n}\log|Q_n(z,w)-1|\right]$$

locally uniformly approximates V_K on $\mathbb{C}^2 \setminus \partial K$; and

$$(dd^c\tilde{u}_n)^2 \to (dd^cV_K)^2$$

weak- as measures in $\mathbb{C}^2$. Moreover, if K is the closure of a strictly pseudoconvex domain (e.g., a ball), then*

$$(dd^cU_n)^2 \to (dd^cV_K)^2.$$

Here, for certain plurisubharmonic (psh) functions u in $\mathbb{C}^2$, the complex Monge-Ampère measure $(dd^cu)^2$ associated to u is well-defined. We discuss this issue in section 4. In particular, for regular compact sets $K \subset \mathbb{C}^2$, $(dd^cV_K)^2$ plays a role analogous to ΔV_K in one variable. In Theorem1.1,

- the function $\tilde{u}_n$ is the extremal function for the set

$$\mathcal{K}_n := \{(z,w) \in \mathbb{C}^2 : |P_n(z,w)| \leqslant 1,\ |Q_n(z,w)| \leqslant 1\}; \tag{1.5}$$

- the Monge-Ampère measure $(dd^cU_n)^2$ is supported on the finite point set (see section 4)

$$K_n := \{(z,w) : P_n(z,w) = Q_n(z,w) = 1\}; \tag{1.6}$$

- the measures $\{(dd^c\tilde{u}_n)^2\}_{n=1,\dots}$, $\{(dd^cU_n)^2\}_{n=1,\dots}$ are supported in a fixed compact set in $\mathbb{C}^2$.

The distinction between the sequences $\{\tilde{u}_n\}$ and $\{U_n\}$ can easily be seen even in one variable: take $K = \mathbb{D} := \{t \in \mathbb{C} : |t| \leqslant 1\}$, the closed unit disk. Then $V_{\mathbb{D}}(t) = \max[\log|t|, 0]$ and, taking $p_n(t) = t^n$, we have

$$\tilde{v}_n(t) := \max\left[\frac{1}{n}\log|p_n(t)|, 0\right] \equiv V_{\mathbb{D}}(t)$$

while

$$V_n(t) := \frac{1}{n}\log|p_n(t)-1| = \frac{1}{n}\log|t^n - 1|$$

converges locally uniformly to $V_{\mathbb{D}}$ in $\mathbb{C}\setminus\{|t|=1\}$ but we clearly do not have $V_n \to V_{\mathbb{D}}$ pointwise, or even "in capacity" (*cf.* [17]) on the circle $\{|t|=1\}$. However, we do have $V_n \to V_{\mathbb{D}}$ in $L^1_{\mathrm{loc}}(\mathbb{C})$. Thus, we can utilize elementary distribution theory to conclude that the normalized counting measure of the zeros of these Fekete polynomials $p_n(t)$ converge weak-* to $\Delta V_{\mathbb{D}}$. Of course, in this example, the convergence of these measures is trivial (and, as mentioned earlier, always holds for Fekete polynomials). We discuss the analogous example of the unit bidisk in $\mathbb{C}^2$ in section 4.

We prove the first part of Theorem 1.1 by reducing it to a one-variable approximation problem in section 2. Given a measure μ in $\mathbb{C}$ with $\mu(\mathbb{C}) = 1$ consider its logarithmic potential

$$V(t) = \int_{\mathbb{C}} \log \left|1 - \frac{t}{\zeta}\right| d\mu(\zeta). \tag{1.7}$$

We assume that

$$\lim_{|t|\to\infty} [V(t) - \log |t|] \text{ exists,} \tag{1.8}$$

$$\int_{\mathbb{C}} |\log |t|| d\mu(t) < \infty, \tag{1.9}$$

and that $V(t)$ is continuous in $\mathbb{C}$. Under these assumptions, we will prove the following theorem, which is of interest in its own right, in section 3:

Theorem 1.2. — *Given V satisfying (1.7), (1.8) and (1.9), for each $\epsilon > 0$ there exist a number N and polynomials $P(t)$ and $Q(t)$ of degree N such that*

$$|V(t) - \frac{1}{N} \max\{\log |P(t)|, \log |Q(t)|\}| < \epsilon, \quad t \in \mathbb{C}. \tag{1.10}$$

The construction is based on techniques developed in [14]. There the authors construct an L^1−approximant to an arbitrary subharmonic function u in $\mathbb{C}$ of the form $\log |f|$ with a (single) entire function f. The proof utilizes a clever partition of $\mathbb{C}$ related to the measure μ and its support, due to Yulmukhametov [19]. The precise version of the result that we use in section 3 is labeled Lemma A. We remark that the genesis of Theorem 1.2 occurred during an Oberwolfach meeting attended by the second and third authors in February 2004.

In the final section of the paper, we turn to the proof of Monge-Ampère convergence, the second part of Theorem 1.1. For the sequence $\{\tilde{u}_n\}$ this convergence is automatic; but for the sequence $\{U_n\}$, which is *not* locally bounded, a non-trivial argument is required. This is given as Theorem 4.3. We would like to thank Urban Cegrell for pointing out an error in our proof of this result in a previous version.

We remark that from Bishop's theorem one can construct sequences of psh functions with the same properties as the sequence $\{U_n\}$ in Theorem 1.1 for general regular, polynomially convex compact sets $K \subset \mathbf{C}^N$ which are not necessarily circled. However, this work of Bishop is technically complicated and the construction may not yield psh functions which are the maximum of exactly N functions of the form $c \log |p|$ where p is a polynomial. Our methods in constructing the polynomials in Theorem 1.1 are purely one-variable in nature and provide, via the sets $\{K_n\}$ in (1.6), discrete approximations to the Monge-Ampère measure $(dd^c V_K)^2$.

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2. Reduction to one-variable

For $N = 1, 2, ...$, let

$$L(\mathbb{C}^N) := \{u \text{ psh in } \mathbb{C}^N : \ u(z) \leqslant \log^+ |z| + C\}$$

denote the class of psh functions of logarithmic growth on $\mathbb{C}^N$ where the constant C can depend on u. For example, given a polynomial p, $u(z) := \frac{1}{\deg p} \log |p(z)| \in L(\mathbb{C}^N)$. We also consider the class

$$L^+(\mathbb{C}^N) := \{u \in L(\mathbb{C}^N) : \log^+ |z| + C_1 \leqslant u(z) \leqslant \log^+ |z| + C_2, \text{ some } C_1, C_2\}.$$

Note functions in this class are locally bounded.

For a bounded Borel set E in $\mathbb{C}^N$, one can define

$$V_E(z) := \sup\{u(z) : u \in L(\mathbb{C}^N), \ u \leqslant 0 \text{ on } E\}. \tag{2.1}$$

The uppersemicontinuous (usc) regularization $V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta)$ is called the global extremal function of E; either $V_E^* \equiv +\infty$ – this occurs precisely when E is *pluripolar*; *i.e.*, $E \subset \{u = -\infty\}$ for some $u \not\equiv -\infty$ psh on a neighborhood of E – or else $V_E^* \in L^+(\mathbb{C}^N)$. It is well-known that if E is a compact set in $\mathbb{C}^N$, then V_E defined in (2.1) coincides with V_E in formula (1.1) (*cf.* [13] Theorem 5.1.7) and hence V_E is lowersemicontinuous. Thus for compact sets E, E is regular if and only if $V_E = V_E^*$.

As well as the classes $L(\mathbb{C}^N)$ and $L^+(\mathbb{C}^N)$, we will consider the class

$$H(\mathbb{C}^N) := \{u \in L(\mathbb{C}^N) : u(\lambda z) = u(z) + \log |\lambda| \text{ for } \lambda \in \mathbb{C}, \ z \in \mathbb{C}^N\}$$

of *logarithmically homogeneous* psh functions.

Given $u : \mathbb{C}^N \to \mathbb{R}$ in $L(\mathbb{C}^N)$ we define the *Robin function* of u to be

$$\rho_u(z) := \limsup_{|\lambda| \to \infty} [u(\lambda z) - \log |\lambda|] .$$

Note that for $\lambda \in \mathbb{C}$, $\rho_u(\lambda z) = \log|\lambda| + \rho_u(z)$; *i.e.*, ρ_u is logarithmically homogeneous. It is known (*cf.* [6], Proposition 2.1) that for $u \in L(\mathbb{C}^N)$, the Robin function $\rho_u(z)$ is plurisubharmonic in $\mathbb{C}^N$; indeed, either $\rho_u \in H(\mathbb{C}^N)$ or $\rho_u \equiv -\infty$. As an example, if p is a polynomial of degree d so that $u(z) := \frac{1}{d}\log|p(z)| \in L(\mathbb{C}^N)$, then $\rho_u(z) = \frac{1}{d}\log|\hat{p}(z)|$ where $\hat{p}$ is the top degree (d) homogeneous part of p. For a compact set K, we denote by ρ_K the Robin function of V_K^*; *i.e.*, $\rho_K := \rho_{V_K^*}$.

