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JENSEN MEASURES AND UNBOUNDED $B-$REGULAR DOMAINS IN $\mathbf{C}^n$

by Quang Dieu NGUYEN & Dau Hoang HUNG

ABSTRACT. — Following Sibony, we say that a bounded domain Ω in C^n is B-regular if every continuous real valued function on the boundary of Ω can be extended continuously to a plurisubharmonic function on Ω. The aim of this paper is to study an analogue of this concept in the category of unbounded domains in C^n. The use of Jensen measures relative to classes of plurisubharmonic functions plays a key role in our work

RÉSUMÉ. — En suivant Sibony, nous dirons qu'un domaine borne Ω de C^n est $B-$ régulier si toute fonction continue à valeurs réelles sur la frontière de Ω peut être prolongée continûment à une fonction plurisousharmonique sur Ω. Le but de ce papier est d'étudier une notion analogue dans la catégorie des domaines non bornés dans C^n. L'usage des mesures de Jensen relatives à des classes de fonctions plurisousharmoniques jouent un rôle clé dans notre travail.

1. Introduction

Let $\Omega \subset \mathbf{C}^n$ be an open set. An upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if the restriction of u on the intersection of Ω with every complex line is subharmonic (the function identically $-\infty$ is considered to be plurisubharmonic). By $\mathrm{PSH}(\Omega)$ we denote the set of plurisubharmonic functions on Ω and by $\mathrm{PSH}^c(\Omega)$ we mean the set of continuous functions on $\overline{\Omega}$ which are plurisubharmonic on Ω. We say that $u \in \mathrm{PSH}(\Omega)$ is *maximal* if for every relatively compact subdomain Ω' of Ω and every $v \in \mathrm{PSH}(\Omega)$ satisfying $v \leqslant u$ on $\partial\Omega$ we have $v \leqslant u$ on Ω'. Denote by $\mathrm{MPSH}(\Omega)$ the class of maximal plurisubharmonic functions

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on Ω. It is well known that in pluripotential theory maximal plurisubharmonic functions play the role as harmonic functions do in classical potential theory.

In this paper, we concern with the following problem: *Given* $h \in \mathcal{C}(\partial\Omega)$, *the space of real valued continuous function* $\partial\Omega$, *we study conditions on* Ω *and* h *to ensure the existence of* $u \in \mathrm{MPSH}(\Omega)$ *which is continuous on* $\overline{\Omega}$ *such that* $u = h$ *on* $\partial\Omega$. This problem, which is usually referred to as *the Dirichlet-Bremermann problem*, in the case Ω is bounded, has been initiated in a classical work of Bremermann (*cf.* [3]). Bremermann proved among other things that if Ω is bounded and $h \in \mathcal{C}(\partial\Omega)$ then there is at most one continuous function φ on $\overline{\Omega}$ with boundary data h which is maximal plurisubharmonic on Ω. This solution, if exists, is given by

$$\varphi_{h,\Omega}(z) = \sup\{u(z) : u \in \mathrm{PSH}(\Omega), u^* \leqslant h \text{ on } \partial\Omega\}, \tag{1.1}$$

where u^* is the *upper regularization* of u which is defined on $\overline{\Omega}$ as

$$u^*(z) = \limsup_{x \to z} \ u(x) \ \forall z \in \overline{\Omega}.$$

Notice that if Ω is *regular in the real sense i.e.,* for every $h \in \mathcal{C}(\partial\Omega)$ there is $H \in C(\overline{\Omega})$ harmonic on Ω such that $H = h$ on $\partial\Omega$, then $\varphi_{h,\Omega} \in \mathrm{PSH}(\Omega)$. On the other hand, the continuity of $\varphi_{h,\Omega}$ is more delicate, Bremermann showed that if Ω is *strictly pseudoconvex* (*e.g.*, Ω is an open ball) then $\varphi_{h,\Omega}$ is continuous at every boundary point of Ω and coincides with h there. Later on, Walsh proved in [16] that $\varphi_{h,\Omega}$ is indeed continuous on Ω. He also gives an example where $\varphi_{h,\Omega}$ is not continuous. More recently, in [14] Sibony characterizes bounded domains Ω in $\mathbf{C}^n$ such that $\varphi_{h,\Omega}$ is continuous on $\overline{\Omega}$ and satisfies $\varphi_{h,\Omega} = h$ on $\partial\Omega$ for every $h \in \mathcal{C}(\partial\Omega)$. This class of domains is called $B-$ *regular* domains. It should be remarked that the class of $B-$ regular domains properly contains the class of strictly pseudoconvex domains.

The Dirichlet-Bremermann problem for *unbounded* domains Ω was studied only recently in [13] and [15]. In [13], the authors pointed out that $\varphi_{h,\Omega}$ may be identically $-\infty$ even if $h \in \mathcal{C}(\partial\Omega)$ and Ω is a strictly convex domain with smooth boundary (*cf.* [13] and [15]). On the other hand, it is shown in Proposition 19 in [15] that if $h \in \mathcal{C}(\partial\Omega)$ is bounded and if Ω is an unbounded strictly convex domain with $\mathcal{C}^2$ smooth boundary, then there exists $u \in \mathcal{C}(\overline{\Omega})$, maximal plurisubharmonic on Ω such that $u = h$ on $\partial\Omega$. In Theorem 4.4 and Proposition 5.1 we generalize this nice result to the case Ω is only *locally* $B-$ regular *i.e.,* for every $z \in \partial\Omega$ there is a bounded neighbourhood U of z such that $U \cap \Omega$ is $B-$ regular.

The principal tool in this paper is the use of Jensen measures relative to classes of plurisubharmonic functions. Roughly speaking, we express upper envelopes of type (1) in terms of lower envelopes of integral relative to different classes of Jensen measures. Then equality of these classes of measures will give continuity of the corresponding upper envelopes. This approach has been used in [17], [11] and [9]. During the course of our development, we also obtain in Theorem 4.1 a global approximation result for bounded plurisubharmonic functions on unbounded domains.

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2. Necessary Background

We first fix notation and terminology that will be used in this paper. The set of (real valued) continuous functions on a subset X of $\mathbf{C}^n$ is denoted by $\mathcal{C}(X)$. Denote by $\mathcal{C}_0(X)$ the set of functions $u \in \mathcal{C}(X)$ that vanishes at infinity *i.e.*, for every $\varepsilon > 0$ there is a compact $K \subset X$ such that $|u| < \varepsilon$ on $X \backslash K$. We write $\mathrm{USC}_*(X)$ for the set of bounded upper semicontinuous functions u on X that is non positive at infinity *i.e.*, for every $\varepsilon > 0$, there is a compact $K \subset X$ such that $u < \varepsilon$ on $X \backslash K$. We also denote by $\mathcal{C}_c(X)$ the subset of $\mathcal{C}(X)$ consisting functions of compact support.

Given a subset $\mathcal{A}$ of $\mathrm{USC}_*(X)$ and $z \in X$ we denote by $J_z(\mathcal{A})$ the class of of positive regular Borel measures μ on X satisfying $\mu(X) \leqslant 1$ and

$$u(z) \leqslant \int_X u d\mu \ \forall u \in \mathcal{A}.$$

It is customary to call $J_z(\mathcal{A})$ the class of Jensen measures relative to $\mathcal{A}$ with barycentre at z.

Now we recall the following result of Sibony (*cf.* Theorem 2.1 in [14]).

THEOREM 2.1. — *Let Ω be a bounded domain in $\mathbf{C}^n$. Then the following conditions are equivalent.*

(a) *For every $h \in \mathcal{C}(\partial\Omega)$ there exists $u \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfying $u = h$ on $\partial\Omega$.*
(b) *For every $z \in \partial\Omega$, there exists $u \in \mathrm{PSH}^c(\Omega)$ satisfying $u(z) = 0$ while $u < 0$ elsewhere.*
(c) *There exists $\varphi \in \mathcal{C}^2(\Omega) \cap \mathrm{PSH}(\Omega)$ and $\lambda > 0$ such that $\{\varphi < c\}$ is relatively compact in Ω for every $c < 0$ and $\varphi(z) - \lambda|z|^2$ is plurisubharmonic on Ω.*
(d) *For every $h \in \mathcal{C}(\partial\Omega)$ there exists $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfying $u = h$ on $\partial\Omega$.*
(e) *For every $z \in \partial\Omega$, there exists a negative function $u \in \mathrm{PSH}(\Omega)$ such that $u^* < 0$ on $\partial\Omega \backslash \{z\}$ while $\lim_{x \to z} u(x) = 0$.*

Following Sibony, a bounded domain Ω in $\mathbf{C}^n$ is called $B-$regular if Ω satisfies one of above equivalent conditions. For more details on bounded $B-$regular domains, the reader is invited to the original article [14] (see also [2] and [10] for more recent developments).

We now extend some notions introduced at the beginning of the article to the context of *unbounded* domains.

DEFINITION 2.2. — *An unbounded domain $\Omega \subset \mathbf{C}^n$ is called $B-$regular if for every bounded function $h \in \mathcal{C}(\partial\Omega)$ there is a bounded function $u \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $u = h$ on $\partial\Omega$.*

By considering the functions $h_\xi(z) := \max(-|z - \xi|, -1)$ for $\xi \in \partial\Omega$ we deduce that if an unbounded domain Ω is $B-$ regular and has an isolated boundary point then it is *not locally* $B-$regular. Recall that Ω is said to be locally $B-$regular if for every $\xi \in \partial\Omega$, there exists a bounded neighbourhood U of ξ such that $U \cap \Omega$ is $B-$regular. Notice also that the restriction on boundedness of h in Definition 2.2 seems natural, in view of the mentioned above example due to Shcherbina and Tomassini. However, it is unclear that a result analogous to Theorem 2.1 still holds in the case where Ω is unbounded.

