ANNALES

DE

L'INSTITUT FOURIER

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Tome 58, nº 3 (2008), p. 989-1022.

<http://aif.cedram.org/item?id=AIF_2008__58_3_989_0>

Ann. Inst. Fourier, Grenoble
58, 3 (2008) 989-1022

HOMOLOGY CLASSES OF REAL ALGEBRAIC SETS

by Wojciech KUCHARZ

Abstract. — There is a large research program focused on comparison between algebraic and topological categories, whose origins go back to 1952 and the celebrated work of J. Nash on real algebraic manifolds. The present paper is a contribution to this program. It investigates the homology and cohomology classes represented by real algebraic sets. In particular, such classes are studied on algebraic models of smooth manifolds.

Résumé. — Il existe un vaste programme de recherche portant sur la comparaison entre catégories topologiques et algébriques, dont l'origine remonte à 1952 avec les travaux célèbres de J. Nash sur les variétés algébriques réelles lisses. Ce papier est une contribution à ce programme. Il contient l'étude des classes d'homologie et de cohomologie représentées par des ensembles algébriques réels. En particulier, de telles classes sont étudiées dans les modèles algébriques de variétés lisses.

1. Introduction and main results

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions (in [12] such objects are called affine real algebraic varieties). By convention, subvarieties are assumed to be closed in the Zariski topology. Morphisms between real algebraic varieties will be called *regular maps*. Basic facts on real algebraic varieties and regular maps can be found in [12]. Every real algebraic variety carries also the Euclidean topology, which is determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given a compact real algebraic variety X (as in [5, 12], nonsingular means that the irreducible components of X are pairwise disjoint, nonsingular

Keywords: Real algebraic variety, algebraic cycles, cohomology.
Math. classification: 14P05, 14P25, 14C25, 14F25.

and of the same dimension), we denote by $H_p^{\mathrm{alg}}(X,\mathbb{Z}/2)$ the subgroup of the homology group $H_p(X,\mathbb{Z}/2)$ generated by the homology classes of p-dimensional subvarieties of X, cf. [5, 11, 12, 16, 17]. For technical reasons it is advantageous to work with cohomology rather than homology. We let $H^q_{\mathrm{alg}}(X,\mathbb{Z}/2)$ denote the inverse image of $H_p^{\mathrm{alg}}(X,\mathbb{Z}/2)$ under the Poincaré duality isomorphism $H^q(X,\mathbb{Z}/2)\to H_p(X,\mathbb{Z}/2)$, where $p+q=\dim X$. The groups $H^q_{\mathrm{alg}}(-,\mathbb{Z}/2)$ of algebraic cohomology classes play the central role in real algebraic geometry [3, 4, 5, 6, 8, 10, 9, 11, 12, 13, 14, 23, 30, 32, 39] (cf. [16] for a short survey of their properties and applications). They have the expected functorial property: if $f: X\to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphism $f^*: H^q(Y,\mathbb{Z}/2)\to H^q(X,\mathbb{Z}/2)$ satisfies

$$f^*(H^q_{\mathrm{alg}}(Y,\mathbb{Z}/2))\subseteq H^q_{\mathrm{alg}}(X,\mathbb{Z}/2).$$

Furthermore, $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2)=\bigoplus_{q\geqslant 0}H^q_{\mathrm{alg}}(X,\mathbb{Z}/2)$ is a subring of the cohomology ring $H^*(X,\mathbb{Z}/2)$. The qth Stiefel-Whitney class $w_q(X)$ of X is in $H^q_{\mathrm{alg}}(X,\mathbb{Z}/2)$ for all $q\geqslant 0$.

Recently a certain subgroup of $H^q_{\mathrm{alg}}(X,\mathbb{Z}/2)$, defined below, proved to be very useful. A cohomology class u in $H^q_{\mathrm{alg}}(X,\mathbb{Z}/2)$ is said to be *algebraically equivalent to* 0 if there exist a compact irreducible nonsingular real algebraic variety T, two points t_0 and t_1 in T, and a cohomology class z in $H^q_{\mathrm{alg}}(X\times T,\mathbb{Z}/2)$ such that $u=i^*_{t_1}(z)-i^*_{t_0}(z)$, where given t in T, we let $i_t: X\to X\times T$ denote the map defined by $i_t(x)=(x,t)$ for all x in X (note analogy with the definition of an algebraic cycle algebraically equivalent to 0 [21, Chapter 10]). The subset $\mathrm{Alg}^q(X)$ of $H^q_{\mathrm{alg}}(X,\mathbb{Z}/2)$ consisting of all elements algebraically equivalent to 0 forms a subgroup [32, p. 114], which is often highly nontrivial [1, 29, 32, 33]. It allows to detect transcendental cohomology classes: the quotient group $H^p(X,\mathbb{Z}/2)/H^p_{\mathrm{alg}}(X,\mathbb{Z}/2)$ maps homomorphically onto $\mathrm{Alg}^q(X)$, where $p+q=\dim X$, cf. [29, Theorem 2.1] or Theorem 4.1(i) in this paper. Some substantial constructions in [32], at the borderline between real algebraic geometry and differential topology, depend on $\mathrm{Alg}^q(-)$. It was R. Silhol [38] who first demonstrated that $\mathrm{Alg}^1(-)$ is important for understanding of $H^1_{\mathrm{alg}}(-,\mathbb{Z}/2)$. In [31] it is proved, among other things, that $\mathrm{Alg}^1(-)$ is a birational invariant (while, obviously, $H^1_{\mathrm{alg}}(-,\mathbb{Z}/2)$ is not). For $f: X\to Y$ as above,

$$f^*(\mathrm{Alg}^q(Y))\subseteq \mathrm{Alg}^q(X).$$

Moreover, $\mathrm{Alg}^*(X) = \bigoplus_{q\geqslant 0} \mathrm{Alg}^q(X)$ is an ideal in the ring $H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$. These last two assertions readily follow from the definition, cf. [32, pp. 114, 115].

The basic properties, listed above, of $H^*_{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $\mathrm{Alg}^*(-)$ will be used without further comments. An alternative description of $H^*_{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $\mathrm{Alg}^*(-)$, relating these groups to algebraic cycles on schemes over $\mathbb{R}$, is given in Section 3.

We will first deal with the groups $H^1_{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $\mathrm{Alg}^1(-)$, for which we have a quite general Noether-Lefschetz type theorem (Theorem 1.4).

Notation. — Unless stated to the contrary, in the remainder of this section, X will denote a compact irreducible nonsingular real algebraic variety.

Definition 1.1. — *Given a nonsingular subvariety Y of X, the groups $H^1_{\mathrm{alg}}(Y, \mathbb{Z}/2)$ and $\mathrm{Alg}^1(Y)$ are said to be determined by X if*

$$H^1_{\mathrm{alg}}(Y, \mathbb{Z}/2) = i^*(H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)) \text{ and } \mathrm{Alg}^1(Y) = i^*(\mathrm{Alg}^1(X)),$$

where $i : Y \hookrightarrow X$ is the inclusion map.

In general it is hard to decide whether or not we have the desirable situation described in Definition 1.1, unless Y is allowed to "move" in X. This is made precise below.

We say that a subset Σ of $\mathbb{R}^k$ is *thin* if it is contained in the union of a countable family of proper subvarieties of $\mathbb{R}^k$. In particular, $\mathbb{R}^k \backslash \Sigma$ is dense in $\mathbb{R}^k$, provided Σ is thin.

Definition 1.2. — *A nonsingular subvariety Y of X is said to be movable if there exist a positive integer k, a nonsingular subvariety Z of $X \times \mathbb{R}^k$, and a thin subset Σ of $\mathbb{R}^k$ such that the family $\{Y_t\}_{t\in\mathbb{R}^k}$ of subvarieties of X defined by*

$$Y_t \times \{t\} = (X \times \{t\}) \cap Z$$

has the following properties:

(i) *$X \times \{0\}$ is transverse to Z in $X \times \mathbb{R}^k$ and $Y_0 = Y$,*

(ii) *if t is in $\mathbb{R}^k \backslash \Sigma$, then $X \times \{t\}$ is transverse to Z in $X \times \mathbb{R}^k$ and either $Y_t = \emptyset$ or else Y_t is irreducible and nonsingular with*

$$H^1_{\mathrm{alg}}(Y_t, \mathbb{Z}/2) = i_t^*(H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)),\ \mathrm{Alg}^1(Y_t) = i_t^*(\mathrm{Alg}^1(X)),$$

where $i_t : Y_t \hookrightarrow X$ is the inclusion map.

Roughly speaking, Definition 1.2 means that Y "moves" in the family $\{Y_t\}_{t\in\mathbb{R}^k}$, and for general t, the subvariety Y_t of X is irreducible and nonsingular, with the groups $H^1_{\mathrm{alg}}(Y_t, \mathbb{Z}/2)$ and $\mathrm{Alg}^1(Y_t)$ determined by X.

Denote by $\mathrm{Diff}(X)$ the space of all smooth (that is, $\mathcal{C}^\infty$) diffeomorphisms of X endowed with the $\mathcal{C}^\infty$ topology. We wish to emphasize the following straightforward consequence of Definition 1.2.

Proposition 1.3. — *With notation as in Definition 1.2, for any neighborhood $\mathcal{U}$ of the identity map in $\mathrm{Diff}(X)$, there exists a neighborhood U of 0 in $\mathbb{R}^k$ such that for each t in $U \backslash \Sigma$, there is a diffeomorphism φ_t in $\mathcal{U}$ satisfying $\varphi_t(Y) = Y_t$.*

Proof. — Given t in $\mathbb{R}^k$, let $j_t : X \to X \times \mathbb{R}^k$ be defined by $j_t(x) = (x,t)$ for all x in X. Note that j_t is transverse to Z for $t = 0$ and for all t in $\mathbb{R}^k \backslash \Sigma$. The proof is complete since $Y_t = j_t^{-1}(Z)$, cf. [2, Theorem 20.2]. □

Our first result asserts that movable subvarieties of X occur in a natural way.

Theorem 1.4. — *Let ξ be an algebraic vector bundle on X with $2 + \operatorname{rank}\xi \leqslant \dim X$. If $s : X \to \xi$ is an algebraic section transverse to the zero section, then the nonsingular subvariety $Y = s^{-1}(0)$ of X is movable.*

Here, as in [12], an algebraic vector bundle on X is, by definition, isomorphic to an algebraic subbundle of the trivial vector bundle $X \times \mathbb{R}^\ell$ for some ℓ (such an object is called a strongly algebraic vector bundle in the earlier literature [10, 9, 11, 13, 14, 44]). Of course, $s^{-1}(0) = \{x \in X \mid s(x) = 0\}$. Theorem 1.4 will be proved in Section 3, whereas now we will derive some consequences.

By an algebraic hypersurface in X we mean an algebraic subvariety of pure codimension 1.

Corollary 1.5. — *Let $Y = Y_1 \cap \ldots \cap Y_c$, where $Y_1, \ldots, Y_c$ are nonsingular algebraic hypersurfaces in X that are in general position (when regarded as smooth submanifolds of X) at each point of Y. If $\dim Y \geqslant 2$, then Y is movable.*

Proof. — It is well known that there are an algebraic line bundle ξ_i on X and an algebraic section $s_i : X \to \xi_i$ such that $Y_i = s_i^{-1}(0)$ and s_i is transverse to the zero section, $1 \leqslant i \leqslant c$, cf. [12, Remarks 12.2.5 and 12.4.3]. Then $Y = s^{-1}(0)$, where $s = s_1 \oplus \cdots \oplus s_c$ is an algebraic section of $\xi_1 \oplus \cdots \oplus \xi_c$. Since s is transverse to the zero section, the conclusion follows from Theorem 1.4. □

We will now examine the problem under consideration from a slightly different point of view. All manifolds in this paper will be without boundary. Submanifolds will be closed subsets of the ambient manifold. Given a compact smooth manifold N, we denote by $[N]$ its fundamental class in $H_n(N,\mathbb{Z}/2)$, $n = \dim N$. If N is a submanifold of a compact smooth manifold M, we write $[N]^M$ for the cohomology class in $H^k(M,\mathbb{Z}/2)$, $k = \dim M - \dim N$, Poincaré dual to the image of $[N]$ under the homomorphism $H_n(N,\mathbb{Z}/2) \to H_n(M,\mathbb{Z}/2)$ induced by the inclusion map $N \hookrightarrow M$.