Suppose now that K is *circled*; *i.e.*, $z \in K$ if and only if $e^{it}z \in K$. Then the extremal function V_K in (1.1) can be gotten via

$$\begin{aligned} V_K(z) &= \max[0, \sup\{u(z) : u \in H(\mathbb{C}^N),\ u \leqslant 0 \text{ on } K\}] \\ &= \max[0, \sup\{\frac{1}{\deg p}\log|p(z)| : p \text{ homogeneous polynomial},\ ||p||_K \leqslant 1\}] \end{aligned}$$

(*cf.* [13], Theorem 5.1.6). Moreover, we have the following.

Lemma 2.1. — *Let $K \subset \mathbb{C}^N$ be compact, circled, and nonpluripolar. Then*

$$V_K^*(z) = \max[0, \rho_K(z)] \tag{2.2}$$

and

$$\operatorname{supp}(dd^cV_K^*)^N \subset \{\rho_K = 0\}. \tag{2.3}$$

Proof. — Equation (2.2) follows from the above equation for V_K, which shows that $V_K^*(\lambda z) = V_K^*(z) + \log|\lambda|$ provided $z, \lambda z \notin \hat{K}$, and the definition of ρ_K: if $V_K^*(z) > 0$, then

$$\begin{aligned} \rho_K(z) &:= \limsup_{|\lambda|\to\infty}[V_K^*(\lambda z) - \log|\lambda|] \\ &= \limsup_{|\lambda|\to\infty}[V_K^*(z) + \log|\lambda| - \log|\lambda|] = V_K^*(z). \end{aligned}$$

We have $\rho_K \in H(\mathbb{C}^N)$ and $\rho_K(z) = V_K^*(z)$ if $V_K^*(z) > 0$; since the set $\{z \in \mathbb{C}^N : \rho_K(z) \leqslant 0\}$ differs from $\hat{K} = \{z \in \mathbb{C}^N : V_K(z) = 0\}$ by at most a pluripolar set, (2.2) follows (*cf.* Corollary 5.2.5 [13]). The Robin function ρ_K is locally bounded away from the origin which implies, by the logarithmic homogeneity, that $(dd^c\rho_K)^N = 0$ on $\mathbb{C}^N \setminus \{0\}$ (see section 4 for a discussion of the complex Monge-Ampère operator). This gives (2.3). ☐

Let $u \in L(\mathbb{C})$ and $d\mu(t) = \frac{i}{4\pi}\Delta u(t) dt \wedge d\bar{t}$ be its Riesz measure. Jensen's formula yields that $\mu(\mathbb{C}) := \int_{\mathbb{C}} d\mu(t) \leqslant 1$. If, in addition, $u(0) = 0$, we have

$$u(t) = \int_{\mathbb{C}} \log|1 - \frac{t}{\zeta}| d\mu(\zeta)$$

([15], p. 37). In the notation introduced in this section, Theorem 1.2 yields the following version of a one-variable approximation result:

Theorem 2.2. — *Let $u \in L^+(\mathbb{C}) \cap C(\mathbb{C})$ with the additional property that*

$$\lim_{|t|\to\infty} [u(t) - \log|t|]$$

exists. Given $\epsilon > 0$, there exist polynomials p_n, q_n of degree $n = n(\epsilon)$ with

$$u(t) - \epsilon \leqslant \max\Big[\frac{1}{n}\log|p_n(t)|, \frac{1}{n}\log|q_n(t)|\Big] \leqslant u(t),\ \ t \in \mathbb{C}. \tag{2.4}$$

Note that $u \in L^+(\mathbb{C})$ implies (1.9) and that (2.4) implies that p_n and q_n have no common zeros; this latter fact will also follow from the proof of the theorem. This immediately gives an approximation result for the class $H(\mathbb{C}^2)$ of logarithmically homogeneous psh functions in $\mathbb{C}^2$.

Corollary 2.3. — *Let $U \in H(\mathbb{C}^2)$ be logarithmically homogeneous with the additional property that $u(t) := U(1,t)$ satisfies the hypotheses of the previous theorem. Given $\epsilon > 0$, there exist homogeneous polynomials P_n, Q_n of degree $n = n(\epsilon)$ with no common factors such that*

$$U(z,w) - \epsilon \leqslant \max\Big[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|\Big] \leqslant U(z,w), \quad (z,w) \in \mathbb{C}^2. \tag{2.5}$$

Proof. — If (2.4) holds, define

$$P_n(z,w) := z^n p_n(w/z) \text{ and } Q_n(z,w) := z^n q_n(w/z).$$

Note that if p_n, q_n are of degree exactly n; *i.e.*, if

$$p_n(t) = a_0 + a_1 t + \cdots + a_n t^n \text{ and } q_n(t) = b_0 + b_1 t + \cdots + b_n t^n$$

with $a_n b_n \neq 0$, then $P_n(0,w) = a_n w^n$ and $Q_n(0,w) = b_n w^n$. Otherwise, we may have $P_n(0,w) \equiv 0$ and/or $Q_n(0,w) \equiv 0$. Then, since $U(1,w/z) + \log|z| = U(z,w)$ for $z \neq 0$, (2.4) implies

$$U(z,w) - \epsilon \leqslant \max\Big[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|\Big] \leqslant U(z,w)$$

for $z \neq 0$. But U is subharmonic on $z = 0$ so

$$U(0,w) = \limsup_{z\to 0} U(z,w);$$

together with the previous inequalities, this yields (2.5) for all $(z,w) \in \mathbb{C}^2$. $\square$

For a regular compact set $K \subset \mathbb{C}^N$, it is known that the Robin function ρ_K is continuous on $\mathbb{C}^N \setminus \{0\}$ (*cf.* [6]). Thus, if $N = 2$, $\rho_K(1,t) \in L^+(\mathbb{C}) \cap C(\mathbb{C})$ and

$$\lim_{|t|\to\infty} [\rho_K(1,t) - \log|t|] = \lim_{|t|\to\infty} \rho_K(1/t,1) = \rho_K(0,1).$$

We can apply the corollary to ρ_K to find homogeneous polynomials P_n, Q_n with

$$\rho_K(z,w) - \epsilon \leqslant \max\Big[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|\Big] \leqslant \rho_K(z,w). \tag{2.6}$$

To prove the first part of Theorem 1.1, for a regular circled set $K \subset \mathbb{C}^2$, using (2.2) from Lemma 2.1 and (2.6), we have

$$V_K(z,w) - \epsilon \leqslant \max\Big[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|, 0\Big] \leqslant V_K(z,w). \tag{2.7}$$

This gives uniform convergence of

$$\tilde{u}_n(z,w) := \max\Big[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|, 0\Big] \to V_K(z,w)$$

in Theorem 1.1.

For regular circled sets $K \subset \mathbb{C}^2$, (2.3) of Lemma 2.1 implies that

$$\operatorname{supp}(dd^cV_K)^2 \subset \{(z,w) : \rho_K(z,w) = 0\}.$$

We now show using (2.6) and (2.7) that

$$U_n(z,w) := \max\Big[\frac{1}{n}\log|P_n(z,w) - 1|, \frac{1}{n}\log|Q_n(z,w) - 1|\Big] \to V_K(z,w) \tag{2.8}$$

locally uniformly on $\mathbb{C}^2 \setminus \{\rho_K = 0\}$.

To prove (2.8), we observe from the inequality $|A - B| \leqslant 2\max[|A|, |B|]$ we have

$$U_n(z,w) \leqslant \max\Big[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|, 0\Big] + \frac{\log 2}{n}. \tag{2.9}$$

Now on a compact set $E \subset \mathbb{C}^2 \setminus \{\rho_K \leqslant 0\}$, by (2.7), given $\epsilon > 0$ with $2\epsilon < \inf_E V_K$, for $n > n_0(\epsilon)$,

$$\max\big[|P_n(z,w)|, |Q_n(z,w)|\big] > \exp\left[n(V_K(z,w) - \epsilon)\right] \text{ on } E.$$

By choosing $n_0(\epsilon)$ larger, if necessary, we may assume

$$\exp\left[n(V_K(z,w) - \epsilon)\right] - 1 > \exp\left[n(V_K(z,w) - 2\epsilon)\right] \text{ on } E$$

so that

$$\max\big[|P_n(z,w) - 1|, |Q_n(z,w) - 1|\big] > \exp\left[n(V_K(z,w) - 2\epsilon)\right] \text{ on } E.$$

Together with (2.7) and (2.9), this proves local uniform convergence outside of $\{\rho_K \leqslant 0\}$. On compact subsets of $\{\rho_K < 0\}$, the story is similar due to the logarithmic homogeneity of ρ_K, $\frac{1}{n}\log|P_n(z,w)|$, and $\frac{1}{n}\log|Q_n(z,w)|$ and (2.6): for $r > 0$, if $E := \{z \in K : \rho_K(z) < -r\}$, by (2.6),

$$\max\big[|P_n(z,w)|, |Q_n(z,w)|\big] < \exp(-nr) \text{ on } E.$$

Thus, $|P_n(z,w)-1|, |Q_n(z,w)-1| > 1-\exp(-nr)$ on E. We conclude that

$$\max\Big[\frac{1}{n}\log|P_n(z,w)-1|, \frac{1}{n}\log|Q_n(z,w)-1|\Big] > \frac{1}{n}\log\big[1-\exp(-nr)\big] \quad \text{on } E.$$

Hence $U_n \to 0$ uniformly on E.