DEFINITION 2.3. — *An unbounded domain Ω is called regular in the real sense if for every $z \in \partial\Omega$, there is an open neighbourhood $z \in U$ such that $U \cap \Omega$ is regular in the classical sense.*

Observe that Ω is regular in the real sense if and only if every $z \in \partial\Omega$ is a *regular boundary* point of Ω *i.e.*, there exists a subharmonic function u on Ω satisfying $u^* < 0$ on $\partial\Omega \backslash \{z\}$ while $\lim_{x \to z} u(x) = 0$. To check the regularity of $z \in \partial\Omega$ we frequently appeal to Theorem 2.11 in [7] which asserts that if $n \geqslant 2$ and if there exists a real cone lying in $\mathbf{C}^n \setminus \overline{\Omega}$ with vertex at z, then z is a regular boundary point of Ω. It is also easy to see that every (unbounded) locally $B-$regular domain is regular in the real sense but the reverse implication does not hold in general.

Throughout this paper, unless otherwise specified, by Ω we always mean an *unbounded* domain in $\mathbf{C}^n$.

The simple fact below explains the role of regular domains in the real sense in the study of Dirichlet-Bremermann problem.

LEMMA 2.4. — *Assume that Ω is regular in the real sense. Let $h \in \mathcal{C}(\partial\Omega)$ be a bounded function. Define $\tilde{h} = h$ on $\partial\Omega$ and $\tilde{h} = M := \sup_{\partial\Omega} h$ on Ω. Set*

$$\varphi(z) := \sup\{u(z) : u \in \mathrm{PSH}(\Omega), u^* \leqslant \tilde{h} \text{ on } \Omega\}.$$

Then $\varphi \in \mathrm{PSH}(\Omega)$ and $\varphi^* \leqslant \tilde{h}$ on $\overline{\Omega}$.

Proof. — Clearly $\varphi^* \leqslant M$ on Ω, so $\varphi = \varphi^* \in \mathrm{PSH}(\Omega)$. It remains to show $\varphi^* \leqslant h$ on $\partial\Omega$. Fix $z \in \partial\Omega$. Choose an open ball $B \subset \mathbf{C}^n$ such that $z \in B$. Let $\Omega' := \Omega \cap B$. Then Ω' is bounded regular in the real sense. Let H be the solution to the (classical) Dirichlet problem on Ω' with boundary data $\tilde{h}$. It follows that $\varphi \leqslant H$ on Ω. Since z is a regular boundary point of Ω' and h is continuous near z, we infer $\varphi^*(z) \leqslant H^*(z) = h(z)$. The proof is complete. □

The following classical fact due to Choquet(*cf.* [8], Lemma 2.3.4) is very useful while working with upper envelopes .

CHOQUET'S LEMMA. — *Let $\{u_\alpha\}_{\alpha\in A}$ be a family of functions which are locally bounded from above on a subset X of $\mathbf{C}^n$. Then there is a countable subfamily $\{\alpha_j\}_{j\geqslant 1} \subset A$ such that*

$$(\sup\{u_\alpha : \alpha \in A\})^* = (\sup\{u_{\alpha_j} : j \geqslant 1\})^*.$$

Furthermore, if u_α is lower semicontinuous for every $\alpha \in A$, then we can choose $\{\alpha_j\}_{j\geqslant 1}$ such that

$$\sup\{u_\alpha : \alpha \in A\} = \sup\{u_{\alpha_j} : j \geqslant 1\}.$$

We also need some elements from pluripotential theory. For more complete treatments on this subject we refer the reader to the excellent accounts [1] and [8]. First recall that maximality for *locally bounded* plurisubharmonic function is a *local* property, and if $\{u_j\}_{j\geqslant 1}$ is a sequence in $\mathrm{MPSH}(\Omega)$ that converges monotonically almost everywhere to $u \in \mathrm{PSH}(\Omega)$

then $u \in \mathrm{MPSH}(\Omega)$. These properties are applications of the highly nontrivial theory of the complex Monge-Ampere operator developed by Bedford and Taylor. Next, a subset F in $\mathbf{C}^n$ is called *pluripolar* if for every $z \in \Omega$ there exists an open neighbourhood U of z and $u \in \mathrm{PSH}(U), u \not\equiv -\infty$ on any connected component of U such that $u = -\infty$ on $F \cap U$. A basic theorem of Josefson states that for every pluripolar set F in $\mathbf{C}^n$ we can find $u \in \mathrm{PSH}(\mathbf{C}^n), u \not\equiv -\infty$ such that $u = -\infty$ on F. In particular, if F is *bounded* then u can be made to be negative on every *fixed* ball containing F. It also follows from Josefson's theorem that a countable union of pluripolar set is pluripolar.

We continue this preparatory section by showing that the class of unbounded locally $B-$regular domains is much richer than that of strictly convex domains. Before formulating it, recall that a compact $K \subset \mathbf{C}^n$ is called $B-$regular if every function in $\mathcal{C}(K)$ can be approximated by continuous plurisubharmonic functions on neighbourhood s of K. This notion is also introduced by Sibony in [14]. We also say that a *closed* set K in $\mathbf{C}^n$ is *locally* $B-$regular if for every $z \in K$ there exists an open bounded neighbourhood U of z such that $\overline{U} \cap K$ is $B-$regular. For a detailed treatment of $B-$regular compact sets, the reader may consult Section 1 in [14]. In this work, we only use the following properties.

(i) A compact $K \subset \mathbf{C}^n$ is $B-$regular if and only if K is locally $B-$regular.
(ii) If the compact K is a countable union of $B-$regular compacts then K is $B-$regular.

Proposition 2.5. — *Ω is locally $B-$regular if the following conditions are verified.*

(a) *For every $z \in \partial\Omega$, there exists an open ball U around z such that $\Omega \cap U$ is hyperconvex i.e., there exists a negative plurisubharmonic exhaustion function on $\Omega \cap U$.*
(b) *There exists a locally $B-$regular closed subset K of $\partial\Omega$ such that Ω is strictly pseudoconvex near every point of $\partial\Omega \backslash K$.*

Here by strict pseudoconvexity of Ω near $z_0 \in \partial\Omega$ we mean the following: there is an open neighbourhood U of z_0 and a $\mathcal{C}^2$ smooth plurisubharmonic function u on U such that $U \cap \Omega = \{z : \rho(z) < 0\}, d\rho(z_0) \neq 0$ and $\rho(z) - \lambda|z|^2 \in \mathrm{PSH}(U)$ for some constant $\lambda > 0$. In this case, it is well known that there is $v \in \mathrm{PSH}^c(U \cap \Omega)$ such that $v(z_0) = 0$ whereas $v < 0$ elsewhere.

Proof. — Fix a point $z_0 \in \Omega$ and an open ball U containing z_0 such that $\Omega' := \Omega \cap U$ is hyperconvex. Set $K' = K \cap \partial\Omega'$. Then the compact K' is

$B-$regular. It follows from (b) that $\partial\Omega'$ is a countable union of $B-$regular compact sets. Therefore, the compact $\partial\Omega'$ is $B-$regular as well. Since Ω' is hyperconvex, we may apply Lemma 2.8 in [10] to get that Ω' is $B-$regular. The proof is complete. □

The following type of unbounded domains is quite important in our study.

Definition 2.6. — *We say that Ω is of bounded type if there exists a real valued function $\psi \in \mathrm{PSH}(\Omega)$ such that $\psi < 0$ and $\lim_{|z|\to\infty} \psi(z) = -\infty$.*

In particular, Ω can not contain any (non constant) biholomorphic image of the complex plane. It is immediate to note that the bounded type property is invariant under biholomorphic maps. Now we construct a class of unbounded domains of bounded type.

Proposition 2.7. — *Assume that there are n complex hyperplane $H_1, \cdots, H_n$ lying in $\mathbf{C}^n\backslash\overline{\Omega}$ such that $\bigcap_{i=1}^n H_i$ is a singleton and that dist $(\bigcup_{i=1}^n H_i, \partial\Omega) > 0$. Then Ω is of bounded type.*

Proof. — After a linear change of coordinates, we may arrange that $H_i = \{z_i = 0\}$ for $i = 1, \cdots, n$. It is easy to check that for large constant A, the function

$$\psi(z_1, \cdots, z_n) = -\log|z_1\cdots z_n| - A$$

is negative plurisubharmonic on Ω. Moreover, $\lim_{|z|\to\infty} \psi(z) = -\infty$. Thus Ω is of bounded type. □

Finally, we introduce a class of (unbounded) domains which enable us to control the growth to infinity of upper envelopes like (1).

Definition 2.8. — *If $n \geqslant 2$ then we say that $\overline{\Omega}$ contains no complex hyperplane at infinity if there is a compact K of $\overline{\Omega}$ such that $\overline{\Omega}\backslash K$ contains no complex hyperplane.*

Consider the domain $\Omega := B \times \mathbf{C}$, where B is the open unit ball in $\mathbf{C}^2$. It is easy to see that $\overline{\Omega}$ containes no *complex hypersurface* whereas Ω is not of bounded type.

