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On entire functions represented by Dirichlet series. IV


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ON ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES (IV)

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1. Let

\[ f(s) = \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n s}, \quad s = \sigma + it \]

represent an entire function, where

\begin{align*}
(1.1) & \quad \lim_{n \to \infty} n/\lambda_n = D < \infty; \\
(1.2) & \quad \lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = h > 0,
\end{align*}

such that ([10], p. 201) \( hD \leq 1 \), and

\[ 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty \]

as \( n \to \infty \). Now \( f(s) \) represents an entire function and so its abscissa of absolute convergence must be infinite, that is

\[ \lim_{n \to \infty} \log |\alpha_n|/\lambda_n = -\infty. \]

Let us define \( \chi_n \) as follows:

\[ \chi_n = \frac{\log |\alpha_{n-1}|/\alpha_n|}{\lambda_n - \lambda_{n-1}}. \]

Then \( \chi_n \) is a non-decreasing function of \( n \) (see [1]) and \( \to \infty \) as \( n \to \infty \). The fact is similar to what G. Valiron describes about rectified ratio in his book ([12], p. 32). So we have:

\[ 0 \leq \chi_1 \leq \chi_2 \leq \cdots \leq \chi_n \leq \cdots; \quad \chi_n \to \infty, \quad n \to \infty. \]

Let \( \mu(s) \) be the maximum term in the representation of \( \Sigma |\alpha_n| e^{\sigma \lambda_n} \) and call it as the maximum term of \( f(S) \). Let \( \lambda_{\mu(s)} \).
be that value of \( \lambda_n \) which makes \(|a_n|e^{\lambda_n}\) the maximum term and call \( \lambda_{\nu(\sigma)} \) as the rank of \( \mu(\sigma) \). Let us similarly correspond \( \nu^{(m)}(\sigma) \) and \( \lambda_{\nu^{(m)}(\sigma)} \) to \( f^{(m)}(S) \), the \( m \)-th derivative of \( f(S) \) as we have done about \( \mu(\sigma) \) and \( \lambda_{\nu(\sigma)} \) connecting them with \( f(S) \), where \( \nu^{(0)}(\sigma) \equiv \mu(\sigma), \lambda_{\nu^{(0)}(\sigma)} \equiv \lambda_{\nu(\sigma)} \). It is well-known that ([13]; [4], pp. 1-2)

\[
\log \mu(\sigma) = \int_{1}^{\nu(\sigma)} \lambda_{\nu(\sigma)} \, dx.
\]

We define the order \((R)p\) and lower order \((R)\lambda\) of \( f(s) \) as follows:

\[
\lim_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \frac{p}{\lambda};
\]

where \( M(\sigma) = \text{l.u.b.} |f(s)| \).

According to Mandelbrojt ([10], p. 216) we call \( p \) as the Ritt order (to be written as order \((R)p\)) of \( f(s) \). We, therefore, naturally call the lower limit in \( \log \log M(\sigma)/\sigma \) as \( \sigma \to \infty \) to be the lower order \((R)\lambda\). However, I shall drop the word \((R)\) in the sequel. The results starting after Theorem C and onwards are expected to be new; Theorems A and B have already appeared but the secretary wishes them to incorporate here. This paper is to be considered as a sequel to my previous papers [6; 7; 8 et 9]. For the sake of completeness I start with the following result ([4], Th. 1).

2. Theorem A. — For an entire function \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \) where \( \{\lambda_n\} \) satisfies (1.2), then

\[
\mu(\sigma) \leq M(\sigma) < \mu(\sigma) \left[ \left( 1 + \frac{1}{L\sigma} \right) \lambda_{\nu(\sigma) + \lambda_{\nu(\sigma)}} + 1 \right],
\]

where \( L = h - \varepsilon, \varepsilon \) being an arbitrarily taken small positive number.

