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Lluís ALSEDÀ & Sylvie RUETTE

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ROTATION SETS FOR GRAPH MAPS OF DEGREE 1

by Lluís ALSEDÀ & Sylvie RUETTE (*)

Abstract. — For a continuous map on a topological graph containing a loop S it is possible to define the degree (with respect to the loop S) and, for a map of degree 1, rotation numbers. We study the rotation set of these maps and the periods of periodic points having a given rotation number. We show that, if the graph has a single loop S then the set of rotation numbers of points in S has some properties similar to the rotation set of a circle map; in particular it is a compact interval and for every rational α in this interval there exists a periodic point of rotation number α.

For a special class of maps called combed maps, the rotation set displays the same nice properties as the continuous degree one circle maps.

Résumé. — Pour une transformation continue sur un graphe topologique contenant une boucle S, il est possible de définir le degré (par rapport à la boucle S) et, quand la transformation est de degré 1, des nombres de rotation. Nous étudions l'ensemble de rotation de ces transformations et les périodes des points périodiques ayant un nombre de rotation donné. Nous montrons que, si le graphe a une unique boucle S, alors l'ensemble des nombres de rotation des points de S a des propriétés similaires à celles de l'ensemble de rotation d'une transformation du cercle ; en particulier, c'est un intervalle compact et pour tout rationnel α dans cet intervalle il existe un point périodique de nombre de rotation α.

Pour une classe particulière de transformations appelées transformations peignées, l'ensemble de rotation possède les mêmes bonnes propriétés que celui des transformations continues de degré 1 sur le cercle.

Introduction

One of the basic problems in combinatorial and topological dynamics is the characterisation of the sets of periods in dimension one. This problem has its roots and motivation in the striking Sharkovskii Theorem [18, 19].

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Since then, a lot of effort has been spent in finding characterisations of the set of periods for more general one dimensional spaces.

One of the lines of generalisation of Sharkovskii Theorem consists on characterising the possible sets of periods of continuous self maps on trees. The first remarkable results in this line after [18] are due to Alsedà, Llibre and Misiurewicz [5] and Baldwin [8]. In [5] it is obtained the characterisation of the set of periods of the continuous self maps of a 3-star with the branching point fixed in terms of three linear orderings, whereas in [8] the characterisation of the set of periods of all continuous self maps of n-stars is given (an n-star is a tree composed of n intervals with a common endpoint). Further extensions of Sharkovskii Theorem are due to:

- Baldwin and Llibre [9] to continuous maps on trees such that all the branching points are fixed,
- Bernhardt [10] to continuous maps on trees such that all the branching points are periodic,
- Alsedà, Juher and Mumbrú [1, 3, 2, 4] to the general case of continuous tree maps.

Another line of generalisation of Sharkovskii Theorem is to consider spaces that are not contractile to a point. In particular topological graphs which are not trees, the circle being the simplest one. This case displays a new feature: While the sets of periods of continuous maps on trees can be characterised using only a finite number of orderings, the sets of periods of continuous circle maps of degree one contain the set of all denominators of all rationals (not necessarily written in irreducible form) in the interior of an interval of the real line. As a consequence, these sets of periods cannot be expressed in terms of a finite collection of orderings. The result which characterises the sets of periods of continuous circle maps of degree one is due to Misiurewicz [16] and uses as a key tool the rotation theory. Indeed, the sets of periods are obtained from the rotation interval of the map.

The characterisation of the sets of periods for circle maps of degree different from one is simpler than the one for the case of degree one. It is due to Block, Guckenheimer, Misiurewicz and Young [12].

Finding a generalisation of the Sharkovskii Theorem for self maps of a topological graph which is not the circle is a big challenge and in general it is not known what the sets of periods may look like. However, in this setting, one expects to find at least sets of periods of all possible types appearing for tree and circle maps.

Two motivating results that give some insight on the kind of sets of periods that one can find in this setting are [14] and [15]. The first of them deals with continuous self maps on a graph σ consisting on a circuit and an interval attached at a unique branching point b such that the maps fix b. The second one studies the continuous self maps of the 2-foil (that is, the graph consisting on two circles attached at a single point).

Our aim is to go forward in the generalisation of [14] by using the ideas and techniques of [16, 12]. To this end we need to develop a rotation theory for continuous self maps of degree one of topological graphs having a unique circuit and, afterwards, we need to apply this theory to the characterisation of the sets of periods of such maps.

In this paper we propose a rotation theory for the above class of maps and we study the relation between the rotation numbers and the periodic orbits. The use of this theory in the characterisation of the sets of periods of such maps will be the goal of a future project.

A rotation theory is usually developed in the universal covering space by using the liftings of the maps under consideration. It turns out that the rotation theory on the universal covering of a graph with a unique circuit can be easily extended to a wider family of spaces. These spaces are defined in detail in Subsection 1.1 and called *lifted spaces*. Each lifted space T has a subset $\widehat{T}$ homeomorphic to the real line $\mathbb{R}$ that corresponds to an "unwinding" of a distinguished circuit of the original space.

In the rest of this section (and in fact in the whole paper) we will abuse notation and denote the set $\widehat{T}$ by $\mathbb{R}$ for simplicity.

Given a lifted space T and a map F from T to itself of degree one, there is no difficulty to extend the definition of rotation number to this setting in such a way that every periodic point still has a rational rotation number as in the circle case. However, the obtained rotation set $\mathrm{Rot}(F)$ may not be connected and we do not know yet whether it is closed. Despite of this fact, the set $\mathrm{Rot}_{\mathbb{R}}(F)$ corresponding to the rotation numbers of all points belonging to $\mathbb{R}$, has properties which are similar to (although weaker than) those of the rotation interval for a circle map of degree one.

Also, there is a special class of degree one continuous maps on lifted spaces that we call *combed maps*, whose rotation set displays the same nice properties as the continuous degree one circle maps.

The paper is organised as follows. Section 1 is devoted to fixing the notation, to defining the notion of rotation number and rotation set in this setting, and to studying the basic properties of this set. In Section 2 we introduce the technical notion of *positive covering* and, by means of its use, we prove a result that will be used throughout the paper.

Section 3 is devoted to studying the basic properties of the rotation set. It is divided into two subsections. In the first one (Subsection 3.1) we study the connectedness and compactness of the rotation set whereas in the second one (Subsection 3.2) we describe the information on the periodic orbits of the map which is carried out by the rotation set.

In Section 4 we define the combed maps and, for this class of maps, we study the special features of the rotation set and its relation with the set of periods.

Section 5 specialises the results obtained previously in the particular case when the lifted space is a graph. Finally, Section 6 is devoted to showing some examples and counterexamples to illustrate some previous comments and results.

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1. Definitions and elementary properties

1.1. Lifted spaces and retractions

The aim of this subsection is to define in detail the class of *lifted spaces* where we will develop the rotation theory. They are obtained from a metric space by unwinding one of its loops. This gives a new space that contains a subset homeomorphic to the real line and that is "invariant by a translation". This construction mimics the process of considering the universal covering space of a compact connected topological graph that has a unique loop.

Before defining lifted spaces we will informally discuss a couple of examples to fix the ideas. Consider the topological graph G represented in Figure 1.1. The unwinding of G with respect to the loop S is the infinite graph $\widehat{G}$ which is made up of infinitely many subspaces $(\widehat{G}_n)_{n\in\mathbb{Z}}$ that are all homeomorphic by a translation τ. Moreover, there is a continuous projection $\pi\colon \widehat{G} \longrightarrow G$ such that $\pi\big|_{\operatorname{Int}(\widehat{G}_n)}$ is a homeomorphism onto $G \setminus \{x_0\}$ for each $n \in \mathbb{Z}$, and $\pi(\tau(y)) = \pi(y)$ for all $y \in \widehat{G}$. The set $\pi^{-1}(S)$ is homeomorphic to the real line. If we imagine that the loop S has length 1 and that x_0 is the origin, then it is natural to consider a homeomorphism $h\colon \mathbb{R} \longrightarrow \pi^{-1}(S)$ such that $\pi^{-1}(x_0) = h(\mathbb{Z})$. In this setting, $\tau(h(x)) = h(x+1)$ for all $x \in \mathbb{R}$.

Note that, since G has more than one loop, $\widehat{G}$ is not the universal covering of G.

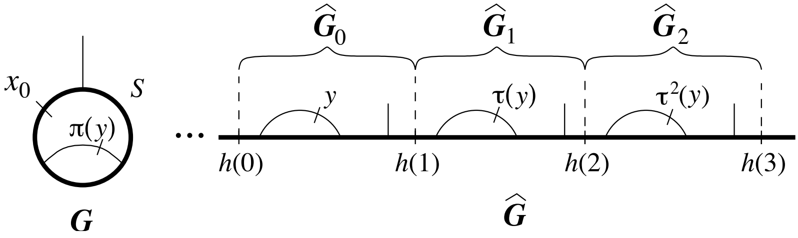

Figure 1.1. The graph G, on the left, is unwound with respect to the bold loop S, on the right. In $\widehat{G}$, the bold line is $\pi^{-1}(S) = h(\mathbb{R})$ and $h(\mathbb{Z}) = \pi^{-1}(x_0)$. $\widehat{G}$ in this example will not be considered as a lifted space since it cannot be retracted to $h(\mathbb{R})$.

In a similar way, we can unwind any connected compact metric X with a loop, as in Figure 1.2. These two examples have a main difference: the space $\widehat{X}$ shown in Figure 1.2 can be "retracted" to $h(\mathbb{R})$ because the closure of any connected component of $\widehat{X} \setminus h(\mathbb{R})$ meets $h(\mathbb{R})$ at a single point, whereas this property does not hold for $\widehat{G}$ in Figure 1.1. Notice that the unwinding of a graph with a single loop always has this property. In this paper, we deal with spaces $\widehat{X}$ of the type shown in Figure 1.2.

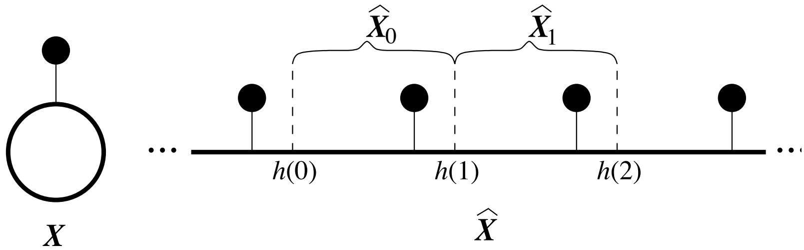

Figure 1.2. The unwinding of a connected compact metric space X with a loop. In this example $\widehat{X}$ can be retracted to $h(\mathbb{R})$.

Now we formalise the definition of this class of spaces.

Definition 1.1. — *Let T be a connected metric space. We say that T is a* lifted space *if there exists a homeomorphism h from $\mathbb{R}$ into T, and a homeomorphism $\tau\colon T \longrightarrow T$ such that*

(i) *$\tau(h(x)) = h(x+1)$ for all $x \in \mathbb{R}$,*

(ii) *the closure of each connected component of $T \setminus h(\mathbb{R})$ is a compact set that intersects $h(\mathbb{R})$ at a single point, and*

(iii) *the number of connected components* C *of* $T \setminus h(\mathbb{R})$ *such that* $\mathrm{Clos}\,(C) \cap h([0,1]) \neq \emptyset$ *is finite.*

The class of all lifted spaces will be denoted by T*.*

Remark 1.2. — By replacing $h(x)$ by $h(x+a)$ for some appropriate a, if necessary, we may assume that $h(\mathbb{Z})$ does not intersect the closure of any connected component of $T \setminus h(\mathbb{R})$. In this situation, for every $n \in \mathbb{Z}$, let T_n denote the closure of the connected component of $T \setminus \{h(n), h(n+1)\}$ intersecting $h((n, n+1))$. Then $\tau\big|_{T_n} : T_n \longrightarrow T_{n+1}$ is a homeomorphism.

To simplify the notation, in the rest of the paper we will identify $h(\mathbb{R})$ with $\mathbb{R}$ itself. In particular, we are implicitly extending the usual ordering, the arithmetic and the notion of intervals from $\mathbb{R}$ to $h(\mathbb{R})$.

Observe that, in the above setting, Definition 1.1(i) gives $\tau(x) = x+1$ for all $x \in \mathbb{R}$. Taking this and Remark 1.2 into account, it is natural to visualise the homeomorphism τ as a "translation by 1" in the whole space T (despite of the fact that such an arithmetic operation need not be defined). Thus, in what follows, to simplify the formulae we will abuse notation and write $x+1$ to denote $\tau(x)$ for all $x \in T$. Then the fact that T is homeomorphic to itself by τ can be rewritten in this notation as: $T + 1 = T$. Note also that, since τ is a homeomorphism, this notation can be extended by denoting $\tau^m(x)$ by $x+m$ for all $m \in \mathbb{Z}$. In what follows, if $A \subset T$ is a set and $m \in \mathbb{Z}$ then $A + m$ will denote $\{x + m : x \in A\}$.

Example 1.3. — To better understand the simplifications introduced above consider the following paradigmatic particular case (see Figure 1.3 for an example): The lifted space T is embedded in $\mathbb{R}^n$ and the map $\tau(\overrightarrow{x})$ is defined as $\overrightarrow{x} + \overrightarrow{e_1}$, where $\overrightarrow{e_1} = (1, 0, \ldots, 0)$ denotes the first vector in the canonical base. Then T must contain the line $t\overrightarrow{e_1}$ for $t \in \mathbb{R}$, and the map h from Definition 1.1 is defined by $h(t) = t\overrightarrow{e_1}$.

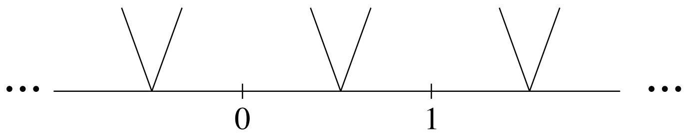

Figure 1.3. An example of a lifted tree that can be embedded in $\mathbb{R}^2$*.*

Next we introduce a tool that will play a crucial role in the rest of the paper. It is the retraction from T to $\mathbb{R}$. It will be used as a measuring tool of displacements to the left or to the right and also to identify the place where the image of a point lies in $T \setminus \mathbb{R}$.

Definition 1.4. — *Given $T \in \mathsf{T}$ there is a natural retraction from T to $\mathbb{R}$ that in the rest of the paper will be denoted by r. When $x \in \mathbb{R}$, then clearly $r(x) = x$. When $x \notin \mathbb{R}$, by definition, there exists a connected component C of $T \setminus \mathbb{R}$ such that $x \in C$ and* $\operatorname{Clos}(C)$ *intersects $\mathbb{R}$ at a single point z. Then $r(x)$ is defined to be, precisely, the point z. In particular, r is constant on* $\operatorname{Clos}(C)$.

A point $x \in \mathbb{R}$ such that $r^{-1}(x) \neq \{x\}$ will be called a branching point of T. *The set of all branching points of T will be denoted by* $\mathrm{B}(T)$. *It is a subset of $\mathbb{R}$ by definition.*

The next lemma recalls the basic properties of the natural retraction. Its proof is a simple exercise and thus it will be omitted.

Lemma 1.5. — *For each $T \in \mathsf{T}$ the following statements hold:*

(a) *If $x \notin \mathbb{R}$, then there exists a neighbourhood U of x such that r is constant in U.*
(b) *The map $r \colon T \longrightarrow \mathbb{R}$ is continuous and verifies $r(x+1) = r(x) + 1$ for all $x \in T$.*

1.2. Maps and orbits on lifted spaces

The aim of this subsection is to study which is the object that corresponds to orbits at the level of lifted spaces. We start by generalising the notion of lifting and degree to this setting.

Suppose that X is a metric space with a loop S and that the unwinding of S gives a lifted space $T \in \mathsf{T}$. Then, there exists a continuous map $\pi \colon T \longrightarrow X$, called the *standard projection from T to X*, such that $\pi([0,1]) = S$ and $\pi(x+1) = \pi(x)$ for all $x \in T$.

Let $f \colon X \longrightarrow X$ be continuous. By using standard techniques (see for instance [20]) it is possible to construct a (non-unique) continuous map $F \colon T \longrightarrow T$ such that $f \circ \pi = \pi \circ F$. Each of these maps will be called a *lifting of f*.

Observe that $f \circ \pi = \pi \circ F$ implies that $F(1) - F(0) \in \mathbb{Z}$ and, as the next lemma states, this number is independent of the choice of the lifting. It is called indistinctly the *degree of f* or the *degree of F* and denoted by $\deg(f)$ and $\deg(F)$.

The next lemma, whose proof is straightforward (see for instance [6, Section 3.1]), summarises the basic properties of lifting maps.

Lemma 1.6. — *Let $f\colon X \longrightarrow X$ be continuous. If the continuous map $F\colon T \longrightarrow T$ is a lifting of f then $F(x+1) = F(x)+\deg(f)$ for every $x \in T$. On the other hand, if $F'\colon T \longrightarrow T$ is continuous, then F' is a lifting of f if and only if $F = F' + k$ for some $k \in \mathbb{Z}$. Moreover, the following statements hold for all $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and $n \geqslant 0$:*

(a) $F^n(x+k) = F^n(x) + k\deg(f)^n$, *and*

(b) $(F+k)^n(x) = F^n(x) + k(1+d+d^2+\cdots+d^{n-1})$, *with* $d = \deg(f)$.

If g is another continuous map from X into itself, then $\deg(g\circ f) = \deg(g)\cdot\deg(f)$.

Next, as we have said, we want to describe how periodic points and periodic orbits of f are seen at the lifting level.

Let F be any lifting of f. A point $x \in T$ is called *periodic* (mod 1) if there exists $n \in \mathbb{N}$ such that $F^n(x) \in x + \mathbb{Z}$. The *period* (mod 1) of x is the least positive integer n satisfying this property; that is, $F^n(x) \in x+\mathbb{Z}$ and $F^i(x) \notin x+\mathbb{Z}$ for all $1 \leqslant i \leqslant n-1$. Observe that x is periodic (mod 1) for F if and only if $\pi(x)$ is periodic for f. Moreover, the F-period (mod 1) of x and the f-period of $\pi(x)$ coincide.

In a similar way, the set

$$\{F^n(x)+m : n \geqslant 0 \text{ and } m \in \mathbb{Z}\},$$

will be called the *orbit* (mod 1) *of* x, and denoted by $\mathrm{Orb}_1(x,F)$. Clearly,

$$\mathrm{Orb}_1(x,F) = \pi^{-1}(\{f^n(\pi(x)) : n \geqslant 0\}) = \pi^{-1}(\mathrm{Orb}(\pi(x),f)).$$

When x is periodic (mod 1) then the orbit (mod 1) of x, $\mathrm{Orb}_1(x,F)$, is also called *periodic* (mod 1). In this case it is not difficult to see that $\mathrm{Card}\big(\mathrm{Orb}_1(x,F)\cap r^{-1}([n,n+1))\big)$ coincides with the f-period of x for all $n \in \mathbb{Z}$.

A standard approach to study the periodic points and orbits of f is to work at the lifting level with the periodic (mod 1) points and orbits instead of the original map and space. This is the approach we will follow in this paper. The results on F can obviously be pulled back to f and X.

As it has been said in the introduction, the aim of this paper is to develop the rotation theory for liftings in lifted spaces and study the relation between rotation numbers and periodic (mod 1) orbits. As it is usual, this theory can only be developed for maps of degree one, that is, for maps verifying $F(x+1) = F(x)+1$ for all $x \in T$. So, in the rest of the paper, we will only consider the class $\mathcal{C}_1(T)$ of all continuous maps of degree 1 from $T \in \mathsf{T}$ into itself.

The following lemma is a specialisation of Lemma 1.6 to maps of $\mathcal{C}_1(T)$. Its last statement follows from the previous one and Lemma 1.5(b).

LEMMA 1.7. — *The following statements hold for $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x \in T$:*

(a) $F^n(x+k) = F^n(x) + k$,

(b) $(F+k)^n(x) = F^n(x) + kn$.

(c) *If G is another map from $\mathcal{C}_1(T)$ then $F \circ G \in \mathcal{C}_1(T)$. In particular $F^n \in \mathcal{C}_1(T)$.*

(d) *The map $r \circ F^n \colon T \longrightarrow \mathbb{R}$ is continuous and verifies*
$$r(F^n(x+1)) = r(F^n(x)) + 1$$
for all $x \in T$.

1.3. Maps of degree 1 and rotation numbers

The aim of this subsection is to introduce the notion of rotation number for our setting and to study its basic properties. We define three types of rotation numbers.

DEFINITION 1.8. — *Let $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$ and $x \in T$. We set*
$$\underline{\rho}_{_F}(x) := \liminf_{n\to+\infty} \frac{r \circ F^n(x) - r(x)}{n} \text{ and } \overline{\rho}_{_F}(x) := \limsup_{n\to+\infty} \frac{r \circ F^n(x) - r(x)}{n}.$$
When $\underline{\rho}_{_F}(x) = \overline{\rho}_{_F}(x)$ then this number will be denoted by $\rho_{_F}(x)$ and called the rotation number of x. *The numbers $\underline{\rho}_{_F}(x)$ and $\overline{\rho}_{_F}(x)$ are called the* lower rotation number of x *and* upper rotation number of x, *respectively.*

Remark 1.9. — If T is embedded in a normed vector field (e.g. $T \subset \mathbb{R}^n$), then one can easily see that the composition with the retraction r can be removed from the above formula without any change and the rotation numbers can be defined simply by using
$$\frac{F^n(x) - x}{n}.$$
The only reason to consider $r \circ F^n$ instead of F^n in the general case is to "project" the point $F^n(x)$ to $\mathbb{R}$ where we have arithmetic, to be able to measure the distance between $F^n(x)$ and x.

We now give some elementary properties of rotation numbers.

LEMMA 1.10. — *Let $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$, $x \in T$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.*

(a) $\overline{\rho}_{F}(x+k)=\overline{\rho}_{F}(x)$.

(b) $\overline{\rho}_{(F+k)}(x)=\overline{\rho}_{F}(x)+k$.

(c) $\overline{\rho}_{F^n}(x)=n\overline{\rho}_{F}(x)$.

The same statements hold with $\underline{\rho}$ *instead of* $\overline{\rho}$.

Proof. — The Statements (a) and (b) follow from Lemma 1.7(a) and (b) respectively. The proof of (c) is similar to [6, Lemma 3.7.1(b)]. □

An important object that synthesises all the information about rotation numbers is the *rotation set* (i.e., the set of *all* rotation numbers). Since we have three types of rotation numbers, we have three kinds of rotation sets.

Definition 1.11. — *For* $T \in \mathbf{T}$ *and* $F \in \mathcal{C}_1(T)$ *we define the following* rotation sets*:*

$$\begin{aligned}\mathrm{Rot}^+(F) &= \{\overline{\rho}_{F}(x) : x \in T\},\\ \mathrm{Rot}^-(F) &= \{\underline{\rho}_{F}(x) : x \in T\},\\ \mathrm{Rot}(F) &= \{\rho_{F}(x) : x \in T \text{ and } \rho_{F}(x) \text{ exists}\},\end{aligned}$$

Similarly we define $\mathrm{Rot}^+_{\mathbb{R}}(F)$*,* $\mathrm{Rot}^-_{\mathbb{R}}(F)$ *and* $\mathrm{Rot}_{\mathbb{R}}(F)$ *by replacing* $x \in T$ *by* $x \in \mathbb{R}$ *in the above three definitions.*

The next simple example helps in better understanding the basic features of rotation numbers and sets. In particular it will show that the rotation set in this framework does not display the nice properties of the rotation sets for continuous degree one circle maps and will justify the study of the sets $\mathrm{Rot}^+_{\mathbb{R}}$, $\mathrm{Rot}^-_{\mathbb{R}}$ and $\mathrm{Rot}_{\mathbb{R}}$.

Example 1.12. — Let T be the lifted space shown in Figure 1.4. This lifted space has two branches A, B between 0 and 1 outside $\mathbb{R}$, joined at a common branching point e. We denote by a and b the endpoints of A and B, respectively.

Observe that T is uniquely arcwise connected. So, given two points x and y, the *convex hull of* $\{x,y\}$ *in* T which is by definition the smallest closed connected subset of T containing x and y coincides with the image of any injective path in T joining x and y. It will be denoted by $\langle x,y\rangle$.

Let $F\colon T \longrightarrow T$ be the continuous map of degree 1 defined by

(i) $F|_{\mathbb{R}}=\mathrm{Id}$,

(ii) $F(A)=\langle e,a-1\rangle$ and $F|_A$ is injective,

(iii) $F(B)=\langle e,b+1\rangle$ and $F|_B$ is injective.

Obviously, $\mathrm{Rot}_{\mathbb{R}}(F)=\{0\}$, $\rho_{F}(a)=-1$ and $\rho_{F}(b)=1$. Let $x\in A$. If there exists $k\geqslant 1$ such that $F^k(x)\in\mathbb{R}$, then $F^n(x)=F^k(x)$ for all $n\geqslant k$

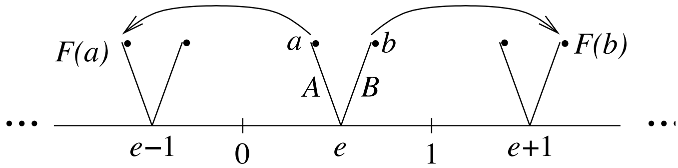

Figure 1.4. The set $\mathrm{Rot}(F) = \{-1, 0, 1\}$ *is not connected and is not equal to* $\mathrm{Rot}_{\mathbb{R}}(F) = \{0\}$.

and $\rho_F(x) = 0$. Otherwise, $F^n(x) \in A - n$ for all $n \geqslant 1$, and $\rho_F(x) = -1$. Similarly, if $x \in B$ then $\rho_F(x)$ equals 0 or 1. Hence $\mathrm{Rot}(F) = \{-1, 0, 1\}$, which is not a connected set. Consequently, $\mathrm{Rot}(F) \neq \mathrm{Rot}_{\mathbb{R}}(F)$ despite of the fact that the set $\bigcup_{n\geqslant 0} F^{-n}(\mathbb{R})$ coincides with $T \setminus (\{a, b\} + \mathbb{Z})$, which is dense in T.

