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AN INTERPOLATION THEOREM IN TORIC VARIETIES

by Martin WEIMANN

ABSTRACT. — In the spirit of a theorem of Wood, we give necessary and sufficient conditions for a family of germs of analytic hypersurfaces in a smooth projective toric variety X to be interpolated by an algebraic hypersurface with a fixed class in the Picard group of X.

RÉSUMÉ. — Dans la lignée d'un théorème de Wood, on donne des conditions nécessaires et suffisantes pour qu'une famille de germes d'hypersurfaces analytiques d'une variété torique projective lisse X s'interpole par une hypersurface algébrique de classe de Picard donnée.

1. Introduction

Let X be a compact algebraic variety over $\mathbb{C}$. We are interested in the following problem:

Let $V_1, \ldots, V_N$ be a collection of germs of smooth analytic hypersurfaces at pairwise distincts smooth points $p_1, \ldots, p_N$ of X, and fix α in the Picard group $Pic(X)$ of X. When does there exist an algebraic hypersurface $\widetilde{V} \subset X$ with class α containing all the germs V_i ?

A natural way to answer this question is to study sums and products of values of rational functions at points of intersection of the germs V_i with a "moving" algebraic curve(1).

Let us recall a theorem of Wood [21] treating the case of N germs in an affine chart $\mathbb{C}^n$ of $X = \mathbb{P}^n$, transversal to the line $l_0 = \{x_1 = \cdots = x_{n-1} = 0\}$. Any line l_a close to l_0 has affine equations $x_k = a_{k0} + a_{k1} x_n$, $k = 1, \ldots, n-1$.

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(1) This idea goes back to Abel in his studies of abelian integrals [1].

The trace on $V = V_1 \cup \cdots \cup V_N$ of any function f holomorphic in an analytic neighborhood of V is the function

$$a \longmapsto Tr_V(f)(a) := \sum_{p \in V \cap l_a} f(p)\,,$$

defined and holomorphic for $a = ((a_{10}, a_{11}), \ldots, (a_{n-1,0}, a_{n-1,1}))$ close enough to $0 \in \mathbb{C}^{2n-2}$.

Theorem (*Wood,* [21]). — *There exists an algebraic hypersurface $\widetilde{V} \subset \mathbb{P}^n$ of degree N which contains V if and only if the function $a \mapsto Tr_V(x_n)(a)$ is affine in the constant coefficients $a_0 = (a_{10}, \ldots, a_{n-1,0})$.*

We show here that Wood's theorem admits a natural generalization to the case of germs $V_1, \ldots, V_N$ in a smooth toric compactification X of $\mathbb{C}^n$ endowed with an ample line bundle. While our proof is constructive, we do not obtain (contrarly to [21]) the explicit construction of the polynomial equation of the interpolating hypersurface in the affine chart $\mathbb{C}^n$. Thus, in that toric context, we need more informations to characterize the class of $\widetilde{V}$ in $Pic(X)$.

For any projective variety X, there exist very ample line bundles $L_1, \ldots, L_{n-1}$ and a global section $s_0 \in \Gamma(X, L_1) \oplus \cdots \oplus \Gamma(X, L_{n-1})$ whose zero locus is a smooth irreducible curve C which intersects transversely each germ V_i at p_i. A generic point a in the associated parameter space

$$X^* := \mathbb{P}(\Gamma(X, L_1)) \times \cdots \times \mathbb{P}(\Gamma(X, L_{n-1}))$$

determines a closed curve C_a in X, which, for a close enough to the class $a^0 \in X^*$ of s_0, is smooth and intersects each germ V_i transversely at a point $p_i(a)$ whose coordinates vary holomorphically with a by the implicit functions theorem. For any function f holomorphic at $p_1, \ldots, p_N$, we define the trace of f on $V := V_1 \cup \cdots \cup V_N$ relatively to $(L_1, \ldots, L_{n-1})$ as the function

$$a \longmapsto Tr_V(f)(a) := \sum_{i=1}^{N} f(p_i(a)),$$

which is defined and holomorphic for a in an analytic neighborhood of a^0.