Definition 1.6. — *A smooth submanifold M of X is said to be admissible if for any neighborhood $\mathcal{U}$ of the identity map in* $\mathrm{Diff}(X)$, *there exists a diffeomorphism φ in $\mathcal{U}$ such that $Y = \varphi(M)$ is an irreducible nonsingular subvariety of X, with the groups $H^1_{\mathrm{alg}}(Y,\mathbb{Z}/2)$ and $\mathrm{Alg}^1(Y)$ determined by X.*

Corollary 1.7. — *Let ξ be an algebraic vector bundle on X with $2+\mathrm{rank}\xi \leqslant \dim X$. If $\sigma : X \to \xi$ is a smooth section transverse to the zero section, then the smooth submanifold $M = \sigma^{-1}(0)$ of X is admissible.*

Proof. — By [12, Theorem 12.3.2], there exists an algebraic section $s : X \to \xi$ arbitrarily close to σ in the $\mathcal{C}^\infty$ topology. Hence there is a diffeomorphism ψ in $\mathrm{Diff}(X)$, close to the identity map, such that $\psi(M) = s^{-1}(0)$, cf. [2, Theorem 20.2]. The conclusion follows in view of Theorem 1.4. and Proposition 1.3. □

Corollary 1.8. — *Let $M = M_1 \cap \ldots \cap M_c$, where $M_1, \ldots, M_c$ are smooth hypersurfaces in X that are in general position at each point of M. If $\dim M \geqslant 2$ and the cohomology class $[M_i]^X$ belongs to $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$ for $1 \leqslant i \leqslant c$, then M is admissible.*

Proof. — There exist a smooth line bundle ξ_i on X and a smooth section $\sigma_i : X \to \xi_i$ such that $M_i = \sigma_i^{-1}(0)$ and σ_i is transverse to the zero section, cf. for example [12, Remark 12.4.3]. Since $[M_i]^X$ belongs to $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$, we may assume that ξ_i is an algebraic line bundle on X, cf. [12, Theorem 12.4.6]. Then $M = \sigma^{-1}(0)$, where $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_c$ is a smooth section of $\xi_1 \oplus \cdots \oplus \xi_c$. Since σ is transverse to the zero section, the proof is complete in virtue of Corollary 1.7. □

Given an arbitrary nonsingular subvariety Y of X, what relationships are there between the following triples of groups:

$$(H^1(X,\mathbb{Z}/2),\ H^1_{\mathrm{alg}}(X,\mathbb{Z}/2),\ \mathrm{Alg}^1(X))$$

$$\text{and } (H^1(Y,\mathbb{Z}/2),\ H^1_{\mathrm{alg}}(Y,\mathbb{Z}/2),\ \mathrm{Alg}^1(Y))?$$

Our next theorem provides a complete answer to this question for X and Y connected with $\dim X > \dim Y \geqslant 3$, assuming that no additional algebraic geometric conditions are imposed on X and Y. First we need some preparation.

For any smooth manifold P, we let

$$SW^*(P) = \underset{k\geqslant 0}{\oplus} SW^k(P)$$

denote the graded subring of the cohomology ring $H^*(P,\mathbb{Z}/2)$ generated by the Stiefel-Whitney classes of P. More generally, if $E_1,\dots,E_r$ are subsets of $H^*(P,\mathbb{Z}/2)$, write

$$SW^*(P;E_1,\dots,E_r) = \underset{k\geqslant 0}{\oplus} SW^k(P;E_1,\dots,E_r)$$

for the graded subring of the cohomology ring $H^*(P,\mathbb{Z}/2)$ generated by the Stiefel-Whitney classes of P and the union of the $E_1,\dots,E_r$. Let

$$\rho_P : H^*(P,\mathbb{Z}) \to H^*(P,\mathbb{Z}/2)$$

denote the reduction modulo 2 homomorphism. As usual, we will use $\cup$ and $\langle\ ,\ \rangle$ to denote the cup product and scalar (Kronecker) product.

Theorem 1.9. — *Let M be a compact connected smooth manifold and let N be a connected smooth submanifold of M, with* $\dim M = m > \dim N = n \geqslant 3$. *Given subgroups $\Gamma_M \subseteq G_M$ of $H^1(M,\mathbb{Z}/2)$ and $\Gamma_N \subseteq G_N$ of $H^1(N,\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist a nonsingular real algebraic variety X, a nonsingular subvariety Y of X, and a smooth diffeomorphism $\varphi : X \to M$ such that $\varphi(Y) = N$ and*

$$\varphi^*(G_M) = H^1_{\mathrm{alg}}(X,\mathbb{Z}/2),\ \varphi^*(\Gamma_M) = \mathrm{Alg}^1(X),$$
$$\psi^*(G_N) = H^1_{\mathrm{alg}}(Y,\mathbb{Z}/2),\ \psi^*(\Gamma_N) = \mathrm{Alg}^1(Y),$$

where $\psi : Y \to N$ is the restriction of φ.

(b) $w_1(M) \in G_M$, $w_1(N) \in G_N$, $\Gamma_M \subseteq \rho_M(H^1(M,\mathbb{Z}))$, $\Gamma_N \subseteq \rho_N(H^1(N,\mathbb{Z}))$, $e^*(G_M) \subseteq G_N$, $e^*(\Gamma_M) \subseteq \Gamma_N$, *where $e : N \hookrightarrow M$ is the inclusion map, and*

(b$_1$) $\langle a\cup w,[M]\rangle = 0$ *for all* $a \in \Gamma_M$, $w \in SW^{m-1}(M;G_M)$,

(b$_2$) $\langle b\cup z,[N]\rangle = 0$ *for all* $b\in\Gamma_N$, $z\in SW^{n-1}(N;G_N,e^*(SW^*(M)))$.

Furthermore, if $m-n=1$, the cohomology class $[N]^M$ belongs to G_M.

Theorem 1.9 will be proved in Section 4. Although the groups $H^k_{\mathrm{alg}}(-,\mathbb{Z}/2)$ and $\mathrm{Alg}^k(-)$, with $k \geqslant 2$, do not appear in the statement of this theorem, they play a crucial role in its proof, which is rather long and involved. Perhaps it is useful to note here that condition (b) becomes less complicated if M and N are stably parallelizable, so that all their Stiefel-Whitney classes are trivial.

If one is interested only in $H^1_{\mathrm{alg}}(-,\mathbb{Z}/2)$ and ignores $\mathrm{Alg}^1(-)$, then Theorem 1.9 can be significantly simplified.

Corollary 1.10. — *Let M be a compact connected smooth manifold and let N be a connected smooth submanifold of M, with $\dim M = m > \dim N = n \geqslant 3$. Given subgroups G_M of $H^1(M,\mathbb{Z}/2)$ and G_N of $H^1(N,\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist a nonsingular real algebraic variety X, a nonsingular subvariety Y of X, and a smooth diffeomorphism $\varphi : X \to M$ such that $\varphi(Y) = N$ and*

$$\varphi^*(G_M) = H^1_{\mathrm{alg}}(X,\mathbb{Z}/2),\ \psi^*(G_N) = H^1_{\mathrm{alg}}(Y,\mathbb{Z}/2)$$

where $\psi : Y \to N$ is the restriction of φ.

(b) $w_1(M) \in G_M,\ w_1(N) \in G_N,$ and $G_N \subseteq e^*(G_M)$, *where $e : N \hookrightarrow M$ is the inclusion map. Moreover, if $m - n = 1$, the cohomology class $[N]^M$ belongs to G_M.*

Proof. — It suffices to apply Theorem 1.9 with $\Gamma_M = 0$ and $\Gamma_N = 0$. □

It is plausible that in Theorem 1.9 and Corollary 1.10 the assumption $\dim N \geqslant 3$ can be replaced by $\dim N \geqslant 2$, but our technique does not allow us to do it.

Theorem 1.11. — *Let N be a compact connected smooth manifold of dimension $n \geqslant 2$. Given subgroups $\Gamma \subseteq G$ of $H^1(N,\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist a nonsingular real algebraic variety Y and a smooth diffeomorphism $\psi : Y \to N$ such that*

$$\psi^*(G) = H^1_{\mathrm{alg}}(Y,\mathbb{Z}/2) \text{ and } \psi^*(\Gamma) = \mathrm{Alg}^1(Y).$$

(b) $w_1(N) \in G,\ \Gamma \subseteq \rho_N(H^1(N,\mathbb{Z}))$, *and for all nonnegative integers $k, \ell, i_1, \dots, i_r$ with $\ell \geqslant 1,\ k + \ell + i_1 + \cdots + i_r = n$, one has*

$$\langle u_1 \cup \ldots \cup u_k \cup v_1 \cup \ldots \cup v_\ell \cup w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N]\rangle = 0$$

for all $u_1, \dots, u_k$ in G and $v_1, \dots, v_\ell$ in Γ.

We postpone the proof of Theorem 1.11 to Section 4. The case $\dim N = 2$ requires special care.

Corollary 1.12. — *Let N be a compact connected smooth manifold of dimension $n \geqslant 2$. Given a subgroup G of $H^1(N,\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist a nonsingular real algebraic variety Y and a smooth diffeomorphism $\psi : Y \to N$ such that*

$$\psi^*(G) = H^1_{\mathrm{alg}}(Y, \mathbb{Z}/2).$$

(b) $w_1(N) \in G$.

Proof. — It suffices to take $\Gamma = 0$ in Theorem 1.11. □

For $\dim N \geqslant 3$ a different proof of Corollary 1.12 can be found in [13, Theorem 1.3]. However, for $\dim N = 2$ only a much weaker result has been known until now [13, Theorem 1.4].

Theorems 1.9 and 1.11 together with Corollaries 1.10 and 1.12 are examples of results belonging to a large research program focused on comparison between algebraic and topological categories. The origins of this program go back to 1973, when A. Tognoli [43], improving upon an earlier work of J. Nash [36], demonstrated that every compact smooth manifold M has an algebraic model, that is, M is diffeomorphic to a nonsingular real algebraic variety. This fundamental theorem has several important generalizations, which allow to realize algebraically not only M alone, but also some objects attached to it, such as submanifolds, vector bundles, certain homology or cohomology classes, etc. [3, 4, 10, 9, 11, 44]. It came as a surprise when R. Benedetti and M. Dedò [8] found a compact smooth manifold, whose each algebraic model has $H^2_{\mathrm{alg}}(-, \mathbb{Z}/2) \neq H^2(-, \mathbb{Z}/2)$. In particular, [8] provided a counterexample to a conjecture of S. Akbulut and H. King [4] that was to be a major step towards a topological characterization of all real algebraic sets. Below we give a generalization of the main result of [8], based on a simple obstruction discovered in a later paper [6]. Although our generalization is easy to prove, it has not been noticed heretofore.

Theorem 1.13. — *Let k be a positive even integer. For any integer m with $m \geqslant 2k + 2$, there exist a compact connected orientable smooth manifold M of dimension m and a cohomology class u_M in $H^k(M, \mathbb{Z}/2)$ such that if X is a nonsingular real algebraic variety and $\varphi : X \to M$ is a homotopy equivalence, then $\varphi^*(u_M)$ does not belong to $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$.*

Proof. — Let X be a compact nonsingular real algebraic variety. By [6, Theorem A(b)], if a is in $H^r_{\mathrm{alg}}(X, \mathbb{Z}/2)$ then $a \cup a$ is in $\rho_X(H^{2r}(X, \mathbb{Z}))$ (in fact, [6] contains a much more precise result).

In [41, Lemmas 1, 2] there are constructed a compact connected orientable smooth manifold N of dimension 6 and a cohomology class u in $H^2(N, \mathbb{Z}/2)$ such that $u \cup u$ is not in $\rho_N(H^4(N, \mathbb{Z}))$. Let $\mathbb{P}^2(\mathbb{C})$ be the complex projective plane and let z be the generator of $H^2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Let $P = \mathbb{P}^2(\mathbb{C}) \times \cdots \times \mathbb{P}^2(\mathbb{C})$ be the ℓ-fold product, where $2\ell = k - 2$, and let $v = z \times \cdots \times z$ in $H^{k-2}(P, \mathbb{Z}/2)$ be the ℓ-fold cross product; if $\ell = 0$, we assume that P consists of one point and $v = 1$. Let Q be the unit $(m - (2k+2))$-sphere; if $m = 2k + 2$, then by convention, Q consists of one point. Set $M = N \times P \times Q$ and $u_M = u \times v \times 1$. Then M is a compact connected orientable smooth manifold of dimension m and u_M is a cohomology class in $H^k(M, \mathbb{Z}/2)$. Making use of Künneth's theorem in cohomology, one readily checks that $u_M \cup u_M$ is not in $\rho_M(H^{2k}(M, \mathbb{Z}))$. Hence the conclusion follows from the opening paragraph in this proof. □

It seems likely that the only restriction on k one needs in Theorem 1.13 is $k \geqslant 2$. However, our proof does not work if k is odd. Indeed, if P is a smooth manifold and b is in $H^r(P, \mathbb{Z}/2)$ with r odd, then $b \cup b$ belongs to $\rho_P(H^{2r}(P, \mathbb{Z}))$. The last assertion holds since $b \cup b = Sq^r(b) = Sq^1(Sq^{r-1}(b))$, where Sq^i is the ith Steenrod square (cf. [40, p. 281; 35, p. 182]), and each class in the image of Sq^1 belongs to $\rho_P(H^*(P, \mathbb{Z}))$ (cf. [35, p. 182]).

2. Other consequences of the main theorems

Recall that real projective n-space $\mathbb{P}^n(\mathbb{R})$ is a real algebraic variety in the sense of this paper [12, Theorem 3.4.4] (in other words, using terminology of [12], $\mathbb{P}^n(\mathbb{R})$ is an affine real algebraic variety). We have

$$H^k_{\mathrm{alg}}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) = H^k(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2, \quad \mathrm{Alg}^k(\mathbb{P}^n(\mathbb{R})) = 0$$

for $0 \leqslant k \leqslant n$ (the first equality is obvious, whereas the second one follows from [29, Theorem 2.1] or Theorem 4.1(i) in this paper). Therefore a nonsingular subvariety Y of $\mathbb{P}^n(\mathbb{R})$ has the groups $H^1_{\mathrm{alg}}(Y, \mathbb{Z}/2)$ and $\mathrm{Alg}^1(Y)$ determined by $\mathbb{P}^n(\mathbb{R})$ precisely when $H^1_{\mathrm{alg}}(Y, \mathbb{Z}/2) = i^*(H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2))$, where $i : Y \hookrightarrow \mathbb{P}^n(\mathbb{R})$ is the inclusion map, and $\mathrm{Alg}^1(Y) = 0$. It is well known that every topological real vector bundle on $\mathbb{P}^n(\mathbb{R})$ is isomorphic to an algebraic vector bundle [12, Example 12.3.7c]. Moreover, if ξ is an algebraic vector bundle on $\mathbb{P}^n(\mathbb{R})$ and $\sigma : \mathbb{P}^n(\mathbb{R}) \to \xi$ is a smooth section transverse to the zero section and such that $Y = \sigma^{-1}(0)$ is a nonsingular subvariety of $\mathbb{P}^n(\mathbb{R})$, then there is an algebraic section $s : \mathbb{P}^n(\mathbb{R}) \to \xi$ transverse to the zero section and with $Y = s^{-1}(0)$, cf. for example [30, p. 571].