In a similar way one can construct examples of lifted spaces and maps F such that $\mathrm{Rot}(F)$ has n connected components for any finite, arbitrarily large n, even when there is a single branch outside $\mathbb{R}$. Or connected components outside $\mathrm{Rot}_{\mathbb{R}}(F)$ which are non degenerate intervals (e.g., $F|_{\mathbb{R}} = \mathrm{Id}$ and $F(A) \supset (A+1) \cup (A+2)$ in the above example). Generally, when the dynamics of parts of the branches has no relation with the dynamics of $\mathbb{R}$, disconnectedness of the rotation set is likely to occur.

To study the sets $\mathrm{Rot}(F)$ and $\mathrm{Rot}_{\mathbb{R}}(F)$ and their relation with the periodic (mod 1) points and orbits of the map F we introduce the following notation. For a continuous map $F \in \mathcal{C}_1(T)$ and $n \in \mathbb{N}$ we set

$$F_n^r := r \circ F^n\big|_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbb{R}.$$

From Lemma 1.7(d) it follows that the map F_n^r is a lifting of a circle map of degree 1, and thus the results on rotation sets for circle maps apply to it straightforwardly.

We also generalise the notion of a *twist orbit* from the context of degree one circle maps to this setting.

Definition 1.13. — *Let* $T \in \mathbf{T}$ *and* $F \in \mathcal{C}_1(T)$. *An orbit* (mod 1) $P \subset \mathbb{R}$ *of* F *will be called* twist *if* $F\big|_P$ *is strictly increasing.*

Remark 1.14. — The following statements are easy to check.

(i) Two points in the same orbit (mod 1) have the same rotation number.

(ii) If $F^q(x) = x + p$ with $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, then $\rho_F(x) = p/q$. Therefore all periodic (mod 1) points have rational rotation numbers.

(iii) Let x be a periodic (mod 1) point of period q and $p \in \mathbb{Z}$ such that $F^q(x) = x + p$. If $\operatorname{Orb}_1(x, F)$ is a twist orbit, then it follows from [6, Corollary 3.7.6] that $(p, q) = 1$.

The following theorem describes the relation between the sets $\operatorname{Rot}(F^n)$ and $\operatorname{Rot}(F_n^r)$.

Theorem 1.15. — *Let $F \in \mathcal{C}_1(T)$ and let $n \geqslant 1$. Assume that $x \in \mathbb{R}$ is such that $\operatorname{Orb}_1(x, F^n) \subset \mathbb{R}$. Then, $\rho_{F_n^r}(x) = \rho_{F^n}(x) = n\rho_F(x)$. Conversely, for each $\alpha \in \operatorname{Rot}(F_n^r)$ there exists $x \in \mathbb{R}$ such that $\alpha = \rho_{F_n^r}(x) = \rho_{F^n}(x) = n\rho_F(x)$, $\operatorname{Orb}_1(x, F^n) = \operatorname{Orb}_1(x, F_n^r) \subset \mathbb{R}$ and $\operatorname{Orb}_1(x, F^n)$ is twist. Moreover, if $\alpha \in \mathbb{Q}$ then x can be chosen to be periodic* (mod 1) *for F. In particular, for each $n \in \mathbb{N}$, $\frac{1}{n}\operatorname{Rot}(F_n^r) \subset \operatorname{Rot}_{\mathbb{R}}(F)$.*

To prove Theorem 1.15 we introduce the notion of $\operatorname{Const}(F)$ and study its basic properties.

Given a continuous map $g\colon X \longrightarrow Y$, we will denote by $\operatorname{Const}(g)$ the set of points $x \in X$ such that g is constant in a neighbourhood of x. Clearly, $\operatorname{Const}(g)$ is open and $g|_{\operatorname{Clos}(C)}$ is constant for each connected component C of $\operatorname{Const}(g)$.

Lemma 1.16. — *Let $T \in \mathbf{T}$ and let $F \in \mathcal{C}_1(T)$. If $x \notin \operatorname{Const}(r \circ F)$ then $r \circ F(x) = F(x)$. Consequently, for each $n \in \mathbb{N}$, $x \notin \operatorname{Const}(F_n^r)$ implies $F_n^r(x) = F^n(x)$.*

Proof. — Suppose that $r \circ F(x) \neq F(x)$. Then, $F(x) \notin \mathbb{R}$. By the continuity of F and Lemma 1.5(a), there exists an open neighbourhood U of x in T such that $r(F(U)) = r(F(x))$. This shows that $x \in \operatorname{Const}(r \circ F)$, which is a contradiction. The second statement of the lemma follows trivially from the first one. □

Now we are ready to prove Theorem 1.15.

Proof of Theorem 1.15. — The first statement of the theorem follows from Lemma 1.10(c). If $\alpha \in \operatorname{Rot}(F_n^r)$, then by [6, Theorem 3.7.20] there exists a point $x \in \mathbb{R}$ such that $\rho_{F_n^r}(x) = \alpha$, $\operatorname{Orb}_1(x, F_n^r) \subset \mathbb{R} \setminus \operatorname{Const}(F_n^r)$ and $\operatorname{Orb}_1(x, F_n^r)$ is twist. Moreover, if $\alpha \in \mathbb{Q}$ then x can be chosen to be a periodic (mod 1) point of F_n^r. Then the theorem follows from Lemma 1.16. □

From Theorem 1.15 we can derive the following consequences.

Corollary 1.17. — *Let $F \in \mathcal{C}_1(T)$ and let $n \in \mathbb{N}$. Then, $\operatorname{Rot}(F_1^r) \subset \frac{1}{n}\operatorname{Rot}(F_n^r)$. Moreover, for each $n \in \mathbb{N}$, $\operatorname{Rot}(F_n^r)$ is a nonempty compact interval. Consequently, the set $\bigcup_{n\geqslant 1} \frac{1}{n}\operatorname{Rot}(F_n^r)$ is a nonempty interval contained in $\operatorname{Rot}_{\mathbb{R}}(F)$.*

Proof. — For each $\alpha \in \mathrm{Rot}(F_1^r)$ let x be the point given by the second statement of Theorem 1.15 for F_1^r. In particular, $\rho_F(x) = \alpha$ and $\mathrm{Orb}_1(x, F) = \mathrm{Orb}_1(x, F_1^r) \subset \mathbb{R}$. Consequently, for all $n \in \mathbb{N}$, $\mathrm{Orb}_1(x, F^n) \subset \mathbb{R}$ and, by the first statement of Theorem 1.15,

$$\rho_{F_n^r}(x) = \rho_{F^n}(x) = n\rho_F(x) = n\alpha.$$

This ends the proof of the first statement of the corollary. The fact that, for each $n \in \mathbb{N}$, $\mathrm{Rot}(F_n^r)$ is a nonempty compact interval follows for instance from [6, Theorem 3.7.20]. Then, the last statement of the corollary follows immediately. $\square$

Remark 1.18. — In general, the interval $\bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F_n^r)$ need not be closed: see Example 6.6.

2. Positive covering

To find periodic points in one-dimensional spaces, the notion of *covering* (introduced in [11]) is often used. If I, J are two compact intervals, I F-covers J if there exists a subinterval $I_0 \subset I$ such that $F(I_0) = J$. It is well known that if I F-covers I then there exists a point $x \in I$ such that $F(x) = x$. If I F-covers I then $F(I) \supset I$ but the latter condition does not ensure the existence of a fixed point (see e.g. [21]).

In this section we are going to introduce a variant of the notion of covering, that we call *positive covering*. Roughly speaking, I positively F-covers J if $F(I) \supset J$ and this inclusion is "globally increasing". Positive covering does not imply covering but we will see that if I positively F-covers I then F has a fixed point in I (Proposition 2.3). This will be a main tool in the rest of the paper.

Definition 2.1. — *Let $T \in \mathbf{T}$, let $F\colon T \longrightarrow T$ be a continuous map and let I, J be two compact subintervals of $\mathbb{R}$. We say that I* positively F-covers *J and we write $I \xrightarrow[F]{+} J$ if there exist $x, y \in I$ such that $x \leqslant y$, $r \circ F(x) \leqslant \min J$ and $r \circ F(y) \geqslant \max J$. We remark that I positively F-covers J if and only if I positively $r \circ F$-covers J. So, we will indistinctly write $I \xrightarrow[F]{+} J$ or $I \xrightarrow[r\circ F]{+} J$.*

In the next lemma we state some basic properties of positive covering. We say that an interval $I \subset \mathbb{R}$ is *non degenerate* if it is neither empty nor reduced to a point.

LEMMA 2.2. — *Let $T \in \mathbf{T}$, let $F, G \colon T \longrightarrow T$ be two continuous maps and let I, J and K be three compact non degenerate subintervals of $\mathbb{R}$.*

(a) *Suppose that $I \xrightarrow[F]{+} J$. If $K \subset J$ then $I \xrightarrow[F]{+} K$. If $K \supset I$ then $K \xrightarrow[F]{+} J$.*

(b) *If $I \xrightarrow[F]{+} J$ and $a, b \in J$ with $a < b$, then there exist $x_0, y_0 \in I$ such that $x_0 \leqslant y_0$, $F(x_0) = a$, $F(y_0) = b$ and $r \circ F(t) \in (a, b)$ for all $t \in (x_0, y_0)$.*

(c) *If $I \xrightarrow[F]{+} J$ and $J \xrightarrow[G]{+} K$ then $I \xrightarrow[G\circ F]{+} K$.*

(d) *Suppose that F is of degree 1. If $I \xrightarrow[F]{+} J$ then $(I+n) \xrightarrow[F-k]{+} (J+n-k)$ for all $n, k \in \mathbb{Z}$.*

Proof. — Statements (a) and (d) follow easily from the definitions.

To prove (b), suppose that $I \xrightarrow[F]{+} J$, that is, there exist $x_1 \leqslant y_1$ in I such that $r \circ F(x_1) \leqslant \min J$ and $r \circ F(y_1) \geqslant \max J$. Since $r \circ F$ is continuous, $r \circ F(I) \supset J$. Let $a, b \in J$ with $a < b$ and set $x_0 = \max\{t \in [x_1, y_1] : r \circ F(t) = a\}$. Then, $x_0 < y_1$ because $a < \max J$. Lemma 1.5(a) implies then that $F(x_0) \in \mathbb{R}$, and thus $F(x_0) = a$. Similarly, let

$$y_0 = \min\{t \in [x_0, y_1] : r \circ F(t) = b\}.$$

The point $F(y_0)$ is in $\mathbb{R}$, and thus $F(y_0) = b$. The choice of x_0, y_0 implies that if $t \in (x_0, y_0)$, then $r \circ F(t) \in (a, b)$. This proves (b).

Now we prove (c). Suppose that $I \xrightarrow[F]{+} J \xrightarrow[G]{+} K$. Let $a, b \in J$ such that $a < b$, $r \circ G(a) \leqslant \min K$ and $r \circ G(b) \geqslant \max K$. According to (b) there exist $x_0, y_0 \in I$ such that $x_0 \leqslant y_0$, $F(x_0) = a$ and $F(y_0) = b$. Then $r \circ G \circ F(x_0) \leqslant \min K$ and $r \circ G \circ F(y_0) \geqslant \max K$; that is, $I \xrightarrow[G\circ F]{+} K$. This shows (c). $\square$

The next proposition will be a key tool to find periodic (mod 1) points.

PROPOSITION 2.3. — *Let $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$ and let $I_0, \dots, I_{k-1}$ be compact non degenerate intervals in $\mathbb{R}$ such that*

$$I_0 \xrightarrow[(F^r_{n_1})^{q_1} - p_1]{+} I_1 \xrightarrow[(F^r_{n_2})^{q_2} - p_2]{+} \cdots I_{k-1} \xrightarrow[(F^r_{n_k})^{q_k} - p_k]{+} I_0,$$

where the numbers n_i and q_i are positive integers and $p_i \in \mathbb{Z}$. For every $i \in \{1, 2, \dots, k\}$ set $m_i := \sum_{j=1}^{i} q_j n_j$ and $\widehat{p}_i := \sum_{j=1}^{i} p_j$. Then, there exists $x_0 \in I_0$ such that $F^{m_k}(x_0) = x_0 + \widehat{p}_k$ and $F^{m_i}(x_0) \in I_i + \widehat{p}_i$ for all $i = 1, 2, \dots, k-1$.

To prove the above proposition we need three technical lemmas.

Lemma 2.4. — *Let $a, b \in \mathbb{R}$ with $a < b$ and let $g\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous map such that $g(a) \leqslant a$ and $g(b) \geqslant b$. Then there exists $x \in [a,b]$ such that $g(x) = x$ and $x \notin \operatorname{Const}(g)$.*

Proof. — Let $b_0 = \min\{x \in [a,b] : g(x) = b\}$. Observe that b_0 cannot belong to $\operatorname{Const}(g)$, the map g is continuous and $g(a) \leqslant a < b_0 \leqslant g(b_0) = b$. Thus there exists $x \in [a, b_0]$ such that $g(x) = x$. Define

$$x_0 = \max\{x \in [a, b_0] : g(x) = x\}.$$

We will show by absurd that $x_0 \notin \operatorname{Const}(g)$.

Suppose that $x_0 \in \operatorname{Const}(g)$ and call J the connected component of $\operatorname{Const}(g)$ containing x_0 and $a_0 = \sup J$. Then, the interval J is relatively open in $[a, b_0]$ and $b_0 \notin \operatorname{Const}(g)$. This implies that $a_0 \notin J$ and hence, $x_0 < a_0$. Since $g(a_0) = g(x_0) = x_0 < a_0$ and $g(b_0) \geqslant b_0$, there exists a fixed point of g in $[a_0, b_0]$ which contradicts the choice of x_0. Consequently, $x_0 \notin \operatorname{Const}(g)$. $\square$

The next lemma is easy to prove.

Lemma 2.5. — *Let F, H be continuous maps from $\mathbb{R}$ into itself. Then,* $\operatorname{Const}(F) \subset \operatorname{Const}(H \circ F)$.

Lemma 2.6. — *Let $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$, $x_0 \in \mathbb{R}$ and let $n_1, n_2, \dots, n_k$ be positive integers. For every $i \in \{1, 2, \dots, k\}$ set $G_i := F^r_{n_i} \circ \cdots \circ F^r_{n_2} \circ F^r_{n_1}$ and $m_i := n_1 + n_2 + \cdots + n_i$. Assume that $x_0 \in \mathbb{R}$ is such that $x_0 \notin \operatorname{Const}(G_k)$. Then, $G_i(x_0) = F^{m_i}(x_0)$ for all $i = 1, 2, \dots, k$.*

Proof. — To prove the lemma assume that on the contrary there exists $i \in \{1, 2, \dots, k\}$ such that $G_i(x_0) \neq F^{m_i}(x_0)$ but $G_j(x_0) = F^{m_j}(x_0) \in \mathbb{R}$ for all $j = 1, 2, \dots, i-1$. To deal with the case $i = 1$ we set $m_0 = 0$, $G_0 = \operatorname{Id}$ and, to simplify the notation, $z = F^{m_{i-1}}(x_0) = G_{i-i}(x_0)$. Then we have

$$F^r_{n_i}(z) = F^r_{n_i}\big(G_{i-1}(x_0)\big) = G_i(x_0) \neq F^{m_i}(x_0) = F^{n_i}(z).$$

Therefore, from Lemma 1.16, it follows that $z \in \operatorname{Const}(F^r_{n_i})$. Since $z = G_{i-i}(x_0)$, by the continuity of G_{i-i} it follows that $x_0 \in \operatorname{Const}(F^r_{n_i} \circ G_{i-i}) = \operatorname{Const}(G_i)$. When $i < k$ we obtain that $x_0 \in \operatorname{Const}(G_k)$ by Lemma 2.5. Thus, in all cases we have shown that $x_0 \in \operatorname{Const}(G_k)$; a contradiction. $\square$

Proof of Proposition 2.3. — For every $i \in \{1, 2, \dots, k\}$ set

$$G_i := (F^r_{n_i})^{q_i} \circ \cdots \circ (F^r_{n_1})^{q_1}.$$

Then, in view of Lemma 2.2(c,d), we have $I_0 \xrightarrow[G_k - \widehat{p_k}]{+} I_0$. Moreover, applying inductively Lemma 2.2(b), we get that there exist $x, y \in I_0$ such that

$x < y$, $G_i([x,y]) \subset I_i + \widehat{p}_i$ for $i = 1,2,\dots,k-1$, $(G_k - \widehat{p}_k)(x) = \min I_0$ and $(G_k - \widehat{p}_k)(y) = \max I_0$. Moreover, by Lemma 2.4 applied to the map $(G_k - \widehat{p}_k)\big|_{[x,y]} \colon [x,y] \longrightarrow \mathbb{R}$ there exists a point $x_0 \in [x,y]$ such that $G_k(x_0) = x_0 + \widehat{p}_k$ and $x_0 \notin \operatorname{Const}(G_k)$. By Lemma 2.6, $G_i(x_0) = F^{m_i}(x_0)$ for all $i = 1,2,\dots,k$. Therefore, by the definition of $[x,y]$, the point $F^{m_i}(x_0)$ belongs to $I_i + \widehat{p}_i$ for all $i = 1,2,\dots,k-1$, and $F^{m_k}(x_0) = G_k(x_0) = x_0 + \widehat{p}_k$. □

To be able to use Proposition 2.3 in an easy way we introduce the following notation. Let

$$\mathcal{P}\colon I_0 \xrightarrow[F^{n_1}-p_1]{+} I_1 \cdots \xrightarrow[F^{n_k}-p_k]{+} I_k, \text{ and}$$
$$\mathcal{P}'\colon I_k \xrightarrow[F^{m_1}-q_1]{+} J_1 \cdots \xrightarrow[F^{m_l}-q_l]{+} J_l$$

be two sequences of positive coverings. Then we will denote by $\mathcal{P}\mathcal{P}'$ the concatenation of $\mathcal{P}$ and $\mathcal{P}'$. That is, $\mathcal{P}\mathcal{P}'$ denotes the sequence:

$$I_0 \xrightarrow[F^{n_1}-p_1]{+} I_1 \cdots \xrightarrow[F^{n_k}-p_k]{+} I_k \xrightarrow[F^{m_1}-q_1]{+} J_1 \cdots \xrightarrow[F^{m_l}-q_l]{+} J_l.$$

In the particular case when $\mathcal{P}$ is a *loop*, that is $I_0 = I_k$, then we will denote by $\mathcal{P}^n$ the sequence $\mathcal{P}$ concatenated with itself $n-1$ times:

$$\overbrace{\mathcal{P}\cdots\mathcal{P}\cdots\mathcal{P}}^{n \text{ times}}.$$

Finally, $\operatorname{Fol}(\mathcal{P})$ will denote the set of points that “follow” $\mathcal{P}$. That is,

$$\operatorname{Fol}(\mathcal{P}) := \{x \in I_0 : \big(F^{n_1+\dots+n_i} - (p_1+\dots+p_i)\big)(x) \in I_i \text{ for all } 1 \leqslant i \leqslant k\}.$$

Clearly, $\operatorname{Fol}(\mathcal{P})$ is a compact set and, in view of Proposition 2.3, it is non-empty.

3. The rotation set

In this section we deepen the study about the rotation set of the maps from $F \in \mathcal{C}_1(T)$, with $T \in \mathbf{T}$. It is divided into two subsections. In the first one we study the connectedness and compactness of the rotation set together with its relation with periodic (mod 1) orbits. In Subsection 3.2 we describe the information on the periodic (mod 1) orbits of the map which is carried out by the rotation set.

3.1. On the connectedness and compactness of the rotation set

The rotation set $\mathrm{Rot}(F)$ may not be connected (see Example 1.12) and in general we do not know whether it is closed. However, the main result of this subsection (Theorem 3.1) shows that the set of rotation numbers of points $x \in \mathbb{R}$ is a non empty compact interval which coincides with $\mathrm{Rot}^+_{\mathbb{R}}(F)$ and $\mathrm{Rot}^-_{\mathbb{R}}(F)$. Its proof is inspired by [13, Lemma 3].

THEOREM 3.1. — *Let $T \in \mathbf{T}$ and $F \in \mathcal{C}_1(T)$. Then $\mathrm{Rot}_{\mathbb{R}}(F)$ is a non empty compact interval and $\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}^+_{\mathbb{R}}(F) = \mathrm{Rot}^-_{\mathbb{R}}(F) = \mathrm{Clos}\left(\bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F^r_n)\right)$. Moreover, if $\alpha \in \mathrm{Rot}_{\mathbb{R}}(F)$, then there exists a point $x \in \mathbb{R}$ such that $\rho_F(x) = \alpha$ and $F^n(x) \in \mathbb{R}$ for infinitely many n. If $p/q \in \mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F))$, then there exists a periodic* (mod 1) *point $x \in \mathbb{R}$ with $\rho_F(x) = p/q$.*

To prove Theorem 3.1 we will use the next lemma which is not difficult to prove.

LEMMA 3.2. — *Let $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$, $x \in \mathbb{R}$ and A a constant. If $\overline{\rho}_F(x) < \alpha$, then there exists a positive integer N such that, for all $n \geqslant N$, $r \circ F^n(x) \leqslant x + n\alpha - A$. If $\overline{\rho}_F(x) > \alpha$, then there exists an increasing sequence of positive integers $\{n_k\}_{k\geqslant 0}$ such that, for all $k \geqslant 0$, $r \circ F^{n_k}(x) \geqslant x + n_k\alpha + A$.*

Similar statements with the inequalities reversed hold for $\underline{\rho}_F(x)$.

Proof of Theorem 3.1. — We are going to show that $\mathrm{Rot}^+_{\mathbb{R}}(F)$ is a non empty compact interval equal to $\mathrm{Rot}_{\mathbb{R}}(F)$, the case with $\mathrm{Rot}^-_{\mathbb{R}}(F)$ being similar.

By definition, $\mathrm{Rot}^+_{\mathbb{R}}(F) \supset \mathrm{Rot}_{\mathbb{R}}(F)$ and by Corollary 1.17 $\mathrm{Rot}_{\mathbb{R}}(F)$ contains the non empty interval $\bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F^r_n)$. If $\mathrm{Rot}^+_{\mathbb{R}}(F)$ is reduced to a single point, then

$$\mathrm{Rot}^+_{\mathbb{R}}(F) = \mathrm{Rot}_{\mathbb{R}}(F) = \bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F^r_n) = \mathrm{Clos}\left(\bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F^r_n)\right).$$

Moreover, again by Corollary 1.17, $\mathrm{Rot}(F^r_1) = \mathrm{Rot}_{\mathbb{R}}(F)$. So, the theorem follows in this case by Theorem 1.15.

In the rest of the proof we assume that $\mathrm{Rot}^+_{\mathbb{R}}(F)$ contains at least two points. This set is bounded by $\max\{|r \circ F(x) - r(x)| : x \in T\}$ and hence there exist $a = \inf \mathrm{Rot}^+_{\mathbb{R}}(F)$ and $b = \sup \mathrm{Rot}^+_{\mathbb{R}}(F)$. Fix $\alpha \in [a, b]$. Since

$a < b$, there exist sequences of integers $p_n \in \mathbb{Z}$, $q_n \in \mathbb{N}$, such that, for every $n \in \mathbb{N}$, we have $\frac{p_n}{q_n} \in (a,b)$,

$$\left|\frac{p_n}{q_n} - \alpha\right| \leqslant \left|\frac{p_1}{q_1} - \alpha\right| \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{p_n}{q_n} = \alpha.$$

By the choice of a, b and $\frac{p_n}{q_n}$, for all $n \geqslant 1$, there exist $x_n, y_n \in \mathbb{R}$ such that $\overline{\rho}_{_F}(x_n) < \frac{p_n}{q_n}$ and $\overline{\rho}_{_F}(y_n) > \frac{p_n}{q_n}$. Moreover, by Lemma 1.10(a), the points x_n and y_n can be chosen so that $x_n \in [0,1]$ and $y_n \in [x_n, x_n+1]$. Set $I = [0,2]$. By the choice of x_n and y_n we have $[x_n, y_n] \subset I \subset [x_n - 1, y_n + 2]$.

Applying Lemma 3.2 to F^{q_n} we see that there exist two positive integers N and $k_n > N$ such that $r \circ F^{kq_n}(x_n) \leqslant x_n + kp_n - 1$ for all $k \geqslant N$, and $r \circ F^{k_n q_n}(y_n) \geqslant y_n + k_n p_n + 2$. Then $[x_n, y_n] \xrightarrow[F^{k_n q_n} - k_n p_n]{+} [x_n - 1, y_n + 2]$ and, hence, $I \xrightarrow[F^{k_n q_n} - k_n p_n]{+} I$ by Lemma 2.2(a).

Let $\{i_n\}_{n\geqslant 1}$ be a sequence of positive integers that will be specified later and let $\mathcal{P}_n := I \xrightarrow[F^{k_n q_n} - k_n p_n]{+} I$. We set

$$X_n = \mathrm{Fol}\big((\mathcal{P}_1)^{i_1}(\mathcal{P}_2)^{i_2}\cdots(\mathcal{P}_n)^{i_n}\big) \qquad \text{and} \qquad X = \bigcap_{n\geqslant 1} X_n.$$

As it has been noticed before, X_n is a non empty compact set and, clearly, $X_{n+1} \subset X_n$. Hence X is not empty. Moreover, if $x \in X$, then $F^n(x) \in \mathbb{R}$ for infinitely many n.