Let us suppose now that X is a toric projective smooth compactification of $U = \mathbb{C}^n$, endowed with a linear action of an algebraic torus $\mathbb{T}$ that preserves the coordinate hyperplanes $x_i = 0,\ i = 1, \ldots, n$ (see [7]). Clearly, any germ V_i contained in the hypersurface at infinity $X \setminus U$ is algebraic. We can thus suppose that V is contained in U and work with the affine coordinates $x = (x_1, \ldots, x_n)$.

Since the Picard group of $U=\mathbb{C}^n$ is trivial, the classes of the irreducible divisors $G_1,\dots,G_s$ supported outside U form a basis for $Pic(X)$. Any globally generated line bundle L on X has thus a unique global section $s_U\in\Gamma(X,L)$ such that $div(s_U)\cap U=\emptyset$. If $s\in\Gamma(X,L)$, the quotient $\frac{s}{s_U}$ defines a rational function without poles on $U\simeq\mathbb{C}^n$, that is, a polynomial in x, which gives the local equation for the divisor $H=div(s)$ in the affine chart U. Since L is globally generated, a generic section $s\in\Gamma(X,L)$ does not vanish at $0\in U$ and the corresponding polynomial in x has a non-zero constant term.

In the context of very ample line bundles $L_1,\dots,L_{n-1}$ on X, we can then use polynomials equations for C_a restricted to the affine chart U:

$$C_a\cap U=\{x=(x_1,\dots,x_n)\in U,\ a_{k0}=q_k(a'_k,x),\ k=1,\dots,n-1\},$$

where $a_k=(a_{k0},a'_k)$ and $q_k(a'_k,.)$ are polynomials in x vanishing at $0\in U$.

Since X is toric, we know from [9] that the Chow groups $A_k(X)$ are isomorphic to the cohomology groups $H^{2n-2k}(X,\mathbb{Z})$, for any $k=0,\dots,n$, and we can identify the Chow group $A_0(X)$ of 0-cycles with $\mathbb{Z}\simeq H^{2n}(X,\mathbb{Z})$. We denote by $[V]$ the class of any closed subvariety V of X, $c_1(L)\in H^2(X,\mathbb{Z})$ the first Chern class of any line bundle L on X, and we denote by $\frown$ the usual cap product. Our first result is

Theorem 1.1. — *The set $V:=V_1\cup\cdots\cup V_N$ is contained in an algebraic hypersurface $\widetilde{V}\subset X$ such that*

$$[\widetilde{V}]\frown\prod_{k=1}^{n-1}c_1(L_k)=N$$

if and only if for all $i=1,\dots,n$ the functions $a\mapsto Tr_V(x_i)(a)$ are affine in the constant coefficients $a_0=(a_{10},\dots,a_{n-1,0})$.

Note that the left hand side in the formula of Theorem 1 is the intersection number, so that it must be at least N if the required algebraic hypersurface $\widetilde{V}$ exists. If the conditions of Theorem 1 are not satisfied, V can nevertheless be contained in a hypersurface $\widetilde{V}$ of X such that $[\widetilde{V}]\frown\prod_{k=1}^{n-1}c_1(L_k)>N$. In this case, traces of affine coordinates are algebraic in a_0 and no longer polynomials.

It is shown in [20] that in the projectice case $X=\mathbb{P}^n$, Wood's theorem can be derived from the Abel-inverse theorem obtained in [13], using some rigidity properties of a particular system of PDE's. Using similar arguments, the following toric Abel-inverse theorem is proved in [19], Chapter 2, as a corollary of Theorem 1.

Theorem. — *Let ϕ be a holomorphic form of maximal degree on V, not identically zero on any germs V_i, for $i = 1, \dots, N$. There exists an algebraic hypersurface $\widetilde{V} \subset X$ containing V such that $[\widetilde{V}] \frown \prod_{k=1}^{n-1} c_1(L_k) = N$ and a rational form Ψ on $\widetilde{V}$ such that $\Psi_{|V} = \phi$, if and only if the trace form $Tr_V\phi(a) := \sum_{i=1}^N p_i^*(\phi)(a)$ is rational in a_0.*

Let us remark that it should be interesting to derive Theorem 1 from the previous theorem by choosing some form ϕ related to the coordinate functions x_i.