Note that since we assume that K is polynomially convex and circled, we have that

$$\partial K = \{(z,w) : \rho_K(z,w) = 0\}. \tag{2.10}$$

Here is an illustrative example of the reduction scheme: let $K = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 \leqslant 1\}$ be the closed unit ball in $\mathbb{C}^2$. Then $V_K(z,w) = \log^+(|z|^2+|w|^2)^{1/2}$ and $\rho_K(z,w) = \log(|z|^2+|w|^2)^{1/2}$ so that $\rho_K(1,t) = \frac{1}{2}\log(1+|t|^2)$. Note that the support of $\Delta\rho_K(1,t)$ is all of $\mathbb{C}$, but that

$$\int_{\mathbb{C}} |\log|t||\Delta\rho_K(1,t) < +\infty.$$

Thus, Theorem 2.2 provides a uniform approximation of the strictly subharmonic function $\frac{1}{2}\log(1+|t|^2)$ by a function of the form

$$\max\big[\frac{1}{n}\log|p_n(t)|, \frac{1}{n}\log|q_n(t)|\big].$$

To summarize: using the results of this section, in order to complete the proof of the first part of Theorem 1.1, it remains to prove the one-variable approximation result, Theorem 1.2.

3. Main approximation result

In this section, we prove Theorem 1.2. We work exclusively in the complex plane $\mathbb{C}$ with variable z. Recall that $V(z)$ is the logarithmic potential of a probability measure μ; V is continuous in $\mathbb{C}$; $\lim_{|t|\to\infty}[V(t) - \log|t|]$ exists; and $\int_{\mathbb{C}} |\log|t||d\mu(t) < \infty$.

In order to prove the theorem we shall prove the following result:

CLAIM 3.1. — *For each $\epsilon > 0$ there exists a number N, polynomials $P(z)$ and $Q(z)$ of degree N, and sets $E, F \subset \mathbb{C}$, $E \cap F = \emptyset$ such that*

$$|V(z) - \frac{1}{N}\log|P(z)|| < \epsilon, \quad z \in \mathbb{C} \setminus E,$$

$$V(z) + \epsilon > \frac{1}{N}\log|P(z)|, \quad z \in E, \tag{3.1}$$

and

$$|V(z) - \frac{1}{N}\log|Q(z)|| < \epsilon, \quad z \in \mathbb{C} \setminus F,$$

$$V(z) + \epsilon > \frac{1}{N}\log|Q(z)|, \quad z \in F. \tag{3.2}$$

3.1. Pattern of the proof

Step 1: It follows from (1.8) and also from continuity of V that V is uniformly continuous in $\mathbb{C}$. Convolving if need be with an appropriate bump function one may assume that μ has the form

$$d\mu(z) = a(z)d\sigma(z), \tag{3.3}$$

where σ is Lebesgue measure and $a \geqslant 0$ is a smooth function in $\mathbb{C}$. It follows from (1.9) that

$$a(z) \to 0 \quad \text{as} \quad |z| \to \infty.$$

Define

$$A := \max_{z\in\mathbb{C}} a(z). \tag{3.4}$$

Step 2: We reduce the problem to the case when μ has compact support. Given a number $R > 0$ we let Q_R denote the square

$$Q_R = \{z = x + iy; |x|, |y| < R\}.$$

Given $\eta > 0$ we find an integer M and a number R so that

$$\int_{\mathbb{C}\setminus Q_R} |\log|\zeta||d\mu(\zeta) < \eta, \tag{3.5}$$

$$\mu(\mathbb{C} \setminus Q_R) = 1/M < \eta, \tag{3.6}$$

and

$$\max_{|z|>R/3} a(z) \leqslant \eta. \tag{3.7}$$

Denote the logarithmic potential from the portion of μ outside Q_R by

$$V_\infty(z) := \int_{\mathbb{C}\setminus Q_R} \log|1 - \frac{z}{\zeta}| d\mu(\zeta).$$

Finally, set

$$\frac{1}{M}\log r_\infty := \int_{\mathbb{C}\setminus Q_R} \log|\zeta| d\mu(\zeta). \tag{3.8}$$

Note that $r_\infty > R$.

LEMMA 3.2. — *Let*

$$w_\infty \in \mathbb{C}, \quad |w_\infty| = 10 r_\infty. \tag{3.9}$$

Then

$$\left| V_\infty(z) - \frac{1}{M}\log\left|1 - \frac{z}{w_\infty}\right|\right| \leqslant C_1\, \eta, \quad z \notin E_{w_\infty},$$

and

$$\frac{1}{M}\log\left|1 - \frac{z}{w_\infty}\right| \leqslant V_\infty(z) + C_2\, \eta,\ z \in E_{w_\infty}, \tag{3.10}$$

where C_1, C_2 are constants independent of w_∞ and

$$E_{w_\infty} = \left\{ z : |z - w_\infty| < \frac{1}{20}|w_\infty| \right\}.$$

Remarks. — **1.** It is clear that $E_{w_\infty} \cap Q_R = \emptyset$ and also that it is possible to choose two different points w'_∞ and w''_∞ satisfying (3.9) so that $E_{w'_\infty} \cap E_{w''_\infty} = \emptyset$.
2. The values of the constants in this lemma depend upon A.
3. We use the notation $a \prec b$ to mean $a \leqslant Cb$ with C a constant independent of all parameters except perhaps A and $a \asymp b$ to mean $a \prec b$ and $b \prec a$.

Step 3. Define

$$V_0(z) := \int_{Q_R} \log|z - \zeta| d\mu(\zeta).$$

Given Lemma 3.2, it remains to approximate V_0 by a function which has the form $\frac{1}{n}\log|P_N(z)|$, where P_N is a polynomial of degree N. In order to construct this approximation we need a special partition of Q_R. Existence of the desired partitions is ensured by a lemma due to R. Yulmuhametov [19], see also [10]. We state this result in a form which is adjusted to our situation. Let $\hat{\mu}$ denote the restriction of μ to Q_R. We have $\hat{\mu}(Q_R) = (M-1)/M$. Given an integer k we split Q_R into $k(M-1)$ pieces each of measure $1/Mk$.

Lemma A. — *Given an integer $k > 0$, there exists a covering of Q_R*

$$Q_R = \cup_{l=1}^{(k-1)M} Q^{(l)},$$

and $\hat{\mu}$,

$$\hat{\mu} = \sum_{l=1}^{(k-1)M} \mu^{(l)},$$

with the following properties:

- *Each $Q^{(l)}$ is a rectangle with sides parallel to the coordinate axes such that the ratio of longest to shortest side does not exceed 3;*
- *each point in Q_R belongs to at most four distinct rectangles $Q^{(l)}$;*
- *supp $\mu^{(l)} \subset Q^{(l)}$;*
-

$$\mu^{(l)}(Q^{(l)}) = \frac{1}{kM}. \tag{3.11}$$

Fix such a partition. We look for a polynomial P_k of degree $N := k(M-1)$ of the form

$$P_k(z) = \prod_{l=1}^{N} (z - \zeta^{(l)}),$$

where the choice of the points $\{\zeta^{(l)}\}_1^{k(M-1)} \subset Q_R$ is related to the partition.

Let $d(l) := \operatorname{diam}(Q^{(l)})$. We then have $\operatorname{Area}(Q^{(l)}) \asymp d(l)^2$. In choosing the points $\{\zeta^{(l)}\}_1^{k(M-1)}$, we first observe that, by (3.3) and (3.4), $d(l)$ cannot be too small:

$$d(l) \geqslant \frac{1}{3(MA)^{1/2}} \frac{1}{k^{1/2}}. \tag{3.12}$$

We split the set of indices into two subsets:

(3.13)

$$\mathcal{I}_k = \left\{ l : 1 \leqslant l \leqslant N, d(l) \leqslant k^{1/3} \frac{1}{3(MA)^{1/2}} \frac{1}{k^{1/2}} \right\}, \ \mathcal{J}_k = \{1, 2, \ldots, N\} \setminus \mathcal{I}_k.$$

We say that $Q^{(l)}$ is a *normal* rectangle if $l \in \mathcal{I}_k$. For such rectangles we set

$$\zeta_0^{(l)} = kM \int_{Q^{(l)}} \zeta d\mu^{(l)}(\zeta), \tag{3.14}$$

the center of mass of $\mu^{(l)}$ in $Q^{(l)}$, and then take

$$\zeta^{(l)} := \zeta_0^{(l)} + \delta^{(l)} \in Q^{(l)}$$

where $\delta^{(l)}$ are any complex numbers satisfying

$$|\delta^{(l)}| \leqslant k^{-5}. \tag{3.15}$$

For $l \in \mathcal{J}_k$ we let $\zeta^{(l)} \in Q^{(l)}$ be any points of Q_R with the property that

$$|\zeta^{(l)} - \zeta^{(m)}| > k^{-5},\ l, m \in \mathcal{J}_k,\ l \neq m.$$

The choice of $\zeta^{(l)}$'s is related to the integer k and to the corresponding partition; hence we write

$$Z_k := \{\zeta^{(l)}\}_1^N, \quad E_k = \{z \in \mathbb{C}; \operatorname{dist}(z, Z_k) < k^{-10}\}.$$

Step 4: We approximate the finite potential V_0.