We will show that if the sequence $\{i_n\}_{n\geqslant 1}$ increases sufficiently fast then, $\rho_{_F}(x) = \alpha$ for all $x \in X$. To do it write $N_n = i_n k_n q_n$. Now we set $i_1 = 1$ and, if $i_1, \ldots, i_{n-1}$ are already fixed, we choose i_n such that

(i) $\frac{N_1+\cdots+N_{n-1}}{i_n k_n q_n} \leqslant \frac{1}{n}$,

(ii) $\frac{k_{n+1} q_{n+1}}{i_n k_n q_n} \leqslant \frac{1}{n}$.

For any $k \in \mathbb{N}$ there exists an integer n such that

$$N_1 + \cdots + N_{n-1} \leqslant k < N_1 + \cdots + N_{n-1} + N_n.$$

Therefore, there exist $0 \leqslant i < i_n$ and $0 \leqslant s < k_n q_n$ so that k can be written as $k = \widetilde{N} + s$ where for simplicity we have set $\widetilde{N} := N_1 + \cdots + N_{n-1} + ik_n q_n$. On the other hand, recall that the map $y \mapsto r \circ F(y) - r(y)$ is 1-periodic on T. Thus, $L = \max\{|r \circ F(z) - r(z)| : z \in T\}$ exists. Consequently, for $x \in X$ and k large enough we have,

$$|r \circ F^k(x) - r \circ F^{\widetilde{N}}(x)| \leqslant sL.$$

Thus,

$$\left|\frac{r \circ F^k(x) - x - k\alpha}{k}\right| \leqslant \frac{s}{k} L + \frac{s}{k}|\alpha| + \left|\frac{F^{\widetilde{N}}(x) - x - \widetilde{N}\alpha}{k}\right|. \tag{3.1}$$

Since $x \in X$ we have that $x \in I$, and $F^{\widetilde{N}}(x) = z + m$ with $z \in I$ and

$$m = \sum_{j=1}^{n-1} i_j k_j p_j + i k_n p_n = \sum_{j=1}^{n-1} N_j \frac{p_j}{q_j} + i k_n q_n \frac{p_n}{q_n}.$$

Therefore, since I has length 2,

$$\begin{aligned}
\left|F^{\widetilde{N}}(x) - x - \widetilde{N}\alpha\right| &\leqslant \left|z - x\right| + \left|m - \widetilde{N}\alpha\right| \\
&\leqslant 2 + \sum_{j=1}^{n-1} N_j \left|\frac{p_j}{q_j} - \alpha\right| + i k_n q_n \left|\frac{p_n}{q_n} - \alpha\right| \\
&\leqslant 2 + \sum_{j=1}^{n-2} N_j \left|\frac{p_1}{q_1} - \alpha\right| + N_{n-1}\left|\frac{p_{n-1}}{q_{n-1}} - \alpha\right| + i k_n q_n \left|\frac{p_n}{q_n} - \alpha\right|
\end{aligned}$$

(where in the last inequality we have used that $\left|\frac{p_j}{q_j} - \alpha\right| \leqslant \left|\frac{p_1}{q_1} - \alpha\right|$ for all j).

Now, observe that

- from Condition (i) we see that,

$$\frac{1}{k}\sum_{j=1}^{n-2} N_j \leqslant \frac{1}{N_{n-1}} \sum_{j=1}^{n-2} N_j \leqslant \frac{1}{n-1},$$

- Condition (ii) gives $\frac{s}{k} < \frac{q_n k_n}{N_{n-1}} \leqslant \frac{1}{n-1}$, and
- $\frac{N_{n-1}}{k} \leqslant 1$ and $\frac{i k_n q_n}{k} \leqslant 1$ because $k \geqslant \widetilde{N} \geqslant N_{n-1} + i k_n q_n$.

Consequently, by replacing all the above in Equation (3.1), we obtain

$$\begin{aligned}
\left|\frac{r \circ F^k(x) - x}{k} - \alpha\right| < \frac{L + |\alpha|}{n-1} + \frac{2}{k} + \frac{1}{n-1}\left|\frac{p_1}{q_1} - \alpha\right| \\
+ \left|\frac{p_{n-1}}{q_{n-1}} - \alpha\right| + \left|\frac{p_n}{q_n} - \alpha\right|.
\end{aligned}$$

Since n goes to infinity when so does k and $\lim_{n \to +\infty} \frac{p_n}{q_n} = \alpha$, we get that the right hand side of the above inequality converges to zero. Hence,

$$\rho_{_F}(x) = \lim_{k \to +\infty} \frac{r \circ F^k(x) - x}{k} = \alpha.$$

This proves that $\mathrm{Rot}^+_{\mathbb{R}}(F) \subset [a,b] \subset \mathrm{Rot}_{\mathbb{R}}(F)$; that is,

$$\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}^+_{\mathbb{R}}(F) = [a,b].$$

When $\alpha = \frac{p}{q} \in (a,b)$, the proof is simpler and gives a periodic (mod 1) point with rotation number p/q. Indeed, by taking $p_1 = p$ and $q_1 = q$, the sequence $\mathcal{P}_1$ gives $I \xrightarrow[F^{k_1 q}-kp]{+} I$. Thus, by Proposition 2.3, there exists a point $x \in I$ such that $F^{k_1 q}(x) = x + k_1 p$. Hence x is periodic (mod 1) and $\rho_{F^{k_1 q}}(x) = k_1 p$. By Lemma 1.10 $\rho_F(x) = p/q$ and $p/q \in \frac{1}{k_1 q}\mathrm{Rot}(F^r_{k_1 q})$. Moreover, by Theorem 1.15, $\frac{1}{k_1 q}\mathrm{Rot}(F^r_{k_1 q}) \subset \mathrm{Rot}_{\mathbb{R}}(F)$. Thus the density of the rational numbers in $[a,b]$ implies that

$$\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Clos}\left(\bigcup_{n \geqslant 1} \frac{1}{n}\mathrm{Rot}(F^r_n)\right).$$

□

Remark 3.3. — The last statement of Theorem 3.1 is weaker than Theorem 3.11. We nevertheless state it here because it is a byproduct of the proof.

Generally $\mathrm{Rot}_{\mathbb{R}}(F)$ is a proper subset of $\mathrm{Rot}(F)$. The next proposition gives an immediate sufficient condition to have $\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F)$. We will see later other sufficient conditions (which include the transitive case) when the lifted space T is an infinite graph (Theorem 5.5).

Proposition 3.4. — *Let $T \in \mathbf{T}$ and $F \in \mathcal{C}_1(T)$. If $\bigcup_{n \in \mathbb{Z}} F^n(\mathbb{R}) = T$ then*

$$\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F) = \mathrm{Rot}^+(F) = \mathrm{Rot}^-(F).$$

Proof. — Let $y \in T$. If $y \in F^n(\mathbb{R})$ with $n \geqslant 0$, let $x \in \mathbb{R}$ such that $y = F^n(x)$. If $y \in F^{-n}(\mathbb{R})$ with $n \geqslant 0$, let $x = F^n(y) \in \mathbb{R}$. In both cases, $\overline{\rho}_F(y) = \overline{\rho}_F(x)$ and $\underline{\rho}_F(y) = \underline{\rho}_F(x)$. Thus, $\mathrm{Rot}^+_{\mathbb{R}}(F) = \mathrm{Rot}^+(F)$, $\mathrm{Rot}^-_{\mathbb{R}}(F) = \mathrm{Rot}^-(F)$ and $\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F)$. On the other hand, by Theorem 3.1, we get that $\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}^+_{\mathbb{R}}(F) = \mathrm{Rot}^-_{\mathbb{R}}(F)$; which ends the proof of the proposition. □

3.2. Relation between the rotation set and the set of periods

In this subsection, we study the set of periods of periodic (mod 1) points with a given (rational) rotation number. To be more precise we need to introduce the appropriate notation.

Definition 3.5. — *Let $T \in \mathbf{T}$ and $F \in \mathcal{C}_1(T)$. The set of periods of all periodic* (mod 1) *points of F in T will be denoted by* $\mathrm{Per}(F)$. *Also, given $\alpha \in \mathbb{R}$,* $\mathrm{Per}(\alpha, F)$ *will denote the set of periods of all periodic* (mod 1) *points of F in T whose F-rotation number is α. Similarly, we denote by* $\mathrm{Per}_{\mathbb{R}}(F)$ *and* $\mathrm{Per}_{\mathbb{R}}(\alpha, F)$ *the same sets as before with the additional restriction that the periodic* (mod 1) *points under consideration must belong to* $\mathbb{R}$ *(we do not require that the whole periodic* (mod 1) *orbits belong to* $\mathbb{R}$*).*

The main results of this section state that, for every $p/q \in \mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F))$, the set $\mathrm{Per}(p/q, F)$ contains $\{nq : \text{for all } n \in \mathbb{N} \text{ large enough}\}$. Moreover, if $\mathrm{Rot}_{\mathbb{R}}(F)$ is not reduced to a single point, then $\mathbb{N} \setminus \mathrm{Per}(F)$ is finite.

The next proposition clarifies the relation between $\mathrm{Per}(F)$ and $\mathrm{Per}(\alpha, F)$. It improves Remark 1.14(ii).

Proposition 3.6. — *Assume that $F \in \mathcal{C}_1(T)$. Then,*

$$\mathrm{Per}(F) = \bigcup_{\alpha \in \mathrm{Rot}(F) \cap \mathbb{Q}} \mathrm{Per}(\alpha, F).$$

On the other hand, if p, q are coprime and $p/q \in \mathrm{Rot}(F)$, then

$$\mathrm{Per}(p/q, F) \subset q\mathbb{N}.$$

Proof. — The first statement of the proposition follows directly from Remark 1.14(ii).

Now assume that $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ are coprime and let $n \in \mathrm{Per}(p/q, F)$. Assume that x is a periodic (mod 1) point of F of period n such that $\rho_F(x) = p/q$. There exists $k \in \mathbb{Z}$ such that $F^n(x) = x + k$. By what precedes, $\rho_F(x) = k/n = p/q$. Then, since p, q are coprime there exists $d \geqslant 1$ such that $k = dp$ and $n = dq$. That is, $n \in q\mathbb{N}$. □

The next proposition gives a sufficient condition to have periodic points of all large enough periods. It is a key tool for Theorem 3.11.

Definition 3.7. — *Let $\chi\colon \mathbb{R}^+ \longrightarrow \mathbb{N}$ be the map defined by*

$$\chi(t) = \begin{cases} \max\{\lceil t\rceil^2, 51\lceil t\rceil\} & \text{if } t > 1,\\ 1 & \text{when } 0 \leqslant t \leqslant 1,\end{cases}$$

where $\lceil\cdot\rceil$ denotes the ceiling function.

Proposition 3.8. — *Let $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$, and let I, J be two disjoint compact non degenerate subintervals of $\mathbb{R}$. Assume that there exists a constant $t > 0$ such that for all integers $n \geqslant t$ both I and J positively F^n-cover I and J. Then, for every positive integer $m \geqslant \chi(t)$, there exists a point $x \in I$ such that $F^m(x) = x$ and $F^i(x) \neq x$ for all $1 \leqslant i \leqslant m - 1$.*

The proof of the proposition entirely relies on the following arithmetical lemma.

Lemma 3.9. — *Let $N \in \mathbb{N}$. Then, for every $m \geqslant \chi(N)$, there exist $n_1, \dots, n_{k_0}$ such that*

(a) $n_1 + n_2 + \cdots + n_{k_0} = m$,

(b) $n_i \geqslant N$ *for all* $1 \leqslant i \leqslant k_0$,

(c) *if d divides m, $d \neq m$, then there exists $1 \leqslant i \leqslant k_0 - 1$ such that d divides $n_1 + \cdots + n_i$.*

Proof. — If $N = 1$, then the result is obvious by taking $k_0 = m$ and $n_i = 1$ for all $1 \leqslant i \leqslant m$, because $\chi(N) = 1$.

Let $m \geqslant N > 1$. We write

$$m = p_1^{\alpha_1} \dots p_k^{\alpha_k}$$

with $\alpha_i \geqslant 1$ and $p_1 > p_2 > \cdots > p_k$ the prime factors of m. We define $d_i = \frac{m}{p_i}$ for all $1 \leqslant i \leqslant k$. If d divides m, $d \neq m$, then d divides d_i for some $1 \leqslant i \leqslant k$. Consequently, it is sufficient to prove the lemma for the divisors $d_1, \dots, d_k$ instead of for any d dividing m and $d \neq m$. The numbers d_i are ordered as follows:

$$d_1 < d_2 < \cdots < d_k.$$

The idea of the proof is the following. A small d_i corresponds to a large prime factor p_i, and thus most of the d_i's are "large". It will be possible to write these large divisors as a sum $n_1 + \cdots + n_i$ with $n_j \geqslant N$. It will remain to deal with a small number of small d_i's. For computational reasons, we fix the boundary between "large" and "small" d_i's at $\frac{\sqrt{m}}{\sqrt{N}}$.

Assume that $m \geqslant N^2$, which is equivalent to $\left(\frac{\sqrt{m}}{\sqrt{N}}\right)^4 \geqslant m$. This implies that m has at most three prime factors $p_i > \frac{\sqrt{m}}{\sqrt{N}}$, which are $\{p_i\}_{1 \leqslant i \leqslant \varepsilon}$ for some $0 \leqslant \varepsilon \leqslant 3$ (ε may be zero).

We first deal with $\{d_i\}_{\varepsilon+1 \leqslant i \leqslant k}$ (the "large" divisors — note that this set is empty when $\varepsilon = k$). For $i \in \{\varepsilon + 1, \dots, k\}$, we have $d_i \geqslant \sqrt{m}\sqrt{N} \geqslant N$ because $p_i \leqslant \frac{\sqrt{m}}{\sqrt{N}}$. Moreover, for all $i \in \{\varepsilon + 1, \dots, k\}$,

$$d_{i+1} - d_i = \frac{m(p_i - p_{i+1})}{p_i p_{i+1}} \geqslant \frac{m}{p_i^2} \geqslant N.$$

We define $n_1 = d_{\varepsilon+1}$ and $n_{i+1} = d_{\varepsilon+i+1} - d_{\varepsilon+i}$ for all $1 \leqslant i \leqslant k - \varepsilon - 1$. In this way, $n_i \geqslant N$ and $n_1 + \cdots + n_i = d_{\varepsilon+i}$ for all $1 \leqslant i \leqslant k - \varepsilon$.

Now we deal with $\{d_i\}_{1 \leqslant i \leqslant \varepsilon}$ (the "small" divisors). For all $1 \leqslant i \leqslant \varepsilon$, we define $n_{k-\varepsilon+i}$ such that d_{k+i} divides $n_1 + \cdots + n_{k-\varepsilon+i}$ and $N \leqslant n_{k-\varepsilon+i} \leqslant N + d_{k+i}$.

Finally, we define $k_0 = k + 1$ and $n_{k_0} = m - (n_1 + \cdots + n_{k_0-1})$. It remains to show that $n_{k_0} \geqslant N$ when m is large enough. To prove it, observe that $n_1 + \cdots + n_{k-\varepsilon} = \frac{m}{p_k} \leqslant \frac{m}{2}$ and, for all $1 \leqslant i \leqslant \varepsilon$, $p_i > \frac{\sqrt{m}}{\sqrt{N}}$. Thus $d_i < \sqrt{m}\sqrt{N}$. This implies that

$$n_{k_0} \geqslant \frac{m}{2} - 3\sqrt{m}\sqrt{N} - 3N.$$

Suppose that $m \geqslant \alpha^2 N$, $\alpha > 0$. Then $n_{k_0} \geqslant \left(\frac{\alpha^2}{2} - 3\alpha - 3\right) N$. To have $n_{k_0} \geqslant N$, it is sufficient to have $\frac{\alpha^2}{2} - 3\alpha - 3 \geqslant 1$, that is, $\alpha \geqslant 3 + \sqrt{17}$. Since $(3 + \sqrt{17})^2 < 51$, it follows that when $N > 1$ then it is sufficient to have m larger than or equal to $\max\{N^2, 51N\}$. This completes the proof of the lemma. □

Remark 3.10. — The values of the function χ specified in Definition 3.7 are not optimal, but this is not important. We only need that there exist positive integers $\chi(N)$ verifying Lemma 3.9, and that $\chi(t) = 1$ if $0 \leqslant t \leqslant 1$.

Proof of Proposition 3.8. — Take $m \geqslant \chi(t)$ and write $m = n_1 + \cdots + n_k$ with $n_1, \ldots, n_k$ satisfying Lemma 3.9 for $N = \lceil t \rceil$. We consider

$$I \xrightarrow[F^{n_1}]{+} J \xrightarrow[F^{n_2}]{+} J \xrightarrow[F^{n_3}]{+} \cdots J \xrightarrow[F^{n_k}]{+} I.$$

By Proposition 2.3 (with $q_i = 1$ and $p_i = 0$), there exists x in I such that $F^m(x) = x$ and $F^{n_1+\cdots+n_i}(x) \in J$ for all $1 \leqslant i \leqslant k-1$. We have to prove that $F^i(x) \neq x$ for all $1 \leqslant i \leqslant m-1$. Let d be the minimal positive integer such that $F^d(x) = x$. Clearly, d divides m. Suppose that $d < m$. Then, in view of Lemma 3.9(c) there exists $1 \leqslant i \leqslant k-1$ such that d divides $n_1 + \cdots + n_i$, which implies that $F^{n_1+\cdots+n_i}(x) = x$. On the other hand, $F^{n_1+\cdots+n_i}(x) \in J$ and $I \cap J = \emptyset$, which leads to a contradiction. Thus, the period of x is $d = m$. □

In the rest of this subsection we use Proposition 3.8 to study the sets $\mathrm{Per}_{\mathbb{R}}(p/q, F)$ and $\mathrm{Per}_{\mathbb{R}}(F)$. Obviously, these sets depend on $\mathrm{Rot}_{\mathbb{R}}(F)$ which, by Theorem 3.1 is a non-empty compact interval of the real line. The next result is the analogue in our setting (although it is somewhat weaker) of [6, Lemma 3.9.1] that, for circle maps of degree one, says that if $p/q \in \mathrm{Int}(\mathrm{Rot}(F))$ with p and q coprime, then $\mathrm{Per}(p/q, F) = q\mathbb{N}$.

Theorem 3.11. — *Let $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$ and $\alpha, \beta \in \mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F))$, $\alpha \leqslant \beta$. There exists a positive integer N (depending on α, β) such that, if $\frac{p}{q} \in [\alpha, \beta]$ with p, q coprime, then*

$$\mathrm{Per}_{\mathbb{R}}(p/q, F) \supset \{mq : m \geqslant \chi(N/q)\}.$$

In particular, if $q \geqslant N$ then $\mathrm{Per}_{\mathbb{R}}(p/q, F) = q\mathbb{N}$.

Proof. — According to Lemma 3.2, there exist a positive integer N and two points $x_0, x_1 \in \mathbb{R}$ such that $\overline{\rho}_F(x_0) < \alpha$, $\overline{\rho}_F(x_1) > \beta$ and, for all $n \geqslant N$, $r \circ F^n(x_0) \leqslant x_0 + n\alpha - 1$ and $r \circ F^n(x_1) \geqslant x_1 + n\beta + 1$ By Lemma 1.7(a) we may translate x_1 by an integer such that $x_0 < x_1 < x_0 + 1$. Set $I = [x_0, x_1]$. Clearly, for every $n \geqslant N$ and $j \in \{n\alpha - 1, \dots, n\beta + 1\} \cap \mathbb{N}$, we have

$$I \xrightarrow[F^n]{+} I + j.$$

In particular, if $nq \geqslant N$ and $i \in \{nq\alpha - np - 1, \dots, nq\beta - np + 1\} \cap \mathbb{N}$,

$$I \xrightarrow[F^{nq} - np]{+} I + i.$$

Thus I positively $(F^q - p)^n$-covers $I - 1$, I and $I + 1$ (notice that $nq\alpha - np \leqslant 0 \leqslant nq\beta - np$ because $p/q \in [\alpha, \beta]$).

Set $J = I + 1$. Then $I \cap J = \emptyset$ and both I and J positively $(F^q - p)^n$-cover I and J for all $n \geqslant N/q$. According to Proposition 3.8, we get that, for all $m \geqslant \chi(N/q)$, there exists a periodic point x of period m for the map $F^q - p$. Hence $F^{qm}(x) = x + mp$ and $\rho_F(x) = \frac{p}{q}$. To end the proof of the first statement of the theorem we have to show that $F^i(x) - x \notin \mathbb{Z}$ for $i = 1, 2, \dots, mq - 1$. Assume that, on the contrary, there exists $1 \leqslant d = \frac{mq}{l}$ with $l \in \mathbb{N}$, $l > 1$ such that $F^d(x) = x + a$ for some $a \in \mathbb{Z}$. Then, in view of Lemma 1.7(a),

$$x + mp = F^{mq}(x) = F^{ld}(x) = x + la = x + \frac{mq}{d} a.$$

Consequently, $a = d\frac{p}{q}$ with $d\frac{p}{q} \in \mathbb{Z}$. Thus d must be a multiple of q because p, q are coprime. Write $d = bq$. Then $F^{bq}(x) = x + bp$, which implies that $b = m$ which, in turn, implies $d = mq$. In other words, x is periodic (mod 1) of period mq for F. Therefore, $\mathrm{Per}_{\mathbb{R}}(p/q, F) \supset \{mq : m \geqslant \chi(N/q)\}$.

The second statement of the theorem follows from the first one and the fact that $\chi(t) = 1$ whenever $t \leqslant 1$. □

Remark 3.12. — In view of Example 6.6, the positive integer N of Theorem 3.11 cannot be taken uniform for the whole interval $\mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F))$.

On the other hand, Theorem 3.11 does not imply that $\mathrm{Per}_{\mathbb{R}}(p/q, F)$ is equal to $\{n \in \mathbb{N} : n \geqslant N\}$ for some positive integer N (see Example 6.5).

In Corollary 3.14 we deduce from Theorem 3.11 that $\mathrm{Per}(F)$ contains all but finitely integers, provided $\mathrm{Rot}_{\mathbb{R}}(F)$ is non-degenerate. Its proof relies on the next arithmetical lemma.

LEMMA 3.13. — *Let N be a positive integer and $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. There exists a positive integer N_0 such that, for all $n \geqslant N_0$, there exists $\frac{p}{q} \in [\alpha, \beta]$ with p, q coprime, such that $q \geqslant N$ and q divides n.*

Proof. — We fix a rational $\frac{a}{b} \in [\alpha, \beta)$ with a, b coprime and $b > 0$, and M a positive integer such that $\frac{a}{b} + \frac{1}{M} \in [\alpha, \beta]$. Let $n \geqslant M$. There exists $r \in \{1, \ldots, b\}$ such that b divides $na + r$. Then $\frac{a}{b} + \frac{r}{bn} = \frac{na+r}{bn}$ belongs to $[\alpha, \beta]$ because $\frac{r}{bn} \leqslant \frac{1}{M}$. Since $(na + r)b - (bn)a = br$, Bézout's theorem implies that $\gcd(na + r, bn)$ divides $br \neq 0$. Thus we can write $\frac{na+r}{bn} = \frac{p}{q}$ with p, q coprime and

$$q = \frac{bn}{\gcd(na + r, bn)} \geqslant \frac{bn}{br} \geqslant \frac{n}{b}.$$

Moreover, $\frac{na+r}{bn} = \frac{(na+r)/b}{n}$ because b divides $na + r$, and hence q divides n. Consequently, the lemma holds by taking $N_0 = \max(M, bN)$. $\square$

Corollary 3.14. — *Let $T \in \mathbf{T}$ and $F \in \mathcal{C}_1(T)$. If $\mathrm{Rot}_{\mathbb{R}}(F)$ is not degenerate to a point, then the set $\mathbb{N} \setminus \mathrm{Per}_{\mathbb{R}}(F)$ is finite.*

Proof. — Let N be the positive integer given by Theorem 3.11 for some $\alpha, \beta \in \mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F))$, $\alpha < \beta$. By Lemma 3.13, there exists an integer N_0 such that, for all $n \geqslant N_0$, there exists $\frac{p}{q} \in [\alpha, \beta]$ with p, q coprime, such that $q \geqslant N$ and q divides n. According to Theorem 3.11, $\mathrm{Per}_{\mathbb{R}}(p/q, F) = q\mathbb{N} \ni n$. Hence $\mathrm{Per}_{\mathbb{R}}(F)$ contains all integers $n \geqslant N_0$. $\square$

4. Combed maps

The aim of this section is to show that the rotation set of all maps from a special subclass of $\mathcal{C}_1(T)$ (with $T \in \mathbf{T}$), called *combed maps*, has nice properties analogous to the ones displayed by the continuous circle maps. To do this we will extend the notions of "lower" and "upper" lifting and "water functions" in the spirit of [6, Section 3.7] to this setting.

In the rest of this section T will denote a space from $\mathbf{T}$.

4.1. General definitions for combed maps

We start our task with the simple observation that, for each $x, y \in T$, the relation $r(x) \leqslant r(y)$ defines a linear pre-ordering on T which, in what follows, will be denoted by $x \preccurlyeq y$ (we recall that a pre-ordering is a reflexive, transitive relation). We will also use the notation $x \prec y$ to denote $r(x) < r(y)$.