Contrarly to the projective case handled in [21], Theorem 1 does not characterize the class of $\widetilde{V}$. To do so, we introduce the norm on V relatively to $(L_1, \dots, L_{n-1})$ of any function f holomorphic at $p_1, \dots, p_N$,

$$a \longmapsto N_V(f)(a) := \prod_{i=1}^{N} f(p_i(a)),$$

which is defined and holomorphic for $a \in X^*$ close to a^0. We then study the degree in a_0 of norms of some rational functions on X whose polar divisors generate $Pic_{\mathbb{Q}}(X)$. As in [20], let us fix very ample effective divisors $E_1, \dots, E_s$ supported by $X \setminus U$, whose classes form a $\mathbb{Q}$-basis of $Pic_{\mathbb{Q}}(X)$. We can now characterize the class of the interpolating hypersurface.

Theorem 1.2. — *Suppose that conditions of Theorem 1 are satisfied. Then the equality $[\widetilde{V}] = \alpha \in Pic(X)$ holds if and only if there exist rational functions $f_j \in H^0(X, \mathcal{O}_X(E_j))$ for $j = 1, \dots, s$, whose norms $N_V(f_j)$ are polynomials in a_{10} of degree exactly*

$$deg_{a_{10}}\ N_V(f_j) = \alpha \cdot [E_j] \frown \prod_{k=2}^{n-1} c_1(L_k) \in \mathbb{Z}_{\geqslant 0}.$$

Note that Bernstein's theorem [4] allows to compute the degrees of intersection in Theorems 1 and 2 as mixed volume of the polytopes associated (up to translation) to the involved line bundles.

If $X = \mathbb{P}^n$, then $Pic(X) \simeq \mathbb{Z}$ and Theorem 2 follows from Theorem 1: if $Tr_V(x_n)$ is affine in a_0, then $N_V(x_n)$ has degree N in a_0.

The proof of Theorem 1 uses a toric generalization of Abel-Jacobi's theorem [14] which gives combinatorial conditions for the vanishing of sums of Grothendieck residues associated to zero-dimensional complete intersections in toric varieties, those conditions being interpreted in terms of affine coordinates.

The difficulty to generalize Theorem 1 to other compactifications X of $\mathbb{C}^n$, as Grassmannians or flag varieties, is that there is no natural choice of affine coordinates, so *a priori* no grading for the algebra of regular functions over $U = \mathbb{C}^n$ naturally associated to X. Such an interpolation result in Grassmannians would be important to generalize Theorem 1 to any projective variety X and to any union of germs of dimension $k \leqslant n-1$, by using a grassmannian embedding of X associated to an adequat rank k ample bundle E on X. Nevertheless, we know now that there exist global intrinsec representations of residue currents, using some Chern connections acting on global sections of some vector bundle instead of usual differentials acting on holomorphic functions [2]. Then it has been recently shown [16] that such a global setting provides directly some generalizations of Abel-Jacobi's theorem obtained in [17]. We could hope that this approach should give an alternative proof for Theorem 1 (at least the direct part) which could admit generalizations to larger class of manifolds than toric varieties, for instance Grassmannians.

Finally, let us mention that we can hope for a generalization to the case of non-projective toric varieties, using blowing-up and essential families of globally generated line bundles, as presented in [18].

Section 2 is devoted to the proof of Theorem 1, and Section 3 to the proof of Theorem 2.

This article is part of my PhD thesis [19] "La trace en géométrie projective et torique", which is available on the web page
`http://tel.archives-ouvertes.fr/tel-00136109`.