Proposition 2.9. — *Assume that $\overline{\Omega}$ contains no complex hyperplane at infinity and $h \in \mathcal{C}(\partial\Omega)$ is bounded. Then for every bounded function $u \in \mathrm{PSH}(\Omega)$ satisfying $u^* \leqslant h$ on $\partial\Omega$ we have*

$$\limsup_{|z|\to\infty} u(z) \leqslant \limsup_{|z|\to\infty} h(z).$$

Proof. — Choose a compact $K \subset \overline{\Omega}$ such that $\overline{\Omega}\backslash K$ contains no complex hyperplane. Denote $\alpha := \limsup_{|z|\to\infty} h(z)$. Fix $\varepsilon > 0$, then there is compact $L \subset \partial\Omega$ such that $h < \alpha + \varepsilon$ on $\partial\Omega\backslash L$. We are going to show that $u(z) < \alpha + \varepsilon$ for all z outside the convex hull of $K \cup L$. Fix such a point z. Then there is a complex hyperplane H passing through z which is disjoint from $K \cup L$. Pick $z' \in H\backslash\overline{\Omega}$. Let l be the complex line connecting z and z'. Set $U = l \cap \Omega$. Then $U \cap L = \emptyset$. Thus $h < \alpha + \varepsilon$ on ∂U. We may view U as a nonempty open subset (possibly unbounded) of $\mathbf{C}$. Since u is bounded, $u^* \leqslant h < \alpha + \varepsilon$ on ∂U and U is *not* dense in $\mathbf{C}$, we may apply the maximum principle to reach $u(z) < \alpha + \varepsilon$. The proof is complete. $\square$

3. Duality theorem and unbounded $B-$ regular domains

We start with the following variation on a basic duality theorem of Edwards (*cf.* [5] and [17]).

THEOREM 3.1. — *Let X be a closed subset of $\mathbf{C}^n$ and $\mathcal{A}$ be a convex cone of* $\mathrm{USC}_*(X)$. *Let $g : X \to (-\infty, \infty]$ be a lower semicontinuous function which is increasing limit of a sequence in $\mathcal{C}_0(X)$. Then for every $z \in X$ we have*

$$\sup\{u(z) : u \leqslant g, u \in \mathcal{A}\} = \inf\Big\{ \int_X g d\mu : \mu \in J_z(\mathcal{A})\Big\}.$$

We require the following elementary fact.

LEMMA 3.2. — *Let $u \in \mathrm{USC}_*(X)$ then there is a uniformly bounded sequence $\{u_j\} \subset \mathcal{C}_c(X)$ such that:*

(a) *$u_j \to u$ pointwise on X.*
(b) *$u_j \geqslant u - 1/j$ for every $j \geqslant 1$.*

Proof. — Define for $j \geqslant 1$ the function

$$v_j(z) = \sup\{u(x) - j|x - z| : x \in X\}.$$

Clearly the sequence $\{v_j\}_{j\geqslant 1}$ is uniformly bounded. It is also well known that $v_j \in \mathcal{C}(X)$ and that $v_j \downarrow u$ on X as $j \to \infty$. Next, we claim that v_j is non positive at infinity for every j. Fix $j \geqslant 1$ then for every $\varepsilon > 0$, we can find a compact $K \subset X$ such that $u < \varepsilon$ on $X\backslash K$. Set

$$K_j = \{z : z \in X, \mathrm{dist}(z, K) \leqslant M/j\},$$

where $M = \sup_X u$. Obviously K_j is a compact subset of X. Furthermore, from the definition of v_j we can check easily that $v_j(z) \leqslant \varepsilon$ for $z \in X\backslash K_j$. This proves the claims.

Now we choose a sequence of compacts sets $P_j \uparrow X$ and a sequence of positive numbers $\lambda_j \uparrow \infty$ such that $Q_j := X \cap \{z : |z| \geqslant \lambda_j\} \cap P_j = \emptyset$. By Tietze's extension theorem, we can find $\tilde{v}_j \in \mathcal{C}_c(X)$ such that $\tilde{v}_j = v_j$ on $P_j, \tilde{v}_j = 0$ on Q_j and $|\tilde{v}_j| \leqslant ||v_j||_X$. Set

$$u_j = \max(\tilde{v}_j, u - 1/j) \ \forall j \geqslant 1.$$

It is easy to check that u_j is the desired sequence. □

Remark. — It follows easily from Lemma 3.2 that $J_z(\mathcal{A})$ is a *closed* convex subset of the space of positive regular Borel measures on X with total mass $\leqslant 1$.

Proof of Theorem 3.1. — Fix $z \in \Omega$. Given $u \in \mathcal{A}, u \leqslant g$ and $\mu \in J_z(\mathcal{A})$ we have

$$u(z) \leqslant \int_X u d\mu \leqslant \int_X g d\mu.$$

To prove the reverse direction, first we consider the case $g \in C_0(X)$. For $\varphi \in \mathrm{USC}_*(X)$ we set

$$S\varphi := \sup\{u(z) : u \leqslant \varphi, u \in \mathcal{A}\}.$$

Clearly $S\varphi$ is of real value for all $\varphi \in \mathrm{USC}_*(X)$. Moreover, since $\mathcal{A}$ is a convex cone we have

(a) $S(\lambda\varphi) = \lambda S(\varphi)$ for $\lambda > 0$ and $\varphi \in \mathrm{USC}_*(X)$.
(b) $S(\varphi_1) + S(\varphi_2) \leqslant S(\varphi_1 + \varphi_2)$ for $\varphi_1, \varphi_2 \in \mathrm{USC}_*(X)$.
In view of Hahn-Banach's theorem we can find a real linear functional $\tilde{S}$ on $\mathcal{C}_0(X)$ such that
(c) $\tilde{S}g = Sg$.
(d) $S(\varphi) \leqslant \tilde{S}\varphi \leqslant -S(-\varphi)$ for all $\varphi \in \mathcal{C}_0(X)$.

Since $|S\varphi| \leqslant ||\varphi||$ we infer that $\tilde{S}$ is a continuous linear functional on $\mathcal{C}_0(X)$ and $||\tilde{S}|| \leqslant 1$. By Riesz's representation theorem (*cf.* Theorem 6.19 in [12]), there is a Borel measure μ on X satisfying

$$\tilde{S}\varphi = \int_X \varphi d\mu, \ \forall \varphi \in \mathcal{C}_0(X).$$

Since $\varphi \geqslant 0$ implies $\tilde{S}(\varphi) \geqslant 0$ we infer that μ is a positive regular Borel measure on X satisfying $\mu(X) \leqslant 1$. It remains to check that $\mu \in J_z(\mathcal{A})$. For this, fix $u \in \mathcal{A}$. By Lemma 3.2, there is a uniformly bounded sequence $u_j \subset \mathcal{C}_c(X)$ satisfying $u_j \to u$ as $j \to \infty$ and $u_j \geqslant u - 1/j$. By Lebesgue's convergence theorem we have

$$\int_X u d\mu = \lim_{j\to\infty} \int_X u_j d\mu \geqslant \lim_{j\to\infty} S u_j(z) \geqslant u(z).$$

Now for general g. Take a sequence $\{g_j\}_{j\geqslant 1} \subset \mathcal{C}_0(X)$ such that $g_j \uparrow g$ on X. By the previous paragraph, we can find a sequence $\mu_j \in J_z(\mathcal{A})$ satisfying

$$\sup\{u(z) : u \leqslant g_j, u \in \mathcal{A}\} = \int_{\overline{\Omega}} g_j d\mu_j. \tag{3.1}$$

Let μ be a weak$^*-$ limit of μ_j, then by the Remark following Lemma 3.2, $\mu \in J_z(\mathcal{A})$. Observe that for any fixed $k \geqslant 1$

$$\limsup_{j\to\infty} \int_{\overline{\Omega}} g_j d\mu_j \geqslant \lim_{j\to\infty} \int_{\overline{\Omega}} g_k d\mu_j = \int_{\overline{\Omega}} g_k d\mu.$$

Letting $k \to \infty$ and using Lebesgue's monotone convergence theorem we get

$$\limsup_{j\to\infty} \int_{\overline{\Omega}} g_j d\mu_j \geqslant \int_{\overline{\Omega}} \tilde{h} d\mu. \tag{3.2}$$

Putting (3.1) and (3.2) together we arrive at the desired conclusion. This completes the proof. $\square$

The following result is an application of the above duality theorem to the case where X is the closure of an unbounded domain in $\mathbf{C}^n$.

Proposition 3.3. — *Let $h \in \mathcal{C}(\partial\Omega)$ be a bounded, non negative function and $\mathcal{A} \subset \mathrm{USC}_*(\overline{\Omega})$ be a convex cone. Let $\tilde{h}$ be the function equal to h on $\partial\Omega$ and to M on Ω, where M is some positive constant larger than $\sup_{\partial\Omega} h$. Then for $z \in \overline{\Omega}$*

$$\sup\{u(z) : u \leqslant \tilde{h}, u \in \mathcal{A}\} = \inf\Big\{ \int_{\overline{\Omega}} \tilde{h} d\mu : \mu \in J_z(\mathcal{A})\Big\}.$$

Proof. — According to Theorem 3.1, it suffices to construct a sequence $\{h_j\} \in \mathcal{C}_c(\overline{\Omega})$ such that $h_j \uparrow \tilde{h}$. To do this, we first choose sequences of compact sets $K_j \uparrow \Omega, L_j \uparrow \partial\Omega$ and a sequence of positive numbers $\lambda_j \uparrow \infty$ such that $P_j \cap (L_j \cup K_j) = \emptyset$, where $P_j = \overline{\Omega} \cap \{z : |z| \geqslant \lambda_j\}$. Define

$$g_j = \begin{cases} h & \text{on } \partial\Omega \backslash P_j \\ 0 & \text{on } P_j, \end{cases}$$

and

$$\tilde{g}_j = \begin{cases} h & \text{on } L_j \\ 0 & \text{on } (\partial\Omega \cup P_j)\backslash L_j. \end{cases}$$

Then g_j is lower semicontinuous on $\partial\Omega \cup P_j$ and $\tilde{g}_j$ is upper semicontinuous on $\partial\Omega\cup P_j$. Notice that $\tilde{g}_j \leqslant g_j$ on $\partial\Omega\cup P_j$. Thus by the Hahn interpolation theorem (*cf.* [4], Proposition 7.2.1), we can find $\tilde{h}_j \in \mathcal{C}(\partial\Omega \cup P_j)$ satisfying $\tilde{g}_j \leqslant \tilde{h}_j \leqslant g_j$. In particular $\tilde{h}_j \leqslant h$ on $\partial\Omega, \tilde{h}_j = h$ on L_j and $\tilde{h}_j = 0$ on P_j.