Definition 4.1. — *A map $F \in \mathcal{C}_1(T)$ such that $F(x) \preccurlyeq F(y)$ whenever $x \preccurlyeq y$ will be called* non-decreasing. *Also, given $F, G \in \mathcal{C}_1(T)$ we write $F \preccurlyeq G$ to denote that $F(x) \preccurlyeq G(x)$ for each $x \in T$.*

Remark 4.2. — When F is non-decreasing and $r(x) = r(y)$, then it easily follows that $r(F(x)) = r(F(y))$. Notice also that the map $r \in \mathcal{C}_1(T)$ is non-decreasing.

The following simple lemma follows in a similar way to [6, Lemma 3.7.19] (and hence we omit its proof).

LEMMA 4.3. — *Assume that $F, G \in \mathcal{C}_1(T)$, $F \preccurlyeq G$ and either F or G is non-decreasing. Then $F^n \preccurlyeq G^n$ for all $n \in \mathbb{N}$.*

Next we define the *upper* and *lower* maps that, as in the circle case, will play a key role in the study of the rotation interval of maps from $\mathcal{C}_1(T)$. Given $F \in \mathcal{C}_1(T)$, we define $F_l, F_u \colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F_u(x) := \sup\{r(F(y)) : y \preccurlyeq x\},$$
$$F_l(x) := \inf\{r(F(y)) : y \succcurlyeq x\}.$$

Remark 4.4. — The following equivalent definitions for the maps F_u and F_l hold:

$$F_u(x) = \max\{r(F(y)) : x - 1 \preccurlyeq y \preccurlyeq x\},$$
$$F_l(x) = \min\{r(F(y)) : x + 1 \succcurlyeq y \succcurlyeq x\}.$$

To prove the above equalities we have to show that

$$\sup\{r(F(y)) : y \preccurlyeq x\} = M := \max\{r(F(y)) : x - 1 \preccurlyeq y \preccurlyeq x\}$$

(we only prove the statement for F_u; the other one follows analogously). Since the map $r \circ F$ is continuous,

$$\sup\{r(F(y)) : y \preccurlyeq x\} = \max\Big\{\sup\{r(F(y)) : y \preccurlyeq x - 1\}, M\Big\}.$$

Thus, it is enough to see that $r(F(y)) \leqslant M$ for all $y \preccurlyeq x - 1$. If, on the contrary, there exists $z \preccurlyeq x - 1$ such that $r(F(z)) > M$, then there exists $k \in \mathbb{N}$ such that $x - 1 \preccurlyeq z + k \preccurlyeq x$ and, by Lemma 1.7(d)

$$r(F(z + k)) = r(F(z)) + k > M + k > \max\{r(F(y)) : x - 1 \preccurlyeq y \preccurlyeq x\};$$

a contradiction.

Now we introduce the notions of combed maps.

DEFINITION 4.5. — *A map $F \in \mathcal{C}_1(T)$ will be called* left-combed *(respectively* right-combed*) at $x \in \mathbb{R}$ if $r \circ F(\{y \in \mathbb{R} : y \leqslant x\}) \supset r \circ F(r^{-1}(x))$ (respectively $r \circ F(\{y \in \mathbb{R} : y \geqslant x\}) \supset r \circ F(r^{-1}(x))$). If F is both left-combed and right-combed at x then it will be simply called* combed *at x (see Figure 4.1 for an example). The map F will be called* combed *if it is combed at every point $x \in \mathbb{R}$.*

Remark 4.6. — If $x \notin \mathrm{B}(T)$ (recall that $\mathrm{B}(T)$ denotes the set of all branching points of T), then $r^{-1}(x) = \{x\}$. Therefore, F is combed at x.

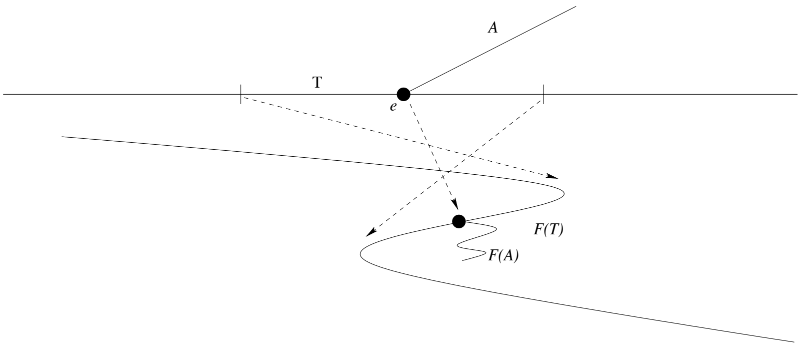

Figure 4.1. The image of the branch A gets "hidden" inside $F(\mathbb{R})$ and thus F is combed at e (actually, $F(\mathbb{R})$ is in T, and the figure shows how it folds up). An observer looking at $F(T)$ from above or below does not distinguish this map from a "pure circle map".

4.2. A characterisation of the upper and lower map for combed maps

The following technical lemma gives a nice characterisation of the maps F_u and F_l for combed maps.

LEMMA 4.7. — *For any map $F \in \mathcal{C}_1(T)$ and $x \in \mathbb{R}$ the following statements hold:*

(a) *If F is left-combed at all $y \in \mathbb{R}$ such that $y \leqslant x$, then*

$$F_u(x) = \sup\{r(F(y)) : y \in \mathbb{R} \text{ and } y \leqslant x\}.$$

(b) *If F is right-combed at all $y \in \mathbb{R}$ such that $y \geqslant x$, then*

$$F_l(x) = \inf\{r(F(y)) : y \in \mathbb{R} \text{ and } y \geqslant x\}.$$

Proof. — We will only prove statement (a). The proof of (b) is analogous. Clearly,

$$\{y \in T : y \preccurlyeq x\} = \{y \in \mathbb{R} : y \leqslant x\} \cup \left(\bigcup_{\substack{z \in \mathrm{B}(T) \\ z \leqslant x}} r^{-1}(z) \right).$$

Then, since F is left-combed at all $y \in \mathbb{R}$ such that $y \leqslant x$, we get

$$r \circ F(r^{-1}(z)) \subset r \circ F(\{y \in \mathbb{R} : y \leqslant z\}) \subset r \circ F(\{y \in \mathbb{R} : y \leqslant x\})$$

for all $z \in \mathrm{B}(T)$, $z \leqslant x$. Consequently,

$$F_u(x) = \sup\{r(F(y)) : y \preccurlyeq x\} = \sup\{r(F(y)) : y \in \mathbb{R} \text{ and } y \leqslant x\}.$$

□

Remark 4.8. — As in Remark 4.4 it follows that if F is left-combed at all $y \in \mathbb{R}$ such that $y \leqslant x$, then

$$F_u(x) = \max\{r(F(y)) : y \in \mathbb{R} \text{ and } x - 1 \leqslant y \leqslant x\}$$

and if F is right-combed at all $y \in \mathbb{R}$ such that $y \geqslant x$, then

$$F_l(x) = \min\{r(F(y)) : y \in \mathbb{R} \text{ and } x + 1 \geqslant y \geqslant x\}.$$

The next result studies the basic properties of the maps F_l and F_u.

LEMMA 4.9. — *For each $F \in \mathcal{C}_1(T)$ the maps F_l and F_u are non-decreasing liftings of (non necessarily continuous) degree one circle maps that satisfy:*

(a) *$F_l(x) \preccurlyeq F(y) \preccurlyeq F_u(x)$ for each $x \in \mathbb{R}$ and $y \in r^{-1}(x)$.*

(b) *If $G \in \mathcal{C}_1(T)$ verifies $F \preccurlyeq G$, then $F_l \leqslant G_l$ and $F_u \leqslant G_u$.*

(c) *If F is non-decreasing, then $F_u = F_l = F_1^r = r \circ F\big|_{\mathbb{R}}$. Moreover,*

$$\{x \in \mathbb{R} : r(F(x)) \neq F(x)\} \subset \mathrm{Const}(F_u) = \mathrm{Const}(F_l).$$

(d) *The map F_u is continuous from the right whereas F_l is continuous from the left.*

(e) *If F is left-combed (respectively right-combed) at $x \in \mathbb{R}$ then F_u (respectively F_l) is continuous at x. In particular, F_u and F_l are continuous in $\mathbb{R} \setminus \mathrm{B}(T)$.*

(f) *If F_u (respectively F_l) is discontinuous at some $x \in \mathbb{R}$, then $x \in \mathrm{B}(T)$ and there exists $\varepsilon > 0$ such that $[x, x+\varepsilon] \subset \mathrm{Const}(F_u)$ (respectively $[x-\varepsilon, x] \subset \mathrm{Const}(F_l)$).*

Proof. — As in the previous lemma, we will only consider the map F_u. The proof for F_l is analogous.

Let $x, z \in \mathbb{R}$ be such that $x \leqslant z$. We have

$$r \circ F(\{y : y \preccurlyeq x\}) \subset r \circ F(\{y : y \preccurlyeq z\}).$$

So, $F_u(x) \leqslant F_u(z)$. On the other hand, by Lemma 1.7(d),

$$\begin{aligned} F_u(x+1) &= \sup\{r(F(y)) : y \preccurlyeq x+1\} = \sup\{r(F(z+1)) : z \preccurlyeq x\} \\ &= \sup\{r(F(z)) + 1 : z \preccurlyeq x\} = F_u(x) + 1. \end{aligned}$$

Thus, F_u is non-decreasing and has degree one.

To prove (a) observe that $F(y) \preccurlyeq F_u(x)$ is equivalent to $r(F(y)) \leqslant F_u(x)$ which, in turn, is equivalent to $r(F(y)) \in \{r(F(z)) : z \preccurlyeq x\}$. On the other hand, $y \in r^{-1}(x)$ implies that $y \preccurlyeq x$ and this last statement implies $r(F(y)) \in \{r(F(z)) : z \preccurlyeq x\}$. So, (a) holds. Statements (b) and (c) follow immediately from the definitions, Remark 4.2 and Lemma 1.16.

To prove (d) take $x \in \mathbb{R}$ and $\delta > 0$. We have

$$F_u(x+\delta) = \max\left\{F_u(x), \sup\{r(F(y)) : x \preccurlyeq y \preccurlyeq x+\delta\}\right\}.$$

Notice that,

$$\begin{aligned}\lim_{\delta \searrow 0}\bigl(\sup\{r(F(y)) : x \preccurlyeq y \preccurlyeq x+\delta\}\bigr) &= \sup\{r(F(y)) : r(y) = x\} \\ &\leqslant \sup\{r(F(y)) : y \preccurlyeq x\} = F_u(x).\end{aligned}$$

Consequently, $\lim_{\delta \searrow 0} F_u(x+\delta) = F_u(x)$.

To prove (e) and (f) notice that, since $r \circ F$ is continuous and $r^{-1}(x)$ is compact,

$$\begin{aligned}F_u(x) &= \sup\{r(F(y)) : y \preccurlyeq x\} \\ &= \max\left\{\sup\{r(F(y)) : y \prec x\}, \max\Bigl\{r \circ F\bigl(r^{-1}(x)\bigr)\Bigr\}\right\}.\end{aligned}$$

Now observe that since the points from $B(T)$ are isolated, if $\delta > 0$ is small enough then $[x-\delta, x) \cap B(T) \neq \emptyset$, and thus $\sup\{r(F(y)) : y \preccurlyeq x-\delta\}$ varies continuously with δ. Consequently,

$$\lim_{\delta \searrow 0}\bigl(\sup\{r(F(y)) : y \preccurlyeq x-\delta\}\bigr) = \lim_{\delta \searrow 0} F_u(x-\delta)$$

exists and coincides with $\sup\{r(F(y)) : y \prec x\}$. In summary,

$$F_u(x) = \max\left\{\lim_{\delta \searrow 0} F_u(x-\delta), \max\Bigl\{r \circ F\bigl(r^{-1}(x)\bigr)\Bigr\}\right\}$$

and hence, in view of (d), the continuity of F_u at x is equivalent to

$$\max\Bigl\{r \circ F\bigl(r^{-1}(x)\bigr)\Bigr\} \leqslant \lim_{\delta \searrow 0} F_u(x-\delta) = \sup\{r(F(y)) : y \prec x\}. \tag{4.1}$$

Since F is left-combed at x we have,

$$r \circ F(r^{-1}(x)) \subset r \circ F(\{y \in \mathbb{R} : y \leqslant x\}) \subset r \circ F(\{y : y \prec x\} \cup \{x\}),$$

which gives (4.1) by the continuity of $r \circ F$. This ends the proof of (e).

To prove (f) assume that the map F_u is discontinuous at $x \in \mathbb{R}$. Then, from (4.1) it follows that $r^{-1}(x) \neq \{x\}$ and there exists $z \in r^{-1}(x) \setminus \{x\}$ such that

$$r(F(z)) > \sup\{r(F(y)) : y \prec x\} \geqslant r(F(x)).$$

In particular, $x \in \mathrm{B}(T)$ and, by continuity, there exists $\varepsilon > 0$ such that $\mathrm{B}(T) \cap (x, x+\varepsilon] = \emptyset$ and $r(F(y)) < r(F(z))$ for all $y \in (x, x+\varepsilon]$. For all such points y we have

$$F_u(y) = \sup\{r(F(y')) : y' \preccurlyeq y\} = \sup\{r(F(y')) : y' \preccurlyeq x\} = F_u(x).$$

This ends the proof of the lemma $\square$

Remark 4.10. — According to Lemma 4.9(e), if F is left-combed at $x \in \mathbb{R}$ then F_u is continuous at x. The converse is not true. From the proof of statements (e) and (f) of this lemma it easily follows that if F_u is continuous at some $x \in \mathbb{R}$ but F is *not* left-combed at x (and, hence, $x \in \mathrm{B}(T)$), then there exists a point $z \in \mathrm{B}(T)$, $z < x$ such that F_u is also not left-combed at z and

$$\max\Big\{r \circ F\big(r^{-1}(x)\big)\Big\} \leqslant \max\Big\{r \circ F\big(r^{-1}(z)\big)\Big\}.$$

Iterating this process if necessary, one can find a point $z' \in \mathrm{B}(T)$, $z' < x$, such that F_u is not continuous at z'. Therefore, F_u is continuous if and only if F is left-combed at all $x \in \mathbb{R}$.

Similar statements with reverse inequalities hold for right-combed and F_l.

Definition 4.11. — *The fact that the maps F_l and F_u are non-decreasing implies* [17, Theorem 1] *that $\rho_{F_l}(x)$ and $\rho_{F_u}(x)$ exist for each $x \in \mathbb{R}$ and are independent of the choice of the point x. These two numbers will be denoted by $\rho(F_l)$ and $\rho(F_u)$ respectively.*

4.3. Rotation sets and water functions for combed maps

The main goal of this subsection (Theorem 4.16) is to show that, as in the case of circle maps, for combed maps the rotation set is a closed interval of the real line. This is achieved with the help of the so called *water functions* that we extend from the circle maps to the setting of combed maps from $\mathcal{C}_1(T)$.

As a consequence of Definition 4.11 and Lemma 4.9 one obtains:

Corollary 4.12. — *For each $F \in \mathcal{C}_1(T)$ it follows that $\rho(F_l) \leqslant \rho(F_u)$, $\mathrm{Rot}^-(F) \subset [\rho(F_l), \rho(F_u)]$, $\mathrm{Rot}^+(F) \subset [\rho(F_l), \rho(F_u)]$ and, consequently, $\mathrm{Rot}(F) \subset [\rho(F_l), \rho(F_u)]$.*

Proof. — By a rewriting of Lemma 4.9(a) we have $F_l \circ r(y) \preccurlyeq F(y) \preccurlyeq F_u \circ r(y)$ for each $y \in T$. From Remark 4.2 and Lemma 4.9 it follows that

$F_l \circ r$ and $F_u \circ r$ are non-decreasing. Hence, since F_l and F_u are self maps of $\mathbb{R}$, by Lemma 4.3,

$$(F_l)^n \circ r(y) = (F_l \circ r)^n(y) \preccurlyeq F^n(y) \preccurlyeq (F_u \circ r)^n(y) = (F_u)^n \circ r(y)$$

for each $n \in \mathbb{N}$. Consequently,

$$\frac{(F_l)^n(r(y)) - r(y)}{n} \leqslant \frac{r(F^n(y)) - r(y)}{n} \leqslant \frac{(F_u)^n(r(y)) - r(y)}{n}$$

for each $y \in T$ and $n \in \mathbb{N}$. Then the corollary follows from the fact that $\rho(F_l) = \rho_{F_l}(x)$ and $\rho(F_u) = \rho_{F_u}(x)$ for all $x \in \mathbb{R}$. □

In what follows we need to introduce a distance in $\mathcal{C}_1(T)$. We will use the usual one, namely the sup distance, which gives the topology of the uniform convergence. But to do this we need to specify before the distance that we will use in T.

Definition 4.13. — *Assume that the metric space T is endowed with a τ-invariant distance δ_T (that is, for all $x, y \in T$, $\delta_T(x+1, y+1) = \delta_T(x, y)$). In this paper, instead of this distance we will use the distance ν defined as follows in the spirit of the taxicab metric (although such a metric, in general, cannot be defined in lifted spaces). Given $x, y \in T$ we set*

$$\nu(x, y) := \delta_T(x, y)$$

if x and y lie in the same connected component of $T \setminus \mathbb{R}$, and

$$\nu(x, y) := \delta_T(x, r(x)) + |r(x) - r(y)| + \delta_T(r(y), y)$$

otherwise.

Note that ν coincides on $\mathbb{R}$ with the natural distance. Observe also that when T is uniquely arcwise connected (in particular, when T is a lifted tree) then the distance ν gives the length of the shortest path (in T) joining x and y and, thus, it is indeed the taxicab metric.

Now we endow the space $\mathcal{C}_1(T)$ with the sup distance with respect to the distance ν. Given two maps $F, G \in \mathcal{C}_1(T)$, we set

$$d(F, G) := \sup_{x \in T} \nu(F(x), G(x)) = \sup_{x \in r^{-1}([0,1])} \nu(F(x), G(x)).$$

Observe that the space of (not necessarily continuous) maps from $\mathbb{R}$ to itself of degree one is also endowed with the sup distance:

$$d(F, G) := \sup_{x \in \mathbb{R}} |F(x) - G(x)| = \sup_{x \in [0,1]} |F(x) - G(x)|.$$

Lemma 4.14. — *The maps r, $F \mapsto r \circ F$, $F \mapsto F_l$ and $F \mapsto F_u$ are Lipschitz continuous with constant 1.*

Proof. — The fact that r is Lipschitz continuous with constant 1 follows easily from the above definitions. Then, this trivially implies that $F \mapsto r \circ F$ is Lipschitz continuous with constant 1. The other two statements follow in a similar way to [6, Proposition 3.7.7(e)]. □

Now we are ready to extend to this setting the so called "water functions", that play a key role in the study of the rotation intervals of circle maps (see [6]). Before defining these maps we notice that, if $F \in \mathcal{C}_1(\mathbb{R})$, then the definition of F_u is simply given by $F_u(x) = \sup\{F(y) : y \leqslant x\}$. We recall that F_1^r denotes the map $r \circ F\big|_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbb{R}$. Given a map $F \in \mathcal{C}_1(T)$ we define the family $F_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$(4.2)\quad F_\mu = \left(\min\{F_1^r, F_l + \mu\}\right)_u \quad \text{for} \quad 0 \leqslant \mu \leqslant \mu_1 = \sup_{x\in\mathbb{R}}\big\{F_1^r(x) - F_l(x)\big\}.$$

The next lemma studies the basic properties of the family F_μ. Its proof basically follows that of [6, Proposition 3.7.17] by using Lemma 4.9 in addition to [6, Proposition 3.7.7]. However, in sake of completeness and clarity, we will outline the proof.

Proposition 4.15. — *Let $F \in \mathcal{C}_1(T)$ be combed. Then, the maps F_μ are non-decreasing continuous liftings of degree one circle maps that satisfy:*

(a) $F_0 = F_l$ *and* $F_{\mu_1} = F_u$.
(b) *If* $0 \leqslant \lambda \leqslant \mu \leqslant \mu_1$, *then* $F_\lambda \leqslant F_\mu$.
(c) $\operatorname{Const}(F_1^r) \subset \operatorname{Const}(F_\mu)$ *for each* μ.
(d) *Each* F_μ *coincides with* F_1^r *outside* $\operatorname{Const}(F_\mu)$.
(e) *The function* $\mu \mapsto F_\mu$ *is Lipschitz continuous with constant 1.*

Proof. — To simplify the notation we denote by G_μ the map

$$\min\{F_1^r, F_l + \mu\} : \mathbb{R} \longrightarrow \mathbb{R}.$$

Then, $F_\mu = (G_\mu)_u$.

Since F is combed, Lemma 4.9(e) implies that F_l, and hence G_μ, are continuous liftings of degree one circle maps for each μ. Then, in view of [6, Proposition 3.7.7(d)], the maps F_μ are non-decreasing continuous liftings of degree one circle maps.

Lemma 4.9(a) and Remark 4.2 tell us that $F_l \leqslant F_1^r$. So, $G_0 = F_l$ and, since F_l is a self-map of $\mathbb{R}$, $F_0 = (F_l)_u = F_l$ by [6, Lemma 3.7.7(c)]. On the other hand, $G_{\mu_1} = F_1^r$. Consequently, for every $x \in \mathbb{R}$,

$$F_{\mu_1}(x) = (F_1^r)_u(x) = \sup\{r(F(y)) : y \in \mathbb{R} \text{ and } y \leqslant x\} = F_u(x)$$

by Lemma 4.7. This ends the proof of (a). Statement (b) follows from [6, Proposition 3.7.7(b)] and the simple observation that $G_\lambda \leqslant G_\mu$.

Again by Lemma 4.7 we see that

$$F_l(x) = \inf\{r(F(y)) : y \in \mathbb{R} \text{ and } y \geqslant x\} = (F_1^r)_l(x).$$

Thus, $\mathrm{Const}(F_l+\mu) = \mathrm{Const}(F_l) \supset \mathrm{Const}(F_1^r)$ by [6, Lemma 3.7.9(b)] and, hence, $\mathrm{Const}(G_\mu) \supset \mathrm{Const}(F_1^r)$. By [6, Lemma 3.7.9(a)] we see that

$$\mathrm{Const}(F_\mu) \supset \mathrm{Const}(G_\mu) \supset \mathrm{Const}(F_1^r); \tag{4.3}$$

and (c) holds.

To prove (d) suppose that $F_1^r(x) \neq F_\mu(x) = (G_\mu)_u(x)$. If $F_l(x) + \mu \geqslant F_1^r(x)$ then

$$G_\mu(x) = F_1^r(x) \neq (G_\mu)_u(x).$$

So, $x \in \mathrm{Const}((G_\mu)_u) = \mathrm{Const}(F_\mu)$ by [6, Lemma 3.7.8(a)]. Now suppose that $F_l(x) + \mu < F_1^r(x)$. This implies that $(F_1^r)_l(x) = F_l(x) < F_1^r(x)$. Then, [6, Lemma 3.7.8(b)] implies that $x \in \mathrm{Const}(F_l) = \mathrm{Const}(F_l + \mu)$. Hence, there exists a neighbourhood $U \subset \mathrm{Const}(F_l + \mu)$ of x in $\mathbb{R}$ such that $F_1^r(y) > F_l(y) + \mu = G_\mu(y)$ for every $y \in U$. Thus, by (4.3), $x \in \mathrm{Const}(G_\mu) \subset \mathrm{Const}(F_\mu)$.

Finally, one can show that $\mu \mapsto G_\mu$ is Lipschitz continuous with constant 1. So, (e) follows from [6, Proposition 3.7.7(e)]. □

The next theorem is the main result of this section. It shows that for maps which are combed, the rotation set has properties similar to the ones displayed by the rotation interval of continuous degree one circle maps.

Theorem 4.16. — *For each map $F \in \mathcal{C}_1(T)$ which is combed the following statements hold*

(a) $\mathrm{Rot}(F) = \mathrm{Rot}(F_1^r) = \mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}^+(F) = \mathrm{Rot}^-(F)$. *Moreover,* $\mathrm{Rot}(F) = [\rho(F_l), \rho(F_u)]$.

(b) *For every $\alpha \in \mathrm{Rot}(F)$, there exists a twist orbit* (mod 1) *of F contained in $\mathbb{R}$, disjoint from* $\mathrm{Const}(F|_{\mathbb{R}})$ *and having rotation number* α.

(c) *For every $\alpha \in \mathbb{Q} \cap \mathrm{Rot}(F)$, the orbit* (mod 1) *given by (b) can be taken periodic* (mod 1).

(d) *The endpoints of the rotation interval, $\rho(F_l)$ and $\rho(F_u)$ depend continuously on F.*

Proof. — It follows along the lines of the proof of [6, Theorem 3.7.20] but using the previous results for combed maps and the family F_μ with $0 \leqslant \mu \leqslant \mu_1$ defined by (4.2). By Proposition 4.15 every F_μ is a continuous non-decreasing lifting of a degree one circle map. Hence, [6, Lemma 3.7.11] implies that $\rho(F_\mu) = \rho(F_\mu(x))$ exists and is independent on x. Also, from

Proposition 4.15(a,b) it follows easily that $\rho(F_l) \leqslant \rho(F_\mu) \leqslant \rho(F_\lambda) \leqslant \rho(F_u)$ whenever $0 \leqslant \mu \leqslant \lambda \leqslant \mu_1$. Notice also that the function $\mu \mapsto \rho(F_\mu)$ is continuous and Statement (d) holds by Proposition 4.15(e), Lemma 4.14 and [6, Lemma 3.7.12].

From Corollary 4.12 and Theorem 1.15 we obtain that the rotation sets $\mathrm{Rot}^+(F)$, $\mathrm{Rot}^-(F)$ and $\mathrm{Rot}(F_1^r) \subset \mathrm{Rot}_{\mathbb{R}} \subset \mathrm{Rot}(F)$ are contained in $[\rho(F_l), \rho(F_u)]$.