2. Proof of Theorem 1

2.1. Direct implication

Let us suppose that V is contained in an algebraic hypersurface $\widetilde{V}$ whose equation in the affine chart U is given by a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$. Since the line bundles $L_1, \ldots, L_{n-1}$ are very ample, the hypothesis on the degree of intersection is equivalent to the fact that for a near a^0, the intersection $\widetilde{V} \cap C_a$ is contained in U and equal to $V \cap C_a$. In particular, the n polynomials $f, a_{10} - q_1(a'_1, \cdot), \ldots, a_{n-1,0} - q_{n-1}(a'_{n-1}, \cdot)$ of x define a complete intersection in $\mathbb{C}^n$. Now, it is well known (see [12], Chapter 5, Section 2) that the trace of x_i is equal, for a close to a^0, to the action of the Grothendieck residue defined by these polynomials on the holomorphic form $x_i df \wedge dq_1 \cdots \wedge dq_{n-1}/(2i\pi)^n$, that is,

$$Tr_V(x_i)(a) = \mathrm{Res}\begin{bmatrix} x_i df \wedge dq_1 \cdots \wedge dq_{n-1} \\ f, a_{10}-q_1, \ldots, a_{n-1,0}-q_{n-1}\end{bmatrix},$$

where we use classical notations (see [3]) for Grothendieck residues(2). This action is given by the integral formula

$$Tr_V(x_i)(a) = \int_{|a_{i0}-q_i|=\epsilon_i,\, i=1,\ldots n-1, |f|=\epsilon_n} \frac{x_i df \wedge dq_1 \cdots \wedge dq_{n-1}}{f(a_{10}-q_1)\cdots(a_{n-1,0}-q_{n-1})},$$

so that differentiation of the trace with respect to a_{k0} gives the equality

$$\partial^{(l)}_{a_{k0}} Tr_V(x_i)(a) = \mathrm{Res}\begin{bmatrix} (-1)^l\, l!\, x_1\cdots x_i^2 \cdots x_n \frac{df\wedge dq_1\cdots\wedge dq_{n-1}}{x_1\cdots x_n} \\ f, a_{10}-q_1, \ldots, (a_{k0}-q_k)^{l+1}, \ldots, a_{n-1,0}-q_{n-1}\end{bmatrix}.$$

If $h, f_1, \ldots, f_n$ are Laurent polynomials in $t = (t_1, \ldots, t_n)$ with Newton polytopes $P, P_1, \ldots, P_n$, the toric Abel-Jacobi theorem [14] asserts that

$$\mathrm{Res}\begin{bmatrix} h\frac{dt_1\cdots\wedge dt_n}{t_1\cdots t_n} \\ f_1, \ldots, f_n\end{bmatrix} = 0$$

as soon as P is contained in the interior of the Minkowski sum $P_1 + \cdots + P_n$. Since L_k is very ample, the support of the polynomial P_k is n-dimensional and it is not difficult to check that the Newton polytope of the Jacobian of the map $(f, q_1, \ldots, q_{n-1})$ translated *via* the vector $(1, \ldots, 2, \ldots, 1)$ (corresponding to multiplication by $x_1 \cdots x_i^2 \cdots x_n$) is stricly contained in the Minkowski sum of the Newton polytopes of polynomials $f, a_{10} - q_1, \ldots, a_{n-1,0} - q_{n-1}$ for $l \geqslant 2$. This shows the direct part of Theorem 1.

Remark 2.1. — If R_k is the unique divisor in $|L_k|$ supported outside U, the previous argument yields the implication

$$h \in H^0(X, \mathcal{O}_X(dR_k)) \Rightarrow deg_{a_{k0}} Tr_V(h) \leqslant d$$

with equality if the zero set of h has a proper intersection with $X \setminus U$ (which is generically the case since L_k is globally generated). See [19], Corollary 3.6 p 127. In particular, the trace of the coordinate function x_i is affine in a_{k0} if the vector $e_i := (0, \ldots, 1, \ldots, 0)$ is a vertice of P_k, and does not depend on a_{k0} otherwise.