Corollary 2.1. — *Let Y (resp. M) be a nonsingular subvariety (resp. a smooth submanifold) of $\mathbb{P}^n(\mathbb{R})$ of dimension at least 2 and of codimension*

$1, 2, 4$ *or* 8. *If the normal vector bundle of* Y *(resp.* M*) in* $\mathbb{P}^n(\mathbb{R})$ *is trivial, then* Y *is movable (resp.* M *is admissible) in* $\mathbb{P}^n(\mathbb{R})$.

Proof. — There are a smooth real vector bundle ξ on $\mathbb{P}^n(\mathbb{R})$ and a smooth section $s : \mathbb{P}^n(\mathbb{R}) \to \xi$ (resp. $\sigma : \mathbb{P}^n(\mathbb{R}) \to \xi$) such that $Y = s^{-1}(0)$ (resp. $M = \sigma^{-1}(0)$) and s (resp. σ) is transverse to the zero section; this is a special case of [15, Theorem 1.5]. We may assume that ξ is an algebraic vector bundle and s is an algebraic section. Hence the conclusion follows from Theorem 1.4 and Corollary 1.7. □

If Y (resp. M) in Corollary 2.1 is of codimension 1, triviality of the normal vector bundle is not necessary, cf. Corollaries 1.5 and 1.8. For Y (resp. M) of codimension 2 one can also prove a stronger result.

Corollary 2.2. — *Let* Y *(resp.* M*) be a nonsingular subvariety (resp. a smooth submanifold) of* $\mathbb{P}^n(\mathbb{R})$, $n \geqslant 4$, *of codimension* 2. *Then* Y *is movable (resp.* M *is admissible) in* $\mathbb{P}^n(\mathbb{R})$ *if and only if* $w_1(Y)$ *(resp.* $w_1(M)$*) belongs to the image of the homomorphism*

$$i_Y^* : H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) \to H^1(Y, \mathbb{Z}/2)$$

$$(\textit{resp. } i_M^* : H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) \to H^1(Y, \mathbb{Z}/2))$$

induced by the inclusion map $i_Y : Y \hookrightarrow \mathbb{P}^n(\mathbb{R})$ *(resp.* $i_M : M \hookrightarrow \mathbb{P}^n(\mathbb{R})$*).*

Proof. — In one direction the required implication is obvious: if Y is movable (resp. M is admissible), then $w_1(Y) \in \operatorname{Im} i_Y^*$ (resp. $w_1(M) \in \operatorname{Im} i_M^*$). To prove the converse, one makes use of a purely topological Lemma 2.3 below (only (b) $\Rightarrow$ (a) in Lemma 2.3 is needed) and argues as in the proof of Corollary 2.1. □

Lemma 2.3. — *Let* P *be a smooth manifold and let* M *be a smooth submanifold of* P *of codimension* 2. *Then the following conditions are equivalent:*

(a) *There exist a smooth real vector bundle* ξ *on* P *and a smooth section* $s : P \to \xi$ *such that* $\operatorname{rank} \xi = 2$, $M = s^{-1}(0)$, *and* s *is transverse to the zero section,*

(b) $w_1(M)$ *belongs to the image of the homomorphism* $i^* : H^1(P, \mathbb{Z}/2) \to H^1(M, \mathbb{Z}/2)$ *induced by the inclusion map* $i : M \hookrightarrow P$.

Proof. — Assume that (a) holds. Denote by Z the image of the zero section $P \to \xi$. We identify the normal vector bundle of Z in the total space of ξ with ξ. Hence $s^*\xi|M$ is isomorphic to the normal vector bundle ν of M in P. Since $s^*\xi|M \cong \xi|M$, we get

$$w_1(\nu) = w_1(s^*\xi|M) = w_1(\xi|M) = i^*(w_1(\xi)).$$

Let τ_M and τ_P denote the tangent bundles to M and P. Making use of $\tau_M \oplus \nu \cong \tau_P|M$, we obtain

$$\begin{aligned} w_1(M) &= w_1(\nu) + w_1(\tau_P|M) \\ &= i^*(w_1(\xi)) + i^*(w_1(P)) \\ &= i^*(w_1(\xi) + w_1(P)) \end{aligned}$$

and hence $w_1(M)$ is in the image of i^*. In other words, (b) is satisfied.

Suppose now that (b) holds, that is, $w_1(M) = i^*(v)$ for some cohomology class v in $H^1(P, \mathbb{Z}/2)$. Let λ be a smooth line bundle on P with $w_1(\lambda) = v + w_1(P)$.

Let $\pi : T \to M$ be a tubular neighborhood of M in P. We identify (T, π, M) with the normal vector bundle ν of M in P. Clearly, there exists a smooth section $\sigma : T \to \pi^*\nu$ such that σ is transverse to the zero section and $\sigma^{-1}(0) = M$. We have

$$\pi^*\nu|T\backslash M = \eta \oplus \epsilon_\sigma, \tag{1}$$

where ϵ_σ is the trivial line subbundle of $\nu|T\backslash M$ generated by σ and η is a smooth line bundle on $T\backslash M$. We assert that

$$w_1(\eta) = w_1(\lambda|T\backslash M). \tag{2}$$

Indeed, we have $\nu \oplus \tau_M = \tau_P|M$ and hence

$$w_1(\nu) = w_1(\tau_M) + w_1(\tau_P|M) = w_1(\lambda|M) = i^*(w_1(\lambda)).$$

Let $j : T \hookrightarrow P$ be the inclusion map. Since $i \circ \pi$ and j are homotopic, we get

$$w_1(\pi^*\nu) = \pi^*(w_1(\nu)) = \pi^*(i^*(w_1(\lambda))) = j^*(w_1(\lambda)) = w_1(\lambda|T).$$

Hence (2) is a consequence of (1).

Let ϵ be the trivial line bundle on P with total space $P \times \mathbb{R}$ and let $\tau : P \to \lambda \oplus \epsilon$ be the smooth section defined by $\tau(x) = (0, (x, 1))$ for all x in P. By (2), η and $\lambda|T\backslash M$ are isomorphic and hence there exists a smooth isomorphism

$$\varphi : \pi^*\nu|T\backslash M \to (\lambda \oplus \epsilon)|T\backslash M$$

such that $\varphi \circ \sigma = \tau$ on $T\backslash M$.

Let ξ be the smooth vector bundle on P obtained by gluing $\pi^*\nu$ and $(\lambda \oplus \epsilon)|P\backslash M$ over $T\backslash M$ using φ. Similarly, let $s : P \to \xi$ be the smooth section obtained by gluing σ and $\tau|P\backslash M$ over $T\backslash M$ using φ. By construction, ξ is of rank 2, $s^{-1}(0) = M$, and s is transverse to the zero section. Thus (a) is satisfied. □

3. Noether-Lefschetz type theorems

To begin with we give an alternative description of the groups $H^k_{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $\mathrm{Alg}^k(-)$. Let V be a reduced quasiprojective scheme over $\mathbb{R}$. The set $V(\mathbb{R})$ of $\mathbb{R}$-rational points of V is contained in an affine open subset of V. Thus if $V(\mathbb{R})$ is dense in V, we can regard $V(\mathbb{R})$ as a real algebraic variety whose structure sheaf is the restriction of the structure sheaf of V; up to isomorphism, each real algebraic variety is of this form.

Assume that V is nonsingular (our convention is that all irreducible components of V have the same dimension) with $V(\mathbb{R})$ compact and dense in V. Then $V(\mathbb{R})$ is a compact nonsingular real algebraic variety and we have the cycle homomorphism:

$$c\ell_{\mathbb{R}} : Z^k(V) \to H^k(V(\mathbb{R}), \mathbb{Z}/2),$$

defined on the group $Z^k(V)$ of algebraic cycles on V of codimension k: for any integral subscheme W of V of codimension k, the cohomology class $c\ell_{\mathbb{R}}(W)$ is Poincaré dual to the homology class in $H_*(V(\mathbb{R}), \mathbb{Z}/2)$ represented by $W(\mathbb{R})$, provided that $W(\mathbb{R})$ has codimension k in $V(\mathbb{R})$, and otherwise $c\ell_{\mathbb{R}}(W) = 0$. By construction,

$$H^k_{\mathrm{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = c\ell_{\mathbb{R}}(Z^k(V)).$$

Moreover, we readily see that

$$\mathrm{Alg}^k(V(\mathbb{R})) = c\ell_{\mathbb{R}}(Z^k_{\mathrm{alg}}(V)),$$

where $Z^k_{\mathrm{alg}}(V)$ is the subgroup of $Z^k(V)$ consisting of all cycles algebraically equivalent to 0 (cf. [21, Chapter 10] for the theory of algebraic equivalence).

It will be convenient to express $H^1_{\mathrm{alg}}(V(\mathbb{R}), \mathbb{Z}/2)$ and $\mathrm{Alg}^1(V(\mathbb{R}))$ using line bundles on V. Given a vector bundle E on V, we denote by $E(\mathbb{R})$ the algebraic vector bundle on $V(\mathbb{R})$ determined by E. The correspondence which assigns to any line bundle L on V the first Stiefel-Whitney class $w_1(L(\mathbb{R}))$ of $L(\mathbb{R})$ gives rise to a canonical homomorphism

$$\omega_V : \mathrm{Pic}(V) \to H^1(V(\mathbb{R}), \mathbb{Z}/2),$$

defined on the Picard group $\mathrm{Pic}(V)$ of isomorphism classes of line bundles on V. When no confusion is possible, we make no distinction between line bundles and their isomorphism classes. If $\mathcal{O}(D)$ is the line bundle associated with a Weil divisor D on V, then $\omega_V(\mathcal{O}(D)) = c\ell_{\mathbb{R}}(D)$, cf. [17, p. 498] (obviously, $Z^1(V)$ is the group of Weil divisors on V). Since every element of $\mathrm{Pic}(V)$ is of the form $\mathcal{O}(D)$ for some D in $Z^1(V)$, we have

$$H^1_{\mathrm{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = \omega_V(\mathrm{Pic}(V)). \tag{3.1}$$

Moreover,

$$\operatorname{Alg}^1(V(\mathbb{R})) = \omega_V(\operatorname{Pic}^0(V)), \tag{3.2}$$

where $\operatorname{Pic}^0(V)$ is the subgroup of $\operatorname{Pic}(V)$ consisting of the isomorphism classes of line bundles of the form $\mathcal{O}(D)$ for D in $Z^1_{\mathrm{alg}}(V)$. The homomorphism ω_V is natural in V. Given another quasiprojective nonsingular scheme W over $\mathbb{R}$ with $W(\mathbb{R})$ compact and dense in W and given a morphism $f : V \to W$ over $\mathbb{R}$, we have the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Pic}(W) & \xrightarrow{f^*} & \operatorname{Pic}(V) \\ {\scriptstyle\omega_W}\big\downarrow & & {\scriptstyle\omega_V}\big\downarrow \\ H^1(W(\mathbb{R}), \mathbb{Z}/2) & \xrightarrow{f(\mathbb{R})^*} & H^1(V(\mathbb{R}), \mathbb{Z}/2), \end{array} \tag{3.3}$$

where $f(\mathbb{R}) : V(\mathbb{R}) \to W(\mathbb{R})$ is the regular map determined by f.

In order to make use of formulas (3.1) and (3.2) we need to study $\operatorname{Pic}(V)$ and $\operatorname{Pic}^0(V)$. To this end we consider the scheme $V_{\mathbb{C}} = V \times_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ and the corresponding groups $\operatorname{Pic}(V_{\mathbb{C}})$ and $\operatorname{Pic}^0(V_{\mathbb{C}})$ on $V_{\mathbb{C}}$. The Galois group $G = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ of $\mathbb{C}$ over $\mathbb{R}$ acts on $\operatorname{Pic}(V_{\mathbb{C}})$ and $\operatorname{Pic}^0(V_{\mathbb{C}})$; denote by $\operatorname{Pic}(V_{\mathbb{C}})^G$ and $\operatorname{Pic}^0(V_{\mathbb{C}})^G$ the subgroups consisting of the elements fixed by G. Given a vector bundle E on V, we write $E_{\mathbb{C}}$ for the corresponding vector bundle on $V_{\mathbb{C}}$. There is a canonical group homomorphism

$$\alpha_V : \operatorname{Pic}(V) \to \operatorname{Pic}(V_{\mathbb{C}})^G, \ \alpha_V(L) = L_{\mathbb{C}}.$$

It is well known that under certain natural assumptions α_V is an isomorphism. Note that if V is irreducible and nonsingular with $V(\mathbb{R})$ nonempty (hence $V(\mathbb{R})$ automatically dense in V), then $V_{\mathbb{C}}$ is irreducible and nonsingular.

THEOREM 3.1. — *Let V be an irreducible nonsingular projective scheme over $\mathbb{R}$. If $V(\mathbb{R})$ is nonempty, then $\alpha_V : \operatorname{Pic}(V) \to \operatorname{Pic}(V_{\mathbb{C}})^G$ is an isomorphism and $\alpha_V(\operatorname{Pic}^0(V)) = \operatorname{Pic}^0(V_{\mathbb{C}})^G$.*

Reference for the proof. — This is a special case of a far more general descent theory [22]. A simple treatment of the case under consideration can also be found in [23]. □

We write $V(\mathbb{C})$ for the set of $\mathbb{C}$-rational points of V and identify it with the set $V_{\mathbb{C}}(\mathbb{C})$ of $\mathbb{C}$-rational points of $V_{\mathbb{C}}$. If $f : V \to W$ is a morphism of schemes over $\mathbb{R}$, then $f_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ will denote the morphism of schemes over $\mathbb{C}$ after the base extension, while $f(\mathbb{C}) : V(\mathbb{C}) \to W(\mathbb{C})$ will denote the map induced by f. The following is a straightforward, but very useful consequence of Theorem 3.1.