We now proceed to prove it. The left-hand inequality in (2.1) is obvious in view of Ritt’s inequality:

\[ |a_n|e^{\lambda_n} \leq M(\sigma). \]

Let

\[ W(\sigma) = \sum_{n=1}^{\infty} e^{-G_n + \lambda_n}, \quad G_n = - \log |a_n|. \]
Suppose $p$ is a positive integer $> \lambda_{\chi(\sigma)}$, such that $\chi_p > \sigma$. Let $q \geq p$. Now

$$e^{-G_p e^{2\lambda_q}} \leq e^{-G_{p-1} e^{2\lambda_{p-1}}} \exp\{ (\sigma - \chi_p) (\lambda_q - \lambda_{p-1}) \} \leq \mu(\sigma) \exp\{ (\sigma - \chi_p) (\lambda_q - \lambda_{p-1}) \}.$$ 

Hence

$$W(\sigma) \leq \mu(\sigma) \left[ p + \sum_{q=p}^{\infty} \left( \frac{e^\sigma}{e^{\chi_p}} \right)^{\lambda_q - \lambda_{p-1}} \right].$$

Hence in view of (1.2), if we write $x = \exp(\chi_p - \sigma)$, then $\chi > 1$ and so

$$\sum_{q=p}^{\infty} \left( \frac{e^\sigma}{e^{\chi_p}} \right)^{\lambda_q - \lambda_{p-1}} < \chi^{-1} + \chi^{-2} + \cdots = \frac{1}{\chi - 1}.$$

Therefore

$$W(\sigma) < \mu(\sigma) \left[ p + \frac{e^{\chi \sigma}}{e^{\chi_p} - e^{\chi \sigma}} \right].$$

Let

$$p = \lambda_{\chi(\sigma + e^{\chi \sigma})} + 1,$$

we find that

$$e^{\chi_p - e^{\chi \sigma}} > e^{\chi \sigma} \{ e^{\chi \sigma} - e^{\chi(\sigma + e^{\chi \sigma})} - 1 \}$$

and therefore the right-hand part in (2.1) follows.

Making use of Theorem A, we prove ([4], Th. 2, p. 5):

**Theorem B.** — Let $f(s)$ be an entire function of order $\rho$ and lower order $\lambda$; $\lambda_n$ satisfies (1.2) in the expansion of $f(s)$. Then

$$(2.2) \quad \lim_{\sigma \to \infty} \frac{\log \lambda_{\chi(\sigma)}}{\sigma} = \frac{\rho}{\lambda}; \quad (0 \leq \rho \leq \infty; \ 0 \leq \lambda \leq \infty).$$

As regards the proof, the upper limit is similar to a result proved by Valiron ([12], p. 33), care is only to be taken that during the course of proof, we use the fact that $\log \mu(\sigma)$ is a convex function of $\sigma$ [2]. From the previous theorem and the fact that if $p$ is finite, we notice that

$$\log M(\sigma) \sim \log \mu(\sigma), \quad \sigma \to \infty.$$ 

Let

$$\lim_{\sigma \to \infty} \frac{\log \lambda_{\chi(\sigma)}}{\sigma} = \rho < \infty,$$
so that from (1.4), for \( \sigma \geq \sigma_0 \)

\[
\log \mu(\sigma) < K + \frac{e^{(\lambda+\epsilon)\sigma}}{\rho + \epsilon}
\]

Therefore

\[
\lim_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} \leq \rho.
\]

Let us suppose now

\[
\lim_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \rho_1 (\leq \rho).
\]

Therefore from (1.4) and the relation \( \mu(\sigma) \leq M(\sigma) \), we find that

\[
2\lambda_{y(\sigma)} \leq \int_{\alpha}^{\alpha+\epsilon} \lambda_{y(\sigma)} \, dx < (1 + \epsilon)\sigma(\alpha+2)(\sigma+\epsilon),
\]

and so we find that

\[
\lim_{\sigma \to \infty} \frac{\log \lambda_{y(\sigma)}}{\sigma} \leq \rho_1.
\]

Therefore \( \rho = \rho_1 \). Therefore the ratios \( \log \log M(\sigma)/\sigma \) and \( \log \lambda_{y(\sigma)}/\sigma \) have the same upper limit. To prove that

\[
\lim_{\sigma \to \infty} \frac{\log \lambda_{y(\sigma)}}{\sigma} = \lambda,
\]

we proceed in some other way. Let

\[
\lim_{\sigma \to \infty} \frac{\log \lambda_{y(\sigma)}}{\sigma} = \alpha.
\]

With the help of (1.4) and \( \mu(\sigma) \leq M(\sigma) \), one easily finds that for any constant \( C > 0 \).

\[
C\lambda_{y(\sigma)} \leq \log \mu(\sigma + c) \leq \log M(\sigma + c) < e^{(\lambda+\epsilon)(\sigma+c)},
\]

for an arbitrarily large value of \( \sigma \). This implies \( \alpha \leq \lambda \). If \( \lambda = 0 \), then \( \alpha = 0 \) and there is nothing to prove. Let \( 0 \leq \alpha < \infty \). Choose \( \beta \) and \( \gamma \), such that \( \alpha < \beta \) and \( \alpha/\beta < \gamma < 1 \). Hence

\[(2.3) \quad \lambda_{y(\sigma)} < e^{\beta\sigma}, \quad (\gamma\sigma_n \leq \sigma \leq \sigma_n)\]
where \( \{ \sigma_n \} \) is a sequence of \( \sigma \), such that \( \sigma_n \to \infty \) as \( n \to \infty \).