By Tietze's extension theorem we may extend $\tilde{h}_j$ to $\hat{h}_j \in \mathcal{C}_c(\overline{\Omega})$ such that $0 \leqslant \hat{h}_j \leqslant M$ on $\overline{\Omega}$ and that $\hat{h}_j = M$ on K_j. Set

$$h_j = \max(\hat{h}_1, \cdots, \hat{h}_j).$$

It is easy to check that h_j is the desired sequence. We are done. □

From now on, we only interested in the case where $X = \overline{\Omega}$ and $\mathcal{A}$ is one of the following convex subcones of $\mathrm{USC}_*(\overline{\Omega})$: $\mathcal{A}_1$, the set of functions having compact support in $\mathrm{PSH}^c(\Omega)$, $\mathcal{A}_2$, the set of functions in $\mathrm{USC}_*(\overline{\Omega})$ which are plurisubharmonic on Ω. For any bounded function φ on $\overline{\Omega}$ and $1 \leqslant i \leqslant 2$ we also define upper envelopes relative to the convex cone introduced above.

$$S_i\varphi(z) = \sup\{u(z) : u \leqslant \varphi, u \in \mathcal{A}_i\},\ z \in \overline{\Omega}.$$

The result below is a geometric interpretation of the situation when the class of Jensen measures at every boundary point of Ω is trivial.

Proposition 3.4. — *The following assertion are equivalent.*

(a) *For every $z \in \partial\Omega$, there is an open neighbourhood U of z and a function $u \in \mathrm{PSH}^c(U \cap \Omega)$ satisfying $u(z) = 0$ whereas $u < 0$ elsewhere.*

(b) *Ω is regular in the real sense and $J_z(\mathcal{A}_1) = \{\delta_z\}$ for every $z \in \partial\Omega$, where δ_z is the Dirac mass at z.*

(c) *Ω is locally $B-$regular.*

Proof. — (a) $\Rightarrow$ (b). Clearly Ω is regular in the real sense. Fix $z \in \partial\Omega$ and $\mu \in J_z(\mathcal{A}_1)$. Let V be an open neighbourhood of z lying compactly in U. Set $\alpha = \sup_{\Omega\cap\partial V} u$. Since $\alpha < 0$, the function v equals to 0 on $\overline{\Omega}\backslash V$ and to $\max(u - \alpha, 0)$ on $\overline{\Omega} \cap V$ belongs to $\mathcal{A}_1$. It follows that

$$-\alpha = v(z) \leqslant \int_{\overline{\Omega}} v d\mu = \int_{\overline{\Omega}\cap V} v d\mu.$$

Since μ is a positive regular Borel measure on $\overline{\Omega}$ with total mass $\leqslant 1$ and since $\alpha < 0$ we deduce that the support of μ is contained in V. Since V is arbitrary μ must be supported at z. Thus $\mu = \delta_z$. The proof is complete.

(b) $\Rightarrow$ (c). In view of Theorem 2.1, it is enough to show that for every $z_0 \in \partial\Omega$ there is a negative function $u \in PSH(\Omega)$ such that $u^* < 0$ on $\partial\Omega\backslash\{z_0\}$ while $\lim_{x\to z_0} u(x) = 0$. For this, fix such a point z_0. Define

$$h(z) = \begin{cases} \max(-|z - z_0|, -1) & z \in \partial\Omega \\ 0 & z \in \Omega. \end{cases}$$

By Proposition 3.3 we have $S_1 h = h$ on $\partial\Omega$ and $S_1 h \leqslant 0$ on Ω. On the other hand, by Lemma 2.4 we obtain $(S_1 h)^* \leqslant h$ on $\partial\Omega$. Since $S_1 h$ is lower semicontinuous on $\overline{\Omega}$, by the choice of h, we can check that $u := (S_1 h)^*$ is the desired function.

(c) $\Rightarrow$ (a) follows immediately from Theorem 2.1. $\square$

Remark. — It is important to note that $J_z(\mathcal{A}_2) = \{\delta_z\}$, for every $z \in \partial\Omega$. Indeed, for $z_0 \in \partial\Omega$, consider the function $h(z) = \max(-|z-z_0|-1, -2)$ for $z \in \partial\Omega$ and $h(z) = -2$ on Ω. Since $h \in \mathcal{A}_2, h(z_0) = -1$ and $h < -1$ elsewhere, we see immediately that $J_{z_0}(\mathcal{A}_2) = \{\delta_{z_0}\}$.

In view of Theorem 3.1, if $J_z(\mathcal{A}_1) = \{\delta_z\}$ for every $z \in \partial\Omega$ then Ω is locally $B-$ regular. Our next result is a partial generalization of Proposition 19 in [15].

Proposition 3.5. — *Assume that Ω is locally $B-$regular. Let $h \in \mathcal{C}(\partial\Omega)$ be bounded. Then there exists $\varphi \in \mathrm{MPSH}(\Omega)$ having the following properties.*

(a) $\inf_{\partial\Omega} h \leqslant \varphi \leqslant \sup_{\partial\Omega} h, \lim_{x\to z} \varphi(x) = h(z)$ *for all* $z \in \partial\Omega$.

(b) *There is a pluripolar subset F of Ω such that φ is continuous on $\Omega\backslash F$.*

Moreover, if $h \in \mathcal{C}_0(\partial\Omega)$ then the pluripolar set F can be constructed to be independent of h.

Proof. — We may assume that $h \geqslant 0$ on $\partial\Omega$. Define the function $\tilde{h}$ as $\tilde{h} = h$ on $\partial\Omega$ and $\tilde{h} = \sup_{\partial\Omega} h$ on Ω. Then by Theorem 3.1 and Proposition 3.4 we have $S_1\tilde{h} = h$ on $\partial\Omega$. Since $\partial\Omega$ is regular in the real sense, Lemma 2.4 implies $\varphi := (S_1\tilde{h})^* \in \mathrm{PSH}(\Omega)$ and $0 \leqslant \varphi \leqslant \tilde{h}$ on $\overline{\Omega}$. Thus φ satisfies (a) of the theorem. For (b), fix an open ball $B \subset \Omega$ it suffices to show the maximality of φ on B. By Choquet's lemma, there is a sequence $\{v_j\}_{j\geqslant 1} \subset \mathrm{PSH}^c(\Omega)$ such that $v_j \uparrow S_1\tilde{h}$ on $\overline{\Omega}$. Let $\tilde{v}_j$ be the solution of the Dirichlet-Bremermann problem on B with boundary data v_j. Then the function $v_j^* = \tilde{v}_j$ on B and $v_j^* = v_j$ on $\overline{\Omega}\backslash B$ belongs to $\mathrm{PSH}^c(\Omega)$ and satisfies $v_j \leqslant v_j^* \leqslant \tilde{h}$ on $\overline{\Omega}$. Thus $\tilde{v}_j \uparrow S_1\tilde{h}$ on $\overline{\Omega}$. It follows that $\tilde{v}_j \uparrow \varphi$ almost everywhere on B. Thus φ is maximal on B. Finally, by the solution to the second problem of Lelong (*cf.* [1]) the set $F := \{z \in \Omega : S_1\tilde{h}(z) < \varphi(z)\}$ is pluripolar, thus the function φ, being lower semicontinuous on $\Omega\backslash F$, must be continuous on $\Omega\backslash F$.

Next, observe that $\mathcal{C}_0(\partial\Omega)$ is a separable Banach space with the sup norm. Choose a countable dense sequence $\{h_j\}_{j\geqslant 1}$ in $\mathcal{C}_0(\partial\Omega)$. For each $j \geqslant 1$ we set $\tilde{h}_j = h_j$ on $\partial\Omega$ and $h_j := \sup_{\partial\Omega} h_j$ on Ω. By the above argument, there exist a pluripolar subset F_j of Ω and $\varphi_j \in \mathrm{MPSH}(\Omega)$ such that

(a) $\lim_{x\to z} \varphi_j(x) = h_j(z)$ for all $z \in \partial\Omega$.

(b) φ_j is continuous on $\Omega\backslash F_j$.

Thus $F := \cup F_j$ is pluripolar. Now given $h \in \mathcal{C}_0(\partial\Omega)$, we can choose a subsequence h_{j_k} that converges to h uniformly on $\partial\Omega$. It is easy to check that $\tilde{h}_{j_k}$ converges to $\tilde{h}$ uniformly on $\overline{\Omega}$. Therefore φ_{j_k} converges uniformly to φ on $\overline{\Omega}$. Thus $\varphi \in \mathrm{MPSH}(\Omega), \lim_{x\to z}\varphi(x) = h(z)$ for all $z \in \partial\Omega$ and φ is continuous on $\Omega\backslash F$.