Lemma 3.3. — *For each $\eta > 0$ one can choose k large enough so that*

$$|V_0(z) - \frac{1}{kM}\log|P_k(z)|| < \eta,\ z \notin E_k;\ V_0(z) + \eta > \frac{1}{kM}\log|P_k(z)|,\ z \in E_k.$$

Together with Lemma 3.2 this statement immediately yields the Claim since it allows us to choose two polynomials of the form

$$\big(1 - \frac{z}{w'_\infty}\big)^k P_k(z) \text{ and } \big(1 - \frac{z}{w''_\infty}\big)^k Q_k(z)$$

such that the corresponding exceptional sets are disjoint.

We now give the proofs of lemmas 3.2 and 3.3. We begin with the atomization of the external part of the potential, V_∞; *i.e.*, we prove Lemma 3.2.

3.2. Proof of Lemma 3.2

The quantity to be estimated

$$D_\infty(z) = V_\infty(z) - \frac{1}{M}\log\left|1 - \frac{z}{w_\infty}\right|,$$

admits two representations:

$$D_\infty(z) = \int_{\mathbb{C}\setminus Q_R}\left(\log\left|1 - \frac{z}{\zeta}\right| - \log\left|1 - \frac{z}{w_\infty}\right|\right) d\mu(\zeta); \tag{3.16}$$

and also

$$\begin{aligned} D_\infty(z) &= \int_{\mathbb{C}\setminus Q_R} (\log|z - \zeta| - \log|z - w_\infty|) d\mu(\zeta) + \frac{\log 10}{M} \\ &= \int_{\mathbb{C}\setminus Q_R} \log\left|1 + \frac{w_\infty - \zeta}{z - w_\infty}\right| d\mu(\zeta) + \frac{\log 10}{M}. \end{aligned} \tag{3.17}$$

The term $\frac{\log 10}{M}$ does not exceed $\eta \log 10$ and does not influence our estimates. We consider the following cases:

Case 1: $|z| \leqslant R/2$.

In this case it suffices to use the representation (3.16) and note that for $\zeta \notin Q_R$,

$$\log 1/2 \leqslant \log \left|1 - \frac{z}{\zeta}\right|, \log \left|1 - \frac{z}{w_\infty}\right| \leqslant \log 3/2.$$

Case 2: $R/2 \leqslant |z| \leqslant 3|w_\infty|$.

Note that the set E_{w_∞} is contained in this annulus. We still use the representation (3.16) and estimate each summand independently. We have

$$\begin{aligned}&\int_{\mathbb{C}\setminus Q_R} \log \left|1 - \frac{z}{\zeta}\right| d\mu(\zeta)\\ &= \left(\int_{\zeta\in\mathbb{C}\setminus Q_R, |\zeta|<4|w_\infty|} + \int_{|\zeta|>4|w_\infty|}\right) \log \left|1 - \frac{z}{\zeta}\right| d\mu(\zeta) = S_1(z) + S_2(z).\end{aligned}$$

We then have

$$\begin{aligned}S_1(z) = \int_{\zeta\in\mathbb{C}\setminus Q_R, |\zeta|<4|w_\infty|} \log|z-\zeta| d\mu(\zeta) - \int_{\zeta\in\mathbb{C}\setminus Q_R, |\zeta|<4|w_\infty|} \log|\zeta| d\mu(\zeta)\\ = S_{11}(z) + S_{12}.\end{aligned}$$

Note that S_{12} is independent of z; from (3.5), $|S_{12}| \prec \eta$. In order to estimate $S_{11}(z)$ we mention that according to (3.3) and (3.5)

$$\int_{|z-\zeta|<1} \log|z-\zeta| d\mu(\zeta) \asymp \eta;$$

this is used for (3.10). In the rest of the set $\{\zeta \in \mathbb{C} \setminus Q_R, |\zeta| < 4|w_\infty|\}$ we have

$$0 < \log|z-\zeta| < 10 \log r_\infty.$$

Using (3.8) and (3.6) we have $|S_{11}| \asymp \eta$.

When estimating $S_2(z)$ it suffices to observe that, since $|z|/|\zeta| \leqslant 3/4$, the integrand is bounded and then apply (3.6).

Case 3: $|z| \geqslant 3|w_\infty|$.

We now use (3.17). We have

$$\begin{aligned}D_\infty(z) = &\left(\int_{\zeta\notin Q_R, |\zeta|<2|w_\infty|} + \int_{2|w_\infty|<|\zeta|<4|z|} + \int_{4|z|<|\zeta|}\right)\\ &\log \left|1 + \frac{w_\infty - \zeta}{z - w_\infty}\right| d\mu(\zeta) + \frac{\log 10}{M}\\ &= T_1(z) + T_2(z) + T_3(z) + \frac{\log 10}{M}.\end{aligned}$$

We have $|T_1| \asymp 1/M$ since the integrand is bounded. When estimating T_2 we observe that the integrand is bounded from above throughout the whole region of integration thus it suffices to estimate the integral over the region $|z-\zeta|<|z|/5$, say, in which the integrand is not bounded from below. In this domain we have

$$\int_{|z-\zeta|<|z|/5}\log\left|1+\frac{w_\infty-\zeta}{z-w_\infty}\right|d\mu(\zeta)=\int_{|z-\zeta|<|z|/5}\log|z-\zeta|d\mu(\zeta)\\ -\int_{|z-\zeta|<|z|/5}\log|z-w_\infty|d\mu(\zeta).$$

The estimate of the right hand side is similar to that of $S_1(z)$. Precisely, to get an *upper bound* on $\int_{|z-\zeta|<|z|/5}\log|z-w_\infty|d\mu(\zeta)$, since $|z|\geqslant 3|w_\infty|$ and $|z-\zeta|<|z|/5$, we have $|z-w_\infty|\leqslant 4|z|/3$ and $4|z|/5\leqslant|\zeta|\leqslant 6|z|/5$. Hence

$$\int_{|z-\zeta|<|z|/5}\log|z-w_\infty|d\mu(\zeta)\leqslant\int_{|z-\zeta|<|z|/5}\log(4|z|/3)d\mu(\zeta)\\ \leqslant\int_{|z-\zeta|<|z|/5}\log(5|\zeta|/3)d\mu(\zeta).$$

From (3.5) and (3.6), $\int_{|z-\zeta|<|z|/5}\log(5|\zeta|/3)d\mu(\zeta)\asymp\eta$. For the other integral,

$$\left|\int_{|z-\zeta|<|z|/5}\log|z-\zeta|d\mu(\zeta)\right|\prec\eta$$

from (3.3) and (3.7).

The estimate of T_3 is also straightforward; we use $|z-w_\infty|>\sqrt{2}|w_\infty|$ and $|\zeta|>12|w_\infty|$ to obtain

$$0\leqslant T_3(z)=\int_{|\zeta|>4|z|}\log\left|\frac{z-\zeta}{z-w_\infty}\right|d\mu(\zeta)\leqslant\int_{|\zeta|>4|z|}\log\frac{5}{8}\frac{|\zeta|}{|w_\infty|}d\mu(\zeta),$$

and apply (3.5).

3.3. Proof of Lemma 3.3

We turn to the atomization of the potential V_0.

We split the proof into several steps.

a. Write

$$D_0(z):=V_0(z)-\frac{1}{kM}\log|P_k(z)|\\ =\sum_{l=1}^{N}\underbrace{\int_{Q^{(l)}}(\log|z-\zeta|-\log|z-\zeta^{(l)}|)d\mu^{(l)}(\zeta)}_{j_l(z)}.$$

We will estimate the contributions from j_l's for $l \in \mathcal{I}_k$ and $l \in \mathcal{J}_k$ separately. The general estimate in **b.** will be used in **c**.

b. *Estimation of $j_l(z)$: Assume $z \notin Q^{(l)}$.*

Then

$$j_l(z) = \Re \int_{Q^{(l)}} (L(\zeta) - L(\zeta^{(l)})) d\mu^{(l)}(\zeta)$$

with

$$L(\zeta) = \log(z - \zeta).$$

Using the Taylor expansion

$$\begin{aligned} L(\zeta) - L(\zeta^{(l)}) &= L'(\zeta^{(l)})(\zeta - \zeta^{(l)}) + \int_{\zeta^{(l)}}^{\zeta} L''(s)(\zeta - s) ds \\ &= L'(\zeta^{(l)})(\zeta - \zeta_0^{(l)}) - L'(\zeta^{(l)})\delta^{(l)} + \int_{\zeta^{(l)}}^{\zeta} L''(s)(\zeta - s) ds \end{aligned}$$

as well as (3.14) and (3.11) we obtain

$$j_l(z) = \Re\Big(\frac{\delta^{(l)}}{Mk} \frac{1}{z - \zeta^{(l)}} + \int_{Q^{(l)}} \int_{\zeta^{(l)}}^{\zeta} \frac{\zeta - s}{(z - s)^2} ds d\mu^{(l)}(\zeta)\Big).$$

Taking (3.15) into account we obtain

$$|j_l(z)| \leqslant \frac{1}{Mk^6} \frac{1}{\operatorname{dist}(z, Q^{(l)})} + \frac{1}{kM} \frac{d(l)^2}{\operatorname{dist}(z, Q^{(l)})^2}. \tag{3.18}$$

c. *Contribution from remote normal rectangles.*

Consider

$$l \in \mathcal{I}_k \quad \text{with} \quad \operatorname{dist}(z, Q^{(l)}) > 3k^{-1/2}. \tag{3.19}$$