Corollary 3.2. — *Let $f : V \to W$ be a morphism of irreducible nonsingular projective schemes over $\mathbb{R}$. Assume that $V(\mathbb{R})$ is nonempty (so $W(\mathbb{R})$ is nonempty too). If $f_{\mathbb{C}}^* : \mathrm{Pic}(W_{\mathbb{C}}) \to \mathrm{Pic}(V_{\mathbb{C}})$ is an isomorphism, then $f^* : \mathrm{Pic}(W) \to \mathrm{Pic}(V)$ is an isomorphism and $f^*(\mathrm{Pic}^0(W)) = \mathrm{Pic}^0(V)$.*

Proof. — Suppose that $f_{\mathbb{C}}^* : \mathrm{Pic}(W_{\mathbb{C}}) \to \mathrm{Pic}(V_{\mathbb{C}})$ is an isomorphism. Consequently, $f_{\mathbb{C}}^*(\mathrm{Pic}^0(W_{\mathbb{C}})) = \mathrm{Pic}^0(V_{\mathbb{C}})$, as one readily sees. Clearly, $f_{\mathbb{C}}^*$ is G-equivariant and the restriction $f_{\mathbb{C}}^* : \mathrm{Pic}(W_{\mathbb{C}})^G \to \mathrm{Pic}(V_{\mathbb{C}})^G$ also is an isomorphism. The proof is complete in view of Theorem 3.1. □

Let H be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. A subset Σ of H is said to be thin if it is contained in the union of a countable family of proper algebraic subsets of H.

Given a vector bundle E on a quasiprojective scheme V over $\mathbb{R}$ and a section s of E, we denote by $Z(s)$ the subscheme of V of zeros of s. Assuming that V is nonsingular, we say that s is transverse to the zero section if the holomorphic section $s(\mathbb{C}) : V(\mathbb{C}) \to E(\mathbb{C})$ of the holomorphic vector bundle $E(\mathbb{C})$ on $V(\mathbb{C})$ is transverse to the zero section (note that then $Z(s)$ is nonsingular). Given a line bundle L on V, we write L^m for the m-fold tensor product $L \otimes \cdots \otimes L$. We will need the following analogue of Max Noether's theorem.

Theorem 3.3. — *Let V be an irreducible nonsingular projective scheme over $\mathbb{R}$. Let E be a vector bundle on V with $2 + \mathrm{rank}\, E \leqslant \dim V$ and let L be an ample line bundle on V. There exists a positive integer m_0 such that for each integer $m \geqslant m_0$, there is a thin subset $\Sigma(m)$ of $H^0(V, E \otimes L^m)$ with the property that each section s in $H^0(V, E \otimes L^m) \backslash \Sigma(m)$ is transverse to the zero section, the subscheme $W = Z(s)$ of zeros of s is irreducible, and whenever $V(\mathbb{R})$ and $W(\mathbb{R})$ are nonempty, the homomorphism $j^* : \mathrm{Pic}(V) \to \mathrm{Pic}(W)$ is an isomorphism with $j^*(\mathrm{Pic}^0(V)) = \mathrm{Pic}^0(W)$, where $j : W \hookrightarrow V$ is the inclusion morphism.*

Proof. — Set $E(m) = E \otimes L^m$. By [20, Theorems 2.2 and 2.4], there exists a positive integer m_0 such that for each integer $m \geqslant m_0$, there is a thin subset $\Sigma(m)_{\mathbb{C}}$ of $H^0(V_{\mathbb{C}}, E(m)_{\mathbb{C}})$ with the property that each section σ in $H^0(V_{\mathbb{C}}, E(m)_{\mathbb{C}}) \backslash \Sigma(m)_{\mathbb{C}}$ is transverse to the zero section, $Z = Z(\sigma)$ is irreducible (note that Z is defined over $\mathbb{C}$), and $i^* : \mathrm{Pic}(V_{\mathbb{C}}) \to \mathrm{Pic}(Z)$ is an isomorphism, where $i : Z \hookrightarrow V_{\mathbb{C}}$ is the inclusion morphism.

The canonical map $H^0(V, E(m)) \to H^0(V_{\mathbb{C}}, E(m)_{\mathbb{C}})$, $s \to s_{\mathbb{C}}$, is injective, and hence we can regard $H^0(V, E(m))$ as a subset of $H^0(V_{\mathbb{C}}, E(m)_{\mathbb{C}})$. Since

$$H^0(V, E(m)) \otimes_{\mathbb{R}} \mathbb{C} \cong H^0(V_{\mathbb{C}}, E(m)_{\mathbb{C}}),$$

it suffices to take $\Sigma(m) = \Sigma(m)_{\mathbb{C}} \cap H^0(V, E(m))$ and apply Corollary 3.2. □

Our next observation is a useful technical fact.

LEMMA 3.4. — *Let ξ be an algebraic vector bundle on a compact irreducible nonsingular real algebraic variety X. Then there exist an irreducible nonsingular projective scheme V over $\mathbb{R}$ with $V(\mathbb{R}) \neq \emptyset$ (hence $V(\mathbb{R})$ dense in V), an isomorphism $\varphi : X \to V(\mathbb{R})$, and a vector bundle E on V such that ξ and $\varphi^* E(\mathbb{R})$ are algebraically isomorphic.*

Proof. — In view of Hironaka's desingularization theorem [26], we may assume that $X = W(\mathbb{R})$, where W is an irreducible nonsingular projective scheme over $\mathbb{R}$. Furthermore, we may assume that $\xi = F(\mathbb{R})$ for some vector bundle F defined on an affine neighborhood W_0 of $W(\mathbb{R})$ in W. Indeed, the category of algebraic vector bundles on X is equivalent to the category of finitely generated projective modules over the ring $\mathcal{R}(X)$ of regular functions on X (cf. [12, Theorem 12.1.7]), while the category of vector bundles on an affine open subset U of W is equivalent to the category of finitely generated projective $\mathcal{O}_W(U)$-modules, where $\mathcal{O}_W$ is the structure sheaf of W. Since $\mathcal{R}(X) = \operatorname{dir\,lim} \mathcal{O}_W(U)$, where U runs through the family of affine neighborhoods of $X = W(\mathbb{R})$ in W, directed by $\supseteq$, the required W_0 and F exist.

Denote by $\mathbb{G}_{n,r}$ the Grassmann scheme over $\mathbb{R}$ corresponding to the r-dimensional vector subspaces of $\mathbb{R}^n$. Let $\Gamma_{n,r}$ be the universal vector bundle on $\mathbb{G}_{n,r}$. Since W_0 is affine, F is generated by global sections on W_0, and hence taking $r = \operatorname{rank} F$ and n sufficiently large, one can find a morphism $f : W_0 \to \mathbb{G}_{n,r}$ over $\mathbb{R}$ such that F is isomorphic to $f^*\Gamma_{n,r}$. Regard f as a rational map from W into $\mathbb{G}_{n,r}$. By Hironaka's theorem on resolution of points of indeterminacy [26], there exist an irreducible nonsingular projective scheme V over $\mathbb{R}$ and two morphisms $\pi : V \to W$, $g : V \to \mathbb{G}_{n,r}$ over $\mathbb{R}$ such that the restriction $\pi : \pi^{-1}(W_0) \to W_0$ is an isomorphism and $g = f \circ \pi$ as rational maps. The conclusion follows if we take $E = g^*\Gamma_{n,r}$ and $\varphi = \pi(\mathbb{R})^{-1} : W(\mathbb{R}) = X \to V(\mathbb{R})$. □

THEOREM 3.5. — *Let X be a compact irreducible nonsingular real algebraic variety. Let ξ be an algebraic vector bundle on X with $2 + \operatorname{rank} \xi \leqslant \dim X$ and let $s : X \to \xi$ be an algebraic section. Then there exist a regular function $f : X \to \mathbb{R}$, algebraic sections $s_i : X \to \xi$, $1 \leqslant i \leqslant k$, and a thin subset Σ of $\mathbb{R}^k$ such that*

(i) $f^{-1}(0) = \emptyset$,

(ii) $s_1, \dots, s_k$ generate ξ, *that is, for each point* x *in* X, *the vectors* $s_1(x), \dots, s_k(x)$ *generate the fiber of* ξ *over* x,

(iii) *the family of algebraic sections* $\{\sigma_t\}_{t\in\mathbb{R}^k}$, *where* $t = (t_1, \dots, t_k)$,

$$\sigma_t = fs + t_1 s_1 + \dots + t_k s_k,$$

has the property that for each t *in* $\mathbb{R}^k \backslash \Sigma$, *the section* σ_t *is transverse to the zero section and the nonsingular subvariety* $Y_t = \sigma_t^{-1}(0)$ *of* X *is either empty or else it is irreducible with the groups* $H^1_{\mathrm{alg}}(Y_t, \mathbb{Z}/2)$ *and* $\mathrm{Alg}^1(Y_t)$ *determined by* X.

Proof. — In view of Lemma 3.4, we may assume that $X = V(\mathbb{R})$ and $\xi = E(\mathbb{R})$, where V is an irreducible nonsingular projective scheme over $\mathbb{R}$ and E is a vector bundle on V. Furthermore, we may assume $V \subseteq \mathbb{P}^n_{\mathbb{R}}$ for some n. There exist an open neighborhood V_0 of X in V and a section $s_0 : V_0 \to E$ such that s_0 is an extension of s, that is, $s_0(\mathbb{R}) : V_0(\mathbb{R}) = X \to E(\mathbb{R}) = \xi$ is equal to s. We have

$$V_0 = V \backslash Z(H_1, \dots, H_\ell),$$

where the H_j are homogeneous polynomials in $\mathbb{R}[T_0, \dots, T_n]$ and $Z(H_1, \dots, H_\ell)$ is the closed subset of $\mathbb{P}^n_{\mathbb{R}}$ described by the equations $H_1 = 0, \dots, H_\ell = 0$. Set $d_j = \deg H_j$, $d = \max\{d_1, \dots, d_\ell\}$, and

$$H = \sum_{j=1}^{\ell} (T_0^2 + \dots + T_n^2)^{d-d_j} H_j^2.$$

Then H is a homogeneous polynomial of degree $2d$, and the closed subset $Z(H)$ of $\mathbb{P}^n_{\mathbb{R}}$ defined by the equation $H = 0$ satisfies

$$X = V(\mathbb{R}) \subseteq V \backslash Z(H) \subseteq V_0.$$

Let $\mathcal{O}(1)$ be the Serre line bundle on $\mathbb{P}^n_{\mathbb{R}}$. Let $h : \mathbb{P}^n_{\mathbb{R}} \to \mathcal{O}(2d)$ be the section determined by the homogeneous polynomial H. Note that $Z(h) = Z(H)$, where $Z(h)$ is the set of zeros of h.

Let $L = \mathcal{O}(2d)|V$ and $u = h|V$. Then L is an ample line bundle on V and $u : V \to L$ is a section. By construction,

$$X \subseteq V \backslash Z(u) = V \backslash Z(H) \subseteq V_0.$$

Note that $L(\mathbb{R})$ is a trivial algebraic line bundle on X. Indeed, since $\mathcal{O}(2d) \cong \mathcal{O}(1)^{2d}$, it immediately follows that $w_1(L(\mathbb{R})) = 0$, which implies that $L(\mathbb{R})$ is topological trivial. Consequently, $L(\mathbb{R})$ is algebraically trivial, as required, cf. [12, Theorem 12.3.1].

Given a positive integer m, we set $E(m) = E \otimes L^m$. There exists a positive integer m_0, such that for each integer $m \geqslant m_0$, the vector bundle $E(m)$ is generated by global sections (cf. [25, p. 153]), the section

$$s_0 \otimes u^m : V \backslash Z(u) \to E(m),$$

where $u^m = u \otimes \cdots \otimes u : V \to L^m$, can be extended to a section $v_m : V \to E(m)$ (cf. [25, Lemma 5.14]), and the conclusion of Theorem 3.3 holds.

Fix $m \geqslant m_0$. Let $w_1, \ldots, w_k$ be a basis for the $\mathbb{R}$-vector space $H^0(V, E(m))$. Given $t = (t_1, \ldots, t_k)$ in $\mathbb{R}^k$, set

$$\tau_t = v_m + t_1 w_1 + \cdots + t_k w_k.$$

By Theorem 3.3, there exists a thin subset Σ of $\mathbb{R}^k$ such that for each t in $\mathbb{R}^k \backslash \Sigma$, the section τ_t is transverse to the zero section, $W_t = Z(\tau_t)$ is irreducible, and whenever $W_t(\mathbb{R})$ is nonempty,

$$(*) \qquad j_t^*(\mathrm{Pic}(V)) = \mathrm{Pic}(W_t) \text{ and } j_t^*(\mathrm{Pic}^0(V)) = \mathrm{Pic}^0(W_t),$$

where $j_t : W_t \hookrightarrow V$ is the inclusion morphism.

Since the line bundle $L(\mathbb{R})$ is algebraically trivial, the algebraic vector bundles $E(m)(\mathbb{R})$ and ξ on X are isomorphic. We may assume $E(m)(\mathbb{R}) = \xi$. Hence

$$v_m(\mathbb{R}) = fs$$

for some regular function $f : X \to \mathbb{R}$ with $f^{-1}(0) = \emptyset$.