It would be interesting to know if every *unbounded* locally $B-$ regular domain is $B-$regular. Unfortunately, we can only prove this statement under an additional assumption on the domain (*cf.* Theorem 4.4). However, if we discard the requirement on maximality of the solution in Definition 2.2 then a satisfactory answer can be obtained. More precisely, we have the following generalization of Theorem 2.1 in [15]. □

Proposition 3.6. — *Assume that Ω is locally $B-$regular. Then for every $h \in \mathcal{C}(\partial\Omega), h \geqslant 0$ and every closed set K of $\overline{\Omega}$ such that $K \cap \partial\Omega = \emptyset$, there exists $u \in PSH^c(\Omega)$ satisfying $u \geqslant 0, u = 0$ on K and $u = h$ on $\partial\Omega$.*

Proof. — We use the same idea as in [15]. Since Ω is locally $B-$regular, we can choose locally finite open coverings $\{U_i\}_i, \{U'_i\}_{i\geqslant 1}$ of $\partial\Omega$ consisting of open balls such that $\Omega_i := \Omega \cap U_i$ is $B-$regular, $U'_i \subset\subset U_i$ and $K \cap U_i = \emptyset$ for all $i \geqslant 1$. Let $\{\chi_i\}_{i\geqslant 1}$ be a partition of unity subordinating to $\{U'_i\}_{i\geqslant 1}$. By Tietze's extension theorem, for every $i \geqslant 1$ there exists $h_i \in \mathcal{C}(\partial\Omega_i)$ such that $0 \leqslant h_i$ on $\partial\Omega_i, h_i = 0$ on $\Omega \cap \partial\Omega_i$ and $h_i = h\chi_i$ on $U'_i \cap \partial\Omega$. Let u_i be the solution to the Dirichlet-Bremermann problem on Ω_i with boundary values h_i. Clearly $u_i \geqslant 0$ on Ω_i and $u_i = 0$ on $\Omega \cap \partial U_i$. Thus we may extend u_i to $\tilde{u}_i \in \mathrm{PSH}^c(\Omega)$ by setting $\tilde{u}_i = 0$ out of Ω_i. Set $u(z) = \sum_{i\geqslant 1}\tilde{u}_i(z)$ for $z \in \overline{\Omega}$. By the construction of $\tilde{u}_i$, *locally* this sum is taken over a *finite* number of indices i. It is not hard to see that u is the desired function. □

4. Equality of Jensen measures

We start with a simple relation between equality of Jensen measures and the possibility of global approximation of bounded plurisubharmonic functions by continuous ones.

Theorem 4.1. — *The following assertions are equivalent.*

(a) $J_z(\mathcal{A}_1) = J_z(\mathcal{A}_2)$ *for every* $z \in \Omega$.
(b) $S_1\varphi = S_2\varphi$ *on* Ω *for every* $\varphi \in \mathcal{C}_0(\overline{\Omega})$.
(c) *For every* $u \in \mathcal{A}_2$ *there is a uniformly bounded sequence* $\{v_j\}_{j\geqslant 1} \in \mathcal{A}_1$ *such that* $v_j \to u$ *on* Ω *and* $\limsup_{j\to\infty} \leqslant u^*$ *on* $\partial\Omega$.

Proof. — We follow the lines of the proof of Theorem 3.2 in [9] (see also Theorem 2.1 in [11]).

(a) $\Rightarrow$ (b). This is a direct application of the duality Theorem 3.1.

(b) $\Rightarrow$ (c). By Lemma 3.2 we can find a sequence $u_j \in \mathcal{C}_0(\overline{\Omega})$ such that $u_j \to u$ and $u_j \geqslant u - 1/j$. Clearly for all $j \geqslant 1$

$$u - 1/j \leqslant S_2 u_j \leqslant u_j \text{ on } \overline{\Omega}.$$

Notice that $(S_2 u_j)^* \leqslant u_j$, so $S_2 u_j \in \mathcal{A}_2$ for all j and $S_2 u_j \to u$ on $\overline{\Omega}$. Moreover, $S_1 u_j = S_2 u_j$ on Ω. In particular $S_2 u_j$ is continuous on Ω for all j. By Choquet's lemma we, can find a sequence $\{v_{j,k}\}_{k\geqslant 1} \subset \mathcal{A}_1$ increasing to $S_2 u_j$ on Ω. Let $K_j \uparrow \Omega$ be an increasing sequence of compact sets. In view of Dini's lemma and continuity of $S_2 u_j$, we can find a sequence $k_j \uparrow \infty$ such that $|v_{j,k_j} - S_2 u_j| < 1/j$ on K_j. It is easy to check that v_{j,k_j} is the desired sequence.

(c) $\Rightarrow$ (a). Given $z \in \Omega$ and $\mu \in J_z(\mathcal{A}_1)$ we must show $\mu \in J_z(\mathcal{A}_2)$. Fix $u \in \mathcal{A}_2$, then there is a sequence $\{v_j\}_{j\geqslant 1} \subset \mathcal{A}_1$ satisfying $v_j \to u$ on Ω and $\limsup_{j\to\infty} v_j \leqslant u^*$ on $\partial\Omega$. Notice that for all $j \geqslant 1$

$$v_j(z) \leqslant \int_{\overline{\Omega}} v_j d\mu.$$

Letting $j \to \infty$ and using Fatou's lemma we get $u(z) \leqslant \int_{\overline{\Omega}} u d\mu$. Thus $\mu \in J_z(\mathcal{A}_2)$. □

Remarks. — (a) By Dini's lemma and the construction of the sequence $\{v_j\}_{j\geqslant 1}$ we see that if u is continuous on a compact subset K of Ω then, in addition, the sequences $\{v_j\}_{j\geqslant 1}$ in (c) can be chosen to converge to u uniformly on K.

(b) Building on a previous example of Fornaess and Wiegerinck (*cf.* [6] p. 260) we will construct a unbounded domain Ω such that any of equivalent assertions in Theorem 4.1 does not hold. Recall that Fornaess and Wiegerinck consider a smoothly bounded domain D in $\mathbf{C}^2$ defined by

$$D = \{(z, w) : |w - e^{i\varphi(|z|)}|^2 < r(|z|)\},$$

where r and φ are $\mathcal{C}^\infty$ smooth functions on $\mathbf{R}$. Moreover, Fornaess and Wiegerinck show that if r and φ are well chosen then D has the following properties.

(i) The projection of D onto the first coordinate is the annulus $\{z : 1 < |z| < 17\}$.

(ii) The compact $A := \{(z, 0) : |z| = 2\} \cup \{(z, 0) : |z| = 9\} \cup \{(z, 0) : |z| = 16\}$ is contained in D.

(iii) The annulus $B := \{(z, 0) : 2 \leqslant |z| \leqslant 15\}$ lies in $\overline{\Omega}$ (but not in Ω).

(iv) There is $u \in \mathrm{PSH}^c(D)$ such that $u(z,w) = 0$ if $|z| \leqslant 4$ or $|z| \geqslant 14$ and that there is no continuous plurisubharmonic function g on a neighbourhood of $\overline{D}$ verifying $|g - u| < 1$ on A, since otherwise there would be a violation to the maximum principle on B.

Now denote by f the holomorphic mapping $f(z,w) = (\frac{1}{z-a}, w)$, where a is any number in $(16, 17)$. Then $\Omega := f(D')$ is an unbounded domain in $\mathbf{C}^2$ with smooth boundary, where $D' = D \backslash \{\{a\} \times \mathbf{C}\}$. Define $\tilde{u} := u \circ f^{-1}$, then $\tilde{u}$ is continuous and plurisubharmonic on Ω. Notice that, in view of (iv), the function u^* has compact support in $\overline{\Omega}$. Assume that (a) of Theorem 4.1 holds on Ω, then by Remark (a) there is a $g \in \mathcal{A}_1$ satisfying $|g - \tilde{u}| < 1$ on $f(A)$. Set $\tilde{g} = g \circ f^{-1}$. Then $|\tilde{g} - u| < 1$ on A. Observe that $\tilde{g}$ is plurisubharmonic on D', continuous at every point of $\partial D \backslash \{\{a\} \times \mathbf{C}\}$, and equals to 0 near the complex line $z = a$. Thus we may extend $\tilde{g}$ to an element of $\mathrm{PSH}^c(D)$ by setting $\tilde{g} = 0$ on $\overline{D} \cap (\{a\} \times \mathbf{C})$. By Theorem 1 in [6], the function $\tilde{g}$ can be approximated uniformly on $\overline{D}$ by continuous plurisubharmonic functions on neighbourhood s of $\overline{D}$. Thus we can find such a function g^* satisfying $|g^* - u| < 1$ on A. This is a contradiction to the property (iv).

(c) Similar statements to the equivalence between (a) and (b) in Theorem 4.1 have been claimed in Section 2 in [9] and at the end of the proof of Theorem 4.3 in [11]. The proofs given in these references contain gaps due to incorrect applications of Hahn-Banach's separation theorem. However, we can give honest (and simpler) proofs by repeating the proof of Theorem 4.1.

It remains to decide when one of the equivalent conditions in Theorem 4.1 holds. For this, we recall the following terminology from Section 3 of [11] (see also [9]).