It follows from the definition of normal rectangle in (3.13) and $l \in \mathcal{I}_k$ that

$$|s - z| \prec k^{1/3} \operatorname{dist}(z, Q^{(l)})$$

for all $s \in Q^{(l)}$. Combining this with (3.18), integrating with respect to Lebesgue measure σ over $Q^{(l)}$, and recalling that $\operatorname{Area}(Q^{(l)}) \asymp d(l)^2$, we obtain

$$|j_l(z)| \prec \frac{k^{1/3}}{k^5} \int_{Q^{(l)}} \frac{d\sigma(s)}{|s - z|} + \frac{k^{2/3}}{k} \int_{Q^{(l)}} \frac{d\sigma(s)}{|s - z|^2}.$$

Therefore

$$\begin{aligned} \sum_{l \in \mathcal{I}_k,\, \operatorname{dist}(z, Q^{(l)}) > 3k^{-1/2}} |j_l(z)| \prec k^{-14/3} &\int_{|s-z|>3k^{-1/2}, |s|<2R} \frac{d\sigma(s)}{|s - z|} \\ &+ \frac{k^{2/3}}{k} \int_{|s-z|>3k^{-1/2}, |s|<2R} \frac{d\sigma(s)}{|s - z|^2}. \end{aligned}$$

We have, uniformly with respect to $z \in \mathbb{C}$:

$$\int_{|s-z|>3k^{-1/2},\, |s|<R} \frac{d\sigma(z)}{|s-z|^2} < \int_{1>|s-z|>3k^{-1/2}} \frac{d\sigma(z)}{|s-z|^2} + 4\pi R^2 = O(\log k),\quad k \to \infty,$$

and similarly

$$\int_{|s-z|>3k^{-1/2},\, |s|<R} \frac{d\sigma(z)}{|s-z|} = O(1),\ k \to \infty.$$

Therefore

$$\sum_{l\in\mathcal{I}_k,\ \mathrm{dist}(z,Q^{(l)})>3k^{-1/2}} |j_l(z)| \prec k^{-1/3} \log k \to 0 \text{ as } k \to \infty.$$

Thus choosing k large enough we can make the contribution from the remote normal rectangles; *i.e.*, those satisfying (3.19), arbitrarily small.

d. *Contribution from normal rectangles which are close to z.*
Set

$$\mathcal{B}_k(z) := \{l \in \mathcal{I}_k : \mathrm{dist}(z, Q^{(l)}) < 3k^{-1/2}\}.$$

In this section we estimate

$$\sum_{l\in\mathcal{B}_k(z)} j_l(z).$$

It follows from the construction that the total number of indices in $\mathcal{B}_k(z)$ is bounded by some constant independent of z and k and also, from the definition of normal rectangle, that all the rectangles $Q^{(l)}$, $l \in \mathcal{B}_k(z)$ are contained in the disk $\{|\zeta - z| \leqslant Ck^{-1/6}\}$, C being independent of z and k. Let $\zeta^{(m)}$ be the point nearest to z among all $\{\zeta^{(l)}\}_{l\in\mathcal{B}_k(z)}$. We then have, using (3.3) and (3.4),

$$\sum_{l\in\mathcal{B}_k(z)} |j_l(z)| \prec \int_{\{|\zeta-z|\leqslant Ck^{-1/6}\}} |\log|z-\zeta||d\sigma(\zeta) + |\log|z-\zeta^{(m)}|| \int_{\{|\zeta-z|\leqslant Ck^{-1/6}\}} d\sigma(\zeta).$$

Assuming now that $z \notin E_k$ (*i.e.*, $|z - \zeta^{(m)}| > k^{-10}$) we obtain

$$\sum_{l\in\mathcal{B}_k(z)} |j_l(z)| \prec k^{-1/3} \log k.$$

Clearly if $z \in E_k$, we get a lower bound:

$$\sum_{l\in\mathcal{B}_k(z)} j_l(z) \geqslant -Ck^{-1/3} \log k.$$

e. *Contribution of non-normal rectangles.*

Define

$$D_n(z) := \sum_{l\in\mathcal{J}_k} j_l(z).$$

Let

$$E = \bigcup_{l\in\mathcal{J}_k} Q^{(l)};\ \tilde{\mu} = \sum_{l\in\mathcal{J}_k} \mu^{(l)}.$$

From (3.13), the area of each non-normal rectangle is at least $(10MA)^{-1}$ $k^{-1/3}$ and the total area they cover does not exceed $16R^2$ (since the multiplicity of the covering is at most 4). Hence we have

$$\sharp\mathcal{J}_k \prec k^{1/3}. \tag{3.20}$$

Therefore

$$\tilde{\mu}(Q_R) \prec k^{-2/3}.$$

We first assume that $|z| < 2R$. Letting ζ_m denote the point which is the nearest to z among all $\zeta^{(l)}$, $l \in \mathcal{J}_k$, we have

$$|D_n(z)| \prec \int_{Q_R} |\log|z-\zeta||d\tilde{\mu}(\zeta) + |\log|z-\zeta_m|| \int_{Q_R} d\tilde{\mu}(\zeta) = A_1(z) + A_2(z).$$

Now by (3.3) and (3.4),

$$|A_1(z)| \prec A \int_{|\zeta-z|<k^{-5}} |\log|z-\zeta||d\sigma(\zeta) + \log k \int_{|\zeta-z|>k^{-5},\zeta\in Q_R} d\tilde{\mu}(\zeta) \prec k^{-2/3}\log k.$$

Assuming $z \notin E_k$ (*i.e.*, $|z-\zeta_m| > k^{-10}$) we have

$$|A_2(z)| \prec \log k\tilde{\mu}(Q_R) \prec k^{-2/3}\log k.$$

Otherwise we get a one-sided bound. These inequalities complete the estimate of D_n in the case $|z| < 2R$.

If $|z| > 2R$ we simply have

$$D_n(z) = \sum_{l\in\mathcal{J}_k} \int_{Q^{(l)}} \left(\log\left|1-\frac{\zeta}{z}\right| - \log\left|1-\frac{\zeta^{(l)}}{z}\right|\right) d\mu^l(\zeta),$$

and since the integrands are bounded we obtain

$$|D_n(z)| \prec k^{-2/3},\ |z| > 2R.$$

This inequality completes our estimates.

4. Convergence of the Monge-Ampère measures

We return to $\mathbb{C}^2$ with variables (z,w). We use the notation $d=\partial+\bar\partial$ and $d^c=i(\bar\partial-\partial)$ where, for a C^1 function u,

$$\partial u := \frac{\partial u}{\partial z}dz+\frac{\partial u}{\partial w}dw,\ \ \bar\partial u := \frac{\partial u}{\partial \bar z}d\bar z+\frac{\partial u}{\partial \bar w}d\bar w$$

so that $dd^c=2i\partial\bar\partial$. For a C^2 function u,

$$(dd^cu)^2=16\Big[\frac{\partial^2 u}{\partial z\partial\bar z}\frac{\partial^2 u}{\partial w\partial\bar w}-\frac{\partial^2 u}{\partial z\partial\bar w}\frac{\partial^2 u}{\partial w\partial\bar z}\Big]\frac{i}{2}dz\wedge d\bar z\wedge\frac{i}{2}dw\wedge d\bar w$$

is, up to a positive constant, the determinant of the complex Hessian of u times the volume form on $\mathbb{C}^2$. Thus if u is also psh, $(dd^cu)^2$ is a positive measure which is absolutely continuous with respect to Lebesgue measure. If u is psh in an open set D and locally bounded there, or, more generally, if the unbounded locus of u is compactly contained in D, then $(dd^cu)^2$ is a positive measure in D (*cf.* [2], [9]). We discuss aspects of this last statement that we need.

A psh function u in D is an usc function u in D which is subharmonic (or identically $-\infty$) on components of $D\cap L$ for complex affine lines L. In particular, u is a locally integrable function in D such that

(4.1)
$$dd^cu=2i\Big[\frac{\partial^2 u}{\partial z\partial\bar z}dz\wedge d\bar z+\frac{\partial^2 u}{\partial w\partial\bar w}dw\wedge d\bar w+\frac{\partial^2 u}{\partial z\partial\bar w}dz\wedge d\bar w+\frac{\partial^2 u}{\partial \bar z\partial w}dw\wedge d\bar z\Big]$$

is a positive $(1,1)$ current (dual to $(1,1)$ forms); *i.e.*, a $(1,1)$ form with distribution coefficients. Thus the derivatives in (4.1) are to be interpreted in the distribution sense. Here, a $(1,1)$ current T on a domain D in $\mathbb{C}^2$ is positive if $T\wedge(i\beta\wedge\bar\beta)$ is a positive distribution for all $(1,0)$ forms $\beta=adz+bdw$ with $a,b\in C_0^\infty(D)$ (smooth functions having compact support in D). Writing the action of a current T on a test form ψ as $<T,\psi>$, this means that

$$<T,\phi(i\beta\wedge\bar\beta)>\geqslant 0 \text{ for all } \phi\in C_0^\infty(D) \text{ with } \phi\geqslant 0.$$

For a discussion of currents and the general definition of positivity, we refer the reader to Klimek [13], section 3.3.