We shall show that

\[
\frac{\log M(\sigma)}{\log \mu(\sigma)} \to 1,
\]
as \( \sigma \to \infty \) through the sequence for which (2.3) holds (it is not assumed that \( \rho \) is finite; if \( \rho \) is finite we cannot claim necessarily that \( \log M(\sigma) \sim \log \mu(\sigma) \)).

Let \( \delta \) and \( \varepsilon' \) be two positive numbers such that

\[
\gamma < \delta < 1; \quad \gamma/\delta < \varepsilon' < 1.
\]

Put \( \delta \sigma_n = \xi_n \). Then for \( n \geq n_0 \), \( \gamma \sigma_n < \varepsilon' \xi_n < \xi_n < \sigma_n - \frac{1}{2} \).

Further, let \( \mu(0) = 1 \), which we may without loss of generality. Then from (1.4)

\[
\log \mu(\xi_n) = \log \mu(\xi_n \varepsilon') + \int_{\xi_n \varepsilon'}^{\xi_n} \lambda_{\nu(x)} \, dx.
\]

But \( \log \mu(\varepsilon' \xi_n) < \varepsilon' \xi_n \lambda_{\nu(\varepsilon' \xi_n)} \), so

\[
\log \mu(\xi_n) > \log \mu(\varepsilon' \xi_n) + (1 - \varepsilon') \xi_n \lambda_{\nu(\varepsilon' \xi_n)} > \frac{1}{\varepsilon'} \log \mu(\varepsilon' \xi_n).
\]

Hence

\[
(1 - \varepsilon') \log \mu(\xi_n) < \int_{\varepsilon' \xi_n}^{\xi_n} \lambda_{\nu(x)} \, dx
\]

(2.4)

\[
< \frac{1}{\beta} \left[ e^{\beta \xi_n} - e^{\beta \varepsilon' \xi_n} \right],
\]

for all \( n \geq n_0 \). But from Theorem A

\[
\log M(\xi_n) < \log \mu(\xi_n) + \log \lambda_{\nu(\xi_n + \xi_n \lambda_{\nu(x_n)})} + 0(1)
\]

\[
< \log \mu(\xi_n) + \log \lambda_{\nu(\xi_n)} + 0(1)
\]

\[
< \log \mu(\xi_n) + 2 \beta \xi_n + 0(1).
\]

Hence we get for all \( n \geq n_0 \).

\[
\log \log M(\xi_n) < (1 + 0(1)) \log \log \mu(\xi_n)
\]

\[
< (1 + 0(1)) \beta \xi_n,
\]

from (2.4). Consequently \( \lambda \leq \beta \) and as \( (\beta - \alpha) \) can be made arbitrarily small we see that \( \lambda \leq \alpha \); and this, when combined with the already established inequality: \( \lambda \geq \alpha \), gives the required result.

Next, I give the following result ([5], p. 45).
THEOREM C. — Let
\[ f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \]
be an entire function, where \( \{\lambda_n\} \) satisfies (1.2), of order \( \rho \) and lower order \( \lambda \) \((0 < \rho \leq \infty; 0 \leq \lambda < \infty)\). Then
\[ \lim_{\sigma \to +\infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \leq 1 - \frac{\lambda}{\rho}. \]