Defining $s_i = w_i(\mathbb{R})$ for $1 \leqslant i \leqslant k$, one readily sees that $f,\ s_1, \ldots, s_k$, and Σ satisfy the required conditions. Indeed, conditions (i) and (ii) are obvious from the construction. It is also clear that $\sigma_t = \tau_t(\mathbb{R}) : X \to \xi$ is transverse to the zero section, and the nonsingular subvariety $Y_t = \sigma_t^{-1}(0) = W_t(\mathbb{R})$ of X is either empty or irreducible. In the latter case, the groups $H^1_{\mathrm{alg}}(Y_t, \mathbb{Z}/2)$ and $\mathrm{Alg}^1(Y_t)$ are determined by X in view of (*) and (3.1), (3.2), (3.3). ☐

Proof of Theorem 1.4. Let X, Y, ξ, s be as in the statement of Theorem 1.4. Choose f, $s_1, \ldots, s_k$, Σ as in Theorem 3.5. Since $s_1, \ldots, s_k$ generate ξ, the map $F : X \times \mathbb{R}^k \to \xi$, defined by

$$F(x,t) = f(x)s(x) + t_1 s_1(x) + \cdots + t_k s_k(x)$$

for all x in X and $t = (t_1, \ldots, t_k)$ in $\mathbb{R}^k$, is transverse to the zero section of ξ. The nonsingular subvariety $Z = F^{-1}(0)$ of $X \times \mathbb{R}^k$ satisfies conditions (i) and (ii) in Definition 1.2. Hence Y is movable. ☐

We conclude this section by describing some consequences of Larsen's generalization [34] of Barth's theorem [7].

Remark 3.6. —

(i) Let X be a nonsingular subvariety of $\mathbb{P}^n(\mathbb{R})$ with $2\dim X \geqslant n+2$. Assume that the Zariski closure of X in $\mathbb{P}^n(\mathbb{R})$ is nonsingular. Then

$$H^1_{\mathrm{alg}}(X,\mathbb{Z}/2) = i^*(H^1(\mathbb{P}^n(\mathbb{R}),\mathbb{Z}/2)),\ \mathrm{Alg}^1(X) = 0,$$

where $i : X \hookrightarrow \mathbb{P}^n(\mathbb{R})$ is the inclusion map. Indeed, let V be the Zariski closure of X in $\mathbb{P}^n(\mathbb{R})$ and let $j : V \hookrightarrow \mathbb{P}^n_{\mathbb{R}}$ be the inclusion morphism. By [34], the induced homomorphism $j^*_{\mathbb{C}} : \mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}}) \to \mathrm{Pic}(V_{\mathbb{C}})$ is an isomorphism (cf. also [24, Corollary 6.5]). Since $X = V(\mathbb{R})$, $\mathbb{P}^n(\mathbb{R}) = \mathbb{P}^n_{\mathbb{R}}(\mathbb{R})$ and $\mathrm{Alg}^1(\mathbb{P}^n(\mathbb{R})) = 0$ (cf. Section 2), the conclusion follows from Corollary3.2 and (3.1), (3.2), (3.3).

(ii) Let M be a compact smooth submanifold of $\mathbb{R}^n$ with $2\dim M \geqslant n+2$. Suppose $w_1(M) \neq 0$, that is, M is nonorientable. Consider $\mathbb{R}^n$ as a subset of $\mathbb{P}^n(\mathbb{R})$. If M is isotopic in $\mathbb{P}^n(\mathbb{R})$ to a nonsingular subvariety X of $\mathbb{P}^n(\mathbb{R})$, then the Zariski closure of X in $\mathbb{P}^n_{\mathbb{R}}$ is singular. This assertion follows from (i) since $w_1(X)$ is a nonzero element of $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$, while $i^*(H^1(\mathbb{P}^n(\mathbb{R}),\mathbb{Z}/2)) = 0$, where $i : X \hookrightarrow \mathbb{P}^n(\mathbb{R})$ is the inclusion map (here we use $M \subseteq \mathbb{R}^n$). Such a result is obtained in [6, Theorem B] under a stronger assumption $w_1(M) \cup w_1(M) \neq 0$.

4. Varieties with prescribed $H^1_{\mathrm{alg}}(-,\mathbb{Z}/2)$ and $\mathrm{Alg}^1(-)$

First we will collect several facts required for the proof of Theorem 1.9. Recall that if M is a smooth manifold, then a cohomology class u in $H^k(M,\mathbb{Z}/2)$, $k \geqslant 1$, is said to be *spherical*, provided $u = f^*(c)$, where $f : M \to S^k$ is a continuous (or equivalently smooth) map into the unit sphere S^k and c is the unique generator of the group $H^k(S^k,\mathbb{Z}/2) \cong \mathbb{Z}/2$.

Theorem 4.1. — *Let X be a compact nonsingular real algebraic variety. Then:*

(i) $\langle u \cup v, [X]\rangle = 0$ *for all u in* $\mathrm{Alg}^k(X)$ *and v in* $H^\ell_{\mathrm{alg}}(X,\mathbb{Z}/2)$, *where* $k+\ell = \dim X$.

(ii) *Every cohomology class in* $\mathrm{Alg}^1(X)$ *is spherical.*

Reference for the proof. — [29, Theorem 2.1], [1, Theorem 1.1] □

Also the next, very particular, observation concerning $\mathrm{Alg}^1(-)$ will be useful. Let B^k be an irreducible nonsingular real algebraic variety with precisely two connected components B^k_0 and B^k_1, each diffeomorphic to the unit sphere S^k, $k \geqslant 1$. One can take, for example,

$$B^k = \{(x_0,\dots,x_k) \in \mathbb{R}^{k+1} | x_0^4 - 4x_0^2 + 1 + x_1^2 + \cdots + x_k^2 = 0\}.$$

Let $B^k(d) = B^k \times \cdots \times B^k$ and $B_0^k(d) = B_0^k \times \cdots \times B_0^k$ be the d-fold products, and let $\delta : B_0^k(d) \hookrightarrow B^k(d)$ be the inclusion map.

Lemma 4.2. — *With notation as above,*

$$H^q(B_0^k(d), \mathbb{Z}/2) = \delta^*(H^q(B^k(d), \mathbb{Z}/2)) = \delta^*(\mathrm{Alg}^q(B^k(d)))$$

for all $q \geqslant 0$.

Reference for the proof. — [32, Example 4.5] □

Let us now recall an important theorem from differential topology, which will be used repeatedly in this section. Given a smooth manifold P, let $\mathcal{N}_*(P)$ denote the unoriented bordism group of P, cf. [18].

Theorem 4.3. — *Let P be a smooth manifold. Two smooth maps $f : M \to P$ and $g : N \to P$, where M and N are compact smooth manifolds of dimension d, represent the same bordism class in $\mathcal{N}_*(P)$ if and only if for every nonnegative integer q and every cohomology class v in $H^q(P, \mathbb{Z}/2)$, one has*

$$\langle f^*(v) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M),\ [M] \rangle = \langle g^*(v) \cup w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N] \rangle$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = d - q$.

Reference for the proof. — [18, (17.3)]. □

If W is a nonsingular real algebraic variety, then a bordism class in $\mathcal{N}_*(W)$ is said to be *algebraic*, provided it can be represented by a regular map $f : X \to W$ of a compact nonsingular real algebraic variety X into W, cf. [5, 10, 44]. Denote by $\mathcal{N}_*^{\mathrm{alg}}(W)$ the subgroup of $\mathcal{N}_*(W)$ consisting of the algebraic bordism classes. Varieties W with $\mathcal{N}_*^{\mathrm{alg}}(W) = \mathcal{N}_*(W)$ will play a special role in various constructions.

The Grassmannian $\mathbb{G}_{n,p}(\mathbb{R})$ of p-dimensional vector subspaces of $\mathbb{R}^n$ is a real algebraic variety in the sense of this paper, cf. [12, Theorem 3.4.4]. (Note, in particular, $\mathbb{G}_{n,1}(\mathbb{R}) = \mathbb{P}^{n-1}(\mathbb{R})$). Furthermore, $\mathbb{G}_{n,p}(\mathbb{R})$ is nonsingular and $H_i^{\mathrm{alg}}(\mathbb{G}_{n,p}(\mathbb{R}), \mathbb{Z}/2) = H_i(\mathbb{G}_{n,p}(\mathbb{R}), \mathbb{Z}/2)$ for all $i \geqslant 0$, cf. [12, Propositions 3.4.3, 11.3.3]. It follows from Künneth's theorem in homology that

$$W = \mathbb{G}_{n_1,p_1}(\mathbb{R}) \times \cdots \times \mathbb{G}_{n_r,p_r}(\mathbb{R})$$

is a nonsingular real algebraic variety with $H_i^{\mathrm{alg}}(W, \mathbb{Z}/2) = H_i(W, \mathbb{Z}/2)$ for all $i \geqslant 0$. This, in view of [5, Lemma 2.7.1], implies

$$\mathcal{N}_*^{\mathrm{alg}}(W) = \mathcal{N}_*(W). \tag{4.1}$$

Given smooth manifolds N and P, we endow the set $\mathcal{C}^\infty(N, P)$ of all smooth maps from N into P with the $\mathcal{C}^\infty$ topology [27](in our applications

N is always compact so it does not matter whether we take the weak $\mathcal{C}^\infty$ or the strong one).

The following approximation theorem will be crucial.

Theorem 4.4. — *Let M be a compact smooth submanifold of $\mathbb{R}^n$ and let W be a nonsingular real algebraic variety. Let $f : M \to W$ be a smooth map, whose bordism class in $\mathcal{N}_*(W)$ is algebraic. Suppose that M contains a (possibly empty) subset L, which is a union of finitely many nonsingular subvarieties of $\mathbb{R}^n$, the restriction $f|L : L \to W$ is a regular map, and the restriction to L of the tangent bundle of M is topologically isomorphic to an algebraic vector bundle on L. If $2\dim M + 1 \leqslant n$, then there exist a smooth embedding $e : M \to \mathbb{R}^n$, a nonsingular subvariety X of $\mathbb{R}^n$, and a regular map $g : X \to W$ such that $L \subseteq X = e(M)$, $e|L : L \to \mathbb{R}^n$ is the inclusion map, $g|L = f|L$, and $g \circ \bar{e}$ (where $\bar{e} : M \to e(M)$ is the smooth diffeomorphism defined by $\bar{e}(x) = e(x)$ for all x in M) is homotopic of f. Furthermore, given a neighborhood $\mathcal{U}$ in $\mathcal{C}^\infty(M, \mathbb{R}^n)$ of the inclusion map $M \hookrightarrow \mathbb{R}^n$ and a neighborhood $\mathcal{V}$ of f in $\mathcal{C}^\infty(M, W)$, the objects e, X, and g can be chosen in such a way that e is in $\mathcal{U}$ and $g \circ \bar{e}$ is in $\mathcal{V}$.*

Reference for the proof. — Precisely this formulation (with L nonsingular), based on very similar results [3, 5, 10, 9, 44] is in [32, Theorem 4.2]. The slightly more general result needed in the present paper follows from the argument given in [32, Theorem 4.2] since a union of finitely many nonsingular subvarieties of $\mathbb{R}^n$ is a nice set, equivalently, a quasiregular subvariety in the terminology used in [5] and [10, 44], respectively, cf. [44, p. 75]. □

For sake of completeness we include here a simple technical lemma.

Lemma 4.5. — *Let M and P be smooth manifolds, with M compact. Let K and L be smooth submanifolds of M that are transverse in M. Let $f : M \to P$ be a smooth map and let $\mathcal{U}$ be a neighborhood of f in $\mathcal{C}^\infty(M, P)$. Then there exists a neighborhood $\mathcal{V}$ of $f|L$ in $\mathcal{C}^\infty(L, P)$ such that for every smooth map $h : L \to P$ in $\mathcal{V}$ with $h|K \cap L = f|K \cap L$, there is a smooth map $g : M \to P$ in $\mathcal{U}$ satisfying $g|K = f|K$ and $g|L = h$.*

Proof. — We may assume that P is a smooth submanifold of $\mathbb{R}^d$ for some d. Since P has a tubular neighborhood in $\mathbb{R}^d$, it suffices to prove the lemma for $P = \mathbb{R}$. Given a smooth submanifold N of M, denote by $I(N)$ the ideal of the ring $\mathcal{C}^\infty(M, \mathbb{R})$ consisting of all smooth functions vanishing on N. Using partition of unity, one readily shows that the ideal $I(N)$ is finitely generated.

Since K and L are transverse in M, the ideal $I(K \cap L)$ is generated by $I(K) \cup I(L)$. Let $\alpha_1, \ldots, \alpha_r$ (resp. $\beta_1, \ldots, \beta_s$) be generators of $I(K)$ (resp. $I(L)$). Note that

$$\Lambda : \mathcal{C}^\infty(M, \mathbb{R})^{r+s} \to I(K \cap L)$$
$$\Lambda(\varphi_1, \ldots, \varphi_r, \psi_1, \ldots, \psi_s) = \sum_{i=1}^{r} \varphi_i \alpha_i - \sum_{j=1}^{s} \psi_j \beta_j$$

is a continuous, surjective $\mathbb{R}$-linear map. Since $\mathcal{C}^\infty(M, \mathbb{R})^{r+s}$ is a Fréchet space, it follows that Λ is an open map, cf. [37, Theorem 2.11].