Following [2], we now define $(dd^cv)^2$ for a psh v in D if $v\in L^\infty_{\mathrm{loc}}(D)$ using the fact that dd^cv is a positive $(1,1)$ current with measure coefficients. First note that if v were of class C^2, given $\phi \in C_0^\infty(D)$, we have

$$\int_D \phi(dd^c v)^2 = -\int_D d\phi \wedge d^c v \wedge dd^c v$$
$$= -\int_D dv \wedge d^c\phi \wedge dd^c v = \int_D v dd^c\phi \wedge dd^c v$$

since all boundary integrals vanish. The applications of Stokes' theorem are justified if v is smooth; for arbitrary psh v in D with $v \in L^\infty_{\text{loc}}(D)$, these formal calculations serve as motivation to *define* $(dd^c v)^2$ as a positive measure (precisely, a positive current of bidegree $(2,2)$ and hence a positive measure) via

$$< (dd^c v)^2, \phi >:= \int_D v dd^c\phi \wedge dd^c v.$$

This defines $(dd^c v)^2$ as a $(2,2)$ current (acting on $(0,0)$ forms; *i.e.*, test functions) since $vdd^c v$ has measure coefficients. We refer the reader to [2] or [13] (p. 113) for the verification of positivity of $(dd^c v)^2$. Also, the use of Stokes' theorem is valid and hence, for simplicity, we will write $< (dd^c v)^2, \phi >$ as $\int_D \phi(dd^c v)^2$.

Despite the fact that $L^1_{\text{loc}}(D)$ might appear to be the natural topology in which to study psh functions, work of Cegrell and Lelong (*cf.* [13] section 3.8) yields that on, *e.g.*, a ball D, for any psh function $v \in L^\infty_{\text{loc}}(D)$, there always exists a sequence of continuous psh functions $\{v_j\}$ with $v_j \to v$ in $L^1_{\text{loc}}(D)$ but $(dd^c v_j)^2 = 0$ for all j. In the locally bounded category, however, the complex Monge-Ampère operator is continuous under (a.e.) monotone limits (*cf.* Bedford-Taylor [3] or Sadullaev [16]). A simpler argument shows that local uniform convergence of a sequence of locally bounded psh functions $\{v_j\}$ to v implies weak-* convergence $(dd^c v_j)^2 \to (dd^c v)^2$: in case v_j, v are smooth, given $\phi \in C_0^\infty(D)$,

$$\int_D \phi(dd^c v_j)^2 = \int_D v_j dd^c v_j \wedge dd^c\phi$$
$$= \int_D v dd^c v_j \wedge dd^c\phi + \int_D (v_j - v) dd^c v_j \wedge dd^c\phi.$$

The first term tends to $\int_D v dd^c v \wedge dd^c\phi = \int_D \phi(dd^c v)^2$ since $dd^c v_j \to dd^c v$ as positive $(1,1)$ currents; from the uniform convergence $v_j \to v$, the family $\{dd^c v_j\}$ is locally uniformly bounded (*cf.* [16]) so that the second term goes to zero. In particular, we obtain the following result.

Proposition 4.1. — *Let $K \subset \mathbb{C}^2$ be a regular, polynomially convex compact set. Suppose $\{u_n\} \subset L^+(\mathbb{C}^2)$ converges uniformly to V_K on $\mathbb{C}^2$.*

Then

$$(dd^c u_n)^2 \to (dd^c V_K)^2$$

weak- as measures in* $\mathbb{C}^2$*. Thus with* K*,* $\{\tilde{u}_n\}$ *as in Theorem 1.1,*

$$(dd^c \tilde{u}_n)^2 \to (dd^c V_K)^2.$$

The functions $\{U_n\}$ of Theorem 1.1 are *not* locally bounded, but they are in the classical Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{C}^2)$. Following [2] as before – but altering the final application of Stokes' theorem – we note that if $v \in W^{1,2}_{\mathrm{loc}}(D)$ for some domain D, and $\phi \in C_0^\infty(D)$, we can formally write

$$\begin{aligned}\int_D \phi (dd^c v)^2 &= -\int_D d\phi \wedge d^c v \wedge dd^c v \\ &= -\int_D dv \wedge d^c \phi \wedge dd^c v = -\int_D dv \wedge d^c v \wedge dd^c \phi\end{aligned}$$

since all boundary integrals vanish. In this case, these calculations serve as motivation to define $(dd^c v)^2$ as a positive measure for a psh function v in $W^{1,2}_{\mathrm{loc}}(D)$ via

$$\int_D \phi (dd^c v)^2 := -\int_D dv \wedge d^c v \wedge dd^c \phi.$$

The functions $u(z,w) := \frac{1}{2}\log\left(|z|^2+|w|^2\right)$ and $\tilde{u}(z,w) = \max[\log|z|, \log|w|]$ are canonical examples of such functions with

$$(dd^c u)^2 = (dd^c \tilde{u})^2 = (2\pi)^2 \delta_{(0,0)} \tag{4.2}$$

([9], Corollary 6.4). More generally, if f and g are holomorphic functions near $(0,0)$, an elementary calculation (*cf.* [2], p. 15) shows that

$$\left(dd^c \frac{1}{2}\log\left(|f|^2+|g|^2|\right)\right)^2 = 0 \text{ on } \{|f|^2+|g|^2>0\}. \tag{4.3}$$

Thus if $f(0,0) = g(0,0) = 0$ and $(0,0)$ is an isolated zero of $\{f = g = 0\}$, in a neighborhood of the origin, the Monge-Ampère measures

$$\left(dd^c \max(\log|f|, \log|g|)\right)^2, \left(dd^c \frac{1}{2}\log\left(|f|^2+|g|^2|\right)\right)^2$$

are supported at $(0,0)$. Indeed, we have

$$\left(dd^c \max(\log|f|, \log|g|)\right)^2 = \left(dd^c \frac{1}{2}\log\left(|f|^2+|g|^2|\right)\right)^2 = m(2\pi)^2\delta_{(0,0)} \tag{4.4}$$

near $(0,0)$ where m is the degree of the mapping $(z,w) \to (f(z,w), g(z,w))$ at $(0,0)$. For example, taking $(z,w) \to (z,w^2)$,

$$\left(dd^c \frac{1}{2}\log\left(|z|^2+|w|^4|\right)\right)^2 = 2(2\pi)^2\delta_{(0,0)}.$$

To see how (4.2) implies (4.4), following [2], p. 16, we observe that with $u(z,w) := \frac{1}{2}\log(|z|^2+|w|^2)$, the form

$$\omega := d^c u \wedge dd^c u$$

restricted to a sphere $S_\epsilon := \{(z,w) : |z|^2+|w|^2 = \epsilon^2\}$ equals $2\epsilon^{-3}d\sigma_\epsilon$ where $d\sigma_\epsilon$ is the volume form on S_ϵ. If we write $F(z,w) := (f(z,w), g(z,w))$ and $v(z,w) := \frac{1}{2}\log(|f|^2+|g|^2|)\big)^2$, then

$$d^c v \wedge dd^c v = F^*\omega = F^*(d^c u \wedge dd^c u).$$

Moreover,

$$\int F^*(\epsilon^{-3}d\sigma_\epsilon) = 2\pi^2 m.$$

Hence

$$\int_{S_\epsilon} d^c v \wedge dd^c v = \int F^*(2\epsilon^{-3}d\sigma_\epsilon) = 4\pi^2 m.$$

From (4.3), $(dd^c v)^2$ is supported at $(0,0)$ and the second equality in (4.4) follows. The first follows from Corollary 6.4 of [9].

Thus for our functions

$$U_n(z,w) = \max\Big[\frac{1}{n}\log|P_n(z,w)-1|, \frac{1}{n}\log|Q_n(z,w)-1|\Big],$$

the Monge-Ampère measures $(dd^c U_n)^2$ are supported on the finite point sets $K_n := \{(z,w) : P_n(z,w) = Q_n(z,w) = 1\}$, and by the local uniform convergence of $U_n \to V_K$ off of $\partial K = \{\rho_K = 0\}$ (see (2.10)), given $\epsilon > 0$, for $n > n_0(\epsilon)$,

$$K_n \subset (\partial K)^\epsilon := \{(z,w) : |\rho_K(z,w)| \leqslant \epsilon\}. \tag{4.5}$$

From Proposition 3.2 of [5], in $\mathbb{C}^2$, convergence of a sequence $\{v_j\}$ of psh functions in the Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{C}^2)$ implies weak-* convergence of the Monge-Ampère measures $\{(dd^c v_j)^2\}$; we will apply this result to prove Theorem 4.3.

A simple example motivated from the one-variable example in the introduction illustrates the distinction between approximation by $\{\tilde{u}_n\}$ and by $\{U_n\}$.