Definition 4.2. — *By an isotopy family of biholomorphic mappings defined on Ω, we mean a continuous map $\Phi : \overline{\Omega} \times [0,1] \to \mathbf{C}^n$ having the following properties.*

(a) *$\Phi_t := \Phi(t, \cdot)$ maps Ω biholomorphically onto its image; moreover, Φ_t is a homeomorphism between $\overline{\Omega}$ and $\overline{\Phi_t(\Omega)}$.*
(b) *$\Phi_t^{-1}(z)$ is real analytic in t on a neighbourhood of 0 for all $z \in \Omega$.*
(c) *Φ_t^{-1} converges uniformly to $\Phi_0^{-1} = \mathrm{Id}$ on compact subsets of $\overline{\Omega}$ when $t \to 0$.*

Definition 4.3. — *Let Φ_t be an isotopy family of biholomorphic mappings on Ω. Then by the boundary cluster set of Φ_t we mean the set of limit points of sequence of elements in $\overline{\Omega} \cap \Phi_t(\partial\Omega)$ when $t \to 0$.*

Theorem 4.4. — *Assume that Ω is of bounded type and that there is an isotopy family Φ_t of biholomorphic mappings on Ω such that for every z lying in the boundary cluster set X of Φ_t we have $J_z(\mathcal{A}_1) = \{\delta_z\}$. Then the following assertions hold.*

(a) $J_z(\mathcal{A}_1) = J_z(\mathcal{A}_2)$ *for all* $z \in \Omega$.

(b) *Assume in addition that Ω is regular in the real sense. Then for every bounded function $h \in \mathcal{C}(\partial\Omega)$ there exists a bounded function $\Psi \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\Omega)$ satisfying the following properties.*

(i) $\Psi^* \leqslant h$ *on* $\partial\Omega$.

(ii) $\lim_{x\to z} \Psi(x) = h(z)$ *for all* $z \in \partial\Omega$ *that satisfies* $J_z(\mathcal{A}_1) = \{\delta_z\}$.

Moreover, if $J_z(\mathcal{A}_1) = \{\delta_z\}$ for every $z \in \partial\Omega$ then there exists a unique bounded function $\Psi \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $u = h$ on $\partial\Omega$.

Proof. — (a) We will use some ideas in the proof of Theorem 4.4 in [9] (see also Theorem 3.5 in [11]). Fix $\varphi \in \mathcal{C}_0(\overline{\Omega})$ and $z_0 \in \Omega$, we will show that $S_1\varphi(z_0) = S_2\varphi(z_0)$. For this, notice that $\inf_\Omega \varphi \leqslant S_2\varphi \leqslant (S_2\varphi)^* \leqslant \varphi$. This implies that $S_2\varphi = (S_2\varphi)^* \in \mathcal{A}_2$. By Choquet's lemma we can choose a sequence $\{v_m\}_{m\geqslant 1} \subset \mathcal{A}_1$ such that $v_m \uparrow S_1\varphi$ on $\overline{\Omega}$. Observe that, by Theorem 3.1 we also have

$$S_1\varphi \equiv S_2\varphi \equiv \varphi \text{ on } X. \tag{4.1}$$

In particular $v_m \uparrow \varphi$ on X. Now for $t \in (0,1)$ we define $u_t = (S_2\varphi) \circ \Phi_t^{-1}$. It is clear that u_t is non negative, plurisubharmonic on $\Phi_t(\Omega)$. For $k > 0$ we set

$$\Omega_k := \{z : z \in \Omega, |z| \leqslant k, \mathrm{dist}(z, \partial\Omega) > 1/k\}.$$

Let $\rho \in \mathcal{C}_0^\infty(\mathbf{C}^n)$ be a nonnegative, radial function with support in the unit ball and satisfies $\int \rho\, dV = 1$, where $dV\mathbf{C}^n$ is the volume form in $\mathbf{C}^n$. Set $\rho_\delta := \delta^{-n}\rho(z/\delta)$ for $\delta > 0$.

Fix $\varepsilon > 0$, we claim that there is $t_0 \in (0,1), k_0 \geqslant 1$ and $m \geqslant 1$ such that for all $t \in (0, t_0)$ and $k \geqslant k_0$ there is

$$0 < \delta_k < a_k := \min\nolimits_{0\leqslant t\leqslant t_0} \mathrm{dist}(\Phi_t(\partial\Omega), \Phi_t(\partial\Omega_k))$$

satisfying

$$u_t * \rho_{\delta_k} + \psi * \rho_{\delta_k} - \varepsilon \leqslant v_m \text{ on } \overline{\Omega} \cap \Phi_t(\partial\Omega_k),$$

where the convolution of a locally integrable function u with ρ_δ is defined by

$$(u * \rho_\delta)(\xi) := \int_{|t|<\delta} u(\xi - t)\rho_\delta(t)dV(t).$$

Assume otherwise; then we obtain sequences $t_j \downarrow 0, k_j \uparrow \infty, m_j \uparrow \infty$, $\{\xi_j\}_{j\geqslant 1} \subset \overline{\Omega} \cap \Phi_t(\partial\Omega_k)$, and $0 < \delta_{k_j} < a_{k_j}$ such that

$$(u_{t_j} * \rho_{\delta_{k_j}})(\xi_j) + (\psi * \rho_{\delta_{k_j}})(\xi_j) - \varepsilon > v_{m_j}(\xi_j)\ \forall j.$$

After passing to a subsequence we may assume that either $|\xi_j| \to \infty$ or $\xi_j \to \xi^* \in X$. The first possibility is excluded since ψ tends to $-\infty$ at infinity and v_{m_j} are uniformly bounded from below. The second possibility can not occur either, since

$$\liminf_{j\to\infty} v_{m_j}(\xi_j) \geqslant \lim_{j\to\infty} v_{m_j}(\xi^*) = \varphi(\xi^*). \tag{4.2}$$

On the other hand, from (6) and the condition (c) in Definition 4.2 we obtain

$$\limsup_{j\to\infty} \big(u_{t_j} * \rho_{\delta_{k_j}}\big)(\xi_j) - \varepsilon \leqslant \varphi(\xi^*) - \varepsilon. \tag{4.3}$$

Combining (4.2) and (4.3) we get a contradiction. Thus the claim follows. By shrinking t_0 and increasing k_0 we may obtain that $z_0 \in \Phi_t(\Omega_k)$ for all $k \geqslant k_0$ and $t \in (0, t_0)$. Consider the function

$$\tilde{v}_{t,k,m} = \begin{cases} \max\{v_m, u_t * \rho_{\delta_k} + \psi * \rho_{\delta_k} - \varepsilon\} & \text{on } \Omega \cap \Phi_t(\Omega_k) \\ v_m & \text{on } \overline{\Omega}\backslash\Phi_t(\Omega_k). \end{cases}$$

By the claim proven above we can check that $\tilde{v}_{t,k,m} \in \mathcal{A}_1$ and that $\tilde{v}_{t,k,m} \leqslant \varphi$ on $\overline{\Omega}$. It follows that $\tilde{v}_{t,k,m} \leqslant S_1\varphi$ on $\overline{\Omega}$, in particular

$$S_1\varphi(z_0) \geqslant \tilde{v}_{t,k,m}(z_0) \geqslant (u_t * \rho_{\delta_k})(z_0) - \varepsilon \geqslant u_t(z_0) - \varepsilon.$$

Taking the limsup of the rightmost term when $t \to 0$, and observing the curve $t \mapsto \Phi_t^{-1}(z_0)$, being real analytic near 0, is not plurithin at 0, we infer that $S_1\varphi(z_0) \geqslant S_2\varphi(z_0) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary we have $S_1\varphi(z_0) = S_2\varphi(z_0)$. Now by Theorem 4.1 we conclude that $J_z(\mathcal{A}_1) = J_z(\mathcal{A}_2)$ for all $z \in \Omega$.

(b) After adding a constant to h, we may assume that $h \geqslant 0$ on $\partial\Omega$. Let $\tilde{h}$ be the function equal to h on $\partial\Omega$ and to $M = \sup_{\partial\Omega} h$ on Ω. Since $J_z(\mathcal{A}_1) = J_z(\mathcal{A}_2)$ on Ω, by Proposition 3.3 we have $S_1\tilde{h} = S_2\tilde{h}$ on Ω and $S_1\tilde{h}(z) = h(z)$ for all $z \in \partial\Omega$ satisfying $J_z(\mathcal{A}_1) = \{\delta_z\}$. We will show that $\Psi := S_2\tilde{h}$ is the desired function. Since Ω is regular in the real sense, by Lemma 2.4 we have $0 \leqslant \Psi^* \leqslant \tilde{h}$ on $\overline{\Omega}$. The proof of Proposition 3.5 also implies that $\Psi^* \in \mathrm{MPSH}(\Omega)$. Now we claim that $\Psi^* = \Psi$ on Ω. For this, fix $\varepsilon > 0$ and define on Ω the function

$$u_\varepsilon := \max(\Psi^* + \varepsilon\psi, 0).$$

Observe that $u_\varepsilon^* \in \mathcal{A}_2$ and satisfies $u_\varepsilon \leqslant \tilde{h}$ on $\overline{\Omega}$. It follows that $u_\varepsilon \leqslant S_2\tilde{h}$ on Ω. Letting $\varepsilon \to 0$, one obtains $\Psi^* \leqslant S_2\tilde{h}$ on Ω. The claim follows. In particular $\Psi \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\Omega)$. Finally, fix a point $z \in \partial\Omega$ such that $J_z(\mathcal{A}_1) = \{\delta_z\}$ then

$$\liminf_{x\to z} \Psi(x) \geqslant \liminf_{x\to z} S_1\tilde{h}(x) \geqslant h(z).$$

Therefore $\lim_{x\to z} \Psi(x) = h(z)$.