Proof. — We have
\[ \log \mu(\sigma) = \sum_{\lambda_n \leq \sigma} (\lambda_n - \lambda_{n-1})(\sigma - \lambda_n) \]
\[ = \sigma \lambda_{\nu(\sigma)} - \sum_{\lambda_n \leq \sigma} (\lambda_n - \lambda_{n-1}) \lambda_n. \]
But for all \( n \geq n_0 \) (from Th. B)
\[ \log \lambda_n < (\rho + \epsilon) \lambda_n. \]
So we find
\[ \sum_{\lambda_n \leq \sigma} (\lambda_n - \lambda_{n-1}) \lambda_n > \sum_{\lambda_n \leq \sigma, n \geq n_0} (\lambda_n - \lambda_{n-1}) \frac{\log \lambda_n}{\rho + \epsilon}. \]
Let \( N \) be the largest integer such that \( \chi \leq \sigma \), then
\[ \sum_{\lambda_n \leq \sigma} (\lambda_n - \lambda_{n-1}) \lambda_n > \frac{1}{\rho + \epsilon} \{ \lambda_N \log \lambda_N + 0(\lambda_n) \} \]
\[ = \frac{1}{\rho + \epsilon} \{ \lambda_{\nu(\sigma)} \log \nu(\sigma) \} + 0(\nu(\sigma)). \]
So that for \( \sigma \geq \sigma_0 \)
\[ \log \mu(\sigma) < \sigma \lambda_{\nu(\sigma)} \left\{ 1 - \frac{\lambda - \epsilon}{\rho + \epsilon} + 0(1) \right\} \]
and the result follows.

3. Below I construct an example to exhibit that the result of Th. C is best possible in view of the fact that if \( \lambda < \infty \), \( \rho = \infty \), then
\[ (3.1) \quad \lim_{\sigma \to +\infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} = 1. \]

Example 1. — Let
\[ f(s) = \sum_{n=1}^{\infty} \left( \frac{e^s}{\Gamma(\lambda_n)} \right)^{\lambda_n}, \]
where \( \lambda_{n+1} = \lambda_n \); \( N \) is a positive integer, such that \( I(\lambda_N) \geq e \) and that
\[
\log I(x) = \int_{x_n}^{x} \frac{dt}{\theta(t) \log t} \to \infty,
\]
as \( x \to \infty \), where further.

(i) \( \theta(x) \) is a positive, continuous and non-decreasing function for \( x \geq x_0 \) and \( \to \infty \) with \( x \), and has a derivative;

(ii) \( \frac{x\theta'(x)}{\theta(x)} \leq \frac{1}{\log x \log \log x \log \log \log x} \), \( x \geq x_0 \).

**Demonstration of the aim.** — According to a result ([10], p. 217, eq. (94)) we see that the order \( \rho \) of \( f(s) \) is
\[
\rho = \lim_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\lambda_n \log I(\lambda_n)} \geq \lim_{n \to \infty} \frac{\log \lambda_n}{A \log \log \lambda_n},
\]
from (ii) and the integral representation of \( I(x) \), \( A \) being a finite number. Therefore the order \( \rho \) of \( f(s) \) is infinite. Let
\[
\chi_n = \log \left\{ \frac{I(\lambda_n)}{I(\lambda_n - 1)} \right\}^{\lambda_n / \lambda_{n-1}} / \{/\right\} / (\lambda_n - \lambda_{n-1}),
\]
then it is easily found that \( \chi_{n+1} > \chi_n (n > n_0) \) and that \( \chi_n \to \infty \), as \( n \to \infty \). Hence for \( \chi_n < \sigma < \chi_{n+1} \),
\[
\log \mu(\sigma) = \{ \sigma - \log I(\lambda_n) \} \lambda_n, \quad \lambda_n = \lambda_{\psi(\sigma)}.
\]
Therefore
\[
\frac{\log \mu(\chi_{n+1})}{\chi_{n+1} \lambda_{\psi(\chi_{n+1})}} = 1 - \frac{(1 + 0(1)) \log I(\lambda_n)}{\log I(\lambda_{n+1}) + 0(\log I(\lambda_n))}.
\]
Further
\[
\log I(\lambda_{n+1}) - \log I(\lambda_n) > (1 + 0(1)) \frac{\lambda_{n+1}}{\theta(\lambda_{n+1})},
\]
and as \( \log I(\lambda_n) < A\theta_{\lambda_n}, A = a \) constant, we find that
\[
\frac{\log I(\lambda_{n+1})}{\log I(\lambda_n)} \to \infty, \quad (n \to \infty)
\]
and so
\[ \log \frac{\mu(\chi_{n+1})}{\chi_{n+1}^{\lambda_{\nu}(\chi_{n+1})}} \to 1, \quad (n \to \infty) \]
and hence
\[ \lim_{n \to \infty} \frac{\log \mu(\sigma)}{\sigma^{\lambda_{\nu}(\sigma)}} \geq 1. \tag{3.2} \]
Further
\[ \log \mu(\chi_{n+1}) = \frac{\lambda_n \lambda_{n+1} \log \{I(\lambda_{n+1})/I(\lambda_n)\}}{\lambda_{n+1} - \lambda_n} \]
\[ = (1 + o(1))\lambda_n \log I(\lambda_{n+1}), \]
and therefore
\[ \log \log \mu(\chi_{n+1}) \sim \log \log I(\lambda_{n+1}) + \log \lambda_n, \]
and as \( \chi_{n+1} \sim \log I(\lambda_{n+1}) \), it follows that
\[ \lambda = \lim_{\sigma \to \infty} \frac{\log \log \mu(\sigma)}{\sigma} = 0. \]
Hence from Theorem C
\[ \lim_{\sigma \to \infty} \frac{\log \mu(\sigma)}{\sigma^{\lambda_{\nu}(\sigma)}} \leq 1. \tag{3.3} \]
Inequalities (3.2) and (3.3) provide the demonstration of our aim.