Example 4.2. — Let $K = \{(z,w) : |z|, |w| \leqslant 1\}$ be the closed unit bidisk. Then

$$V_K(z,w) = \max[\log|z|, \log|w|, 0] = \max[\rho_K(z,w), 0]$$

so we can trivially take $P_n(z,w) = z^n$ and $Q_n(z,w) = w^n$ in Theorem 1.1. Then $\tilde{u}_n = V_K$ for all n while

$$U_n(z,w) = \max\Big[\frac{1}{n}\log|z^n-1|, \frac{1}{n}\log|w^n-1|\Big].$$

Thus K_n consists of ordered pairs $\zeta_{jk}^{(n)} := (\omega_n^j, \omega_n^k),\ j,k = 1,...,n$ where $\omega_n = \exp(2\pi i/n)$ is a primitive $n-$th root of unity. It is standard that

- $t \to t^n - 1$ is a Fekete polynomial of degree n for the closed unit disk in $\mathbb{C}$;
- $(dd^cV_K)^2 = d\theta_z \times d\theta_w$, the standard measure on the torus $T := \{|z| = 1\} \times \{|w| = 1\}$ (of mass $(2\pi)^2$);
- $U_n \to V_K$ locally uniformly in $\mathbb{C}^2 \setminus K$ and $U_n \to 0$ locally uniformly in $K^o = \{\rho_k < 0\}$, but $\{U_n\}$ does not converge pointwise on T; however,
- $(dd^cU_n)^2 = \frac{(2\pi)^2}{n^2} \sum_{j,k=1}^n \delta_{\zeta_{jk}^{(n)}} \to (dd^cV_K)^2$.

The assumption in Theorem 1.1 that K is circled, regular and polynomially convex implies that K is *balanced*; *i.e.*, $(z,w) \in K$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$ imply $(\lambda z, \lambda w) \in K$; moreover $K = \bar{D}$ where $D = \{(z,w) : \phi(z,w) < 1\}$ is a balanced, pseudoconvex domain determined by $\phi(z,w) := \exp \rho_K(z,w)$, the *Minkowski functional* of D.

THEOREM 4.3. — *If $K = \bar{D}$ with D strictly pseudoconvex, then*

$$(dd^cU_n)^2 \to (dd^cV_K)^2$$

weak- as measures in $\mathbb{C}^2$.*

Proof. — We first note that all of the functions U_n and V_K have the same total Monge-Ampère mass:

$$\int_{\mathbb{C}^2} (dd^cU_n)^2 = \int_{\mathbb{C}^2} (dd^cV_K)^2 = (2\pi)^2. \tag{4.6}$$

This is a standard fact about psh functions $u \in L^+(\mathbb{C}^2)$; *cf.* [18].

Using [12], Theorem 4.1.8, we can find a subsequence $\{U_{n_j}\}$ of $\{U_n\}$ with $U_{n_j} \to U$ in $L^p_{\text{loc}}(\mathbb{C}^2)$ for some psh U for all $p \in [1,\infty)$. Since $U_n \to V_K$ locally uniformly on $\mathbb{C}^2 \setminus \{\rho_K = 0\}$, estimates (2.6), (2.7) and (2.9) imply conditions i)-iii) in Theorem 2.2 of [7]; and we conclude that $U_n \to V_K$ in $L^1_{\text{loc}}(\mathbb{C}^2)$. Hence $U = V_K$ and the full sequence $\{U_n\}$ converges; *i.e.*, we have, in particular, that $U_n \to V_K$ in both $L^2_{\text{loc}}(\mathbb{C}^2)$ and $L^1_{\text{loc}}(\mathbb{C}^2)$. From this latter convergence, ∇U_n converges weakly (as distributions) to ∇V_K. Using Blocki's result, to show that $(dd^cU_n)^2 \to (dd^cV_K)^2$ weak-* as measures, it thus suffices to show that $\nabla U_n \to \nabla V_K$ in $L^2_{\text{loc}}(\mathbb{C}^2)$. Note that $U_n, V_K \in W^{1,2}_{\text{loc}}(\mathbb{C}^2)$ (*e.g.*, from [5], Theorem 1.1).

Fix a strictly pseudoconvex domain $B = \{(z,w) : \psi(z,w) < 0\}$ containing K where ψ is strictly psh. We want to show that $\nabla U_n \to \nabla V_K$ in

$L^2(B)$. It suffices to show that the norms converge; *i.e.*,

$$||\nabla U_n||^2 := \int_B |\nabla U_n|^2 \to \int_B |\nabla V_K|^2 = ||\nabla V_K||^2.$$

That is, by standard Hilbert space theory, weak convergence plus convergence *of* the norms imply convergence *in* the norm. Note that by the weak convergence of ∇U_n to ∇V_K (or simply Fatou's lemma) we have

$$\liminf_{n\to\infty} ||\nabla U_n|| \geqslant ||\nabla V_K||; \tag{4.7}$$

we want to show the limit exists and equals $||\nabla V_K||$.

Let $V_n := \max[U_n, 0]$. From the proof of the first part of Theorem 1.1 in section 2, $V_n \to V_K$ uniformly on $\mathbb{C}^2$ and hence, from Proposition 4.1, $(dd^cV_n)^2 \to (dd^cV_K)^2$ weak-* as measures on $\mathbb{C}^2$. By an observation of Cegrell, $V_n \to V_K$ in $W^{1,2}_{\text{loc}}(\mathbb{C}^2)$. Precisely, *if* $\{u_j\}$, u *are subharmonic functions in* $W^{1,2}_{\text{loc}}(\mathbb{R}^m)$ *and* $u_j \to u$ *locally uniformly, then* $u_j \to u$ *in* $W^{1,2}_{\text{loc}}(\mathbb{R}^m)$. To see this, we may assume that u_j, u are of class C^2 and we use the identity

$$\frac{1}{2}\Delta(v^2) = v\Delta v + |\nabla v|^2$$

for such functions. Take $\Omega' \subset\subset \Omega \subset\subset \mathbb{R}^m$ and $\eta \in C_0^\infty(\Omega)$ with $0 \leqslant \eta \leqslant 1$ and $\eta = 1$ on $\bar{\Omega}'$. Then

$$\begin{aligned}\int_{\Omega'} |\nabla(u_j - u)|^2 \leqslant \int_\Omega \eta|\nabla(u_j-u)|^2 &= \frac{1}{2}\int_\Omega \eta\Delta[(u_j-u)^2] \\ &\quad - \int_\Omega \eta(u_j-u)\Delta(u_j-u) \\ &\leqslant \left|\frac{1}{2}\int_\Omega (u_j-u)^2\Delta\eta\right| + \left|\int_\Omega \eta(u_j-u)\Delta(u_j-u)\right| \\ &\leqslant C\int_\Omega (u_j-u)^2 + \left|\int_\Omega \eta(u_j-u)\Delta(u_j-u)\right|\end{aligned}$$

(here C depends on η) which tends to zero as $j \to \infty$ since $u_j \to u$ uniformly on $\bar{\Omega}$ and $\Delta u_j \to \Delta u$ as measures.

We will work in an equivalent L^2-norm using a weight function. To construct this function, we are assuming that $K = \bar{D}$ with $D = \{(z,w) : \rho_K(z,w) < 0\}$ strictly pseudoconvex; hence $\exp \rho_K$ is strictly psh and we work on the sub-level sets $B = B_R := \{(z,w) : \exp \rho_K(z,w) < e^R\}$ for $R > 0$. For each set B we define

$$\psi(z,w) := \exp \rho_K(z,w) - e^R.$$

The (semi-) norm in our new L^2-space is

$$||\nabla u||^2_\psi := \int_B dd^c\psi \wedge d^cu \wedge du.$$

If $\psi(z) = A_1|z|^2 + A_2$ then $||\nabla u||_\psi^2 = 4A_1||\nabla u||^2$; in general, due to strict plurisubharmonicity and smoothness of ψ, we have

$$c_1||\nabla u|| \leqslant ||\nabla u||_\psi \leqslant c_2||\nabla u||$$

for constants c_1, c_2 depending only on ψ. The same argument as before gives a version of (4.7) in our new norm:

$$\liminf_{n\to\infty} ||\nabla U_n||_\psi \geqslant ||\nabla V_K||_\psi. \tag{4.8}$$

Now via integration by parts, we get

$$\int_B dd^c\psi \wedge dU_n \wedge d^cU_n = \int_B (-\psi)(dd^cU_n)^2$$

modulo boundary integrals $\pm \int_{\partial B} dU_n \wedge d^cU_n \wedge d^c\psi \pm \int_{\partial B} \psi d^cU_n \wedge dd^cU_n$. Since $\psi = 0$ on ∂B, this last term vanishes. Similarly,

$$\int_B dd^c\psi \wedge dV_K \wedge d^cV_K = \int_B (-\psi)(dd^cV_K)^2$$

modulo boundary integrals $\pm \int_{\partial B} dV_K \wedge d^cV_K \wedge d^c\psi \pm \int_{\partial B} \psi d^cV_K \wedge dd^cV_K$; again, this latter term vanishes. Thus we must show that

$$\int_{\partial B} dU_n \wedge d^cU_n \wedge d^c\psi \to \int_{\partial B} dV_K \wedge d^cV_K \wedge d^c\psi \tag{4.9}$$

and

$$\int_B (-\psi)(dd^cU_n)^2 \to \int_B (-\psi)(dd^cV_K)^2. \tag{4.10}$$

Using (4.5), given $\epsilon > 0$, for $n > n_0(\epsilon)$ we have $(dd^cU_n)^2$ is supported in $(\partial K)^\epsilon$, and

$$1 - 2\epsilon - e^R \leqslant \psi(z,w) \leqslant 1 + 2\epsilon - e^R$$

on this set so that

$$(2\pi)^2(1 - 2\epsilon - e^R) \leqslant \int_B \psi(dd^cU_n)^2 \leqslant (2\pi)^2(1 + 2\epsilon - e^R).$$

Since $(dd^cV_K)^2$ is supported on ∂K and, from (4.6), the total Monge-Ampère mass of V_K is $(2\pi)^2$, we have $\int_B(-\psi)(dd^cV_K)^2 = (2\pi)^2(e^R - 1)$ so that

$$\left| \int_B (-\psi)(dd^cU_n)^2 - \int_B (-\psi)(dd^cV_K)^2 \right| \leqslant (2\pi)^2 2\epsilon$$

for $n > n_0(\epsilon)$. This gives (4.10).