From above we see that for all $\alpha \in [\rho(F_l), \rho(F_u)]$ there exists an $a \in [0, \mu_1]$ such that $\rho(F_a) = \alpha$. Since F_a is the lifting of a continuous degree one circle map, by [6, Lemmas 3.7.15 and 3.7.16], F_a has an orbit (mod 1) $P \subset \mathbb{R}$, disjoint from $\mathrm{Const}(F_a)$ and whose F_a-rotation number is α. Moreover, if $\alpha \in \mathbb{Q}$, then P can be taken periodic (mod 1). Since F_a is non-decreasing, P is twist.

Proposition 4.15(c,d) tell us that P is disjoint from

$$\mathrm{Const}(F_a) \supset \mathrm{Const}(F_1^r) \supset \mathrm{Const}(F\big|_{\mathbb{R}})$$

and $F_a\big|_P = r \circ F\big|_P$. Then, since $P \subset \mathbb{R}$, $F_a\big|_P = F\big|_P$. Consequently, P is a twist (mod 1) orbit of F with F-rotation number α and, if $\alpha \in \mathbb{Q}$, then P is periodic (mod 1). This ends the proof of the theorem. □

4.4. The set of periods for combed maps

This subsection is devoted to characterising the set of periods (mod 1) for combed maps. Its main result (Theorem 4.17) is the analogue of [6, Theorem 3.9.6] for circle maps. To state it we need to introduce some notation.

Given two real numbers $a \leqslant b$ we denote by $M(a,b)$ the set $\{n \in \mathbb{N} : a < k/n < b \text{ for some integer } k\}$. Clearly $M(a,b) = \emptyset$ whenever $a = b$ and, if $a \neq b$, $M(a,b) \supset \{n \in \mathbb{N} : n > \frac{1}{b-a}\}$.

Theorem 4.17. — *If $F \in \mathcal{C}_1(T)$ is combed and* $\mathrm{Rot}(F) = [a,b]$, *then the following statements hold:*

(a) *If p, q are coprime and $p/q \in (a,b)$, then* $\mathrm{Per}(p/q, F) = q\mathbb{N}$.

(b) $\mathrm{Per}(F) = \mathrm{Per}(a,F) \cup M(a,b) \cup \mathrm{Per}(b,F)$.

Proof. — If $a = b$ there is nothing to prove. So, in the rest of the proof we assume that $a \neq b$.

Assume that p, q are coprime and $a < p/q < b$, and let $n \in \mathbb{N}$. We have to show that $qn \in \mathrm{Per}(p/q, F)$. By Theorem 4.16(a) we see that $p/q \in \mathrm{Rot}(F_1^r)$ and observe that F_1^r is a degree one circle map. To simplify the notation, let us denote by G the map $(F_1^r)^q - p$. By [6, Lemma 3.7.1], $\operatorname{Rot}(G) = [qa-p, qb-p]$ which contains 0 in its interior. Then, from the proof of [6, Lemma 3.9.1], there exist points $t', z, t, z' \in \mathbb{R}$ such that $t' < z < t < z'$, $G(t') < t'$, $G(z) \geqslant (z')$, $G(t) \leqslant t'$ and $G(z') > z'$.

Let us denote the interval $[t', z]$ by I and the interval $[t, z']$ by J. Then

$$I \xrightarrow[G]{+} I, J \qquad \text{and} \qquad J \xrightarrow[G]{+} I, J.$$

For $n = 1$ take the loop $I \xrightarrow[G]{+} I$ of length 1 and for $n \geqslant 2$ let us consider the following loop of length n:

$$\left(I \xrightarrow[G]{+} J\right)\left(J \xrightarrow[G]{+} J\right)^{n-2}\left(J \xrightarrow[G]{+} I\right).$$

Then, in view of Proposition 2.3, for each $n \in \mathbb{N}$, there exists $x \in I$ such that $F^{nq}(x) = x + np$ and $F^{qi}(x) \in J + ip$ for all $i = 1, 2, \ldots, n-1$. By setting $\widetilde{G} := F^q - p$ this can be rewritten as $\widetilde{G}^q(x) = x$ and $\widetilde{G}^i(x) \in J$ for all $i = 1, 2, \ldots, n-1$. Consequently, x is a periodic point of $\widetilde{G}$ of period n because $I \cap J \neq \emptyset$ or, in other words, x is a periodic (mod 1) point of F^q of period n such that $\rho_{F^q}(x) = p$. Then, from the proof of [6, Lemma 3.9.3] it follows that x is a periodic (mod 1) point of F of period qn such that $\rho_F(x) = p/q$. Since $\operatorname{Per}(p/q, F) \subset q\mathbb{N}$ by Proposition 3.6, this ends the proof of (a).

According to Proposition 3.6,

$$\operatorname{Per}(F) = \operatorname{Per}(a, F) \cup \operatorname{Per}(b, F) \cup \bigcup_{\alpha \in (a,b) \cap \mathbb{Q}} \operatorname{Per}(\alpha, F).$$

On the other hand, $M(a, b)$ can be written as the union of $q\mathbb{N}$ for all pairs p, q such that $a < p/q < b$ and $(p, q) = 1$. Consequently, $M(a, b) = \bigcup_{\alpha \in (a,b) \cap \mathbb{Q}} \operatorname{Per}(\alpha, F)$ by (a), which proves (b). □

Remark 4.18. — In this situation, contrary to the case of circle maps, the characterisation of the sets $\operatorname{Per}(a, F)$ and $\operatorname{Per}(b, F)$ (where a and b are the endpoints of $\operatorname{Rot}(F)$) is not possible without completely knowing the lifted space T.

5. Additional results for infinite graphs

This section is devoted to improving the study of the rotation set and the set of periods (mod 1) for the subclass of $\mathcal{C}_1(T)$ consisting of continuous maps on infinite graph maps defined as follows.

We recall that a *(topological) finite graph* is a compact connected set G containing a finite subset V such that each connected component of $G \setminus V$ is homeomorphic to an open interval. A finite tree is a finite graph with no loops, i.e. with no subset homeomorphic to a circle.

When we unwind a finite graph G with respect to a loop, we obtain an infinite graph T that may or may not be in $\mathbf{T}$ (see Figure 1.1 for an infinite graph not in $\mathbf{T}$ and Figure 1.3 for an infinite tree that belongs to $\mathbf{T}$). Notice that if G has exactly one loop, then T is an infinite tree and $T \in \mathbf{T}$.

Definition 5.1. — *Let $\mathbf{T}^\circ$ denote the subfamily of spaces $T \in \mathbf{T}$ such that*

$$r^{-1}([0,1]) = \{x \in T : 0 \leqslant r(x) \leqslant 1\}$$

is a finite graph. The elements of $\mathbf{T}^\circ$ will be informally called infinite graphs.

A point $x \in T$ is called a vertex *if there exists a neighbourhood U of x such that $U \setminus \{x\}$ has at least 3 connected components. Note that all branching points of T are vertices. Also, a point $x \in T$ is called an* endpoint *if $T \setminus \{x\}$ has a unique connected component.*

5.1. $\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F)$ for transitive $(\mathrm{mod}\ 1)$ infinite graph maps

A map $F \in \mathcal{C}_1(T)$ is said *transitive* $(\mathrm{mod}\ 1)$ if it is the lifting of a transitive map, that is, for every non empty open sets U, V in T, there exists $n \geqslant 0$ such that $(F^n(U) + \mathbb{Z}) \cap V \neq \emptyset$. In other words, for every non empty open set $U \subset T$, $\left(\bigcup_{n\geqslant 0} F^n(U)\right) + \mathbb{Z}$ is dense in T. In particular, $\bigcup_{n\geqslant 0} F^n(\mathbb{R})$ is dense in T if F is transitive $(\mathrm{mod}\ 1)$.

Theorem 5.5 gives a sufficient condition, which includes the case when F is transitive $(\mathrm{mod}\ 1)$, to have $\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F)$ when $T \in \mathbf{T}^\circ$. In this situation, the study of $\mathrm{Rot}_{\mathbb{R}}(F)$ done in the rest of the paper gives indeed information on the whole rotation set. We start with some preliminary results.

In what follows we will set

$$T_{\mathbb{R}} := \bigcup_{n\geqslant 0} F^n(\mathbb{R}).$$

Lemma 5.2. — *Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1(T)$. Then*

(a) *For all $n \geqslant 0$, $F^n(\mathbb{R})$ is a closed set.*

(b) *For all $n \geqslant 0$, $F^{n+1}(\mathbb{R}) \supset F^n(\mathbb{R})$ and $T_{\mathbb{R}}$ is connected. Consequently,* $\mathrm{Clos}\left(T_{\mathbb{R}}\right) \in \mathbf{T}^\circ$.

(c) *We have $F(T_{\mathbb{R}}) = T_{\mathbb{R}}$ and consequently, $F(\mathrm{Clos}\left(T_{\mathbb{R}}\right)) = \mathrm{Clos}\left(T_{\mathbb{R}}\right)$.*

Proof. — If $G \in \mathcal{C}_1(T)$, then $G([0,1])$ contains the non empty interval $[r(G(0)), r(G(0))+1] \subset \mathbb{R}$. Thus there is $x_0 \in [0,1]$ such that $G(x_0+\mathbb{Z}) \subset \mathbb{R}$. For $k \in \mathbb{Z}$ set $R_k = G([x_0+k-1, x_0+k])$. By the continuity of G and Definition 1.1(ii), $R_k \cap \mathbb{R} \supset [G(x_0+k-1), G(x_0+k)]$. Moreover, R_k is compact and

$$G(\mathbb{R}) = \bigcup_{k\in\mathbb{Z}} R_k \supset \bigcup_{k\in\mathbb{Z}} [G(x_0+k-1), G(x_0+k)].$$

Since $R_{k+1} \cap R_k \supset \{G(x_0+k)\}$, the set $G(\mathbb{R})$ contains $\mathbb{R}$.

To prove that $G(\mathbb{R})$ is closed, we proceed as follows. Let $\{x_n\}_{n\in\mathbb{N}} \subset G(\mathbb{R})$ be a sequence converging to a point $x \in T$. We will prove that $x \in G(\mathbb{R})$. The fact that it is convergent implies that it is bounded. The sets R_k are also bounded and $R_{k+1} = R_k + 1$ because G has degree one. This implies that $\{x_n\}_{n\in\mathbb{N}} \subset \bigcup_{k\in E} R_k$ where $E \subset \mathbb{Z}$ is a finite set. Since $\bigcup_{k\in E} R_k$ is compact, we see that $x \in \bigcup_{k\in E} R_k \subset G(\mathbb{R})$.

Now, Statement (a) follows from above by taking $G = F^n$. Also, by taking $G = F$ above we obtain $F(\mathbb{R}) \supset \mathbb{R}$. Therefore, $F^{n+1}(\mathbb{R}) \supset F^n(\mathbb{R})$ for all $n \geqslant 0$. Since $F^n(\mathbb{R})$ is connected by continuity this implies that $T_{\mathbb{R}}$ is connected. Hence $\operatorname{Clos}\left(T_{\mathbb{R}}\right) \in \mathbf{T}^\circ$. This proves (b).

To end the proof of the lemma we only have to show that $F(T_{\mathbb{R}}) = T_{\mathbb{R}}$. The inclusion $F(T_{\mathbb{R}}) \subset T_{\mathbb{R}}$ is obvious. Now we prove the other inclusion. That is, for each $x \in T_{\mathbb{R}}$ there exists $y \in T_{\mathbb{R}}$ such that $F(y) = x$. Since $x \in T_{\mathbb{R}}$ there exists $l \geqslant 0$ such that $x \in F^l(\mathbb{R})$ but $x \notin F^j(\mathbb{R})$ for $j = 0, 1, \ldots, l-1$. If $l > 0$ then, clearly, we can take $y \in F^{l-1}(\mathbb{R})$ and we are done. Otherwise, $x \in \mathbb{R} = \bigcup_{m\in\mathbb{Z}} [F(x_0-m), F(x_0+m)]$ for some $x_0 \in \mathbb{R}$ such that $F(x_0) \in \mathbb{R}$. Hence, there exists $m \in \mathbb{Z}$ such that $x \in [F(x_0-m), F(x_0+m)]$. So, $F(y) = x$ for some $y \in [x_0-m, x_0+m]$. $\square$

LEMMA 5.3. — *Let $T \in \mathbf{T}^\circ$ and $F \in \mathcal{C}_1(T)$ and assume that* $\operatorname{Clos}\left(T_{\mathbb{R}}\right) = T$. *Then there exists a finite set A such that $T \setminus T_{\mathbb{R}} = A + \mathbb{Z}$, the sets $\{A+n\}_{n\in\mathbb{Z}}$ are pairwise disjoint and every point of $T \setminus T_{\mathbb{R}}$ is periodic* (mod 1).

Proof. — By Remark 1.2 we may assume that 0 is not a branching point. Let $X = r^{-1}([0,1]) = \{x \in T : r(x) \in [0,1]\}$. By definition, X is a finite graph. Set $A := X \setminus T_{\mathbb{R}}$. Since 0 is not a branching point,

$$X = r^{-1}((0,1)) \cup \{0,1\},$$

and thus the sets $\{A+n\}_{n\in\mathbb{Z}}$ are pairwise disjoint. Clearly, $T \setminus T_{\mathbb{R}} = A + \mathbb{Z}$. By Lemma 5.2(b), the set $T_{\mathbb{R}} \supset \mathbb{R}$ is connected, and by assumption it is dense in T. Thus A is a finite subset of X.

By Lemma 5.2(c), we have $F(T_{\mathbb{R}}) = T_{\mathbb{R}}$. This implies $F(T \setminus T_{\mathbb{R}}) = T \setminus T_{\mathbb{R}}$ and, since $T \setminus T_{\mathbb{R}} = A + \mathbb{Z}$ with A finite it follows that for each $a \in A$ there exist integers $n \geqslant 1$ and $k \in \mathbb{Z}$ such that $F^n(a) = a + k$. This means that all points in A are periodic (mod 1). □

Remark 5.4. — While Lemma 5.2 holds for any lifted space except for the statement that $\operatorname{Clos}\left(T_{\mathbb{R}}\right) \in \mathbf{T}^{\circ}$, Lemma 5.3 is only true for infinite graphs from $\mathbf{T}^{\circ}$. The fact that $T \setminus T_{\mathbb{R}} = A + \mathbb{Z}$ *being A finite* is not true in general, when we remove the assumption that $T \in \mathbf{T}^{\circ}$. If T is an infinite tree, then A is a subset of the endpoints of T, but this may not be the case for any infinite graph.

Now we are ready to state the main result of this subsection.

Theorem 5.5. — *Let $T \in \mathbf{T}^{\circ}$ and $F \in \mathcal{C}_1(T)$. If $\operatorname{Clos}\left(T_{\mathbb{R}}\right) = T$ then $\operatorname{Rot}_{\mathbb{R}}(F) = \operatorname{Rot}(F) = \operatorname{Rot}^+(F) = \operatorname{Rot}^-(F)$.*

Proof. — By Lemma 5.3, we can write $T \setminus T_{\mathbb{R}}$ as $A + \mathbb{Z}$ where A is finite, the sets $\{A + n\}_{n \in \mathbb{Z}}$ are pairwise disjoint, and there exist $k \in \mathbb{N}$ which is common to all elements of A and integers $\{i_a\}_{a \in A}$ such that for all $a \in A$, $F^k(a) = a + i_a$. Hence, the rotation number of every $a \in A$ exists and we have $\rho_F(a) = i_a/k$.

Clearly,

$$\operatorname{Rot}(F) = \{\rho_F(x) : x \in T_{\mathbb{R}}\} \cup \{\rho_F(x) : x \in A\},$$

and the same holds for the upper and lower rotation numbers. If $y \notin A$, then there exist $x \in \mathbb{R}$ and $n \geqslant 0$ such that $F^n(x) = y$. Thus $\underline{\rho}_F(x) = \underline{\rho}_F(y)$ and $\overline{\rho}_F(x) = \overline{\rho}_F(y)$. This implies that $\operatorname{Rot}_{\mathbb{R}}(F) = \{\rho_F(x) : x \in T_{\mathbb{R}}\}$, and the same holds for the upper and lower rotation numbers. According to Theorem 3.1, $\operatorname{Rot}_{\mathbb{R}}(F) = \operatorname{Rot}^+_{\mathbb{R}}(F) = \operatorname{Rot}^-_{\mathbb{R}}(F)$. Hence

$$\operatorname{Rot}(F) = \operatorname{Rot}^+(F) = \operatorname{Rot}^-(F) = \operatorname{Rot}_{\mathbb{R}}(F) \cup \{\rho_F(x) : x \in A\}.$$

Therefore it remains to prove that for every $a \in A$ there exists $x \in \mathbb{R}$ such that $\rho_F(x) = \rho_F(a)$. We are going to find a point $x \in \mathbb{R}$ whose orbit is attracted by a.

In the rest of the proof the map F^k will be denoted by G so that $G(a) = a + i_a$ for all $a \in A$ (in particular $\rho_G(a) = i_a$). For each $a \in A$ choose neighbourhoods $V_a \subset W_a$ of a such that $(W_a + \mathbb{Z}) \cap (W_{a'} + \mathbb{Z}) = \emptyset$ whenever $a \neq a'$ (this is possible because the sets $\{A + n\}_{n \in \mathbb{Z}}$ are pairwise disjoint) and $G(V_a) \subset W_a + i_a$ for all $a \in A$.

Since $T_{\mathbb{R}}$ is an increasing union by Lemma 5.2(b), we also have

$$\mathrm{Clos}\left(\bigcup_{n\geqslant 0} G^n(\mathbb{R})\right) = T.$$

Thus, there exists a positive integer N such that $T\setminus G^n(\mathbb{R}) \subset \bigcup_{a\in A}(V_a+\mathbb{Z})$ for all $n \geqslant N$. Let $V_a^n = V_a \setminus G^n(\mathbb{R})$. Again by Lemma 5.2(b), $\{V_a^n\}_{n\geqslant N}$ is a decreasing sequence of sets containing a and $\bigcap_{n\geqslant N} V_a^n = \{a\}$ because a is in $\mathrm{Clos}\left(\bigcup_{n\geqslant 0} G^n(\mathbb{R})\right)$ but not in $\bigcup_{n\geqslant 0} G^n(\mathbb{R})$. For all $n \geqslant N$, we have

$$G^n(\mathbb{R}) = G^{n-1}(\mathbb{R}) \amalg \left(\coprod_{a\in A} \left(V_a^{n-1} \setminus V_a^n\right) + \mathbb{Z}\right), \tag{5.1}$$

where $\amalg$ denotes disjoint union. If we apply G once more to Equation (5.1) we get

$$G^{n+1}(\mathbb{R}) = G^n(\mathbb{R}) \cup G\left(\coprod_{a\in A} \left(V_a^{n-1} \setminus V_a^n\right) + \mathbb{Z}\right). \tag{5.2}$$

From Equation (5.1) for $n+1$ and Equation (5.2), we deduce:

$$G\left(\bigcup_{a\in A} \left(V_a^{n-1} \setminus V_a^n\right) + \mathbb{Z}\right) \supset \coprod_{a\in A} \left(V_a^{n} \setminus V_a^{n+1}\right) + \mathbb{Z}.$$

Since $G(V_a) \subset W_a + i_a$, the images by G of the sets $\{V_a^{n-1} \setminus V_a^n\}_{a\in A}$ are pairwise disjoint and $G(V_a^{n-1} \setminus V_a^n)$ is the only one that intersects $W_a + \mathbb{Z}$. Moreover $V_a^n \setminus V_a^{n+1} \subset W_a$. Hence $G(V_a^{n-1} \setminus V_a^n) \supset \left(V_a^n \setminus V_a^{n+1}\right) + i_a$ for every $a \in A$. By compactness we have

$$G\left(\mathrm{Clos}\left(V_a^{n-1} \setminus V_a^n\right)\right) \supset \mathrm{Clos}\left(V_a^n \setminus V_a^{n+1}\right) + i_a \tag{5.3}$$

for all $n \geqslant N$ and $a \in A$.

Let $a \in A$. If $V_a^{k-1} \setminus V_a^k = \emptyset$ for some $k \geqslant N$, then $V_a^{n-1} \setminus V_a^n = \emptyset$ for all $n \geqslant k$, by Equation (5.3). Thus $V_a^n = V_a^k$ for all $n \geqslant k$ and $V_a^k = \bigcap_{n\geqslant k} V_a^n = \{a\}$. This contradicts the fact that $G^k(\mathbb{R})$ is closed by Lemma 5.2(a). Consequently, $V_a^{n-1} \setminus V_a^n \neq \emptyset$ for all $n \geqslant N$ and, by Equation (5.3), there exists $x \in \mathbb{R}$ such that

$$(G - i_a)^n(x) \in \mathrm{Clos}\left(V_a^{n-1} \setminus V_a^n\right) \quad \text{for all} \quad n \geqslant N+1.$$

This implies that $\rho_G(x) = i_a = \rho_G(a)$, that is, $\rho_F(x) = \rho_F(a)$ in view of Lemma 1.10(c). This shows that $\{\rho_F(a) : a \in A\} \subset \mathrm{Rot}_{\mathbb{R}}(F)$ which concludes the proof. $\square$

Remark 5.6. — If $\bigcup_{n\geqslant 1} F^{-n}(\mathbb{R}) \cup \operatorname{Clos}\left(T_{\mathbb{R}}\right) = T$ then the conclusion of Theorem 5.5 remains valid. However, the theorem does not hold with the assumption that $\operatorname{Clos}\left(\bigcup_{n\in\mathbb{Z}} F^{n}(\mathbb{R})\right) = T$ (see Example 1.12).

We deduce from Theorem 5.5 that for infinite graphs, $\operatorname{Rot}_{\mathbb{R}}(F)$ is the rotation set of an F-invariant infinite graph contained in T.

Corollary 5.7. — *Let $T \in \mathbf{T}^{\circ}$ and $F \in \mathcal{C}_1(T)$. Then* $\operatorname{Rot}_{\mathbb{R}}(F) = \operatorname{Rot}(F|_{\operatorname{Clos}\left(T_{\mathbb{R}}\right)})$.

Proof. — Set $\overline{T} := \operatorname{Clos}\left(T_{\mathbb{R}}\right)$. By Lemma 5.2(c) $\overline{T} \in \mathbf{T}^{\circ}$ and the map $F|_{\overline{T}} \colon \overline{T} \longrightarrow \overline{T}$ belongs to $\mathcal{C}_1(\overline{T})$. Then, Theorem 5.5 implies that the sets $\operatorname{Rot}(F|_{\overline{T}})$ and $\operatorname{Rot}_{\mathbb{R}}(F|_{\overline{T}})$ coincide. Also, $\operatorname{Rot}_{\mathbb{R}}(F|_{\overline{T}}) = \operatorname{Rot}_{\mathbb{R}}(F)$ and the corollary follows. □

5.2. Periodic (mod 1) points associated to the endpoints of $\operatorname{Rot}_{\mathbb{R}}(F)$ for infinite graph maps

In Subsection 3.2, we dealt with the rational rotation numbers in the interior of $\operatorname{Rot}_{\mathbb{R}}(F)$. In this subsection we are going to show that for an infinite graph map there exist periodic (mod 1) points whose rotation numbers are equal to $\min \operatorname{Rot}_{\mathbb{R}}(F)$ (resp. $\max \operatorname{Rot}_{\mathbb{R}}(F)$) provided that it is a rational number. This will be proved in the main result of this subsection (Theorem 5.18). However, before stating and proving this result in detail, we will introduce the necessary machinery. It will consist in the notion of a *direct path to* $+\infty$. One of the crucial points of the notation that we will introduce is the construction of a direct version of a given (non direct) path going to $+\infty$. Then we will devote to three technical lemmas to study the properties of this kind of paths and to prepare the proof of the basic technical result of this subsection (Lemma 5.17) that gives sufficient conditions to assure that all points in the rotation set are positive. This is the key tool in proving Theorem 5.18.

In the rest of this subsection, we fix an infinite graph $T \in \mathbf{T}^{\circ}$ and we let

$$X = \operatorname{Clos}\left(\{x \in T : 0 \leqslant r(x) < 1\} \setminus \mathbb{R}\right).$$

Since X is a finite union of finite graphs, we can write $X = \bigcup_{\lambda\in\Lambda} I_{\lambda}$ where Λ is a finite set of indices, I_{λ} is a set homeomorphic to a closed non degenerate interval of the real line, I_{λ} contains no vertex except maybe its endpoints and the intersection of two different sets I_{λ}, $I_{\lambda'}$ contains at most one point. Each interval of the form $I_{\lambda} + n$ with $\lambda \in \Lambda$ and $n \in \mathbb{Z}$ will be called a *basic interval*.

We first formalise the idea that in T there are only finitely many "direct ways" to go from a basic interval towards $+\infty$. A *direct path* is a path without loops or returns backwards. For technical reasons, a direct path is allowed to remain constant on an interval.

DEFINITION 5.8. — *A* path from $x_0 \in T$ to $+\infty$ *is a continuous map* $\gamma\colon [0,+\infty) \longrightarrow T$ *such that* $\gamma(0) = x_0$ *and* $\lim_{t\to+\infty} r \circ \gamma(t) = +\infty$. *Such a path is called* direct *if, in addition, it verifies the following condition*

(DP) *if* $\gamma(t) = \gamma(t')$ *for some* $t \in [0, t']$, *then* $\gamma\big|_{[t,t']}$ *is constant.*

Remark 5.9. — If $\gamma\colon [0,+\infty) \longrightarrow T$ is a direct path to $+\infty$, then there exists t such that $\gamma(t') \in \mathbb{R}$ for all $t' \geqslant t$. This is due to the fact that if a path leaves $\mathbb{R}$ at some point z then $z \in \mathrm{B}(T)$ and, by Definition 1.1(ii), the path must return to $\mathbb{R}$ through the same point z. Then, clearly, such a path does not verify Condition (DP) and, hence, it is not direct.