**Example 2.** — Let us consider the function defined by (see Theorem 6 [3], p. 22 where I put \( \beta = 1 \))
\[ f(s) = \sum_{n=1}^{\infty} \left( \frac{e^{s \lambda_n}}{\lambda_n} \right) \lambda_n, \quad \lambda_{n+1} = \alpha^{\lambda_n}; \quad \alpha \geq \epsilon; \quad \lambda_1 = \alpha. \]
The function \( f(s) \) is certainly an entire function on account of (1.3)''. The order \( \rho \) of \( f(s) \) is in this case
\[ \rho = \lim_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\lambda_n \log \lambda_n} = 1. \]
Also
\[ \mu(\sigma) = \left\{ e^{\sigma / \lambda_n} \right\} \lambda_n, \quad \lambda_n = \lambda_{\nu}(\sigma), \]
for \( \chi_n \leq \sigma < \chi_{n+1} \), where
\[ \chi_n = \frac{\lambda_n \log \lambda_n - \lambda_{n-1} \log \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}. \]
Then
\[ \log \mu(\chi_n) = \lambda_n (\lambda_n - \log \lambda_n) \]
\[ = \frac{\lambda_n \lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \log (\lambda_n / \lambda_{n-1}) \]
\[ = (1 + O(1)) \log \lambda_n; \]
\[ \log \log \mu(\chi_n) = (1 + O(1)) + \log \lambda_{n-1} + \log \log \lambda_n. \]

Also \( \chi_n \to \infty \) as \( n \to \infty \), we see that
\[ (3.5) \quad \frac{\log \log \mu(\chi_n)}{\chi_n} = O(1) + \frac{1}{\chi_n} (\log \lambda_{n-1} + \log \log \lambda_n). \]

Now
\[ \frac{\log \lambda_{n-1}}{\chi_n} = \frac{\log \lambda_{n-1} (\lambda_n - \lambda_{n-1})}{\chi_n} \]
\[ = \frac{\lambda_n \log \lambda_n - \lambda_{n-1} \log \lambda_{n-1}}{\lambda_n \log \lambda_{n-1} + O(\lambda_n)} \]
\[ = \frac{\lambda_n \lambda_{n-1} \log \alpha + O(\lambda_n)}{\lambda_n \lambda_{n-1} \log \alpha + O(\lambda_n)} \]
\[ = (1 + O(1)) \frac{\log \lambda_{n-1}}{\lambda_{n-1} \log \lambda_n} \to 0 \quad (n \to \infty). \]

Also \( \log \log \lambda_n = (1 + O(1)) \log \lambda_{n-1} \) and so the right-hand term in (3.5) \( \to 0 \) as \( n \to \infty \) in view of (3.6). Therefore the lower order \( \lambda \) of \( f(s) \) is zero on account of (3.5). Hence from Theorem C
\[ (3.7) \quad \lim_{\sigma \to \pm \infty} \frac{\log \mu(\sigma)}{\sigma \lambda(\sigma)} \leq 1. \]

Also
\[ \frac{\log \mu(\chi_{n+1})}{\chi_{n+1} \lambda_{\chi(\chi_{n+1})}} = 1 - \frac{\log \lambda_n}{\chi_{n+1} \lambda_{\chi(\chi_{n+1})}} \]
\[ = 1 - \frac{(\lambda_{n+1} - \lambda_n) \log \lambda_n}{\lambda_{n+1} \log \lambda_{n+1} - \lambda_n \log \lambda_n} \to 1 \quad (n \to \infty), \]
for the above solution see the technique used in getting (3.6).