(2) From a more conceptual point of view, the Grothendieck residue action on the form $x_i df \wedge dq_1 \cdots \wedge dq_{n-1}/(2i\pi)^n$ coincides with the action of the logarithmic residue

$$dd^c\log|f| \wedge \cdots \wedge dd^c\log|a_{n-1,0} - q_{n-1}|$$

on the function x_i. It is well known that this logarithmic residue, considered as an (n,n)-current, is equal to the sum of the point masses at the points of intersection so that its action on x_i produces the trace of x_i.

2.2. Converse implication

Let us show that $Tr_V(x_i)$ being affine in a_0 implies that $Tr_V(x_i^l)$ is polynomial of degree at most l in a_0 for any $l \geqslant 1$. We need an auxiliary lemma generalizing to the toric case the "Wave-shock equation" used in [13] to show the Abel-inverse theorem. We give a weak version of this lemma, which will be sufficient for our purpose. See [19], Proposition 3.8 p 128, for a stronger version.

For a near a^0, we use affine coordinates $(x_1^{(j)}(a), \ldots, x_n^{(j)}(a))$ for the unique point of intersection $p_j(a)$ of V_j with C_a. Since L_k is very ample, the monomial x_i occurs in the polynomial q_k with a generically non zero coefficient denoted by a_{ki}, for $i = 1, \ldots, n$.

Lemma 2.2. — *For any $i \in \{1, \ldots, n\}$, and any $j \in \{1, \ldots, N\}$, the function $a \mapsto x_i^{(j)}(a)$ (holomorphic at a^0) satisfies the following P.D.E:*

$$\partial_{a_{ki}} x_i^{(j)}(a) = -x_i^{(j)} \partial_{a_{k0}} x_i^{(j)}(a)$$

for any $k = 1, \ldots, n-1$ and any a close to a^0.

Proof. — Let us fix $i = 1$ for simplicity. Trivially, the equality $a_{k0} = q_k(a'_k, x)$ holds for all $k = 1, \ldots, n-1$ if and only if $x \in C_a \cap U$, and the complex number

$$x_1^{(j)}((q_1(a'_1, x), a'_1), \ldots, (q_{n-1}(a'_{n-1}, x), a'_{n-1}))$$

thus represents the x_1-coordinate of the unique point of intersection of V_j with the curve C_a passing through x. If $x = (x_1, \ldots, x_n)$ belongs to V_j, this complex number, seen as a function of $a' = (a'_1, \ldots, a'_{n-1})$ is thus constant, equal to x_1. Differentiating according to the x_1-coefficient a_{k1} of q_k gives

$$\begin{aligned} 0 = \partial_{a_{k1}} x_1^{(j)}&((q_1(a'_1, x), a'_1), \ldots, (q_{n-1}(a'_{n-1}, x), a'_{n-1})) \\ &+ x_1^{(j)}((q_1(a'_1, x), a'_1), \ldots, (q_{n-1}(a'_{n-1}, x), a'_{n-1})) \\ &\times \partial_{a_{k0}} x_1^{(j)}((q_1(a'_1, x), a'_1), \ldots, (q_{n-1}(a'_{n-1}, x), a'_{n-1})). \end{aligned}$$

We can replace $x \in V_j$ with $(x_1^{(j)}(a), \ldots, x_n^{(j)}(a)) \in V_j$, and the desired relation follows from the equality $q_k(a'_k, (x_1^{(j)}(a), \ldots, x_n^{(j)}(a))) = a_{k0}$. □