Note also that $\gamma([0,+\infty))$ is homeomorphic to $[0,+\infty)$.

In view of the previous remark, when γ is a direct path we can define an ordering $<_\gamma$ on the path $\gamma([0,+\infty))$ such that it coincides with the order of $\mathbb{R}$ on the half-line $\gamma([0,+\infty)) \cap \mathbb{R}$ as follows. If $x, x' \in \gamma([0,+\infty))$, $x \neq x'$, then we write $x <_\gamma x'$ if and only if $x = \gamma(t)$ and $x' = \gamma(t')$ with $t \in [0, t')$. The symbols $\leqslant_\gamma$, $>_\gamma$, and $\geqslant_\gamma$ are then defined in the obvious way.

Remark 5.10. — If γ is a direct path then $\gamma\colon [0,+\infty) \longrightarrow \gamma([0,+\infty))$ is a non decreasing map with respect to the ordering $\leqslant_\gamma$ in the image $\gamma([0,+\infty))$.

Let $x_0, x'_0 \in T$ and let γ and γ' be two direct paths from x_0, x'_0 to $+\infty$. We say that γ, γ' are *comparable* if, either $\gamma'([0,+\infty)) \subset \gamma([0,+\infty))$, or $\gamma([0,+\infty)) \subset \gamma'([0,+\infty))$. In the first situation, $x_0 \leqslant_\gamma x'_0$, that is, x'_0 is "on the way" between x_0 and $+\infty$. The second situation is symmetric.

We will be interested in comparing direct paths starting in the same basic interval.

LEMMA 5.11. — *The relation of comparability is an equivalence relation among all direct paths to* $+\infty$ *starting in the same basic interval. Moreover, the set of equivalence classes of such paths for the comparability relation is finite.*

Proof. — Let $I_\lambda + n$ be a basic interval and assume that γ is a direct path from some $x_0 \in I_\lambda + n$ to $+\infty$.

Set $\lambda_0(\gamma) = \lambda$, $t_0 = 0$ and $t_1 = \max\{t \geqslant t_0 : \gamma([t_0, t]) \subset I_{\lambda_0(\gamma)} + n\}$. Clearly, $\gamma(t_1)$ is an endpoint of $I_\lambda + n$. Now we define inductively two finite sequences $\{\lambda_i(\gamma)\}_{0\leqslant i\leqslant k}$ and $\{t_i\}_{0\leqslant i\leqslant k+1}$ in the following way. If $\gamma(t_i) = r(x_0)$ then $\gamma([t_i,+\infty)) \subset \mathbb{R}$ and we stop the construction. Otherwise, there exists $\lambda_i(\gamma) \neq \lambda_{i-1}(\gamma)$ and $\varepsilon > 0$ such that $\gamma(t_i) \in (I_{\lambda_{i-1}(\gamma)} \cap I_{\lambda_i(\gamma)}) + n$ and $\gamma([t_i, t_i+\varepsilon]) \subset I_{\lambda_i(\gamma)} + n$. Then we can define

$$t_{i+1} = \max\{t \geqslant t_i : \gamma([t_i,t]) \subset I_{\lambda_i(\gamma)} + n\}.$$

Since γ is a direct path, each $\lambda \in \Lambda$ appears at most once in the sequence of $\lambda_i(\gamma)$'s. Therefore the construction ends and the number of possible sequences $\{\lambda_i(\gamma)\}_i$ is finite.

The set $\gamma([0,t_1])$ is a subinterval of $I_\lambda + n$ with endpoints x_0 and

$$(I_\lambda \cap I_{\lambda_1(\gamma)}) + n.$$

Moreover, for every

$$i = 1,\dots,k, \gamma([t_i,t_{i+1}]) = I_{\lambda_i(\gamma)} + n,$$

and

$$\gamma([t_{k+1},+\infty)) = [r(x_0),+\infty).$$

If γ' is another direct path from some point in $I_\lambda + n$ to $+\infty$, then γ' is comparable with γ if and only if

$$\{\lambda_i(\gamma)\}_{1\leqslant i\leqslant k} = \{\lambda_i(\gamma')\}_{1\leqslant i\leqslant k'}.$$

Therefore, comparability is an equivalence relation among the direct paths starting in $I_\lambda + n$, and the number of equivalence classes of direct paths to $+\infty$ starting at $I_\lambda + n$ by the comparability relation is finite. $\square$

In the next definition, we associate to a path γ to $+\infty$ a direct path $\widetilde{\gamma}$ to $+\infty$ by cutting all loops and returns backwards of γ. In some sense, $\widetilde{\gamma}$ "globally follows" the path γ but goes directly towards $+\infty$. Figure 5.1 illustrates this definition.

Definition 5.12. — *Let $x_0 \in X + n$, $n \in \mathbb{Z}$ and let γ be a path from x_0 to $+\infty$. We define a path $\widetilde{\gamma}$ as follows.*

First we set

$$t^* = \min\{t \in [0,+\infty) : r\circ\gamma(t) = r(x_0)\}$$

and we define

$$\widetilde{\gamma}(t) = \max r\circ\gamma([t^*,t]) \text{ for all } t \in [t^*,+\infty).$$

Observe that $\widetilde{\gamma}\big|_{[t^*,+\infty)}$ is a non decreasing map from the interval $[t^*,+\infty)$ onto $[r(x_0),+\infty) \subset \mathbb{R}$. If $x_0 \in \mathbb{R}$ then $t^* = 0$ and $\widetilde{\gamma}$ is already defined. Otherwise, $t^* > 0$ and $\gamma([0,t^*])$ is a path contained in the connected component of $X + n$ containing x_0. Now we inductively define $\widetilde{\gamma}\big|_{[0,t^*]}$.

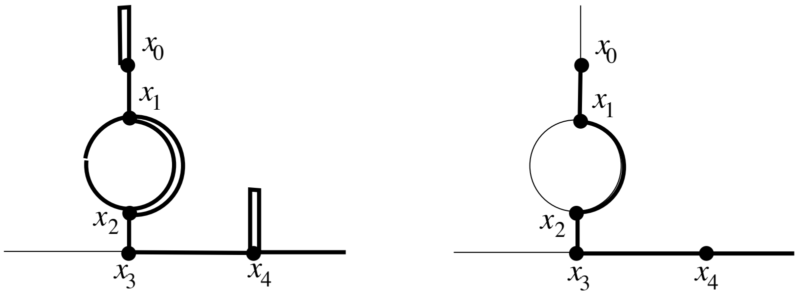

Figure 5.1. A path γ on the left, and the associated direct path $\widetilde{\gamma}$ on the right. With the notation of Definition 5.12, one has $x_0 = \gamma(t_0) = \gamma(t'_0) = \widetilde{\gamma}\big|_{[t_0,t'_0]}$, $x_1 = \gamma(t_1) = \gamma(t'_1) = \widetilde{\gamma}\big|_{[t_1,t'_1]}$, $x_2 = \gamma(t_2) = \widetilde{\gamma}(t_2)$, $x_3 = r(x_0) = \gamma(t^*) = \widetilde{\gamma}(t^*)$, with $0 = t_0 < t'_0 < t_1 < t'_1 < t_2 = t'_2 < t_3 = t^*$.

Step 0. Set $t_0 = 0$ and let $\lambda_0 \in \Lambda$ be such that $x_0 \in I_{\lambda_0} + n$. We define

$$t'_0 := \max\{t \in [t_0, t^*] : \gamma(t) = \gamma(t_0)\}$$

and

$$\widetilde{\gamma}(t) := \gamma(t_0) = \gamma(t'_0) \text{ for all } t \in [t_0, t'_0].$$

Observe that $t'_0 < t^*$ (otherwise $x_0 = \gamma(t'_0) = r(x_0) \in \mathbb{R}$). Let

$$t_1 := \max\{t \in [t'_0, t^*] : \gamma([t'_0, t]) \subset I_{\lambda_0} + n\}.$$

In this situation $\gamma(t_1)$ is an endpoint of $I_{\lambda_0} + n$. Since $\gamma([t'_0, t_1]) \subset I_{\lambda_0} + n$ we can define a linear ordering $\trianglelefteq_{\lambda_0}$ in $I_{\lambda_0} + n$ such that $\gamma(t'_0) \trianglelefteq_{\lambda_0} \gamma(t_1)$. Now, for all $t \in [t'_0, t_1]$, we define

$$\widetilde{\gamma}(t) := \max \gamma([t'_0, t]),$$

where the maximum is taken with respect to the ordering $\trianglelefteq_{\lambda_0}$. The map $\widetilde{\gamma}\big|_{[t'_0,t_1]} : I_{\lambda_0} + n \longrightarrow I_{\lambda_0} + n$ is non decreasing for $\trianglelefteq_{\lambda_0}$. Moreover, by the choice of t_1, $\widetilde{\gamma}(t_1) = \gamma(t_1)$.

If $t_1 = t^*$ then $\widetilde{\gamma}$ is already defined for all $t \geqslant 0$. Otherwise we proceed to the step $k = 1$.

Step k. Suppose that we have already defined $t_0 \leqslant t'_0 \leqslant t_1 \leqslant t'_1 < t_2 \leqslant t'_2 \leqslant \cdots t'_{k-1} < t_k < t^*$ and $\lambda_0, \lambda_1, \ldots, \lambda_{k-1} \in \Lambda$ verifying the following properties:

(i) $t'_i = \max\{t \in [t_i, t^*] : \gamma(t) = \gamma(t_i)\}$ for all $0 \leqslant i \leqslant k-1$,

(ii) $\gamma(t_{i+1})$ is an endpoint of $I_{\lambda_i} + n$ for all $0 \leqslant i \leqslant k-1$,

(iii) $\gamma([t'_i, t_{i+1}]) = I_{\lambda_i} + n$ for all $1 \leqslant i \leqslant k-1$,

(iv) $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$ are all different.

First, let t'_k be the real number given by (i) for k. We define

$$\widetilde{\gamma}(t) := \gamma(t_k) = \gamma(t'_k) \text{ for all } t \in [t_k, t'_k].$$

The definition of t'_k implies that there exists a unique $\lambda_k \in \Lambda$ such that $\gamma([t'_k, t'_k + \varepsilon]) \subset I_{\lambda_k} + n$ for $\varepsilon > 0$ small enough. Since $\gamma(t_k)$ is an endpoint of $I_{\lambda_{k-1}} + n$ by (ii), we get in addition that $\lambda_k \neq \lambda_{k-1}$ and $\gamma(t'_k) = \gamma(t_k)$ is an endpoint of $I_{\lambda_k} + n$. Let

$$t_{k+1} := \max\{t \in [t'_k, t^*] : \gamma([t'_k, t]) \subset I_{\lambda_k} + n\}.$$

The choice of λ_k implies that $t_{k+1} > t'_k$. This definition implies that $\gamma(t_{k+1})$ is an endpoint of $I_{\lambda_k} + n$, which gives (ii) for k. In addition, $\gamma(t_{k+1})$ is not equal to the other endpoint $\gamma(t'_k)$, and thus we get (iii) for k.

In this situation, we can define, as in step (0), a linear ordering $\trianglelefteq_{\lambda_k}$ in $I_{\lambda_k} + n$ such that $\gamma(t'_k) \triangleleft_{\lambda_k} \gamma(t_{k+1})$. Now, for all $t \in [t'_k, t_{k+1}]$, we define

$$\widetilde{\gamma}(t) := \max \gamma([t'_k, t]),$$

where the maximum is taken with respect to the ordering $\trianglelefteq_{\lambda_k}$. As in Step 0, $\widetilde{\gamma}\big|_{[t'_k, t_{k+1}]}$ is non decreasing for $\trianglelefteq_{\lambda_k}$. Also, the the choice of t_{k+1} implies that $\widetilde{\gamma}(t_{k+1}) = \gamma(t_{k+1})$.

Suppose that $\lambda_k = \lambda_i$ for some $0 \leqslant i \leqslant k-2$. Since $\gamma(t'_k)$ and $\gamma(t_{k+1})$ are the two endpoints of $I_{\lambda_k} + n$, one of them is equal to $\gamma(t_{i+1}) = \gamma(t'_{i+1})$ by (i-ii). Then the definition of t'_{i+1} gives a contradiction because $t'_{i+1} < t'_k \leqslant t_{k+1}$. Since we have shown above that $\lambda_k \neq \lambda_{k-1}$, this gives (iv) for k and ends the step k. If $t_{k+1} = t^*$ then $\widetilde{\gamma}$ is already defined for all $t \geqslant 0$. Otherwise we proceed to the step $k+1$.

According to the property (iv), this construction comes to an end because Λ *is finite.*

Remark 5.13. — A construction related with the one performed in Definition 5.12 but in a topological framework can be found in [7].

The next lemma can be easily deduced from the above construction.

Lemma 5.14. — *Given a path* γ *to* $+\infty$, *it follows that the path* $\widetilde{\gamma}$ *constructed in Definition 5.12 is a direct path to* $+\infty$.

Lemma 5.15. — *Let* γ *be a path to* $+\infty$. *If* $\widetilde{\gamma}(a) \neq \gamma(a)$, *then there exist* s_1, s_2 *such that* $s_1 < a < s_2$ *and* $\gamma(s_1) = \gamma(s_2) = \widetilde{\gamma}(a)$ *for all* $t \in [s_1, s_2]$.

Proof. — We use the same notation as in the definition of $\widetilde{\gamma}$ above. There are three cases when $\widetilde{\gamma}(t)$ and $\gamma(t)$ can differ:

Case 1. *$t_i < a < t_i'$ for some integer $i \geqslant 0$.*
Then $\widetilde{\gamma}(t) = \gamma(t_i) = \gamma(t_i')$ for all $t \in [t_i, t_i']$. In this case we take $s_1 = t_i$ and $s_2 = t_i'$.

Case 2. *$t'_{i-1} < a < t_i$ for some integer $i \geqslant 1$.*

There exists $z \in [t'_{i-1}, a)$ such that $\gamma(a) <_{\widetilde{\gamma}} \gamma(z) = \widetilde{\gamma}(a)$, and thus $\widetilde{\gamma}\big|_{[z,a]}$ is constant. Recall that, by Definition 5.12, $\gamma(t'_{i-1}) = \widetilde{\gamma}(t'_{i-1})$, $\gamma(t_i) = \widetilde{\gamma}(t_i)$ and $\widetilde{\gamma}(t) = \max_{\trianglelefteq_{\lambda_i}} \gamma([t_i', t])$ for all $t \in [t'_{i-1}, t_i]$. Taking all this and the continuity of γ and $\widetilde{\gamma}$ into account, it follows that there exists a maximal interval $[s_1, s_2] \subset [t'_{i-1}, t_i]$ containing a such that $\widetilde{\gamma}\big|_{[s_1,s_2]}$ is constant, $\gamma(s_1) = \widetilde{\gamma}(s_1)$ and $\gamma(s_2) = \widetilde{\gamma}(s_2)$. Since $\gamma(a) \neq \widetilde{\gamma}(a)$, we have $s_1 < a < s_2$. This ends the proof of the lemma in this case.

Case 3. *$a > t^*$.*

By Definition 5.12, $\widetilde{\gamma}(t^*) = \gamma(t^*) \in \mathbb{R}$ and

$$\widetilde{\gamma}(t) = \max r \circ \gamma([t^*, t])$$

for all $t \geqslant t^*$. Since $\lim_{t\to+\infty} r \circ \gamma(t) = +\infty$, there exists $t' > a$ such that $r \circ \gamma(t') \geqslant \widetilde{\gamma}(a)$. A similar argument as before shows that there exist a maximal interval $[s_1, s_2] \subset [t^*, +\infty)$ such that $s_1 < a < s_2$, $\widetilde{\gamma}\big|_{[s_1,s_2]}$ is constant, $r \circ \gamma(s_1) = \widetilde{\gamma}(s_1)$ and $r \circ \gamma(s_2) = \widetilde{\gamma}(s_2)$. The maximality of the interval $[s_1, s_2]$ implies that $r \circ \gamma(s_i) = \gamma(s_i) \in \mathbb{R}$ for $i = 1, 2$. This concludes the proof of the lemma. $\square$

Suppose that γ_0 is a direct path such that $\gamma_1 = \widetilde{F \circ \gamma_0}$ is comparable with γ_0 and $\gamma_1(\mathbb{R}^+) \supset \gamma_0(\mathbb{R}^+)$. If there exist $x \in \gamma_0(\mathbb{R}^+)$ and $y \in \mathbb{R}$, $y > r(x)$, such that $F(x) \leqslant_{\gamma_1} x$ and $y \leqslant r \circ F(y)$, then this looks much like a positive covering of $[x, y]$ by itself ($[x, y]$ being seen as a subinterval inside the half-line $\gamma_0(\mathbb{R}^+)$). This situation does indeed imply the existence of a fix point: the next lemma states this result in the more general setting of successive iterations of F.

Recall that F_1^r denotes the map $r \circ F\big|_{\mathbb{R}}$.

Lemma 5.16. — *Let $F \in \mathcal{C}_1(T)$ and let γ be a path from x_0 to $+\infty$. Define $\gamma_0 = \widetilde{\gamma}$ and $\gamma_{n+1} = \widetilde{F \circ \gamma_n}$ for all $n \geqslant 0$. Suppose that for some $n \geqslant 1$ the paths γ_n and γ_0 are comparable and $F^n(x_0) \leqslant_{\gamma_n} x_0$, and suppose that there exists $y \in \mathbb{R}$ such that $(F_1^r)^j(y) \geqslant y$ for all $1 \leqslant j \leqslant n$. Then there exists z such that $F^n(z) = z$.*

Proof. — Since F has degree one, by taking $y + k$ with $k \in \mathbb{Z}$ sufficiently large instead of y, we may assume that $r \circ F^j(x_0) < y$ for all $0 \leqslant j \leqslant n$. There exists $t \in (0, +\infty)$ such that $\gamma_0(t) = y$.

We show by induction that

$$\gamma_j(t) \geqslant (F_1^r)^j(y) \text{ for all } 0 \leqslant j \leqslant n. \tag{5.4}$$

This is clearly true for $j = 0$ by the choice of t. Suppose now that $\gamma_j(t) \geqslant (F_1^r)^j(y)$ for some $j \in \{0, \ldots, n-1\}$ and prove it for $j+1$. By assumption, $(F_1^r)^j(y) \geqslant y > r \circ F^j(x_0)$, and thus there exists $t' \leqslant t$ such that $\gamma_j(t') = (F_1^r)^j(y)$. Since γ_{j+1} is a direct path, we have $\gamma_{j+1}(t) \geqslant \gamma_{j+1}(t') := \widetilde{F \circ \gamma_j}(t')$ by Remark 5.10. Also,

$$r \circ F(\gamma_j(t')) = F_1^r((F_1^r)^j(y)) = (F_1^r)^{j+1}(y).$$

Moreover, observe that $F \circ \gamma_j$ is a path starting at $F^{j+1}(x_0)$ and, by the assumptions,

$$r \circ F(\gamma_j(t')) = (F_1^r)^{j+1}(y) \geqslant y > r \circ F^{j+1}(x_0).$$

Thus, we are in the part $[t^*, +\infty)$ of Definition 5.12, and hence $\widetilde{F \circ \gamma_j}(t') \geqslant r \circ F(\gamma_j(t'))$. Summarising we have shown that $\gamma_{j+1}(t) \geqslant (F_1^r)^{j+1}(y)$; which ends the proof of the induction step.

Set $I = \gamma_0([0,t])$ and $J = \gamma_n([0,t])$. Clearly both sets are homeomorphic to closed intervals of the real line, I has endpoints x_0 and y while the endpoints of J are $F^n(x_0)$ and $\gamma_n(t) \geqslant (F_1^r)^n(y) \geqslant y$ (Equation (5.4) for $i = n$). By assumption, $F^n(x_0) \leqslant_{\gamma_n} x_0$, and thus $I \subset J$. We define a map $G \colon I \longrightarrow J$ as follows. Given a point $x \in I$ take $t \in \mathbb{R}^+$ such that $\gamma_0(t) = x$ and then set $G(x) = \gamma_n(t)$. We have to show that the map G is well defined. Let U_x denote $\{t \in \mathbb{R}^+ : \gamma_0(t) = x\}$. If $\operatorname{Card}(U_x) > 1$ then, since $\gamma_0 = \widetilde{\gamma}$ is a direct path, U_x is an interval where it is constant. Consequently, one can easily prove inductively that $\gamma_{i+1} = \widetilde{F \circ \gamma_i}$ is also constant on U_x for $0 \leqslant i \leqslant n-1$. In particular γ_n is constant on U_x.

The map G is continuous. Then, by identifying J with an interval on $\mathbb{R}$ we can use Lemma 2.4 to prove that there exists $z \in I$ such that $G(z) = z$ and $z \notin \operatorname{Const}(G)$.

It remains to show that $G(z) = F^n(z)$. Let a be such that $z = \gamma_0(a)$. Then, we have to show that $\gamma_n(a) = F^n \circ \gamma_0(a)$. Suppose that, for some $j < n$, $\gamma_{j+1}(a) \neq F \circ \gamma_j(a)$ and $\gamma_j(a) = F^j \circ \gamma_0(a)$. Applying Lemma 5.15 to the path $F \circ \gamma_j$, we find that there exist $s_1 < a < s_2$ such that $F \circ \gamma_j(s_1) = F \circ \gamma_j(s_2) = \gamma_{j+1}(s)$ for all $s \in [s_1, s_2]$. Then, $\gamma_0(s_1) \neq \gamma_0(a)$ because $F \circ \gamma_j(s_1) = \gamma_{j+1}(a) \neq F \circ \gamma_j(a)$, and for the same reason $\gamma_0(a) \neq \gamma_0(s_2)$. Since the map γ_0 is non decreasing, $z = \gamma_0(a)$ is in the interior of $[\gamma_0(s_1), \gamma_0(s_2)]$. Moreover, G is constant on this interval, and thus $z \in \operatorname{Const}(G)$, which is a contradiction. We conclude that $\gamma_{j+1}(a) = F \circ \gamma_j(a)$ for all $0 \leqslant j < n$, and thus $\gamma_n(a) = F^n \circ \gamma_0(a)$. □

The next lemma is the key tool in the proof of Theorem 5.18.

Lemma 5.17. — *Let $F \in \mathcal{C}_1(T)$ be such that* $\operatorname{Clos}\left(T_{\mathbb{R}}\right) = T$ *and* $0 \leqslant \min \operatorname{Rot}(F)$*. Suppose that there exists $y \in \mathbb{R}$ such that $(F_1^r)^n(y) \geqslant y$ for all $n \geqslant 1$, and $F^n(x) \neq x$ for all $x \in T$ and $n \in \mathbb{N}$. Then* $\min \operatorname{Rot}(F) > 0$.

Proof. — By Lemma 5.3, there exists a finite subset A such that $T \setminus T_{\mathbb{R}} = A + \mathbb{Z}$ and there exist a $s \in \mathbb{N}$ which is common to all elements of A and integers $\{i'_a\}_{a\in A}$ such that for all $a \in A$, $F^s(a) = a + i'_a$. Hence, the rotation number of every $a \in A$ exists and we have $\rho_F(a) = i'_a/s$. Moreover, for each $a \in A$, $i'_a \geqslant 0$ because $\min \operatorname{Rot}(F) \geqslant 0$. Moreover, $i'_a \geqslant 1$ because otherwise $a = F^s(a)$, which contradicts our assumptions.

For every $a \in A$, let V_a be a neighbourhood of a such that $F^s(V_a) \subset X + i'_a$. By Lemma 5.2(b), $\{F^n(\mathbb{R})\}_{n\geqslant 0}$ is an increasing sequence of connected sets whose union is $T_{\mathbb{R}}$. By assumption, $T = \operatorname{Clos}\left(T_{\mathbb{R}}\right)$. Thus, there exists an integer $\ell \geqslant 0$ such that $F^{\ell s}(\mathbb{R})$ contains $X \setminus \bigcup_{a\in A} V_a$. Set $G = F^{\ell s}$. We have $T \setminus G(\mathbb{R}) \subset \left(\bigcup_{a\in A} V_a\right) + \mathbb{Z}$ and $G(a) = a + i_a$ for all $a \in A$, where $i_a = \ell i'_a \geqslant 1$.

In the rest of the proof we will use the distance ν on T introduced in Definition 4.13. With this notation we set

$$\delta_n := \min\{\nu(G^n(x), x) : x \in T\} = \min\{\nu(G^n(x), x) : 0 \leqslant r(x) \leqslant 1\}.$$

Notice that $\delta_n > 0$ for all $n \geqslant 1$ because $G^n(x) \neq x$ for all $x \in T$.

Given a point $x \in T$ and a path γ from x to $+\infty$ we iteratively define the following directed paths: $\gamma_0 = \widetilde{\gamma}$ and $\gamma_{i+1} = \widetilde{G \circ \gamma_i}$ for all $i \geqslant 0$. Then, by Lemma 5.16, the following property holds for each $n \in \mathbb{N}$.

(5.5) Either the paths γ_0 and γ_n are not comparable or the inequality $G^n(x) \leqslant_{\gamma_n} x$ does not hold.