Hence
\[ (3.8) \quad \lim_{\sigma \to \pm \infty} \frac{\log \mu(\sigma)}{\sigma \lambda(\sigma)} \geq 1. \]

Therefore from (3.7) and (3.8) one gets
\[ \lim_{\sigma \to \pm \infty} \frac{\log \mu(\sigma)}{\sigma \lambda(\sigma)} = 1, \]
giving thereby again a best possible nature of Theorem C in case $\lambda = 0$ and $\rho < \infty$.

4. Results involving derivatives of $f(s)$:

I have already spoken in the article 1 about $\mu_{(m)}(\sigma)$ and $\lambda_{s(m)}(\sigma)$. I first prove:

**Theorem D.** — For all $\sigma \geq \sigma_0$ ($\sigma_0$ is a fixed large number) one should have:

$$
\mu_{(m)}(\sigma) > \mu(\sigma) \left[ \frac{\log \mu(\sigma)}{\sigma} \right]^m,
$$

$m$ is an integer $\geq 0$. This result I stated in a previous paper ([6], p. 235) without proof.

**Proof.** — We have:

$$
\lambda_{\nu(m)}(\sigma) \leqslant \frac{\mu_{(m+1)}(\sigma)}{\mu_{(m)}(\sigma)} \leqslant \lambda_{\nu(m+1)}(\sigma), \quad m = 0, 1, \ldots
$$

When $m = 0$ in (4.1), it reduces to a result which I have proved in ([3], p., Theorem 2) as follows

$$
\begin{align*}
\mu_{(0)}(\sigma) &= |a_{\nu}(\gamma)| \lambda_{\nu}(\gamma) \exp (\sigma \lambda_{\nu}(\gamma)) \leqslant \lambda_{\nu}(\gamma) \mu(\sigma); \\
\mu_{(0)}(\sigma) &= |a_{\nu}(\gamma)| \lambda_{\nu}(\gamma) \exp (\sigma \lambda_{\nu}(\gamma)) \geqslant |a_{\nu}(\gamma)| \lambda_{\nu}(\gamma) \exp (\sigma \lambda_{\nu}(\gamma)) \\
&= \lambda_{\nu}(\gamma) \mu(\sigma).
\end{align*}
$$

The case $m \geq 1$ can also be treated by simple definitions, for let

$$
f^{(m)}(S) = \sum A_n e^{\lambda_n}, \quad \lambda_{\nu(m)}(\gamma) = \lambda_n; \quad \lambda_{\nu(m+1)}(\gamma) = \lambda_n,
$$

then

$$
\mu_{(m+1)}(\sigma) = \lambda_n |A_n| \exp (\sigma \lambda_n) \leqslant \lambda_n |A_n| \mu_{(m)}(\sigma),
$$

and

$$
\mu_{(m)}(\sigma) = \frac{1}{\lambda_n} (\lambda_n |A_n| \exp (\sigma \lambda_n)) \leqslant \frac{\mu_{(m+1)}(\sigma)}{\mu_{(m)}(\sigma)}
$$

and so these two inequalities complete (4.1) and from which we have:

$$
\lambda_{\nu(0)} \leqslant \frac{\mu_{(0)}(\sigma)}{\mu(\sigma)} \leqslant \lambda_{\nu(1)}(\sigma) \leqslant \frac{\mu_{(2)}(\sigma)}{\mu_{(1)}(\sigma)} \leqslant \cdots \leqslant \lambda_{\nu(m)}(\sigma) \leqslant \frac{\mu_{(m)}(\sigma)}{\mu_{(m-1)}(\sigma)} = \lambda_{\nu(m)}(\sigma).
$$
Multiplying the ratios involving these \( \mu \)'s one finds that
\[
\frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} \geq \lambda_{\nu(m-1)(\sigma)} \ldots \lambda_{\nu(\sigma)} \geq (\lambda_{\nu(\sigma)})^m.
\] (4.2)

Now from (1.3)' we get, for \( K \) to be sufficiently large,
\[
\log |a_{\nu(\sigma)}| < -K\lambda_{\nu(\sigma)}; \quad \sigma \geq \sigma_0
\]
(4.3) \(|a_{\nu(\sigma)}| < \exp(-k\lambda_{\nu(\sigma)}) < 1, \quad \sigma \geq \sigma_0.
\]

Again
\[
\log \mu(\sigma) = \log |a_{\nu(\sigma)}| + \sigma\lambda_{\nu(\sigma)} < \sigma\lambda_{\nu(\sigma)}, \quad \sigma \geq \sigma_0
\] (4.4)
from (4.3). The inequalities (4.2) and (4.4) result in for \( \sigma \geq \sigma_0
\]
\[
\frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} > \left(\frac{\log \mu(\sigma)}{\sigma}\right)^m.
\]

The above theorem is useful in deducing the following interesting.