Let $\mathcal{U}_0$ be a neighborhood of 0 in $\mathcal{C}^\infty(M, \mathbb{R})$ satisfying $f - \mathcal{U}_0 \subseteq \mathcal{U}$. Since Λ is an open map, there is a neighborhood $\mathcal{W}$ of 0 in $\mathcal{C}^\infty(M, \mathbb{R})$ such that every function in $I(K \cap L) \cap \mathcal{W}$ can be written as $f_1 - f_2$, where f_1 is in $I(K) \cap \mathcal{U}_0$ and f_2 is in $I(L) \cap \mathcal{U}_0$ (the fact that f_2 is in $\mathcal{U}_0$ will not be important). If $\mathcal{V}$ is a sufficiently small neighborhood of $f|L$ in $\mathcal{C}^\infty(L, \mathbb{R})$ and $h : L \to \mathbb{R}$ is in $\mathcal{V}$, then we can find a function φ in $\mathcal{C}^\infty(M, \mathbb{R})$ with $\varphi|L = h$ and $f - \varphi$ in $\mathcal{W}$. Thus $f - \varphi$ is in $I(K \cap L) \cap \mathcal{W}$, and hence $f - \varphi = f_1 - f_2$ for some f_1 in $I(K) \cap \mathcal{U}_0$ and f_2 in $I(L)$. Setting $g = f - f_1 = \varphi - f_2$, we get $g|K = (f - f_1)|K = f|K$ and $g|L = (\varphi - f_2)|L = \varphi|L = h$. Moreover, g is in $\mathcal{U}$ since f_1 is in $\mathcal{U}_0$. □

Given a smooth manifold P and subsets $E_1, \ldots, E_r$ of the cohomology ring $H^*(P, \mathbb{Z}/2)$, we write

$$[E_1, \ldots, E_r]^* = \oplus_{k \geqslant 0} [E_1, \ldots, E_r]^k$$

for the graded subring of $H^*(P, \mathbb{Z}/2)$ generated by the union of the subsets $E_1, \ldots, E_r$. Using also notation introduced in Secton 1, we get

$$SW^*(P; E_1, \ldots, E_r) = [SW^*(P), E_1, \ldots, E_r]^*.$$

Clearly, if E is a subgroup of $H^\ell(P, \mathbb{Z}/2)$, then

$$[E]^\ell = E.$$

Proof of Theorem 1.9. — Assume that (a) holds. It follows from Theorem 4.1(ii) that $\Gamma_M \subseteq \rho_M(H^1(M, \mathbb{Z}))$ and $\Gamma_N \subseteq \rho_N(H^1(N, \mathbb{Z}))$. Since $\mathrm{Alg}^*(-)$ and $H^*_{\mathrm{alg}}(-, \mathbb{Z}/2)$ are functors, $\mathrm{Alg}^*(-) \subseteq H^*_{\mathrm{alg}}(-, \mathbb{Z}/2)$, $w_k(-) \in H^*_{\mathrm{alg}}(-, \mathbb{Z}/2)$ for all $k \geqslant 0$, and $H^*_{\mathrm{alg}}(-, \mathbb{Z}/2)$ is a ring, one just needs to apply Theorem 4.1(i) to see that (b) is satisfied.

We now prove that (b) implies (a); the proof is rather long and involved. Suppose then that (b) holds. First we need several auxiliary constructions. We may assume that M is a smooth submanifold of $\mathbb{R}^d$, where $d \geqslant 2m + 1$.

Denote by τ_M the tangent bundle to M and choose a smooth map $h : M \to \mathbb{G}_{d,m}(\mathbb{R})$ such that

$$h^*\gamma_{d,m} \text{ is isomorphic to } \tau_M, \tag{1}$$

where $\gamma_{d,m}$ is the universal vector bundle on $\mathbb{G}_{d,m}(\mathbb{R})$.

Let K be a sufficiently large positive integer such that if $A_M = \mathbb{P}^K(\mathbb{R}) \times \cdots \times \mathbb{P}^K(\mathbb{R})$ is the $(\dim_{\mathbb{Z}/2} G_M)$-fold product and $A_N = \mathbb{P}^K(\mathbb{R}) \times \cdots \times \mathbb{P}^K(\mathbb{R})$ is the $(\dim_{\mathbb{Z}/2} G_N)$-fold product, then there are smooth maps $f_M : M \to A_M$ and $f_N : N \to A_N$ with

$$f_M^*(H^1(A_M, \mathbb{Z}/2)) = G_M, \tag{2}$$

$$f_N^*(H^1(A_N, \mathbb{Z}/2)) = G_N. \tag{3}$$

Since $e^*(G_M) \subseteq G_N$, the restriction $f_M|N : N \to A_M$ satisfies

$$(f_M|N)^*(H^1(A_M, \mathbb{Z}/2)) \subseteq G_N. \tag{4}$$

Set

$$A = \mathbb{G}_{d,m}(\mathbb{R}) \times A_M \times A_N,\ f = (h|N,\ f_M|N,\ f_N) : N \to A.$$

In view of (1), we have $w_q(M) = h^*(w_q(\gamma_{d,m}))$ and hence $e^*(w_q(M)) = (h|N)^*(w_q(\gamma_{d,m}))$ for all $q \geqslant 0$. Recall that $H^*(\mathbb{G}_{d,m}(\mathbb{R}), \mathbb{Z}/2)$ is generated (as a ring) by $w_q(\gamma_{d,m})$, $q \geqslant 0$, cf. [35]. It therefore follows from (3), (4), and Künneth's theorem in cohomology that

$$f^*(H^p(A, \mathbb{Z}/2)) = [e^*(SW^*(M)), G_N]^p \text{ for all } p \geqslant 0. \tag{5}$$

Taking $p = 1$ and making use of $w_1(M) \in G_M$ and $e^*(G_M) \subseteq G_N$, we get

$$f^*(H^1(A, \mathbb{Z}/2)) = G_N. \tag{6}$$

Since $\Gamma_M \subseteq \rho_M(H^1(M, \mathbb{Z}))$ and $\Gamma_N \subseteq \rho_N(H^1(N, \mathbb{Z}))$, it follows that Γ_M and Γ_N consist of spherical cohomology classes, cf. [28, p. 49, Theorem 7.1]. Hence if

$$d_M = \dim_{\mathbb{Z}/2} \Gamma_M \text{ and } d_N = \dim_{\mathbb{Z}/2} \Gamma_N,$$

there exist smooth maps $g_M : M \to B^1(d_M)$ and $g_N : N \to B^1(d_N)$ (notation as in Lemma 4.2) such that

$$g_M(M) \subseteq B_0^1(d_M),\ g_M^*(H^1(B^1(d_M), \mathbb{Z}/2)) = \Gamma_M, \tag{7}$$

$$g_N(N) \subseteq B_0^1(d_N),\ g_N^*(H^1(B^1(d_N), \mathbb{Z}/2)) = \Gamma_N. \tag{8}$$

Making use of $e^*(\Gamma_M) \subseteq \Gamma_N$, we conclude that the restriction $g_M|N : N \to B^1(d_M)$ satisfies

$$(g_M|N)^*(H^1(B^1(d_M), \mathbb{Z}/2)) \subseteq \Gamma_N. \tag{9}$$

Set

$$\bar{\Gamma}_M = \{u \in H^{m-1}(M, \mathbb{Z}/2) | \langle a \cup u, [M]\rangle = 0 \text{ for all } a \in G_M\},$$
$$\bar{\Gamma}_N = \{v \in H^{n-1}(N, \mathbb{Z}/2) | \langle b \cup v, [N]\rangle = 0 \text{ for all } b \in G_N\}.$$

Since M is connected, given u in $H^{m-1}(M, \mathbb{Z}/2)$ with $\langle w_1(M) \cup u, [M]\rangle = 0$, we get $w_1(M) \cup u = 0$. The last equality implies that the homology class in $H_1(M, \mathbb{Z}/2)$ Poincaré dual to u can be represented by a compact smooth curve in M with trivial normal vector bundle, cf. for example [13, p. 599]. This in turn implies that u is a spherical cohomology class [42, Théorème II.1]. By assumption, $w_1(M) \in G_M$ and hence $\bar{\Gamma}_M$ consists of spherical cohomology classes. An analogous argument shows that $\bar{\Gamma}_N$ also consists of spherical cohomology classes. Therefore, if

$$\bar{d}_M = \dim_{\mathbb{Z}/2} \bar{\Gamma}_M \text{ and } \bar{d}_N = \dim_{\mathbb{Z}/2} \bar{\Gamma}_N,$$

there exist smooth maps $\bar{g}_M : M \to B^{m-1}(\bar{d}_M)$ and $\bar{g}_N : N \to B^{n-1}(\bar{d}_N)$ (notation as in Lemma 4.2) such that

$$\bar{g}_M(M) \subseteq B_0^{m-1}(\bar{d}_M),\ \bar{g}_M^*(H^{m-1}(\bar{d}_M), \mathbb{Z}/2) = \bar{\Gamma}_M, \tag{10}$$

$$\bar{g}_N(N) \subseteq B_0^{n-1}(\bar{d}_N),\ \bar{g}_N^*(H^{n-1}(\bar{d}_N), \mathbb{Z}/2) = \bar{\Gamma}_N. \tag{11}$$

If

$$B = B^1(d_M) \times B^1(d_N) \times B^{m-1}(\bar{d}_M) \times B^{n-1}(\bar{d}_N),$$
$$B_0 = B_0^1(d_M) \times B_0^1(d_N) \times B_0^{m-1}(\bar{d}_M) \times B_0^{n-1}(\bar{d}_N),$$
$$g = (g_M|N, g_N, \bar{g}_M|N, g_N) : N \to B,$$

then

$$g(N) \subseteq B_0. \tag{12}$$

Moreover, since $m - 1 > n - 1 > 1$, making use of (8), (9), and Künneth's theorem in cohomology, we get

$$g^*(H^q(B, \mathbb{Z}/2)) = [\Gamma_N]^q \text{ for } 1 \leqslant q \leqslant n-2. \tag{13}$$

Similarly, taking into account also (11), we obtain

$$g^*(H^{n-1}(B, \mathbb{Z}/2)) = [\Gamma_N, \bar{\Gamma}_N]^{n-1}, \tag{14}$$

$$g^*(H^n(B, \mathbb{Z}/2)) = [\Gamma_N, \bar{\Gamma}_N]^n \text{ if } m-1 > n, \tag{15}$$

while (10) yields

$$g^*(H^n(B, \mathbb{Z}/2)) = [\Gamma_N, \bar{\Gamma}_N, e^*(\bar{\Gamma}_M)]^n \text{ if } m-1 = n. \tag{16}$$

Set

$$\bar{G}_M = \{u \in H^{m-1}(M, \mathbb{Z}/2) | \langle a \cup u, [M] \rangle = 0 \text{ for all } a \in \Gamma_M\},$$
$$\bar{G}_N = \{v \in H^{n-1}(N, \mathbb{Z}/2) | \langle b \cup v, [N] \rangle = 0 \text{ for all } b \in \Gamma_N\}.$$

Choose smooth submanifolds (curves) S_i of M and T_j of N such that

$$\bar{G}_M = \{[S_1]^M, \dots, [S_k]^M\},\ \bar{G}_N = \{[T_1]^N, \dots, [T_\ell]^N\}.$$

We may assume that $S_1, \dots, S_k, T_1, \dots, T_\ell$ are pairwise disjoint. Furthermore, we may choose S_i so that it is transverse to N in M for $1 \leqslant i \leqslant k$. By definition of $[S_i]^M$, we have

$$\epsilon_{i*}([S_i]) = [S_i]^M \cap [M],$$

where $\epsilon_i : S_i \hookrightarrow M$ is the inclusion map and $\cap$ stands for the cap product. Note that

$$\langle \epsilon_i^*(a), [S_i] \rangle = \langle a \cup [S_i]^M, [M] \rangle \text{ for all } a \in H^1(M, \mathbb{Z}/2). \tag{17}$$

Indeed, standard properties of $\cup, \cap,\ \langle\ ,\rangle$ (cf. for example [19]) yield

$$\begin{aligned}\langle \epsilon_i^*(a), [S_i] \rangle &= \langle a, \epsilon_{i*}([S_i]) \rangle \\ &= \langle a, [S_i]^M \cap [M] \rangle \\ &= \langle a \cup [S_i]^M, [M] \rangle,\end{aligned}$$

as required. By Poincaré duality (cf. the version given in [19, p. 300, Proposition 8.13]),

$$\Gamma_M = \{a \in H^1(M, \mathbb{Z}/2) | \langle a \cup u, [M] \rangle = 0 \text{ for all } u \in \bar{G}_M\},$$

and hence (17) implies

$$\Gamma_M = \{a \in H^1(M, \mathbb{Z}/2) | \langle \epsilon_i^*(a), [S_i] \rangle = 0 \text{ for } 1 \leqslant i \leqslant k\}. \tag{18}$$

An analogous argument yields

$$\Gamma_N = \{b \in H^1(N, \mathbb{Z}/2) | \langle \delta_j^*(b), [T_j] \rangle = 0 \text{ for } 1 \leqslant j \leqslant \ell\}. \tag{19}$$

where $\delta_j : T_j \hookrightarrow N$ is the inclusion map.

We have completed now the basic setup necessary for the proof of (b) $\Rightarrow$ (a). In what follows we will successively modify the smooth submanifolds $T_1, \dots, T_\ell, N, S_1, \dots, S_k, M$ of $\mathbb{R}^d$ to ensure that they satisfy some additional desirable conditions. Here "modify" means that a given smooth submanifold of $\mathbb{R}^d$ is replaced by an isotopic copy, via a smooth isotopy close in the $\mathcal{C}^\infty$ topology to the appropriate inclusion map (such an isotopy can be extended to a smooth isotopy of $\mathbb{R}^d$, cf. [27, Chapter 8]; this fact will be used repeatedly without an explicit reference). Eventually, after modifications, all the submanifolds listed above will become nonsingular subvarieties of $\mathbb{R}^d$, and the subvarieties corresponding to N and M will satisfy (a). The main tool which enables us to perform the required task is Theorem 4.4.