To prove (4.9), we observe that for any fixed $R > 0$, for n sufficiently large, $U_n = V_n$ on $\partial B = \partial B_R$. Thus we may replace U_n by V_n in (4.9). Now

$(dd^cV_n)^2 \to (dd^cV_K)^2$ weak-* and the support of $(dd^cV_n)^2$ is compactly contained in B for n large so

$$\int_B (-\psi)(dd^cV_n)^2 \to \int_B (-\psi)(dd^cV_K)^2.$$

Since $V_n \to V_K$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{C}^2)$,

$$\int_B dd^c\psi \wedge dV_n \wedge d^cV_n \to \int_B dd^c\psi \wedge dV_K \wedge d^cV_K.$$

Via the previously described integration by parts, (4.9) follows. □

Remark 4.4. — If K is not strictly pseudoconvex, if we can find $\tilde{K} = \bar{\tilde{D}}$ balanced with $\tilde{D}$ strictly pseudoconvex and with $\mathrm{supp}(dd^cV_K)^2 \subset \tilde{K}$, the same argument works using the function $\tilde{\psi}(z,w) = \exp \rho_{\tilde{K}}(z,w) - e^R$. For example, for the bidisk K, $\mathrm{supp}(dd^cV_K)^2$ is the unit torus which is contained in the ball $\tilde{K} = \{(z,w) : |z|^2 + |w|^2 \leqslant 2\}$.

Remark 4.5. — Let Ω be a bounded hyperconvex domain in $\mathbf{C}^N$; *i.e.*, there exists a negative psh function ψ in Ω with $\{z \in \Omega : \psi(z) \leqslant -c\} \subset\subset \Omega$ for all $c > 0$. A bounded psh function v belongs to the class $\mathcal{E}_0(\Omega)$ if $\lim_{z'\to z} v(z') = 0$ for all $z \in \partial\Omega$ and $\int_\Omega (dd^cv)^N < +\infty$. Finally, a psh function v in Ω belongs to the class $\mathcal{F}(\Omega)$ if there exists a sequence of functions $v_j \in \mathcal{E}_0(\Omega)$ with $\sup_j \int_\Omega (dd^cv_j)^N < +\infty$ which decreases to v on Ω. A recent result of Cegrell [8] states the following: for a sequence $\{u_n\} \subset \mathcal{F}(\Omega)$, if $u_n \to u \in \mathcal{F}(\Omega)$ in $L^1_{\mathrm{loc}}(\Omega)$ and if there exists a strictly psh function $v \in \mathcal{E}_0(\Omega)$ such that $\lim_{n\to\infty} \int_\Omega v(dd^cu_n)^N = \int_\Omega v(dd^cu)^N$, then $(dd^cu_n)^N$ converges weak-* to $(dd^cu)^N$. The sequence $\{u_n\}$ must lie in $\mathcal{F}(\Omega)$ in order that certain integration by parts formulae are valid. Note that functions in $\mathcal{E}_0(\Omega)$ have zero boundary values; moreover, if $u_n \in \mathcal{F}(\Omega)$ then $\limsup_{z'\to z} u_n(z') = 0$ for all $z \in \partial\Omega$ (*cf.* [1]). It might appear that (4.10) would suffice (without (4.9)) to prove Theorem 4.3. However, the functions U_n do not lie in the class $\mathcal{F}(B)$ since $\limsup_{z'\to z} U_n(z') \not\equiv 0$ for all $z \in \partial B$.

As mentioned in the introduction, from Bishop's construction, one obtains the following result.

Proposition 4.6. — *Let $K \subset \mathbb{C}^N$ be a regular, polynomially convex compact set. Then there exists a sequence of special polynomial polyhedra $\{\kappa_n\}$ where κ_n is the closure of a union of a finite number of connected components of*

$$\mathcal{K}_n := \{(z_1, ..., z_N) : |P_{n,1}(z_1, ..., z_N)| < 1,\ |P_{n,N}(z_1, ..., z_N)| < 1\}$$

with $\{P_{n,1},...,P_{n,N}\}$ polynomials having degree n, such that the extremal functions $\{V_{\kappa_n}\}$ converge uniformly to V_K and $(dd^cV_{\kappa_n})^N \to (dd^cV_K)^N$ weak-.*

However, it is not known how one can construct full component sets of the form $\mathcal{K}_n$ approximating K as we have in Theorem 1.1 using (1.5) nor how to construct functions u_n of the form

$$u_n(z_1,...,z_N) := \max\left[\frac{1}{n}\log|\tilde{P}_{n,1}(z_1,...,z_N)|,...,\frac{1}{n}\log|\tilde{P}_{n,N}(z_1,...,z_N)|\right]$$

for some polynomials $\tilde{P}_{n,1},...,\tilde{P}_{n,N}$ so that, with

$$K_n := \{(z_1,...,z_N) : u_n(z_1,...,z_N) = -\infty\}$$

we have $(dd^cu_n)^N$ is supported in K_n as in (1.6) and

- $u_n \to V_K$ locally uniformly in $\mathbb{C}^N \setminus K$;
- $u_n \to V_K$ in $L^1_{\text{loc}}(\mathbb{C}^N)$; and
- $(dd^cu_n)^N \to (dd^cV_K)^N$ weak-*.

As a step in this direction, we can achieve a partial result in $\mathbb{C}^2$.

Proposition 4.7. — *Let $K \subset \mathbb{C}^2$ be a regular, polynomially convex compact set. Then there exists a sequence of pairs of polynomials $\{\tilde{P}_n, \tilde{Q}_n\}$ with $\deg \tilde{P}_n = \deg \tilde{Q}_n = n$ such that the functions*

$$v_n(z,w) := \max\left[\frac{1}{n}\log|\tilde{P}_n(z,w)|, \frac{1}{n}\log|\tilde{Q}_n(z,w)|\right]$$

converge to V_K in $L^1_{\text{loc}}(\mathbb{C}^2 \setminus K)$ and $\rho_{v_n} \to \rho_K$ uniformly on $\mathbb{C}^2$. In particular, if K has empty interior (e.g., if $K \subset \mathbb{R}^2$), then $v_n \to V_K$ in $L^1_{\text{loc}}(\mathbb{C}^2)$.

Proof. — Form the Robin function ρ_K of V_K (see section 2) and construct the regular, polynomially convex, circled set

$$K_\rho := \{(z,w) \in \mathbb{C}^2 : \rho_K(z,w) \leqslant 0\}.$$

Apply Theorem 1.1 to obtain a sequence of pairs $\{P_n, Q_n\}$ of homogeneous polynomials such that if $\epsilon > 0$ is given, then

$$\rho_K(z,w) - \epsilon \leqslant \max\left[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|\right] \leqslant \rho_K(z,w)$$

for all $(z,w) \in \mathbb{C}^2$ if $n > n(\epsilon)$. Construct

$$\tilde{P}_n = \text{Tch}_K P_n,\ \tilde{Q}_n = \text{Tch}_K Q_n$$

where, for a homogeneous polynomial H_n of degree n,

$$\text{Tch}_K H_n := H_n + H_{n-1}$$

with $\deg H_{n-1} \leqslant n-1$ and $||\mathrm{Tch}_K H_n||_K \leqslant ||H_n + R_{n-1}||_K$ for all polynomials R_{n-1} of degree at most $n-1$. By Theorem 3.2 of [6],

$$\limsup_{n\to\infty} ||\tilde{P}_n||_K^{1/n} \leqslant 1, \ \limsup_{n\to\infty} ||\tilde{Q}_n||_K^{1/n} \leqslant 1.$$

Thus, given $\epsilon > 0$, for $n > n(\epsilon)$ we have

$$\max[||\tilde{P}_n||_K, ||\tilde{Q}_n||_K] \leqslant (1+\epsilon)^n$$

so that the the functions

$$v_n(z,w) := \max\left[\frac{1}{n}\log|\tilde{P}_n(z,w)|, \frac{1}{n}\log|\tilde{Q}_n(z,w)|\right]$$

satisfy

- $v_n \in L(\mathbb{C}^2)$;
- given $\epsilon > 0$, there exist $N = N(\epsilon)$ with $v_n \leqslant \epsilon$ on K for $n > N(\epsilon)$;
- $\rho_{v_n} \to \rho_K$ uniformly on $\mathbb{C}^2$.

This last item follows since

$$\rho_{v_n} = \max\left[\frac{1}{n}\log|P_n(z,w)|, \frac{1}{n}\log|Q_n(z,w)|\right].$$

By Theorem 2.2 of [7], we conclude that $v_n \to V_K$ in $L^1_{\mathrm{loc}}(\mathbb{C}^2 \setminus K)$. $\square$

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