**Theorem E.** — One has (with the terms involved in to be known):
\[
\lim_{\sigma \to \infty} \log \frac{(\mu_{(m)}(\sigma)/\mu(\sigma))^{1/m}}{\sigma} = \frac{\rho}{\lambda};
\]

**Proof.** — We have:
\[
\frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} \leq \lambda_{\nu(m)(\sigma)} \ldots \lambda_{\nu(\sigma)} \leq (\lambda_{\nu(\sigma)})^m.
\]
Now \( f^{(m)}(s) \) also posses the same order \( \rho \) and lower order \( \lambda \) as \( f(s) \) has, and so (cf. Theorem B)
\[
\lim_{\sigma \to \infty} \frac{\log \lambda_{\nu(m)(\sigma)}}{\sigma} = \frac{\rho}{\lambda};
\]
consequently
\[
(4.5) \quad \lim_{\sigma \to \infty} \frac{\log (\mu_{(m)}(\sigma)/\mu(\sigma))^{1/m}}{\sigma} \leq \frac{\rho}{\lambda};
\]
But Theorem D provides us the inequality (to be deduced with the help of Theorem B and (1.4) (1)  

\[(4.6) \lim_{\sigma \to \infty} \frac{\log (\mu(m)(\sigma)/\mu(\sigma))^{1/m}}{m} \geq \frac{\lambda}{\rho};\]

The inequalities (4.5) and (4.6) yield the desired result.

**Remark.** — Theorem D has been stated without any proof by Srivastav ([11], p. 89 (i)) and that too under the restrictive condition that \( \lambda > 0 \). The proof of Theorem D removes this superfluous restriction which Srivastav asserts. Secondly, Srivastav claims to prove Theorem E but to the best my surprise there is no clue available to its proof in his paper wherever he mentions it. I wish to add that I have stated Theorem D without proof in a recent paper of mine ([6], Theorem 1).

5. Towards the end of this paper, I would like to add a new result on the mean values of entire Dirichlet functions. To the best of my knowledge I introduced these means and discovered their properties relating to the order and lower order of \( f(S) \) in a recent paper [9]. I do here a little more. I define

\[ A_k(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(S)|^k dt, \]

where the sequence \( \{\lambda_n\} \) satisfies (1.1)-(1.3); \( 0 < k < \infty \).

**Theorem F.** — If \( f(S) \) satisfies the conditions stated in § 5, then we have:

\[ \lim_{\sigma \to \infty} \frac{\log \log A_k(\sigma)}{\sigma} = \frac{\rho}{\lambda}; \]

\[(1)\text{ From (1.4), (i)} \]

\[ \log \mu(\sigma) \leq (1 + 0(1))\sigma \lambda_{(\sigma)} \quad \text{and so} \quad \log \log \mu(\sigma)/\sigma \leq 0(1) + \log \lambda_{(\sigma)}/\sigma; \]

and (ii) for \( k > 0 \), \( \log \mu(\sigma + k) \geq k\lambda_{(\sigma)} \) and so

\[ \log \log \mu(\sigma + k)/(1 + 0(1)) (\sigma + k) \geq 0(1) + \log \lambda_{(\sigma)}/\sigma. \]

From (i) et (ii) one deduces that

\[ \lim_{\sigma \to \infty} \log \log \mu(\sigma)/\sigma = \lim_{\sigma \to \infty} \log \lambda_{(\sigma)}/\sigma. \]
Remark. — If \( k = 2 \), I have got the above result in a recent paper ([9], Theorem 1) where I supposed further that \( \chi_n \) was non-decreasing. Here we need not, as one will soon find, make this supposition.