In particular, Lemma 1 implies that

$$(l+1)\partial_{a_{ki}} Tr(x_i^l) = -l \partial_{a_{k0}} Tr(x_i^{l+1})$$

for any $i = 1, \ldots, n$, any $k = 1, \ldots, n-1$, and all integers $l \in \mathbb{N}$, from which we easily deduce

$$\deg_{a_{k0}} Tr(x_i^l) \leqslant l.$$

More generally, let

$$(y_1, \dots, y_n)^t = C(x_1, \dots, x_n)^t, \quad C \in GL_n(\mathbb{C})$$

be any linear change of coordinates in U. Then, we have equality

$$a_{k1}x_1 + \cdots + a_{kn}x_n = \alpha_{k1}y_1 + \cdots + \alpha_{kn}y_n$$

where $\alpha_k = (\alpha_{k1}, \dots, \alpha_{kn})^t = (C^t)^{-1}(a_{k1}, \dots, a_{kn})^t$, so that

$$q_k(a'_k, x) = \alpha_{k1}y_1 + \cdots + \alpha_{kn}y_n + Q_k(a''_k, x)$$

where $a'_k = (a_{k1}, \dots, a_{kn}, a''_k)$ and the polynomial

$$Q_k(a''_k, x) := q_k(a'_k, x) - (a_{k1}x_1 + \cdots + a_{kn}x_n)$$

does not depend on $(a_{k1}, \dots, a_{kn})$. The proof of the previous lemma can be obviously adapted when differentiating with respect to the new parameter α_{ki} (linear combination of $a_{k1}, \dots, a_{kn}$, coding for the new variable y_i) instead of a_{ki}, and we obtain equality

$$\partial_{\alpha_{ki}} y_i(p_j(a)) = -y_i(p_j(a)) \times \partial_{a_{k0}} y_i(p_j(a)) = -\frac{1}{2}\partial_{a_{k0}}[y_i(p_j(a))]^2$$

for $k = 1, \dots, n-1$, $i = 1, \dots, n$ and $j = 1, \dots, N$.

Now, if $y = c_1x_1 + \cdots + c_nx_n$ is any linear combination of the affine coordinates x_i, its trace $\mathrm{Tr}\, y = \sum_{i=1}^n c_i \mathrm{Tr}\, x_i$ is affine in a_0, and the previous equality implies that

$$\deg_{a_{k0}} Tr(y^l) \leqslant l \qquad (*)$$

for any $l \in \mathbb{N}$.

To any such holomorphic function $y = y(x)$, we can associate its caracteristic polynomial

$$P_y(X, a) := \prod_{j=1}^{N} (X - y(p_j(a))),$$

whose coefficients are holomorphic functions near a^0. Using Newton's formulas relating coefficients of P_y to the trace of the powers of y, we deduce from $(*)$ that P_y is polynomial in $a_0 = (a_{01}, \dots, a_{0,n-1})$. For any a near a^0, the function

$$Q_{y,a'}(x) := P_y(y(x), (q_1(x, a'_1), a'_1), \dots, (q_{n-1}(x, a'_{n-1}), a'_{n-1}))$$

is thus a polynomial in $(x_1, \dots, x_n)$, which, by construction, vanishes on V independently of a' and y. Let us consider the algebraic set

$$W_{a'} := \bigcap_{y = c_1x_1 + \cdots + c_nx_n} \{x \in U,\, Q_{y,a'}(x) = 0\}.$$

Then $V \subset W_{a'}$ and $x \in W_{a'} \cap C_a$ if and only if

$$y \in \{y(p_1(a), \ldots, y(p_N(a))\}$$

for any linear combination y of the affine coordinates x_i. This implies that $x \in \{p_1(a), \ldots, p_N(a)\}$ by duality so that $W_{a'} \cap C_a = V \cap C_a$ for any a near a^0. Consider now

$$\widetilde{V} := \bigcap_{a \text{ near } a^0} \bar{W_{a'}},$$

where $\bar{W_{a'}}$ denotes the Zariski closure in X of the affine algebraic hypersurface $W_{a'}$. Then

$$\mathrm{codim}_X \widetilde{V} \cap (X \setminus U) \geqslant 2,$$

so that the intersection $\widetilde{V} \cap C_a$ is generically contained in U. For a near a^0, there is thus equality $\widetilde{V} \cap C_a = V \cap C_a$, so that $[\widetilde{V}] \frown \prod_{k=1}^{n-1} c_1(L_k) = N$. $\square$