Finally, assume that there are functions $\Psi, \Psi' \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $\Psi = \Psi' = h$ on $\partial\Omega$. Let $\{\Omega_k\}_{k\geqslant 1}$ be a sequence of subdomains of Ω such that $\Omega_k \uparrow \Omega$. Fix $\varepsilon > 0$. By the assumptions on boundary values of Ψ and Ψ' we see that there is k_0 so large that $\Psi + \varepsilon\psi \leqslant \Psi' + \varepsilon$ on $\partial\Omega_k$ for all $k \geqslant k_0$. Using maximality of Ψ' one obtains $\Psi + \varepsilon\psi \leqslant \Psi'$ on Ω_k for all $k \geqslant k_0$. Letting $k \to \infty$ and $\varepsilon \to 0$ we have $\Psi \leqslant \Psi'$ on Ω. By changing the role of Ψ and Ψ' we get $\Psi = \Psi'$ on Ω. The proof is thereby concluded. □

Remarks. — (a) If $J_z(\mathcal{A}_1) = \{\delta_z\}$ for every $z \in \partial\Omega$ then by considering the family $\Phi_t = Id$ for all t, we deduce from Theorem 4.4 that $J_z(\mathcal{A}_1) = J_z(\mathcal{A}_2)$ for all $z \in \Omega$. The reverse implication does not hold in general. Indeed, consider the following pseudoconvex domain of bounded type $\Omega := \{(z,w) : 1 < |z| < |w|\}$. Since through every boundary point of $\partial\Omega$ we can find a non constant complex analytic disk lying in $\partial\Omega$ and passes through this point, by Proposition 3.4 we see that $J_z(\mathcal{A}_1) \neq \{\delta_z\}$ for *every* $z \in \partial\Omega$. On the other hand, by considering the family $\Phi_t(z,w) := (z, (1+t)w)$ and applying Theorem 4.4 we infer that $J_z(\mathcal{A}_1) = J_z(\mathcal{A}_2)$ for all $z \in \Omega$. It is an open problem if such an example can be found in the class of unbounded pseudoconvex domains with *smooth* boundaries.

(b) It is proved in Theorem 26 of [15] that an unbounded strictly pseudoconvex domain Ω is $B-$regular if Ω satisfies the following condition:

(L') there exists a holomorphic polynomial p such that :

$$|p(z)|^2 > (1+|z|^2)^{\deg p}\ \forall z \in \Omega.$$

We claim that if the domain Ω satisfies the condition (L') then it is of bounded type. Indeed, pick a polynomial p such that

$$|p(z)|^2 > (1+|z|^2)^{\deg p}\ \forall z \in \Omega.$$

Then we have $\log|p| > 0$ on Ω and $\log|p(z)| \to \infty$ as $|z| \to \infty$. It follows that the function

$$\psi(z) := \frac{1}{4}\log(1+|z|^2) - \frac{1}{\deg p}\log|p(z)|$$

is negative, plurisubharmonic on Ω and tends to $-\infty$ when $|z|$ goes to ∞. This proves the claim. We own this elegant proof to the referee.

In the next result, we deal with domains which are not necessarily of bounded type but stronger conditions on boundary data are imposed.

PROPOSITION 4.5. — *Assume that Ω satisfies the following conditions.*

(i) *Ω is regular in the real sense.*
(ii) *There is an isotopy family Φ_t of biholomorphic mappings on Ω such that for every z lying in the boundary cluster set X of Φ_t we have $J_z(\mathcal{A}_1) = \{\delta_z\}$.*
(iii) *$\overline{\Omega}$ contains no complex hyperplane at infinity.*

Then for every $h \in \mathcal{C}_0(\partial\Omega), h \geqslant 0$, there is $u \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\Omega)$ satisfying (i) and (ii) in Theorem 4.4 (b).

Proof. — Set $M = \sup_{\partial\Omega} h$. Let $\tilde{h} = h$ on $\partial\Omega$ and $\tilde{h} = M$ on Ω. By the *proof* of Proposition 3.5, $S_2\tilde{h} \in \mathrm{MPSH}(\Omega)$ and satisfies $(S_2\tilde{h})^* \leqslant h$ on $\partial\Omega$. We will show that $u := S_2\tilde{h}$ is the desired function. For this, we first observe that by Proposition 2.9

$$\lim_{|z|\to\infty} u(z) = 0.$$

Next, we apply Choquet's lemma to get a sequence $\{v_m\}_{m\geqslant 1} \subset \mathcal{A}_1$ such that $0 \leqslant v_m \leqslant \tilde{h}$ and $v_m \uparrow S_1\tilde{h}$ on $\overline{\Omega}$. Define for $t \in [0,1]$ the function

$$u_t := u \circ \Phi_t^{-1}.$$

Given $\varepsilon > 0$, using (10) and the same reasoning as in the proof of Theorem 4.4 (a), we can find $t_0 \in (0,1), k_0 \geqslant 1$ and $m \geqslant 1$ such that for all $t \in (0, t_0)$ and $k \geqslant k_0$ there is

$$0 < \delta_k < a_k := \min_{0\leqslant t\leqslant t_0} \operatorname{dist}(\Phi_t(\partial\Omega), \Phi_t(\partial\Omega_k))$$

satisfying

$$u_t * \rho_{\delta_k} - \varepsilon \leqslant v_m \text{ on } \overline{\Omega} \cap \Phi_t(\partial\Omega_k),$$

where Ω_k is defined as in Theorem 4.4. Now we proceed exactly as in the proof of Theorem 4.4.(a) to reach $u = S_1\tilde{h}$ on Ω. Thus $u \in \mathcal{C}(\Omega)$ and satisfies $\lim_{x\to z} u(x) = h(z)$ for every $z \in \partial\Omega$ satisfying $J_z(\mathcal{A}_1) = \{\delta_z\}$. The proof is complete. ☐

5. Examples of unbounded $B-$regular domains

We start with the following generalization of Proposition 19 in [15].

Proposition 5.1. — *Assume that Ω is an unbounded convex domain in $\mathbf{C}^n$ with $\mathcal{C}^1$ smooth boundary. For every $z \in \partial\Omega$, denote by K_z the intersection between $T_z\partial\Omega$ the real tangent space at z and $\partial\Omega$. Assume that for every $z \in \partial\Omega$ there is a real hyperplane $L_z \subset T_z\partial\Omega$ satisfying $L_z \cap K_z = \{z\}$. Then Ω is $B-$regular. In particular, every strictly convex domain with $\mathcal{C}^2$ smooth boundary is $B-$regular.*

Proof. — First we must show that Ω is locally $B-$regular. Fix $z \in \partial\Omega$ and an open ball U around z. Let μ be a Jensen measure relative to $PSH^c(U \cap \Omega)$ with barycentre at z. Since Ω is convex, we see that the support of μ is contained in $K_z \cap \partial\Omega \cap U$. From the existence of L_z, we can find a pluriharmonic function u on $\mathbf{C}^n$ such that $u(z) = 0$ whereas $u < 0$ on $K_z\backslash\{0\}$. It follows that $\mu = \{\delta_z\}$. Thus Ω is locally $B-$regular. Now we claim that Ω is of bounded type. Fix a point $z_0 \in \partial\Omega$. Let ρ be a local defining function for Ω on a neighbourhood U of z_0. Define the map

$$\Phi(z) := \left(\frac{\partial\rho}{\partial z_1}(z), \cdots, \frac{\partial\rho}{\rho z_n}(z)\right),\ \forall z \in \partial\Omega \cap U.$$

The following claim is crucial. There exists $a_1, \cdots, a_n \in \partial\Omega \cap U$ such that n vectors $\Phi(a_1), \cdots, \Phi(a_n)$ are linearly independent. Assume otherwise, then $\Phi(\partial\Omega \cap U)$ is contained in a complex hyperplane. Therefore, there is a vector $\lambda := (\lambda_1, \cdots, \lambda_n)$ satisfying $\sum_{k=1}^n \lambda_k \frac{\partial\rho}{\partial z_1}(z) = 0$ for all $z \in \partial\Omega \cap U$. This means that $\lambda \in T_z\Omega$ for all $z \in \partial\Omega$. If $\lambda \notin \partial\Omega$, then there is some point $\lambda' \in \partial\Omega$ which is closest to λ. Obviously $\lambda \notin T_{\lambda'}\partial\Omega$. A contradiction. Thus $\lambda \in \partial\Omega$. It follows that $T_\lambda\partial\Omega$ contains an open piece of $\partial\Omega$ which is absurd. The claim follows. Then we can push those points a_i slightly *into* the half spaces separated by $T_{a_i}\partial\Omega$ and disjoint from $\overline{\Omega}$ to get points a_i'. Let H_i be the complex hyperplane passing through a_i' and parallel to the complex tangent space at a_i. Clearly $H_i \cap \overline{\Omega} = \emptyset$ and dist $(H_i, \partial\Omega) > 0$. By the choices of a_i' we also have $\bigcap_{i=1}^n H_i$ is a singleton. Now the claim follows from Proposition 2.7. Finally, the proof is completed by applying Theorem 4.4. □

The next result is another easy application of Theorem 4.4.

Proposition 5.2. — *Assume that Ω is of bounded type and regular in the real sense and that there is an open subset A of ∂D having the following properties.*

(a) $tA \cap \overline{\Omega} = \emptyset$ *for every* $t > 1$.
(b) $J_z(\mathcal{A}_1) = \{\delta_z\}$ *for every* $z \in \partial\Omega\backslash A$.

Then for every bounded function $h \in \mathcal{C}(\partial\Omega)$, there is $\Psi \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\Omega)$ such that $\Psi^* \leqslant h$ on $\partial\Omega$ and $\lim_{x\to z} \Psi(x) = h(z)$ for all $z \in \partial\Omega$ satisfying $J_z(\mathcal{A}_1) = \{\delta_z\}$.