Proof of Theorem F. — One does have

\[
A_k(\sigma) \leq \left\{ M_\sigma(\sigma) \right\}^k,
\]
where

\[
M_\sigma(\sigma) = \max_{\mid t \mid \leq T} |f(\sigma + it)|.
\]

But (see for references [9] and also [10])

\[
\lim_{\sigma \to \infty} \frac{\log \log M_\sigma(\sigma)}{\sigma} = \frac{\rho}{\lambda};
\]

So we find that

\[
\lim_{\sigma \to \infty} \frac{\log \log A_k(\sigma)}{\sigma} \leq \frac{\rho}{\lambda}; \tag{5.1}
\]

To get the other part, it is sufficient to consider \( f(S) \) in the representation given by:

\[
f(S) = \sum_{n=0}^{\infty} a_n e^{\lambda_n}.
\]

Then, if \( S' = \Delta + ix; \quad a_n = \alpha_n + i\beta_n \), we have

\[
f(\Delta + ix) = \sum_{n=0}^{\infty} [(\alpha_n \cos \lambda_n x - \beta_n \sin \lambda_n x) + i(\alpha_n \sin \lambda_n x + \beta_n \cos \lambda_n x)] e^{\lambda_n};
\]

\[
\text{Rl}\{f(\Delta + ix)\} = \sum_{n=0}^{\infty} (\alpha_n \cos \lambda_n x - \beta_n \sin \lambda_n x) e^{\lambda_n}.
\]

Therefore

\[
\alpha_m e^{\lambda_m} = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \text{Rl}\{f(\Delta + ix)\} \cos \lambda_m x \, dx, \quad m > 0.
\]

\[
\beta_m e^{\lambda_m} = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \text{Rl}\{f(\Delta + ix)\} \sin \lambda_m x \, dx, \quad m > 0.
\]

\[
\alpha_0 = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \text{Rl}\{f(\Delta + ix)\} \, dx.
\]
Therefore from (\(\ast\)) and (\(\ast\ast\))

\[
Rf(\sigma + it) = \sum_{n=0}^{\infty} (\alpha_n \cos \lambda_n t - \beta_n \sin \lambda_n t) e^{\sigma \lambda_n} \tag{5.2}
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} Rf(\Delta + ix) \left\{ 1 + \sum_{n=1}^{\infty} \cos \{x - t\} \lambda_n e^{(\sigma - \Delta) \lambda_n} \right\} dx.
\]

We can treat (5.2) as an analogue to Poisson’s formula in power series. Therefore, if we start our series for \(f(s)\) from \(n = 1\) to \(\infty\), then

\[
|f(s)| \leq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\Delta + ix)|^2 \sum_{n=1}^{\infty} \exp\{(\sigma - \Delta) \lambda_n\} dx,
\]

and since the right-hand side is independent of \(t\), one finds that

\[
M(\sigma) \leq 2A(\Delta) \left( \sum_{n=0}^{n_0 - 1} + \sum_{n=n_0}^{\infty} \right) \exp\{(\sigma - \Delta) \lambda_n\}
\]

\[
< 2A(\Delta) \left( (n_0 - 1) \exp\{(\sigma - \Delta) \lambda_1\} + \sum_{n=n_0}^{\infty} \exp\{(\sigma - \Delta) \lambda_n\} \right). \tag{5.3}
\]

But

\[
\sum_{n=n_0}^{\infty} \exp\{(\sigma - \Delta) \lambda_n\} < \exp\{(\sigma - \Delta) \lambda_1\} \{1 + \exp(\sigma - \Delta)L + \exp(\sigma - \Delta)2L + \cdots\}.
\]

Therefore

\[
M(\sigma) < 2A(\Delta) \left[ (n_0 - 1) \exp(\sigma - \Delta) \lambda_1 + \frac{\exp\{(\sigma - \Delta) \lambda_1\} \exp(\Delta L)}{\exp(\Delta L) - \exp(\sigma L)} \right].
\]

Let \(\Delta = \sigma + \eta, \eta > 0\). Then on simplifications, one gets

\[
M(\sigma) < 0(1) A(\sigma + \eta). \tag{5.4}
\]

Similarly taking \(\{f(s)\}^k\) instead of \(f(s)\), one can prove that

\[
(M(\sigma))^k < 0(1) A_k(\sigma + \eta), \tag{5.5}
\]

where the constants \(0(1)\) in (5.4) and (5.5) might not be the same, and so

\[
\lim_{\sigma \to \infty} \frac{\log \log A_k(\sigma)}{\sigma} \geq \lim_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \rho; \tag{5.6}
\]
The inequalities (5.1) and (5.6) yield the required result. I might like to discuss further results on the means defined by $A_k(\sigma)$ in a next sequel of my work.

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