Since $\mathcal{N}_*^{\mathrm{alg}}(A) = \mathcal{N}_*(A)$ (cf. (4.1)), Theorem 4.4 can be applied to $f|T_j : T_j \to A$ (with $L = \emptyset$), and hence we may assume that T_j is a nonsingular subvariety of $\mathbb{R}^d$ and $f|T_j : T_j \to A$ is a regular map for $1 \leqslant j \leqslant \ell$.

Let $c : N \to B$ be a constant map sending N to a point in B_0.

Claim 1. — The maps $(f, g)|T_j : T_j \to A \times B$ and $(f, c)|T_j : T_j \to A \times B$ represent the same bordism class in $\mathcal{N}_*(A \times B)$.

In order to prove Claim 1 we argue as follows. Since $\dim T_j = 1$, we have $w_1(T_j) = 0$, and hence in view of Theorem 4.3 and Künneth's theorem in cohomology, it suffices to show that

$$\langle ((f,g)|T_j)^*(\xi \times \eta), [T_j] \rangle = \langle ((f,c)|T_j)^*(\xi \times \eta), [T_j] \rangle$$

for all ξ in $H^p(A, \mathbb{Z}/2)$ and η in $H^q(B, \mathbb{Z}/2)$ with $p + q = 1$. There are two cases to deal with: $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$. Observing

$$((f,g)|T_j)^*(\xi \times \eta) = (f|T_j)^*(\xi) \cup (g|T_j)^*(\eta),$$
$$((f,c)|T_j)^*(\xi \times \eta) = (f|T_j)^*(\xi) \cup (c|T_j)^*(\eta),$$

we conclude that the equality under consideration holds when $(p, q) = (1, 0)$ ((12) implies $(g|T_j)^*(\eta) = (c|T_j)^*(\eta)$), while for $(p, q) = (0, 1)$ it is equivalent to

$$\langle (g|T_j)^*(\eta), [T_j] \rangle = 0.$$

The last equality follows from (13) and (19) since $(g|T_j)^*(\eta) = (g \circ \delta_j)^*(\eta) = \delta_j^*(g^*(\eta))$. Claim 1 is proved.

Since $(f, c)|T_j : T_j \to A \times B$ is a regular map, Claim 1 allows us to apply Theorem 4.4 to $(f, g)|T_j : T_j \to A \times B$ (with $L = 0$). Hence modifying T_j once again, we may assume that T_j is a nonsingular subvariety of $\mathbb{R}^d$ and $(f, g)|T_j : T_j \to A \times B$ is a regular map for $1 \leqslant j \leqslant \ell$. Henceforth $T_1, \ldots, T_\ell$ will remain unchanged, but we will modify N in a suitable way.

Note that $T = T_1 \cup \ldots \cup T_\ell$ is a nonsingular subvariety of $\mathbb{R}^d$ and $(f, g)|T : T \to A \times B$ is a regular map. Since $\dim T = 1$, it follows that $\tau_N|T$ is isomorphic to an algebraic vector bundle on T, cf. [12, Theorem 12.5.1]. In view of $\mathcal{N}_*^{\mathrm{alg}}(A) = \mathcal{N}_*(A)$, Theorem 4.4 can be applied to $f : N \to A$ (with $L = T$). Therefore we may assume that N is a nonsingular subvariety of $\mathbb{R}^d$, T and $f|T : T \to A$ remain unchanged, and $f : N \to A$ is a regular map.

Claim 2. — The maps $(f, g) : N \to A \times B$ and $(f, c) : N \to A \times B$ represent the same bordism class in $\mathcal{N}_*(A \times B)$.

The proof of Claim 2 is similar to that of Claim 1, but technically more complicated. In view of Theorem 4.3 and Künneth's theorem in cohomology, it suffices to show that given cohomology classes ξ in $H^p(A,\mathbb{Z}/2)$ and η in $H^q(B,\mathbb{Z}/2)$ with $p+q \leqslant n$, we have

$$\langle (f,g)^*(\xi \times \eta) \cup w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N]\rangle = \langle (f,c)^*(\xi \times \eta) \cup w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N]\rangle$$

for all nonnegative integers $i_1,\ldots,i_r$ satisfying $i_1+\cdots+i_r = n-(p+q)$. Since $(f,g)^*(\xi\times\eta) = f^*(\xi)\cup g^*(\eta)$ and $(f,c)^*(\xi\times\eta) = f^*(\xi)\cup c^*(\eta)$, the equality under consideration holds if $q=0$ ((12) implies $g^*(\eta)=c^*(\eta)$), whereas for $q \geqslant 1$ it is equivalent to

$$\langle f^*(\xi)\cup g^*(\eta) \cup w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N]\rangle = 0. \tag{20}$$

In the proof of (20) we distinguish three cases: $1 \leqslant q \leqslant n-2$, $q=n-1$, and $q=n$.

If $1 \leqslant q \leqslant n-2$, then in view of (5), (13), and $\Gamma_N \subseteq G_N$, the cohomology class

$$f^*(\xi)\cup g^*(\eta) \cup w_{i_1}(N) \cup \ldots \cup w_{i_r}(N)$$

is a sum of finitely many elements of the form $b \cup z$, where $b \in \Gamma_N$ and $z \in SW^{n-1}(N; G_N, e^*(SW^*(M)))$. Hence (20) follows from (b_2), which appears in (b) in Theorem 1.9.

If $q = n-1$, then (14) implies that $g^*(\eta)$ is a finite sum of elements of the form v_1+v_2, where $v_1 \in [\Gamma_N]^{n-1}$ and $v_2 \in \bar{\Gamma}_N$, There are two subcases to consider: $p=0$ and $p=1$.

Suppose $p=0$. Then (20) is equivalent to

$$\langle g^*(\eta)\cup w_1(N), [N]\rangle = 0. \tag{20$'$}$$

Since $\Gamma_N \subseteq G_N$, we conclude that $v_1 \cup w_1(N)$ is a finite sum of elements of the form $b\cup z$, where $b \in \Gamma_N$ and $z \in SW^{n-1}(N;G_N)$, and hence (b_2) yields $\langle v_1 \cup w_1(N), [N]\rangle = 0$. On the other hand, $w_1(N) \in G_N$ and the definition of $\bar{\Gamma}_N$ imply $\langle v_2 \cup w_1(N), [N]\rangle = 0$. Thus (20$'$) holds when $p=0$.

Suppose now $p=1$. Then (20) is equivalent to

$$\langle f^*(\xi)\cup g^*(\eta), [N]\rangle = 0. \tag{20$''$}$$

In view of (6), we have $f^*(\xi) \in G_N$. Hence $f^*(\xi)\cup g^*(\eta)$ is a finite sum of elements of the form $(f^*(\xi)\cup v_1) + (f^*(\xi)\cup v_2)$. Applying (b_2), we get $\langle f^*(\xi)\cup v_1, [N]\rangle = 0$, while the definition of $\bar{\Gamma}_N$ implies $\langle f^*(\xi)\cup v_2, [N]\rangle = 0$. Thus (20$''$) holds when $p=1$. The proof in case $q=n-1$ is complete.

If $q = n$, then $p = 0$ and (20) is equivalent to

$$(20''') \qquad \langle g^*(\eta), [N] \rangle = 0.$$

Once again, we consider two subcases: $m-1>n$ and $m-1=n$.

Suppose $m-1>n$. Then (15) implies that $g^*(\eta)$ is a finite sum of elements of the form $b \cup z$, where $b \in \Gamma_N$ and $z \in [\Gamma_N, \bar{\Gamma}_N]^{n-1}$. Clearly, $z = z_1 + z_2$, where $z_1 \in [\Gamma_N]^{n-1}$ and $z_2 \in \bar{\Gamma}_N$. Since $\Gamma_N \subseteq G_N$, applying (b_2), we get $\langle b \cup z_1, [N] \rangle = 0$, while the definition of $\bar{\Gamma}_N$ yields $\langle b \cup z_2, [N] \rangle = 0$. Thus (20‴) holds when $m-1>n$.

Suppose $m-1=n$. In view of (16), $g^*(\eta)$ is a finite sum of elements of the form $b_1 \cup v_1 + b_2 \cup v_2 + e^*(u)$, where $b_1, b_2 \in \Gamma_N$, $v_1 \in [\Gamma_N]^{n-1} \subseteq [G_N]^{n-1}$, $v_2 \in \bar{\Gamma}_N$, and $u \in \bar{\Gamma}_M$. It follows from (b_2) that $\langle b_1 \cup z_1, [N] \rangle = 0$. Since $\Gamma_N \subseteq G_N$, the definition of $\bar{\Gamma}_N$ yields $\langle b_2 \cup z_2, [N] \rangle = 0$. In order to complete the proof of (20‴) it remains to justify $\langle e^*(u), [N] \rangle = 0$. To this end observe

$$\langle e^*(u), [N] \rangle = \langle u, e_*([N]) \rangle = \langle u, [N]^M \cap [M] \rangle = \langle u \cup [N]^M, [M] \rangle.$$

By assumption, $[N]^M \in G_M$ and hence the definition of $\bar{\Gamma}_M$ implies $\langle u \cup [N]^M, [M] \rangle = 0$. Thus (20‴) holds when $m-1=n$. Claim 2 is proved.

We are now ready to construct the final modification of N. We already know that $(f,g)|T : T \to A \times B$ is a regular map and $\tau_N|T$ is isomorphic to an algebraic vector bundle on T. Since $(f,c) : N \to A \times B$ is a regular map, Claim 2 allows us to apply Theorem 4.4 to the map $(f,g) : N \to A \times B$ (with $L = T$). We may therefore assume that N is a nonsingular subvariety of $\mathbb{R}^d$, T and $(f,g)|T : T \to A \times B$ remain unchanged, and $(f,g) : N \to A \times B$ is a regular map.

Recall that

$$f = (h|N, f_M|N, f_N) \text{ and } g = (g_M|N, g_N, \bar{g}_M|N, \bar{g}_N).$$

In particular, $f_N : N \to A_N$ is a regular map, and hence (3) and $H^1_{\mathrm{alg}}(A_N, \mathbb{Z}/2) = H^1(A_N, \mathbb{Z}/2)$ imply

$$(21) \qquad G_N = f_N^*(H^1(A_N, \mathbb{Z}/2)) \subseteq H^1_{\mathrm{alg}}(N, \mathbb{Z}/2).$$

Since $\bar{g}_N : N \to B^{n-1}(\bar{d}_N)$ is a regular map, it follows from (11) and Lemma 4.2 that

$$(22) \qquad \bar{\Gamma}_N = \bar{g}_N^*(H^{n-1}(B^{n-1}(\bar{d}_N), \mathbb{Z}/2)) \subseteq \mathrm{Alg}^{n-1}(N).$$

Making use of (21), (22), Theorem 4.1(i), and the definition of $\bar{\Gamma}_N$, we obtain

$$(23) \qquad H^1_{\mathrm{alg}}(N, \mathbb{Z}/2) = G_N.$$

Since $g_N : N \to B^1(d_N)$ is a regular map, (8) and Lemma 4.2 imply

$$\Gamma_N = g_N^*(H^1(B^1(d_N), \mathbb{Z}/2)) \subseteq \mathrm{Alg}^1(N).$$

Suppose there is an element b in $\mathrm{Alg}^1(N)\backslash\Gamma_N$. By (19), one can find j, $1 \leqslant j \leqslant \ell$, for which $\langle \delta_j^*(b), [T_j]\rangle \neq 0$. This contradicts Theorem 4.1(i) since $\delta_j^*(b)$ belongs to $\mathrm{Alg}^1(T_j)$, the map $\delta_j : T_j \hookrightarrow N$ being regular. Thus

$$\mathrm{Alg}^1(N) = \Gamma_N. \tag{24}$$

Henceforth N will remain unchanged, but $S_1, \dots, S_k, M$ will be successively modified. Set

$$C = \mathbb{G}_{d,m}(\mathbb{R}) \times A_M,$$
$$\alpha = (h, f_M) : M \to C,$$
$$D = B^1(d_M) \times B^{m-1}(\bar{d}_M),\ D_0 = B_0^1(d_M) \times B_0^{m-1}(\bar{d}_M),$$
$$\beta = (g_M, \bar{g}_M) : M \to D.$$

Using the same argument which justified (5), we get

$$\alpha^*(H^p(C, \mathbb{Z}/2)) = SW^p(M; G_M) \text{ for all } p \geqslant 0. \tag{25}$$

In particular, since $w_1(M) \in G_M$, for $p = 1$ we have

$$\alpha^*(H^1(C, \mathbb{Z}/2)) = G_M. \tag{26}$$

Similarly, in view of (7) and (10), the argument which justified (13), (14), (15), (16) yields

$$\beta^*(H^q(D, \mathbb{Z}/2)) = [\Gamma_M]^q \text{ for } 1 \leqslant q \leqslant m-2, \tag{27}$$

$$\beta^*(H^q(D, \mathbb{Z}/2)) = [\Gamma_M, \bar{\Gamma}_M]^q \text{ for } q = m-1 \text{ or } q = m. \tag{28}$$

By construction, we also have

$$\beta(M) \subseteq D_0. \tag{29}$$

Recall that S_i is transverse to N in M. In particular, $S_i \cap N$ is a finite set, and hence a nonsingular subvariety of $\mathbb{R}^d$. Since $\mathcal{N}_*^{\mathrm{alg}}(C) = \mathcal{N}_*(C)$ (cf. (4.1)), Theorem 4.4 can be applied to $\alpha|S_i : S_i \to C$ (with $L = S_i \cap N$). Thus there exist a smooth embedding $e_i : S_i \to \mathbb{R}^d$, a nonsingular subvariety X_i of $\mathbb{R}^d$, and a regular map $\alpha_i : X_i \to C$ such that $S_i \cap N \subseteq X_i = e_i(S_i)$, $e_i|S_i \cap N : S_i \cap N \to \mathbb{R}^d$ is the inclusion map, $\alpha_i|S_i \cap N = \alpha|S_i \cap N$, e_i is close in the $\mathcal{C}^\infty$ topology to the inclusion map $S_i \hookrightarrow \mathbb{R}^d$, and $\alpha_i \circ \bar{e}_i$ is close in the $\mathcal{C}^\infty$ topology to $\alpha|S_i$, where $\bar{e}_i : S_i \to X_i$ is defined by $\bar{e}_i(x) = e_i(x)$ for all x in S_i. Note that $e_i : S_i \to \mathbb{R}^d$ can be extended to a smooth embedding $E_i : M \to \mathbb{R}^d$ such that $E_i(y) = y$ for all y in $N \cup S_1 \cup \ldots \cup S_{i-1} \cup S_{i+1} \cup \ldots \cup S_k$ (cf. the standard proofs of the isotopy extension theorems [27, Chapter 8]). Hence replacing M by $E_i(M)$ and S_i by $X_i = E_i(S_i)$, and making use of Lemma 4.5, we may assume that S_i is a nonsingular subvariety of $\mathbb{R}^d$ and $\alpha|S_i : S_i \to C$ is a regular map for $1 \leqslant i \leqslant k$, while N and $\alpha|N : N \to C$ remain unchanged.