Proof. — Denote by $\Phi_t(z) = (1+t)z$ for $t \geqslant 0$ and $z \in \mathbf{C}^n$. Clearly Φ_t is an isotopy of biholomorphic mappings on $\overline{\Omega}$. We can check, in view of the assumptions (a) that the boundary cluster of Φ_t is contained $\partial D\backslash A$. Thus using (b), we conclude the proof by invoking Theorem 4.4. □

Remarks. — (a) As an a concrete application of Proposition 5.2, consider the following unbounded domain in $\mathbf{C}^2$

$$\Omega = \{(z,w) : \Re\, z \leqslant 0, -1 < \Re\, w < 1\} \cup \{(z,w) : |z|^2 + |w|^2 < 1, \Re\, z > 0\}.$$

It is easy to check that Ω is convex and satisfies $t\partial\Omega \cap \overline{\Omega} = \emptyset$ for $t > 1$. In particular, Ω is regular in the real sense. Since the hyperplanes $z = 2$ and $w = 2$ have positive distances to $\overline{\Omega}$, by Proposition 2.7, Ω is of bounded type. Notice also that $A := \partial\Omega \cap \{\Re\, z > 0\}$ is strictly pseudoconvex. Thus for every bounded function $h \in \mathcal{C}(\partial\Omega)$, there exists $u \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfying $u = h$ on A. By pushing A slightly inside the unit ball, we may construct similar examples where Ω is not convex.

(b) It is not hard to check that the domain Ω given in the above remark also satisfies the assumptions of Proposition 4.5. Thus, for every $h \in \mathcal{C}_0(\partial\Omega), h \geqslant 0$, there exists a non negative function $u \in \mathrm{MPSH}(\Omega) \cap \mathcal{C}(\Omega)$ such that $\lim_{x\to z} u(x) = h(z)$ for all $z \in A$ and $\lim_{|z|\to\infty} u(z) = 0$.

The final result of the section is an invariant property for a class of unbounded $B-$regular domains. Before formulating it, we introduce the following terminology. A surjective holomorphic map f between domains Ω and Ω' in $\mathbf{C}^n$ is said to have the property (P) if the following conditions hold.

(a) f extends continuously to $\partial\Omega$ and $f : \overline{\Omega} \to \overline{\Omega'}$ is an open surjective continuous map.

(b) There is (possibly empty) a complex subvariety E of Ω' such that $f : \Omega\backslash f^{-1}(E) \to \Omega'\backslash E$ is a holomorphic covering *i.e.*, for every $w \in \Omega'\backslash E$, there is a neighbourhood U of w such that $f^{-1}(U)$ is a union of at most countably disjoint open sets V_i and h is a biholomorphism from V_i onto U.

It is immediate to check that if $f : \Omega \to \Omega'$ is a holomorphic proper map which extends holomorphically to a neighbourhood of $\overline{\Omega}$ then f has the property (P).

Proposition 5.3. — *Let Ω, Ω' be unbounded domains in $\mathbf{C}^n$ and $f : \Omega \to \Omega'$ be a holomorphic map having the property (P). Assume that Ω is $B-$regular and that $\overline{\Omega}$ does not contain any complex hyperplane at infinity.*

Then Ω' is locally $B-$regular. Moreover, if we assume in addition that Ω is of bounded type then Ω' is $B-$regular.

Proof. — (a) Fix $z_0 \in \partial\Omega$. Define

$$h(z) = \max(-|f(z) - f(z_0)|, -1) \ \forall z \in \partial\Omega.$$

Since Ω is $B-$regular, we can find $u \in \mathrm{PSH}^c(\Omega)$ such that $u = h$ on $\partial\Omega$. Since $\overline{\Omega}$ contains no complex hyperplane at infinity, by Proposition 2.9 we have $u < 0$ on Ω and

$$\limsup_{|z|\to\infty} u(z) \leqslant -1.$$

Define

$$\tilde{u}(w) = \sup\{u(\xi) : \xi \in f^{-1}(w)\} \ \forall w \in \Omega' \backslash E.$$

We claim that $\tilde{u}^* < 0$ on $\partial\Omega' \backslash \{f(z_0)\}$ while $\lim_{x\to f(z_0)} \tilde{u}(x) = 0$. Indeed, given $w^* \in \partial\Omega\backslash\{f(z_0)\}$, then $w^* = f(\xi^*)$ where $\xi^* \in \partial\Omega\backslash\{z^*\}$. Consider an arbitrary sequence $w_j \in \Omega'\backslash E, w_j \to w^*$ such that $\tilde{u}(w_j) \to \alpha$. It suffices to check that $\alpha < 0$. To this end, we choose for $j \geqslant 1$ a point $\xi_j \in \Omega$ such that

$$f(\xi_j) = w_j \text{ and } \alpha \leqslant u(\xi_j) + 1/j.$$

After passing to a subsequence we may assume that either $|\xi_j| \to \infty$ or $\xi_j \to \xi' \in \overline{\Omega}$. In either case, it is easy to check by using properties of u that $\alpha < 0$. Now we deal with the point $f(z_0)$. Since f is open, we can find a small open neighbourhood U of z_0 in $\overline{\Omega}$ such that $f(U)$ is an open neighbourhood of $f(z_0)$ in $\overline{\Omega'}$. By the choice of $u, \tilde{u}$ and h, it is not hard to show

$$\lim_{x\to f(z_0), x\in f(U)\backslash E} \tilde{u}(x) = 0.$$

Putting all this together, the claim follows.

Next, since $f : \Omega\backslash f^{-1}(E) \to \Omega'\backslash E$ is a holomorphic covering, we see that *locally* on $\Omega'\backslash E$, the function $\tilde{u}$ is supremum of a family of negative plurisubharmonic functions. Thus $\tilde{u}^*$ is plurisubharmonic on $\Omega'\backslash E$. Notice that $f^{-1}(E)$ is pluripolar in Ω, so $\tilde{u}^*$ is in fact plurisubharmonic on Ω. This statement is implicit in the proof of the theorem on extending locally bounded plurisubharmonic functions through pluripolar sets (*cf.* Chapter 2 in [8]). By the claim proven above, we have also

$$\limsup_{x\to f(z_0), x\in\Omega'} \tilde{u}^*(x) < 0 \text{ on } \partial\Omega'\backslash\{f(z_0\}.$$

Since z_0 is arbitrary, we may apply Theorem 2.1 to find that Ω is locally $B-$regular.

Finally, by Theorem 4.4, it is enough to check that if Ω is of bounded type then so is Ω'. Take a negative real valued function $\psi \in \mathrm{PSH}(\Omega)$ such that $\lim_{|z|\to\infty} \psi(z) = -\infty$. Define

$$\tilde{\psi}(w) = \sup\{\psi(\xi) : \xi \in f^{-1}(w)\} \ \forall w \in \Omega' \setminus E.$$

Reasoning as above, $\tilde{\psi}^*$ is a negative real valued plurisubharmonic function on Ω' and satisfies $\lim_{|w|\to\infty} \tilde{\psi}^*(w) = -\infty$. Thus Ω' is of bounded type. The proof is complete. □

BIBLIOGRAPHY

[1] E. Bedford & A. Taylor, "A new capacity for plurisubharmonic functions", *Acta Math.* **149** (1982), p. 1-40.

[2] Z. Blocki, "The complex Monge-Ampère operator in hyperconvex domains", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **23** (1996), p. 721-747.

[3] H. Bremermann, "On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Silov boundaries", *Trans. Amer. Math. Soc.* **91** (1959), p. 246-276.

[4] A. Brown & C. Pearcy, *An Introduction to Analysis*, new york ed., Graduate Texts in Mathematics, vol. 154, Springer-Verlag, 1995.

[5] D. Edwars, "Choquet boundary theory for certain spaces of lower semicontinuous functions", in *Function algebras* (Chicago) (Scott-Foresman, ed.), Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965, 1966, p. 300-309.

[6] J. Fornaess & J. Wiegerinck, "Approximation of plurisubharmonic functions", *Ark. Mat.* **27** (1989), p. 257-272.

[7] K. W. K. Hayman & B. Kennedy, "Subharmonic function", in *London Mathematical Society Monographs, I*, no. 9, Academic Press, Harcourt Brace Jovanovich, London-New York, 1976, p. xvii+284.

[8] M. Klimek, *Pluripotential theory*, London Mathematical Society Monographs. New Series, vol. 6, The Clarendon Press Oxford University Press, New York, 1991, Oxford Science Publications, xiv+266 pages.

[9] Q. D. Nguyen, "Approximation of plurisubharmonic functions on bounded domains in $\mathbf{C}^n$", *Michigan Math. J.* **54** (2006), no. 3, p. 697-711.

[10] Q. D. Nguyen, N. T. Dung & D. H. Hung, "$B-$regularity of certain domains in $\mathbf{C}^n$,", *Annales Polon. Math.* **86** (2005), no. 2, p. 137-152.

[11] Q. D. Nguyen & F. Wikström, "Jensen measures and approximation of plurisubharmonic functions", *Michigan Math. J.* **53** (2005), p. 529-544.

[12] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

[13] N. Shcherbina & G. Tomassini, "The Dirichlet problem for Levi-flat graphs over unbounded domains", *Internat. Math. Res. Notices* **3** (1999), p. 111-151.

[14] N. Sibony, "Une classe des domaines pseudoconvex", *Duke Math. J.* **55** (1987), p. 299-319.

[15] A. Simioniuc & G. Tomassini, "The Bremermann-Dirichlet problem for unbounded domains in $\mathbf{C}^n$", http://it.arxiv.org/math/pdf/0702/0702179v1.pdf, preprint, 2007, to be published in Manuscripta Mat.

[16] J. Walsh, "Continuity of envelopes of plurisubharmonic functions", *J. Math. Mech.* **18** (1968/1969), p. 143-148.

[17] J. Wikström, "Jensen measures and boundary values of plurisubharmonic functions", *Ark. Mat.* **39** (2001), p. 181-200.

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