3. Proof of Theorem 2

We can associate to any codimension 2 closed subvariety $W \subset X$ its dual set $W^* \subset X^*$ associated to the line bundles $(L_1, \ldots, L_{n-1})$, defined by

$$W^* := \{a \in X^*,\ C_a \cap V \neq \emptyset\}.$$

From [10], this is an hypersurface in the product of projective spaces X^*, irreducible if W is, whose multidegree $(d_1, \ldots, d_{n-1})$ in X^* is given by the intersection numbers

$$d_i = [W] \frown \prod_{i=1, i \neq j}^{n-1} c_1(L_i), \ \ j = 1, \ldots, n-1.$$

We call the $(L_1, \ldots, L_{n-1})$-resultant of W, noted $\mathcal{R}_W$, the multihomogeneous polynomial of multidegree $(d_1, \ldots, d_{n-1})$ vanishing on W^* (it is defined up to a non zero scalar, but this has no consequence here). By linearity, we generalize this situation to the case of cycles:

$$\mathcal{R}_{\sum c_i W_i} := \prod (\mathcal{R}_{W_i})^{c_i}.$$

Duality respects rational equivalence so that the degree of the resultant of a cycle W only depends of the class of W in the Chow group of X (see [19], Proposition 7 p 100).

A generic rational function $f_j \in H^0(X, \mathcal{O}_X(E_j))$ defines a principal divisor $H_j - E_j$, where the zero divisor H_j intersects properly $\widetilde{V}$ and $X \setminus U$. In that case, the product formula [15] gives rise to the equality :

$$N_{\widetilde{V}}(f_j) = \frac{\mathcal{R}_{\widetilde{V}\cdot H_j}}{\mathcal{R}_{\widetilde{V}\cdot E_j}}.$$

Since the constant coefficents $a_0 = (a_{10}, \ldots, a_{n-1,0})$ do not influence the asymptotic behavior of the curves C_a outside the affine chart U, the resultant $\mathcal{R}_{\widetilde{V}\cdot E_j}(a)$ does not depend on a_0. We thus obtain

$$\deg_{a_{10}} N(f_j) = \deg_{a_{10}} \mathcal{R}_{\widetilde{V}\cdot H_j} \leqslant \deg_{a_1} \mathcal{R}_{\widetilde{V}\cdot H_j} = \deg_{a_1} \mathcal{R}_{\widetilde{V}\cdot E_j}.$$

Since we deal with homogeneous polynomials in a_1, strict inequality in the previous expression is equivalent to the equality

$$\mathcal{R}_{\widetilde{V}\cdot E_j}((a_{10}, 0, \ldots, 0), a_2, \ldots, a_{n-1}) \equiv 0.$$

This happens if and only if all subvarieties $C = \{s = 0\}$ given by sections $s \in \Gamma(X, \oplus_{k=2}^{n-1} L_k)$ intersect the set $\widetilde{V} \cap H_j \cap (X \setminus U)$. By a dimension argument, this would imply that $\widetilde{V}$ has an irreducible branch contained in $X \setminus U$, and this can not occur since $\widetilde{V} \cap C_a = V \cap C_a \subset U$ for a close to a^0. Thus we have proved the equality:

$$\deg_{a_{10}} N_{\widetilde{V}}(f_j) = [\widetilde{V}] \frown [E_j] \frown \prod_{k=2}^{n-1} c_1(L_k)$$

Since the classes $[E_j]$, $j = 1, \ldots, s$ determine a basis for $A_{n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, the non degenerated natural pairing between the Chow groups $A_1(X)$ and $A_{n-1}(X)$ shows that the hypothesis of Theorem 2 is equivalent to

$$[\widetilde{V}] \frown \prod_{k=2}^{n-1} c_1(L_k) = \alpha \frown \prod_{k=2}^{n-1} c_1(L_k).$$

Since the Strong Lefschetz theorem remains valid for any very ample algebraic vector bundle on a mooth projective variety X (see Prop. 1.1 in [5]), the preceding equality is equivalent to $[\widetilde{V}] = \alpha$. □

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