Suppose in addition that $x \in \mathbb{R}$ and $x \geqslant r \circ G^n(x)$ for some $n \geqslant 1$. The path γ_n goes from $G^n(x)$ to $+\infty$, and thus its image contains $[r(G^n(x)), +\infty) \supset [x, +\infty)$. Therefore, γ_n and γ_0 are comparable and $G^n(x) \leqslant_{\gamma_n} x$, which contradicts (5.5). Consequently, we have:

$$\forall x \in \mathbb{R},\ \forall n \geqslant 1,\ x < r \circ G^n(x). \tag{5.6}$$

Now we prove the following claim, that means that if the orbit of some point y remains in $X + \mathbb{Z}$ then it cannot go too much to the left and has to go to the right of y in bounded time.

Claim: There exists $\widetilde{N} \in \mathbb{N}$ such that if $y \in X$ verifies that

$$\text{there exists } k_n \geqslant 0 \text{ such that } G^n(y) \in X - k_n,$$

for all $n = 1, \ldots, N-1$, then $N < \widetilde{N}$.

Proof of the claim. We know that $G(\mathbb{R}) \supset X \setminus \bigcup_{a\in A} V_a$. Hence, since $G \in \mathcal{C}_1(T)$, there exists a point $x_0 \in \mathbb{R}$, $x_0 < 1$, such that $G([x_0, +\infty)) \supset X \setminus \bigcup_{a\in A} V_a$. We denote the number $1 - \lfloor x_0 \rfloor$ by p, where $\lfloor x_0 \rfloor$ denotes the integer part of x_0. Clearly, p bounds the number of copies of X of the form $X - n$ fitting between x_0 and 1.

Fix a path ξ from y to $+\infty$ and define $\xi_0 = \widetilde{\xi}$ and $\xi_{n+1} = \widetilde{G \circ \xi_n}$ for all $n \geqslant 0$. We are going to show that a large proportion of $\{G^n(y)\}_{0\leqslant n<N}$ lie in the same basic interval and then a large proportion of paths $\{\xi_n\}_{0\leqslant n<N}$ start in this particular interval and are comparable.

The point y does not belong to $\bigcup_{a\in A} V_a$ because $G(V_a) \subset X + i_a$ with $i_a \geqslant 1$. Hence $y \in G([x_0, +\infty))$. Let $x \in \mathbb{R}$, $x \geqslant x_0$ be such that $G(x) = y$. Equation (5.6) implies that $x < r(G^{n+1}(x)) = r(G^n(y))$ for all $n \geqslant 0$, and thus $x_0 < r \circ G^n(y)$ for all $0 \leqslant n \leqslant N-1$, and hence $0 \leqslant k_n \leqslant p-1$. In other words, the number of basic intervals that contain one of the points $(G^n(y))_{0\leqslant n<N}$ is at most $p\,\mathrm{Card}(\Lambda)$. By the drawers principle, there exists a basic interval Y such that

$$\mathrm{Card}(\{0 \leqslant n \leqslant N-1 : G^n(y) \in Y\}) \geqslant \frac{N}{p\,\mathrm{Card}(\Lambda)}.$$

By Lemma 5.11, the number of equivalence classes of comparable paths starting from Y is finite. Let q be this number. Let $[\sigma]_Y$ denote the equivalence class of a path σ starting in Y. Again by the drawers principle, there exists a path σ such that the number of elements of the set

$$\mathcal{N} = \{0 \leqslant n \leqslant N-1 : G^n(y) \in Y \text{ and } [\xi_n]_Y = [\sigma]_Y\}$$

is at least $\frac{N}{m}$, where m denotes $pq\,\mathrm{Card}(\Lambda)$. Write

$$\mathcal{N} = \{n_1, n_2, \ldots, n_k\} \text{ with } k \geqslant \frac{N}{m} \text{ and } n_1 < n_2 < \cdots < n_k.$$

We will show that there cannot be too many elements in $\mathcal{N}$ because of (5.5). If we choose σ to be maximal with respect to the inclusion relation of images then, for every $n \in \mathcal{N}$, the ordering $\leqslant_{\xi_n}$ is a restriction of $\leqslant_\sigma$. Since $\xi_n(0) = G^n(y)$, all the points $\{G^n(y)\}_{n\in\mathcal{N}}$ belong to Y and are ordered by $\leqslant_\sigma$. If $n_i, n_j \in \mathcal{N}$ with $n_i > n_j$ then $G^{n_i}(y) \leqslant_\sigma G^{n_j}(y)$ contradicts (5.5) applied to the path ξ_{n_i} and initial path ξ_{n_j} (notice that, by definition of $\mathcal{N}$, ξ_{n_i} and ξ_{n_j} are equivalent and hence comparable). Therefore

$$G^{n_1}(y) <_\sigma G^{n_2}(y) <_\sigma \cdots <_\sigma G^{n_k}(y). \tag{5.7}$$

This implies that $\nu(G^{n_k}(y), G^{n_1}(y)) = \sum_{i=1}^{k-1} \nu(G^{n_i}(y), G^{n_{i+1}}(y))$. This observation will be used to find a lower bound of $\nu(G^{n_k}(y), G^{n_1}(y))$.

For $i = 1, \dots, k-1$ set $j_i = n_{i+1} - n_i$. We have $j_i \geqslant 1$ and

$$j_1 + \cdots + j_{k-1} = n_k - n_1 \leqslant N.$$

Define also $M := \operatorname{Card}(\{1 \leqslant i \leqslant k-1 : j_i \leqslant 2m\})$. Then, there are $k-1-M$ integers i such that $j_i \geqslant 2m+1$ and for the rest we have $j_i \geqslant 1$. Thus,

$$N \geqslant j_1 + \cdots + j_{k-1} \geqslant M + (2m+1)(k-1-M).$$

Hence

$$M \geqslant \frac{(2m+1)(k-1) - N}{2m} \geqslant k - 1 - \frac{N}{2m}.$$

Since $k \geqslant \frac{N}{m}$ it follows that $M \geqslant \frac{N}{2m} - 1$. According to the definition of δ_j, we have $\nu(G^{n_{i+1}}(y), G^{n_i}(y)) \geqslant \delta_{j_i}$. There are M integers i such that $j_i \leqslant 2m$. Thus, in view of Equation (5.7), we get that $\nu(G^{n_k}(y), G^{n_1}(y)) \geqslant M\kappa$, where $\kappa = \min\{\delta_1, \dots, \delta_{2m}\} > 0$. Let L denote the maximal length of all the basic intervals. It follows that $L \geqslant \nu(G^{n_k}(y)), G^{n_1}(y)) \geqslant M\kappa$, and thus $N \leqslant 2m\left(\frac{L}{\kappa} + 1\right)$. This concludes the proof of the claim by setting $\widetilde{N} > 2m\left(\frac{L}{\delta} + 1\right)$ (recall that $m = pq \operatorname{Card}(\Lambda)$ and that L and κ depend only on T and G).

To end the proof of the lemma it is enough to show that $\overline{\rho}_{_G}(x) > 0$ for all $x \in T$. Indeed, by Lemma 1.10 in that case we will have $0 < \overline{\rho}_{_G}(x) = \ell s \overline{\rho}_{_F}(x)$ which implies $\overline{\rho}_{_F}(x) > 0$ for all $x \in T$. Since $\operatorname{Rot}^+(F) \supset \operatorname{Rot}(F)$ the lemma holds.

To prove that $\overline{\rho}_{_G}(x) > 0$ for all $x \in T$ we consider three cases.

Case 1. *$G^n(x) \in \mathbb{R}$ for all $n \geqslant N$.*

Then $G^{n+1}(x) \geqslant G^n(x) + \delta_1$ for all $n \geqslant N$ and $\overline{\rho}_{_G}(x) \geqslant \delta_1 > 0$.

Case 2. *$G^n(x) \in X + \mathbb{Z}$ for all $n \geqslant N$.*

By the Claim there exist two sequences $\{n_i\}_{i \geqslant 0}$ and $(k_i)_{i \geqslant 0}$ such that $n_{i+1} - n_i \leqslant \widetilde{N}$, $k_{i+1} \geqslant k_i + 1$ and $G^{n_i}(x) \in X + k_i$ for all $i \geqslant 0$. Hence $\overline{\rho}_{_G}(x) \geqslant \frac{1}{\widetilde{N}} > 0$.

Case 3. *Assume that we are not in the first two cases.*

Then, there exists an increasing sequence $\{n_i\}_{i \geqslant 0}$ such that, for all $i \geqslant 0$, $G^{n_i}(x) \in \mathbb{R}$ and $G^j(x) \notin \mathbb{R}$ for all $j \in \{n_i + 1, \dots, n_{i+1} - 1\}$. For these j, let $q_j \in \mathbb{Z}$ be such that $G^j(x) \in X + q_j$. Let $\widetilde{N}$ be the integer given by the Claim, and define:

$$\begin{aligned}
C &:= \max\{|r \circ G^n(x) - r(x)| : x \in T,\ n \leqslant \widetilde{N}\} \\
N_1 &:= \lceil 2\widetilde{N}(C+2) + 2 \rceil \\
\delta &:= \min\{\delta_1, \dots, \delta_{N_1}\} > 0
\end{aligned}$$

(recall that $\lceil\cdot\rceil$ denotes the ceiling function). Observe that all direct paths going to $+\infty$ and starting at some point $G^{n_i}(x)$ are comparable because $G^{n_i}(x) \in \mathbb{R}$ for all i. Consequently, by (5.5), $G^{n_{i+1}}(x) > G^{n_i}(x)$.

If $n_{i+1} - n_i \leqslant N_1$ then

$$\frac{G^{n_{i+1}}(x) - G^{n_i}(x)}{n_{i+1} - n_i} \geqslant \frac{\delta}{N_1}. \tag{5.8}$$

If $n_{i+1} - n_i > N_1$, by the Claim, there exist $n_i + 1 = j_1 < j_2 < \cdots < j_k < n_{i+1}$ such that $j_{i+1} - j_i \leqslant \widetilde{N}$, $n_{i+1} - j_k \leqslant \widetilde{N}$ and $q_{j_{t+1}} - q_{j_t} \geqslant 1$ for all $1 \leqslant t \leqslant k-1$. Hence $k \geqslant \frac{n_{i+1}-n_i-1}{\widetilde{N}}$. Since $q_{j_{t+1}} - q_{j_t} \geqslant 1$, the point $G^{j_k}(x)$ belongs to $X + q_{j_1} + m$ for some $m \geqslant k-1$, and thus $r \circ G^{j_k}(x) - r \circ G^{j_1}(x) \geqslant k-2$. Moreover, $G^{n_i}(x) < r \circ G^{j_1}(x)$ because of (5.6). Therefore,

$$\begin{aligned}
G^{n_{i+1}}(x) - G^{n_i}(x) =&(G^{n_{i+1}}(x) - r \circ G^{j_k}(x)) + (r \circ G^{j_k}(x) - r \circ G^{j_1}(x)) \\
&+ (r \circ G^{j_1}(x) - G^{n_i}(x)) \geqslant -C + (k-2) + 0 \\
\geqslant&\frac{n_{i+1} - n_i - 1}{\widetilde{N}} - C - 2 \\
=&\frac{n_{i+1} - n_i}{\widetilde{N}} - (C + 2 + 1/\widetilde{N}).
\end{aligned}$$

The choice of N_1 implies that $N_1 \geqslant 2\widetilde{N}(C+2) + 2$ which is equivalent to $\frac{N_1}{2\widetilde{N}} \geqslant C + 2 + 1/\widetilde{N}$. Consequently, since $n_{i+1} - n_i > N_1$

$$\begin{aligned}
G^{n_{i+1}}(x) - G^{n_i}(x) \geqslant&\frac{n_{i+1} - n_i}{\widetilde{N}} - (C + 2 + 1/\widetilde{N}) \geqslant \frac{2(n_{i+1} - n_i)}{2\widetilde{N}} - \frac{N_1}{2\widetilde{N}} \\
\geqslant&\frac{n_{i+1} - n_i}{2\widetilde{N}},
\end{aligned}$$

which is equivalent to

$$\frac{G^{n_{i+1}}(x) - G^{n_i}(x)}{n_{i+1} - n_i} \geqslant \frac{1}{2\widetilde{N}}. \tag{5.9}$$

Summarising, Equations (5.8) and (5.9) imply that

$$\overline{\rho}_G(x) \geqslant \min\left\{\frac{\delta}{N_1}, \frac{1}{2\widetilde{N}}\right\} > 0.$$

This ends the proof of the lemma. □

Now we are ready to prove the main result of this section.

Theorem 5.18. — *Let $T \in \mathbf{T}^\circ$ and let $F \in \mathcal{C}_1(T)$. If $\min \operatorname{Rot}_{\mathbb{R}}(F) = p/q$ (resp. $\max \operatorname{Rot}_{\mathbb{R}}(F) = p/q$), then there exists a periodic* (mod 1) *point $x \in T$ such that $\rho_F(x) = p/q$.*

Proof. — We deal only with the case $p/q = \min \operatorname{Rot}_{\mathbb{R}}(F)$. The other one follows similarly.

If $\operatorname{Rot}_{\mathbb{R}}(F) = \{p/q\}$ then $\operatorname{Rot}(F_1^r) = \{p/q\}$ by Corollary 1.17 and, in view of Theorem 1.15, there exists a periodic (mod 1) point of rotation number p/q. In the rest of the proof we suppose that $\max \operatorname{Rot}_{\mathbb{R}}(F) > p/q$.

Set $\overline{T} = \operatorname{Clos}\left(T_{\mathbb{R}}\right)$ and $G = (F^q - p)\big|_{\overline{T}}$. By Lemma 5.2(b), $\overline{T} \in \mathbf{T}^\circ$ and $G \in \mathcal{C}_1(\overline{T})$. By Lemma 1.10 we have $0 = \min \operatorname{Rot}(G)$ and $\max \operatorname{Rot}(G) > 0$. Also, $\overline{T} = \operatorname{Clos}\left(\bigcup_{n\geqslant 0} G^n(\mathbb{R})\right)$ by Lemma 5.2(b). According to Theorem 3.1, there exists a positive integer N such that $\max \frac{1}{N} \operatorname{Rot}(G_N^r) > 0$, and, by Theorem 1.15, there exists $y \in \mathbb{R}$ such that $\rho_{G_N^r}(y) > 0$ and the orbit of y for G^N is twist. In particular, $(G_N^r)^n(y) \geqslant y$ for all $n \geqslant 0$. Let $H = G^N$ (hence $H_1^r = G_N^r$). If $H^n(x) \neq x$ for all $x \in T$ and $n \in \mathbb{N}$, then $\min \operatorname{Rot}(H) > 0$ by Lemma 5.17, which is a contradiction. Therefore, there exist $x \in T$ and $n \in \mathbb{N}$ such that $H^n(x) = x$, and thus x is periodic (mod 1) for F and $\rho_F(x) = p/q$. □

Remark 5.19. — Unfortunately, the periodic (mod 1) point given by Theorem 5.18 may be in $T \setminus \mathbb{R}$ and there may not exist a periodic (mod 1) point x in $\mathbb{R}$ with $\rho_F(x)$ being an endpoint of the rotation interval (see Example 6.6).

6. Examples

This section is devoted to showing some examples to help understanding the theoretical results of the previous sections. Attention is payed to the differences between this case and the circle one. For easiness the first two examples will be Markov. To be able to compute the periods (mod 1) and rotation numbers of these examples we need to develop the appropriate machinery. So we will divide this section into two subsections. In the first one we will introduce the theoretical results to make the computations in the examples whereas in the second one we provide the examples themselves. Some of the properties of rotation sets for symbolic systems used here already appear in [22].

6.1. Preliminary results on Markov lifted graph maps

We say that a subset of T is an *interval* if it is homeomorphic to an interval of the real line and does not contain vertices (except maybe at its endpoints). In other words, a subset of T is an interval if it is still homeomorphic to an interval after removing the vertices of T.

An interval of T can be endowed with two opposite linear orderings compatible with its structure of interval. If I, J are two intervals of T, we choose arbitrarily one of these two orderings for each interval, and we say that a map $f\colon I \longrightarrow J$ is *monotone* if it is monotone with respect to these orderings. Notice that this is independent of the choice of the orderings.

Let $T \in \mathbf{T}^\circ$ and let ν be the distance on T introduced in Definition 4.13. When T is an infinite tree (i.e., it is uniquely arcwise connected), then $\nu(x, y)$ coincides with the *taxicab metric* which gives the length of the shortest path in T from x to y. We say that $f\colon I \longrightarrow J$ is *affine* if there exists $\lambda \in \mathbb{R}$ such that for all $x, y \in I$, $\nu(f(x), f(y)) = \lambda\nu(x, y)$. Observe that if f is affine then it is also monotone.

Now we adapt the well known notion of Markov map to the context of lifting graphs.

Definition 6.1. — *Let $T \in \mathbf{T}^\circ$ and let $F \in \mathcal{C}_1(T)$. We say that F is a* Markov map *if there exist compact intervals $P_1, \dots, P_k$ such that*

(i) *the vertices of T are included in $\bigcup_{i=1}^k \partial P_i + \mathbb{Z}$,*
(ii) *$(P_1 \cup \cdots \cup P_k) + \mathbb{Z} = T$,*
(iii) *if $i \neq j$ then $P_i \cap P_j$ contains at most one point,*
(iv) *for all $1 \leqslant i \leqslant k$, $F(P_i)$ is an interval, $F\big|_{P_i} : P_i \longrightarrow F(P_i)$ is monotone, and $F(P_i)$ is a finite union of sets $\{P_j + n\}_{1\leqslant j\leqslant k, n\in\mathbb{Z}}$.*

When we will need to specify it, we will say that F is a Markov map with respect to the partition *$(P_1, \dots, P_k)$, or that $(P_1, \dots, P_k)$* is the Markov partition of *F.*

The Markov map F is called affine *if $F\big|_{P_i}$ is affine for all $1 \leqslant i \leqslant k$.*

If $F(P_i) \supset P_j + n$, we write $P_i \xrightarrow{n} P_j$. This gives a finite labelled oriented graph, which is called the Markov graph of *F and denoted by $\mathcal{G}(F)$. If $B = \{B_1, \dots, B_p\}$ and $A \xrightarrow{n} B_i$ for all $1 \leqslant i \leqslant p$, we also write (or picture) $A \xrightarrow{n} B$ for short.*

We now give some notations about paths in graphs that we will need later.

Let $\mathcal{G}$ be a finite labelled oriented graph. A *(finite) path* is a sequence of labelled arrows in $\mathcal{G}$ of the form

$$\mathcal{A} := A_0 \xrightarrow{n_0} A_1 \xrightarrow{n_1} \cdots A_{p-1} \xrightarrow{n_{p-1}} A_p.$$

The *length* of $\mathcal{A}$ is $L(\mathcal{A}) = p$ and its *weight* is $W(\mathcal{A}) = n_0 + \cdots + n_{p-1}$.

If $\mathcal{B} := B_0 \xrightarrow{m_0} B_1 \xrightarrow{m_1} \cdots B_{q-1} \xrightarrow{m_{q-1}} B_q$ is another path with $B_0 = A_p$, we define the concatenated path as

$$A_0 \xrightarrow{n_0} A_1 \xrightarrow{n_1} \cdots A_{p-1} \xrightarrow{n_{p-1}} A_p \xrightarrow{m_0} B_1 \xrightarrow{m_1} \cdots B_{q-1} \xrightarrow{m_{q-1}} B_q.$$

Such a path will be denoted by $\mathcal{A}\mathcal{B}$. A path $\mathcal{A}$ is called a *loop* if $A_0 = A_p$. In such a case, $\mathcal{A}^0$ denotes the empty path and, for $n \geqslant 1$, $\mathcal{A}^n$ denotes the path

$$\overbrace{\mathcal{A}\mathcal{A}\cdots\mathcal{A}}^{n \text{ times}}.$$

Also, $\mathcal{A}^\infty$ denotes the loop $\mathcal{A}$ concatenated with itself infinitely many times, which gives an infinite path.

A loop $\mathcal{A}$ is called *simple* if it is not of form $\mathcal{B}^n$, $\mathcal{B}$ being a shorter loop and $n \geqslant 2$. A loop is *elementary* if it cannot be formed by concatenating two loops, up to a circular permutation. Equivalently, $\mathcal{A} := A_0 \xrightarrow{n_0} A_1 \xrightarrow{n_1} \cdots A_{p-1} \xrightarrow{n_{p-1}} A_0$ is elementary if $A_0, \dots, A_{p-1}$ are all pairwise different. Observe that the number of distinct elementary loops in $\mathcal{G}$ is finite.

If $\mathcal{A} := A_0 \xrightarrow{n_0} A_1 \xrightarrow{n_1} \cdots A_{p-1} \xrightarrow{n_{p-1}} A_p \xrightarrow{n_p} \cdots$ is an infinite path, let $\mathcal{A}_i^j$ denote the truncated path $A_i \xrightarrow{n_i} \cdots \xrightarrow{n_{j-1}} A_j$, where $0 \leqslant i < j$.

Suppose that $\mathcal{G}$ is the Markov graph of a Markov map $F \in \mathcal{C}_1(T)$ and let $x \in T$. We say that an infinite path

$$\mathcal{A} := A_0 \xrightarrow{n_0} A_1 \xrightarrow{n_1} \cdots A_{p-1} \xrightarrow{n_{p-1}} A_p \xrightarrow{n_p} \cdots$$

is an *itinerary* of x if there exists $n(x) \in \mathbb{Z}$ such that $F^i(x) \in A_i + n(x) + W(\mathcal{A}_0^i)$ for all $i \geqslant 0$. If in addition there exists a loop $\mathcal{B}$ such that $\mathcal{A} = \mathcal{B}^\infty$, then we say that $\mathcal{B}$ is a *periodic itinerary* of x.

The following proposition is a version for lifted graph maps of folk knowledge properties of Markov maps on finite topological graphs.

Proposition 6.2. — *Let $T \in \mathbf{T}^\circ$ and let $F \in \mathcal{C}_1(T)$ be a Markov map with respect to the partition $(P_1, \dots, P_k)$.*

(a) *If F is an affine Markov map such that $\mathcal{G}(F)$ is connected and is not reduced to a unique loop, then F is transitive* (mod 1).

(b) *For every $x \in T$, there exists an infinite path in $\mathcal{G}(F)$ which is an itinerary of x.*

(c) *If $x \in T$ is a periodic* (mod 1) *point, then there exists a simple loop $\mathcal{B}$ in $\mathcal{G}(F)$ which is a periodic itinerary of x. Moreover, if the period* (mod 1) *of x is p and $F^i(x) \notin \bigcup_{j=1}^k \partial P_i + \mathbb{Z}$ for all $0 \leqslant i < p$, then $p = L(\mathcal{B})$ and $F^p(x) = x + W(B)$.*

(d) *Every infinite path in $\mathcal{G}(F)$ is an itinerary of some point $x \in T$. Every loop in $\mathcal{G}(F)$ is a periodic itinerary of some periodic* (mod 1) *point x.*

The next two lemmas show how the rotation numbers and the rotation set can be deduced from the Markov graph.

LEMMA 6.3. — *Let $F \in \mathcal{C}_1(T)$ be a Markov map with $T \in \mathbf{T}^\circ$ and let $x \in T$ be such that $\rho_F(x)$ exists. If the infinite path $\mathcal{A} := A_0 \xrightarrow{n_0} \cdots A_i \xrightarrow{n_i} A_{i+1} \cdots$ is an itinerary of x in $\mathcal{G}(F)$, then*

$$\rho_F(x) = \lim_{i \to +\infty} \frac{W(\mathcal{A}_0^i)}{i}.$$

If $\mathcal{B}$ is a loop in $\mathcal{G}(F)$ which is a periodic itinerary of x, then $\rho_F(x) = \frac{W(\mathcal{B})}{L(\mathcal{B})}$.

Proof. — Let $(P_1, \ldots, P_k)$ be the Markov partition of F. By definition of an itinerary, $F^i(x) - n(x) - W(\mathcal{A}_0^i) \in A_i$ for all $i \geqslant 0$. The set $\{r(y) : y \in P_1 \cup \cdots \cup P_k\}$ is bounded, and $A_i \in \{P_1, \ldots, P_k\}$ for all $i \geqslant 0$. Therefore $r \circ F^i(x) - W(\mathcal{A}_0^i)$ is bounded too, and thus

$$\rho_F(x) = \lim_{i \to +\infty} \frac{W(\mathcal{A}_0^i)}{i}.$$

Suppose that the loop $\mathcal{B}$ is a periodic itinerary of x. Then $\mathcal{A} = \mathcal{B}^\infty$ is an itinerary of x, and $W(A_0^{iL(\mathcal{B})}) = iW(\mathcal{B})$ for all $i \geqslant 0$. What precedes implies that $\rho_F(x) = \frac{W(\mathcal{B})}{L(\mathcal{B})}$. □

LEMMA 6.4. — *Let $F \in \mathcal{C}_1(T)$ be a transitive* (mod 1) *Markov map with $T \in \mathbf{T}^\circ$ and set*

$$m := \min_{\mathcal{E}} \frac{W(\mathcal{E})}{L(\mathcal{E})} \quad \text{and} \quad M := \max_{\mathcal{E}} \frac{W(\mathcal{E})}{L(\mathcal{E})},$$

where $\mathcal{E}$ ranges over the set of all elementary loops in $\mathcal{G}(F)$. Then $\mathrm{Rot}(F) = [m, M]$.

Proof. — By Theorem 3.1 $\mathrm{Rot}_{\mathbb{R}}(F)$ is a compact interval, and by Theorem 5.5 $\mathrm{Rot}(F) = \mathrm{Rot}_{\mathbb{R}}(F)$ because F is transitive (mod 1). By Proposition 6.2(d) and Lemma 6.3, m and M belong to $\mathrm{Rot}(F)$, and hence $[m, M] \subset \mathrm{Rot}(F)$.