Let $\gamma : M \to D$ be a constant map sending M to a point in D_0.

Claim 3. — The maps $(\alpha, \beta)|S_i : S_i \to C \times D$ and $(\alpha, \gamma)|S_i : S_i \to C \times D$ represent the same bordism class in $\mathcal{N}_*(C \times D)$.

The proof of Claim 3 is entirely analogous to that of Claim 1. A minor difference is that instead of (13) and (19) one uses (27) and (18). Details are left to the reader.

Since $(\alpha, \gamma)|S_i : S_i \to C \times D$ is a regular map, it follows from Claim 3 that Theorem 4.4 can be applied to $(\alpha, \beta)|S_i : S_i \to C \times D$ (with $L = S_i \cap N$). Arguing as in the paragraph preceding Claim 3, we may assume that S_i is a nonsingular subvariety of $\mathbb{R}^d$ and $(\alpha, \beta)|S_i : S_i \to C \times D$ is a regular map for $1 \leqslant i \leqslant k$, while N and $(\alpha, \beta)|N : N \to C \times D$ remain unchanged. Henceforth $N, S_1, \ldots, S_k$ will remain unchanged, but we still have to modify M.

Note that $S = S_1 \cup \ldots \cup S_k$ is a nonsingular subvariety of $\mathbb{R}^d$. Since S is transverse to N in M, $(\alpha, \beta) : M \to C \times D$ is continuous, and $(\alpha, \beta)|N : N \to C \times D$, $(\alpha, \beta)|S : S \to C \times D$ are regular maps, it follows (cf. for example [10, Lemme 5] or [44, Lemma 6]) that $(\alpha, \beta)|(N \cup S) : N \cup S \to C \times D$ is a regular map. Furthermore, in view of (1) and the definition of α, the restriction $\tau_M|(N \cup S)$ is isomorphic to an algebraic vector bundle on $N \cup S$. The last two facts together with $\mathcal{N}_*^{\mathrm{alg}}(C) = \mathcal{N}_*(C)$ imply that Theorem 4.4 can be applied to $\alpha : M \to C$ (with $L = N \cup S$). Hence we may assume that M is a nonsingular subvariety of $\mathbb{R}^d$, $N \cup S$ and $\alpha|(N \cup S) : N \cup S \to C$ remain unchanged, and $\alpha : M \to C$ is a regular map.

Claim 4. — The maps $(\alpha, \beta) : M \to C \times D$ and $(\alpha, \gamma) : M \to C \times D$ represent the same bordism class in $\mathcal{N}_*(C \times D)$.

As in the proof of Claim 2, it suffices to show that given cohomology classes κ in $H^p(C, \mathbb{Z}/2)$ and λ in $H^q(D, \mathbb{Z}/2)$ with $p + q \leqslant m$, we have

$$\langle (\alpha, \beta)^*(\kappa \times \lambda) \cup w_{j_1}(M) \cup \ldots \cup w_{j_s}(M), [M] \rangle = \langle (\alpha, \gamma)^*(\kappa \times \lambda) \cup w_{j_1}(M) \cup \ldots \cup w_{j_s}(M), [M] \rangle$$

for all nonnegative integers $j_1, \ldots, j_s$ satisfying $j_1 + \cdots + j_s = m - (p + q)$. Since $(\alpha, \beta)^*(\kappa \times \lambda) = \alpha^*(\kappa) \cup \beta^*(\lambda)$ and $(\alpha, \gamma)^*(\kappa \times \lambda) = \alpha^*(\kappa) \cup \gamma^*(\lambda)$, the equality under consideration holds if $q = 0$ ((29) implies $\beta^*(\lambda) = \gamma^*(\lambda)$), whereas for $q \geqslant 1$ it is equivalent to

$$(30) \qquad \langle \alpha^*(\kappa) \cup \beta^*(\lambda) \cup w_{j_1}(M) \cup \ldots \cup w_{j_s}(M), [M]\rangle = 0.$$

In the proof of (30) we distinguish three cases: $1 \leqslant q \leqslant m-2$, $q = m-1$, and $q = m$.

If $1 \leqslant q \leqslant m-2$, then in view of (25), (27), and $\Gamma_M \subseteq G_M$, the cohomology class

$$\alpha^*(\kappa) \cup \beta^*(\lambda) \cup w_{j_1}(M) \cup \ldots \cup w_{j_s}(M)$$

is a finite sum of elements of the form $a \cup w$, where $a \in \Gamma_M$ and $w \in SW^{m-1}(M; G_M)$. Hence (30) follows from (b_1), which appears in (b) in Theorem 1.9.

If $q = m-1$, then (28) implies that $\beta^*(\lambda)$ is a finite sum of elements of the form $u_1 + u_2$, where $u_1 \in [\Gamma_M]^{m-1}$ and $u_2 \in \bar{\Gamma}_M$. There are two subcases to consider: $p = 0$ and $p = 1$.

Suppose $p = 0$. Then (30) is equivalent to

$$(30') \qquad \langle \beta^*(\lambda) \cup w_1(M), [M]\rangle = 0.$$

Since $\Gamma_M \subseteq G_M$, we conclude that $u_1 \cup w_1(M)$ is an finite sum of elements of the form $a \cup w$, where $a \in \Gamma_M$ and $w \in SW^{m-1}(M; G_M)$, and hence (b_1) yields $\langle u_1 \cup w_1(M), [M]\rangle = 0$. On the other hand, $w_1(M) \in G_M$ and the definition of $\bar{\Gamma}_M$ imply $\langle u_2 \cup w_1(M), [M]\rangle = 0$. Thus (30′) holds when $p = 0$.

Suppose now $p = 1$. Then (30) is equivalent to

$$(30'') \qquad \langle \alpha^*(\kappa) \cup \beta^*(\lambda), [M]\rangle = 0.$$

In view of (26), we have $\alpha^*(\kappa) \in G_M$. Hence $\alpha^*(\kappa) \cup \beta^*(\lambda)$ is a finite sum of elements of the form $(\alpha^*(\kappa) \cup u_1) + (\alpha^*(\kappa) \cup u_2)$. Applying (b_1), we get $\langle \alpha^*(\kappa) \cup u_1, [M]\rangle = 0$, while the definition of $\bar{\Gamma}_M$ implies $\langle \alpha^*(\kappa) \cup u_2, [M]\rangle = 0$. Thus (30″) holds when $p = 1$. The proof in case $q = m-1$ is complete.

If $q = m$, then $p = 0$ and (30) is equivalent to

$$(30''') \qquad \langle \beta^*(\lambda), [M]\rangle = 0.$$

By (28), $\beta^*(\lambda)$ is a finite sum of elements of the form $a_1 \cup u_1 + a_2 \cup u_2$, where $a_1, a_2 \in \Gamma_M$, $u_1 \in [\Gamma_M]^{m-1} \subseteq [G_M]^{m-1}$, $u_2 \in \bar{\Gamma}_M$. It follows from (b_1) that $\langle a_1 \cup u_1, [M]\rangle = 0$. On the other hand, $\Gamma_M \subseteq G_M$ and the definition of $\bar{\Gamma}_M$ imply $\langle a_2 \cup u_2, [M]\rangle = 0$. Thus (30‴) holds when $q = m$. Claim 4 is proved.

Now the final modification of M will be constructed. We already know that

$(\alpha,\beta)|(N\cup S) : N\cup S\to C\times D$ is a regular map and $\tau_M|(N\cup S)$ is isomorphic to an algebraic vector bundle on $N\cup S$. Since $(\alpha,\gamma): M\to C\times D$ is a regular map, Claim 4 allows us to apply Theorem 4.4 to the map $(\alpha,\beta): M\to C\times D$ (with $L=N\cup S$). We may therefore assume that M is a nonsingular subvariety of $\mathbb{R}^d$ and $(\alpha,\beta): M\to C\times D$ is a regular map, while $N\cup S$ and $(\alpha,\beta)|(N\cup S): N\cup S\to C\times D$ remain unchanged.

Recall that

$$\alpha=(h,f_M) \text{ and } \beta=(g_M,\bar{g}_M).$$

In particular, $f_M : M\to A_M$ is a regular map, and hence (2) and $H^1_{\mathrm{alg}}(A_M,\mathbb{Z}/2)=H^1(A_M,\mathbb{Z}/2)$ imply

$$G_M=f_M^*(H^1(A_M,\mathbb{Z}/2))\subseteq H^1_{\mathrm{alg}}(M,\mathbb{Z}/2) \tag{31}$$

Since $\bar{g}_M : M\to B^{m-1}(\bar{d}_M)$ is a regular map, it follows from (10) and Lemma 4.2 that

$$\bar{\Gamma}_M=\bar{g}_M^*(H^{m-1}(B^{m-1}(\bar{d}_M),\mathbb{Z}/2))\subseteq \mathrm{Alg}^{m-1}(M). \tag{32}$$

Making use of (31), (32), Theorem 4.1(i), and the definition of $\bar{\Gamma}_M$, we obtain

$$H^1_{\mathrm{alg}}(M,\mathbb{Z}/2)=G_M. \tag{33}$$

Since $g_M : M\to B^1(d_M)$ is a regular map, (7) and Lemma 4.2 imply

$$\Gamma_M=g_M^*(H^1(d_M),\mathbb{Z}/2)\subseteq \mathrm{Alg}^1(M).$$

Suppose there is an element a in $\mathrm{Alg}^1(M)\backslash\Gamma_M$. By (18), one can find $i,\ 1\leqslant i\leqslant k$, for which $\langle\epsilon_i^*(a),[S_j]\rangle\neq 0$. This contradicts Theorem 4.1(i) since $\epsilon_i^*(a)$ belongs to $\mathrm{Alg}^1(S_i)$, the map $\epsilon_i : S_i\hookrightarrow M$ being regular. Thus

$$\mathrm{Alg}^1(M)=\Gamma_M. \tag{34}$$

In view of (23), (24), (33), (34), condition (a) holds. We proved that (b) implies (a). □

Proof of Theorem 1.11. — As in the proof of Theorem 1.9, one readily sees that (a) implies (b). Assume then that (b) is satisfied. Below we show that (a) holds. Let y_0 be a point in the unit circle S^1 and let $M=N\times S^1$. Note that $w_q(M)=w_q(N)\times 1$ for $q\geqslant 0$, where 1 is the identity element in $H^0(S^1,\mathbb{Z}/2)$ and $\times$ stands for the cross product in cohomology. Set $G_N=G$ and $\Gamma_N=\Gamma$. Define G_M to be the subgroup of $H^1(M,\mathbb{Z}/2)$ generated by $[N\times\{y\}]^M$ and all elements of the form $u\times 1$, where u is in G_N. Similarly, let Γ_M be the subgroup of $H^1(M,\mathbb{Z}/2)$ generated by all elements of the form $v\times 1$, where v is in Γ_N. Identify N with $N\times\{y_0\}$ and write $e : N\hookrightarrow M$ for the inclusion map. By construction, $e^*(G_M)=G_N$,

$e^*(\Gamma_M) = \Gamma_N$, and $e^*(w_q(M)) = w_q(N)$. It follows that condition (b) of Theorem 1.9 is satisfied.

If $\dim N \geqslant 3$, then (a) immediately follows from Theorem 1.9.

Suppose then that $\dim N = 2$. Since $\dim M = 3$, it follows from what we already proved that there exist a nonsingular real algebraic variety X and a smooth diffeomorphism $\varphi : X \to M$ such that

$$\varphi^*(G_M) = H^1_{\mathrm{alg}}(X, \mathbb{Z}/2),\ \varphi^*(\Gamma_M) = \mathrm{Alg}^1(X).$$

Since $[\varphi^{-1}(N)]^X = \varphi^*([N]^M)$ is in $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$, Corollary 1.8 implies that the smooth submanifold $\varphi^{-1}(N)$ of X is admissible. Taking into account $e^*(G_M) = G_N$ and $e^*(\Gamma_M) = \Gamma_N$, we conclude that (a) also holds when $\dim N = 2$. The proof is complete.

□

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Manuscrit reçu le 16 novembre 2006,
accepté le 15 mars 2007.

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