Let $x \in T$ such that $\rho_F(x)$ exists and let

$$\mathcal{A} := A_0 \xrightarrow{n_0} A_1 \xrightarrow{n_1} \cdots A_k \xrightarrow{n_k} \cdots$$

be an itinerary of x, which exists by Proposition 6.2(b). Since the number of vertices in $\mathcal{G}(F)$ is finite, there exists P, an element of the partition, and an increasing sequence k_i such that $A_{k_i} = P$ for all $i \geqslant 0$. By Lemma 6.3, $\rho_{F}(x) = \rho_{F}(F^{k_0}(x))$ is equal to

$$\lim_{i\to\infty} \frac{W(\mathcal{A}_{k_0}^{k_i})}{k_i - k_0}.$$

If we decompose the loop $\mathcal{A}_{k_0}^{k_i}$ into elementary loops, we see that the above quantity is a barycentre of

$$\left\{\frac{W(\mathcal{E})}{L(\mathcal{E})} : \mathcal{E} \text{ is an elementary loop of } \mathcal{G}(F).\right\}$$

Hence, $\rho_{F}(x) \in [m, M]$. $\square$

6.2. The examples

Example 6.5. — $\mathrm{Rot}(F) = [-1/2, 1/2]$, $\mathrm{Per}(0, F) = \{1\} \cup \{n \geqslant 4\}$ *and if* $p/q \in \mathrm{Rot}(F)$ *with* p, q *coprime,* $p \neq 0$ *then* $\mathrm{Per}(p/q, F) = q\mathbb{N}$.

Let $F \in \mathcal{C}_1(T)$ be the affine Markov map represented in Figure 6.1. By Proposition 6.2(a), F is transitive (mod 1).

We define:

$$\begin{aligned}
\mathcal{A} :=& C_4 \xrightarrow{0} A \xrightarrow{1} C_4,\\
\mathcal{B} :=& C_2 \xrightarrow{-1} B \xrightarrow{0} C_2 \text{ and}\\
\mathcal{C} :=& C_3 \xrightarrow{0} C_3.
\end{aligned}$$

The loops $\mathcal{B}$ and $\mathcal{A}$ correspond respectively to $\min \frac{W(\mathcal{E})}{L(\mathcal{E})}$ and $\max \frac{W(\mathcal{E})}{L(\mathcal{E})}$, where $\mathcal{E}$ describes the set of elementary loops. Thus $\mathrm{Rot}(F) = [-1/2, 1/2]$ by Lemma 6.4. Notice that

$$\mathrm{Rot}(F) = \tfrac{1}{2}\,\mathrm{Rot}(F_2^r) \quad \text{but} \quad \tfrac{1}{3}\,\mathrm{Rot}(F_3^r) = [-1/3, 1/3] \neq \mathrm{Rot}(F),$$

and thus $\left\{\frac{1}{n}\,\mathrm{Rot}(F_n^r)\right\}_{n\geqslant 1}$ is not an increasing sequence of sets.

We are going to show that 0 is the unique rational $p/q \in \mathrm{Rot}(F)$ such that $\mathrm{Per}(p/q, F) \neq q\mathbb{N}$. Moreover there is a “gap” in $\mathrm{Per}(0, F)$: it contains 1 and 4 but not 2 and 3 (it is also possible to construct examples with more than one gap in $\mathrm{Per}(0, F)$). This shows that $\mathrm{Per}(p/q, F)$ is not necessarily of the form $\{nq : n \geqslant N\}$ when $p/q \in \mathrm{Int}\,\mathrm{Rot}(F)$ and p, q coprime.

We compute $\mathrm{Per}(p/q, F)$ by using Proposition 6.2 and Lemma 6.3. If x is an endpoint of one of the intervals of the Markov partition then, either x is not periodic (mod 1), or $x = e$ (mod 1) and $F(x) = x$.

From one side $1 \in \mathrm{Per}(0, F)$ because $F(e) = e$. Also, there are no simple loops of length 2 or 3 and weight 0. Thus $2, 3 \notin \mathrm{Per}(0, F)$. For all $k \geqslant 0$, the loop

$$(C_2 \xrightarrow{-1} B \xrightarrow{0} C_4 \xrightarrow{0} A \xrightarrow{1} C_3)\mathcal{C}^k$$

is simple, its length is $k+4$ and its weight is 0 (if $k = 0$, take C_2 instead of C_3 to get a loop). Thus there exists a point x such that $F^{k+4}(x) = x$ and $k + 4 \in \mathrm{Per}(0, F)$.

If $p \geqslant 1$, $q > 2p$ and $n \geqslant 1$, we consider the loop

$$\mathcal{A}^{np-1}(C_4 \xrightarrow{0} A \xrightarrow{1} C_3)\mathcal{C}^{n(q-2p)-1}(C_3 \xrightarrow{0} C_4).$$

It is simple, its length is nq and its weight is np. Thus there exists a periodic (mod 1) point of period nq and rotation number p/q. For $q = 2p$ and $np \geqslant 2$ consider the simple loop

$$\mathcal{A}^{np-2}(C_4 \xrightarrow{0} A \xrightarrow{1} C_5 \xrightarrow{0} A \xrightarrow{1} C_4).$$

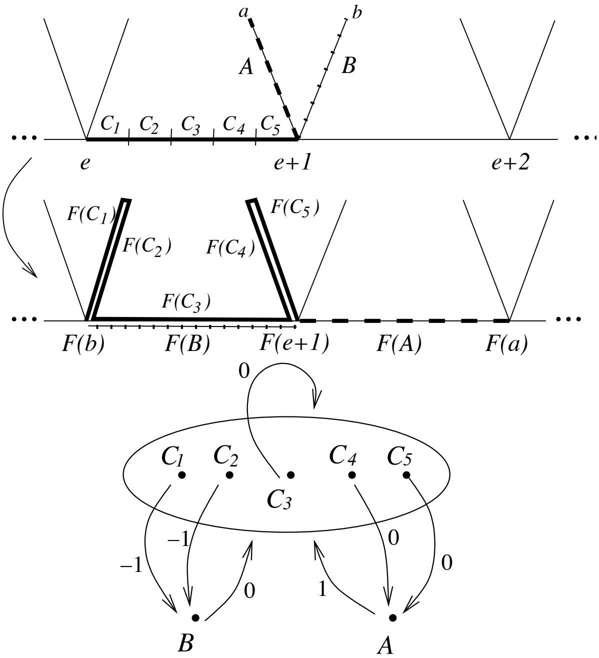

Figure 6.1. In the picture above the affine Markov map F. Below it is displayed its Markov graph. In this example $\mathrm{Rot}(F) = [-1/2, 1/2]$, $\mathrm{Per}(0, F) = \{1\} \cup \{n \geqslant 4\}$ *and if* $p/q \in \mathrm{Rot}(F)$ *with* p, q *coprime,* $p \neq 0$ *then* $\mathrm{Per}(p/q, F) = q\mathbb{N}$.

For $p = n = 1$ and $q = 2$ we consider the loop $\mathcal{A}$.

If $p < 0$ then the same arguments hold with $\mathcal{B}$ instead of $\mathcal{A}$.

Therefore, if $p/q \in [-1/2, 1/2]$, $p \neq 0$ then $\mathrm{Per}(p/q, F) \supset q\mathbb{N}$. To conclude, we use that, if p, q are coprime, then $\mathrm{Per}(p/q, F) \subset q\mathbb{N}$ by Proposition 3.6.

Example 6.6. — $\mathrm{Rot}(F) = [0,1]$, $\bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F_n^r) = (0,1]$ *is not closed and there exist infinitely many* $p/q \in (0,1)$ *with* p, q *coprime such that* $\mathrm{Per}(p/q, F) \neq q\mathbb{N}$.

Let $F \in \mathcal{C}_1(T)$ be the affine Markov map represented in Figure 6.2. By Proposition 6.2(a), F is transitive (mod 1).

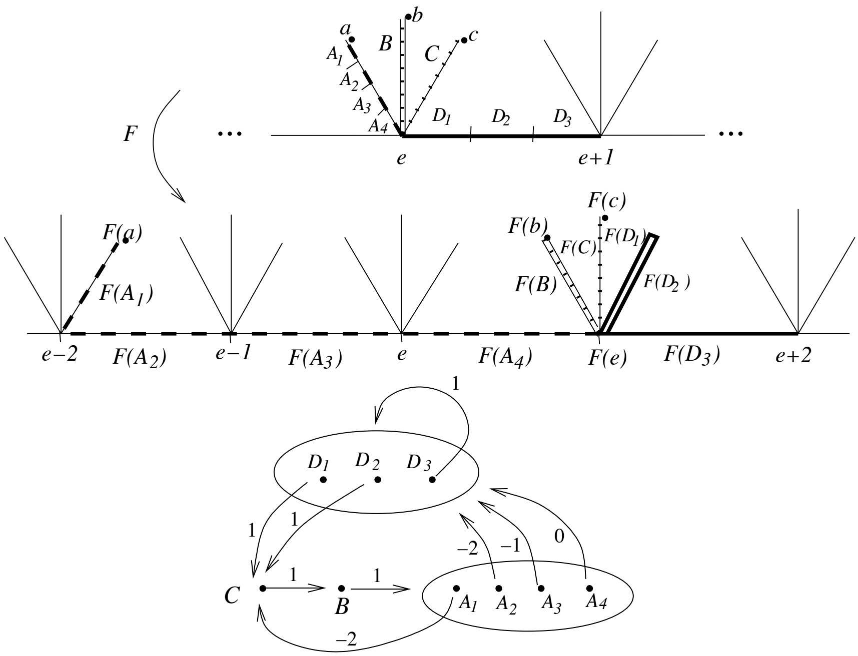

Figure 6.2. In the picture above the affine Markov map F*. Below it is displayed its Markov graph. In this example* $\mathrm{Rot}(F) = [0,1]$, $\bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F_n^r) = (0,1]$ *is not closed and there exist infinitely many* $p/q \in (0,1)$ *with* p, q *coprime such that* $\mathrm{Per}(p/q, F) \neq q\mathbb{N}$.

We define the loops

$$\begin{aligned}\mathcal{A} :=& C \xrightarrow{1} B \xrightarrow{1} A_1 \xrightarrow{-2} C,\\ \mathcal{B}_\varepsilon :=& C \xrightarrow{1} B \xrightarrow{1} A \xrightarrow{\varepsilon} D_2 \xrightarrow{1} C, \text{ and}\\ \mathcal{D} :=& D_3 \xrightarrow{1} D_3,\end{aligned}$$

where $\varepsilon \in \{0, -1, -2\}$ and A represents either A_2, A_3 or A_4 depending on ε.

The weights of $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{B}_\varepsilon$ are respectively 0, 1 and $3+\varepsilon$. Modifying $\mathcal{D}$ into $D_2 \xrightarrow{1} D_3$ and $\mathcal{B}_\varepsilon$ into $D_3 \xrightarrow{1} C \xrightarrow{1} B \xrightarrow{1} A \xrightarrow{\varepsilon} D_2$, we can concatenate them. For short, we will write $\mathcal{D}\mathcal{B}_\varepsilon$ as the concatenated loop.

The only periodic (mod 1) points which are endpoints of intervals of the Markov partition are e (mod 1), with $F(e) = e+1$, and a, b, c, that form a periodic (mod 1) orbit of period 3 and rotation number 0. They correspond respectively to the loops $\mathcal{D}$ and $\mathcal{A}$. Therefore, by Proposition 6.2(c,d), there is a correspondence between periodic (mod 1) points of periods p and simple loops of length p.

The loops $\mathcal{A}$ and $\mathcal{D}$ correspond respectively to $\min \frac{W(\mathcal{E})}{L(\mathcal{E})}$ and $\max \frac{W(\mathcal{E})}{L(\mathcal{E})}$, where $\mathcal{E}$ describes the set of elementary loops. Thus $\mathrm{Rot}(F) = \mathrm{Rot}_{\mathbb{R}}(F) = [0,1]$ by Lemma 6.4. According to Theorem 3.1, $(0,1) \subset \bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F_n^r)$. The only simple loop of weight 0 is $\mathcal{A}$. It is the periodic itinerary of c, which is of period 3, and thus $\mathrm{Per}(0,F) = \{3\}$. Moreover, $F^n(c) \notin \mathbb{R}$ for all $n \geqslant 0$. Thus $0 \notin \mathrm{Rot}(F_n^r)$ by Theorem 1.15. The only simple loop $\mathcal{L}$ with $W(\mathcal{L})/L(\mathcal{L}) = 1$ is $\mathcal{D}$. It is the periodic itinerary of $e+1 \in \mathbb{R}$, and $F(e+1) = (e+2)$. Thus $\mathrm{Per}(1,F) = \{1\}$ and $\bigcup_{n\geqslant 1} \frac{1}{n}\mathrm{Rot}(F_n^r) = (0,1]$.

We are going to compute $\mathrm{Per}(p/q, F)$ for all $p/q \in (0,1)$, p, q coprime. The final results are given in Table 6.1.

		$\mathrm{Per}(p/q,F)$
$p=1$	$q \equiv 0 \bmod 3$	$\{nq : n \geqslant 3\}$
	$q \equiv 1 \bmod 3$	$q\mathbb{N}$
	$q \equiv 2 \bmod 3$	$\{nq : n \geqslant 2\}$
$p=2$	$q \equiv 0 \bmod 3$	$\{nq : n \geqslant 2\}$
	$q \equiv 1,2 \bmod 3$	$q\mathbb{N}$
$p \geqslant 3$		$q\mathbb{N}$

Table 6.1. Values of $\mathrm{Per}(p/q,F)$ *for* $p/q \in (0,1)$ *and* p,q *coprime.*

• The only loops of weight 1 are $\mathcal{D}$ (length 1) and $\mathcal{B}_{-2}\mathcal{A}^k$ (which is of length $4+3k$), for all $k \geqslant 0$. Thus there exists a periodic (mod 1) point of period n and rotation number $1/n$ if and only if $n \equiv 1 \bmod 3$.

• The only simple loops of weight 2 are $\mathcal{B}_{-2}^2\mathcal{A}^k$ (length $8+3k$), $\mathcal{B}_{-1}\mathcal{A}^k$ (length $4+3k$), and $\mathcal{B}_{-2}\mathcal{D}\mathcal{A}^k$ (length $5+3k$), for all $k \geqslant 0$. Thus there exists a periodic (mod 1) point of period n and rotation number $2/n$ if and only if $n \equiv 1$ or $2 \bmod 3$ and $n \geqslant 4$.

• Considering the simple loops $\mathcal{B}_0\mathcal{A}^k$, $\mathcal{D}\mathcal{B}_{-1}\mathcal{A}^k$ and $\mathcal{D}^2\mathcal{B}_{-2}\mathcal{A}^k$ of weight 3, we see that, for all $n \geqslant 4$, there exists a periodic (mod 1) point of period n and rotation number $3/n$. For all $n \geqslant 4$, we call $\mathcal{L}_n$ the loop of length n among the above loops. We notice that $\mathcal{L}_n$ passes through D_2.

• If $m \geqslant 4$ and $n > m$, then $n - m + 3 \geqslant 4$. The loop $\mathcal{L}_{(n-m+3)}\mathcal{D}^{m-3}$ is of length n and weight m, and thus it gives a periodic (mod 1) point of period n and rotation number m/n. This completes Table 6.1.

This example shows that there may exist infinitely many rationals p/q, with p and q coprime, in the interior of the rotation interval such that $\mathrm{Per}(p/q, F) \neq q\mathbb{N}$, and the integer N of Theorem 3.11 cannot be taken the same for the whole interval $\mathrm{Rot}_{\mathbb{R}}(F)$. Moreover, the interval

$$\bigcup_{n\geqslant 1} \tfrac{1}{n}\,\mathrm{Rot}(F_n^r)$$

may not be closed and, if $0 \in \partial\,\mathrm{Rot}_{\mathbb{R}}(F)$, there may not exist a periodic (mod 1) point $x \in \mathbb{R}$ with $\rho_F(x) = 0$ (although there exists $x \in T$ with this property).

Compare also this situation with the one for combed maps. In view of Theorem 4.16, the rotation interval of a combed map is a closed interval and coincides with $\mathrm{Rot}(F_1^r)$. Moreover, in view of Theorem 4.17, $\mathrm{Per}(p/q, F) = q\mathbb{N}$ for every $p/q \in \mathrm{Int}(\mathrm{Rot}(F))$ with p, q coprime. This example shows that both statements can fail for a non combed map lifted graph map.

Example 6.7. — $\mathrm{Rot}_{\mathbb{R}}(F) = [0,1]$ *but there is no periodic* (mod 1) *point* $x \in T$ *such that* $\rho_F(x) = 0$.

We define T as the following subset of $\mathbb{R}^3$:

$$T = \{(x,0,0) : x \in \mathbb{R}\} \cup \{((n,y,z) : n \in \mathbb{Z} \text{ and } y^2 + z^2 \leqslant 1\}.$$

Clearly, $T \in \mathbf{T}$. To be able to define a map F on T we will identify the z, y-plane with $\mathbb{C}$ taking the y axis as the real axis in $\mathbb{C}$. Then we define the

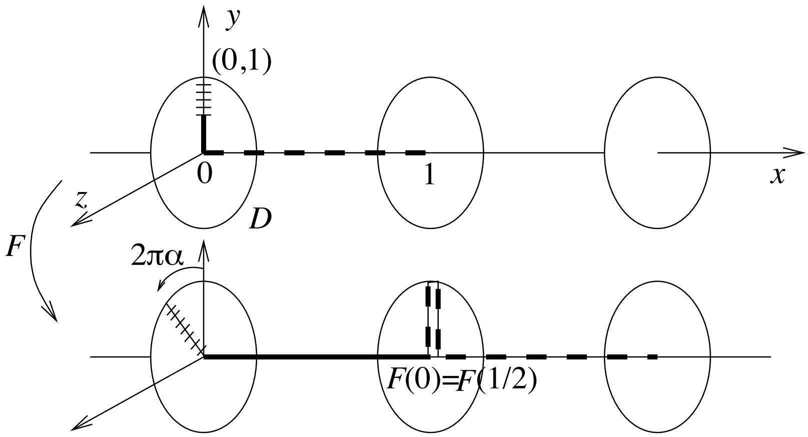

Figure 6.3. $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$, $\mathrm{Rot}_{\mathbb{R}}(F) = [0,1]$ *but there is no periodic* (mod 1) *point* $x \in T$ *such that* $\rho_F(x) = 0$.

sets

$$D = \{z \in \mathbb{C} : |z| \leqslant 1\} \text{ and}$$
$$\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}.$$

We identify the x-axis with $\mathbb{R}$ and we denote $D + (n,0,0)$ by $D + n$ when $n \in \mathbb{Z}$. Note that to define a map $F \in \mathcal{C}_1(T)$ it is enough to define it on $D \cup (x,0,0)$ with $x \in [0,1]$ and extend the definition to the whole T by using that $F(z+1)$ must be $F(z)+1$. Thus, we construct our map by choosing $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ defining (see Figure 6.3 for a representation of T and F):

(1) If $z \in D$ with $\frac{1}{2} \leqslant |z| \leqslant 1$ then $F(z) = z' \in D$ with $|z'| = 2|z| - 1 \in [0,1]$ and $\arg(z') = \arg(z) + 2\pi\alpha$.
(2) If $z \in D$ with $0 \leqslant |z| \leqslant \frac{1}{2}$ then $F(z) = (1 - 2|z|, 0, 0) \in \mathbb{R}$.
(3) If $x \in [0, 1/2]$, $F(x,0,0) = (1, 1 - 4|x - 1/4|, 0)$.
(4) if $x \in [1/2, 1]$, $F(x,0,0) = (2x, 0, 0)$.

The map $F\big|_{\mathcal{C}}$ is the rotation of angle $2\pi\alpha$.

We are going to show that $\mathrm{Rot}_{\mathbb{R}}(F) = [0,1]$ but there is no periodic (mod 1) point $x \in T$ such that $\rho_F(x) = 0$.

It is clear that, if $x \in D + \mathbb{Z}$, then $r(x) \leqslant r(F(x)) \leqslant r(x) + 1$. And, if $x \in \mathbb{R}$, then

$$r(x) + \tfrac{1}{2} \leqslant r(F(x)) \leqslant r(x) + 1.$$

Hence $\mathrm{Rot}(F) \subset [0,1]$. Moreover $F(0,0,0) = (1,0,0)$ and

$$F^{n+1}(1/4,0,0) = F^n(1,1,0) = (1, e^{i2\pi n\alpha}, 0).$$

Thus $\rho_F(0,0,0) = 1$, $\rho_F(1/4,0,0) = 0$ and $\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F) = [0,1]$ by Theorem 3.1.

Suppose that $x \in T$ is a periodic (mod 1) point such that $\rho_F(x) = 0$. Because of the properties stated above, x cannot belong to $\mathbb{R}$, and there exists $k \in \mathbb{Z}$ such that $F^n(x) \in D + k$ for all $n \geqslant 0$. By definition of $F|_D$, the point $x = (k,z)$ with $k \in \mathbb{Z}$ and $z \in D$ must belong to $\mathcal{C} + k$. Thus $F^n(x) = (k, ze^{i2\pi n\alpha}) \neq x$ for all $n \geqslant 1$. This is a contradiction and, hence, $\mathrm{Per}(0,F) = \emptyset$.

BIBLIOGRAPHY

[1] L. Alsedà, D. Juher & P. Mumbrú, "Sets of periods for piecewise monotone tree maps", *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (2003), no. 2, p. 311-341.

[2] ———, "On the preservation of combinatorial types for maps on trees", *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 7, p. 2375-2398.

[3] ———, "Periodic behavior on trees", *Ergodic Theory Dynam. Systems* **25** (2005), no. 5, p. 1373-1400.

[4] ———, "Minimal dynamics for tree maps", *Discrete Contin. Dyn. Syst. Ser. A* **3** (2006), no. 20, p. 511-541.

[5] L. Alsedà, J. Llibre & M. Misiurewicz, "Periodic orbits of maps of Y", *Trans. Amer. Math. Soc.* **313** (1989), no. 2, p. 475-538.

[6] ———, *Combinatorial dynamics and entropy in dimension one*, Advanced Series in Nonlinear Dynamics, 5, World Scientific Publishing Co. Inc., River Edge, NJ, 1993.

[7] L. Alsedà, F. Mañosas & P. Mumbrú, "Minimizing topological entropy for continuous maps on graphs", *Ergodic Theory Dynam. Systems* **20** (2000), no. 6, p. 1559-1576.

[8] S. Baldwin, "An extension of Sharkovskiĭ's theorem to the n-od", *Ergodic Theory Dynam. Systems* **11** (1991), no. 2, p. 249-271.

[9] S. Baldwin & J. Llibre, "Periods of maps on trees with all branching points fixed", *Ergodic Theory Dynam. Systems* **15** (1995), no. 2, p. 239-246.

[10] C. Bernhardt, "Vertex maps for trees: algebra and periods of periodic orbits", *Discrete Contin. Dyn. Syst.* **14** (2006), no. 3, p. 399-408.

[11] L. Block, "Homoclinic points of mappings of the interval", *Proc. Amer. Math. Soc.* **72** (1978), no. 3, p. 576-580.

[12] L. Block, J. Guckenheimer, M. Misiurewicz & L. S. Young, "Periodic points and topological entropy of one dimensional maps", in *Global Theory of Dynamical Systems*, Lecture Notes in Mathematics, no. 819, Springer-Verlag, 1980, p. 18-34.

[13] R. Ito, "Rotation sets are closed", *Math. Proc. Cambridge Philos. Soc.* **89** (1981), no. 1, p. 107-111.

[14] M. C. Leseduarte & J. Llibre, "On the set of periods for σ maps", *Trans. Amer. Math. Soc.* **347** (1995), no. 12, p. 4899-4942.

[15] J. Llibre, J. Paraños & J. A. Rodríguez, "Periods for continuous self-maps of the figure-eight space", *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (2003), no. 7, p. 1743-1754, Dynamical systems and functional equations (Murcia, 2000).

[16] M. Misiurewicz, "Periodic points of maps of degree one of a circle", *Ergodic Theory Dynamical Systems* **2** (1982), no. 2, p. 221-227 (1983).

[17] F. Rhodes & C. L. Thompson, "Rotation numbers for monotone functions on the circle", *J. London Math. Soc. (2)* **34** (1986), no. 2, p. 360-368.

[18] A. N. Sharkovskiĭ, "Co-existence of cycles of a continuous mapping of the line into itself", *Ukrain. Mat. Ž.* **16** (1964), p. 61-71 (in Russian).

[19] ———, "Coexistence of cycles of a continuous map of the line into itself", in *Thirty years after Sharkovskiĭ's theorem: new perspectives (Murcia, 1994)*, World Sci. Ser. Nonlinear Sci. Ser. B Spec. Theme Issues Proc., vol. 8, World Sci. Publ., River Edge, NJ, 1995, Translated by J. Tolosa, Reprint of the paper reviewed in MR1361914 (96j:58058), p. 1-11.

[20] C. T. C. Wall, *A geometric introduction to topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1972, vii+168 pages.

[21] F. Zeng, H. Mo, W. Guo & Q. Gao, "ω-limit set of a tree map", *Northeast. Math. J.* **17** (2001), no. 3, p. 333-339.

[22] K. Ziemian, "Rotation sets for subshifts of finite type", *Fund. Math.* **146** (1995), no. 2, p. 189-201.

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Lluís ALSEDÀ
Universitat Autònoma de Barcelona
Departament de Matemàtiques
08913 Cerdanyola del Vallès, Barcelona (Spain)
alseda@mat.uab.cat

Sylvie RUETTE
Université Paris-Sud 11
Laboratoire de Mathématiques
CNRS UMR 8628
Bâtiment 425
91405 Orsay cedex (France)
sylvie.ruette@math.u-psud.fr