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Algebraic complete integrability of an integrable system of Beauville

ALGEBRAIC COMPLETE INTEGRABILITY OF AN INTEGRABLE SYSTEM OF BEAUVILLE

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ABSTRACT. — We show that the Beauville's integrable system on a ten dimensional moduli space of sheaves on a K3 surface constructed via a moduli space of stable sheaves on cubic threefolds is algebraically completely integrable, using O'Grady's construction of a symplectic resolution of the moduli space of sheaves on a K3.

RÉSUMÉ. — Nous montrons que le système intégrable de Beauville sur un espace de dimension dix de modules de faisceaux sur une surface K3 construit par un espace de modules de faisceaux stables sur les cubiques de dimension trois est algébriquement complètement intégrable. Nous utilisons la construction d'O'Grady d'une résolution symplectique de l'espace des modules de faisceaux sur une surface K3.

Introduction

Beauville [1] constructed an interesting completely integrable Hamiltonian system using moduli spaces of stable vector bundles on a K3 surface and cubic threefolds.

The moduli space M_S^v of stable bundles with rank 2, $c_1 = 0$, $c_2 = 4$ on a K3 surface S is a (non-compact) symplectic algebraic manifold by the theorem of Mukai [6]. Assume $S \subset \mathbb{P}^4$ is a K3 surface of degree 6 and let $X \subset \mathbb{P}^4$ be a smooth cubic threefold containing S. Consider the moduli space M_X^v of stable bundles on X with rank 2, $c_1 = 0$, $c_2 = 2$. Then Beauville showed that the restriction of the bundles on X to S actually defines an embedding $r_X : M_X^v \to M_S^v$ and the image $r_X(M_X^v)$ is a Lagrangian submanifold.

Consider the space Π of cubic threefolds containing S. If we vary smooth cubics $X_t \in \Pi$, we get a family of Lagrangian submanifolds $M_{X_t}^v \subset M_S^v$.

Beauville observed that there is an open dense subset $M_S^\circ \subset M_S^v$ on which $\{M_{X_t}^v\}$ form a complete integrable Hamiltonian system. We call this the *Beauville system*.

On the other hand, Druel [2] showed that the moduli space $\overline{M}_X$ of semi-stable sheaves on a cubic X which compactifies M_X^v is a blowing-up of the intermediate Jacobian $J(X)$ of X whose center is the Fano surface $F(X)$ of lines on X. By Druel's description, it turns out that M_X^v is actually an open subset of the intermediate Jacobian $J(X)$ and the complement contains a divisor.

Beauville raised the following question at the end of his article ([1], Remark 9.5 b)): *Are the Hamiltonian vector fields on the fibers of the Beauville system linearized?* This is equivalent to asking if the Hamiltonian vector fields on M_X^v extend to the whole of $J(X)$ for each X. A complete integrable system with this property is often called *algebraically completely integrable*. As Beauville already mentioned there, in order to show the algebraically complete integrability, it is enough to construct some partial compactification of Beauville system such that the fiber is isomorphic to the complement of a subvariety of codimension at least 2 in the intermediate Jacobian. In this article, we solve Beauville's question affirmatively. In other words, we prove the following theorem (for a more precise statement, see §1).

MAIN THEOREM. — *Let $\overline{M}_S$ is the moduli space of semi-stable sheaves which compactifies M_S^v and $\widetilde{M}_S$ be the O'Grady's resolution of $\overline{M}_S$ [7]. Then there exists an open subset U of $\widetilde{M}_S$ such that the Beauville system extends to U and any fiber is the complement of a subvariety of codimension 2 in the intermediate Jacobian $J(X)$. In particular, the Beauville system is an algebraically completely integrable system.*

One may be able to extend the Beauville system even to a *projective* Lagrangian fibration using O'Grady's space. However, as was also mentioned by Beauville ([1] Remark 9.5 a)), O'Grady's space itself does not seem to allow such an extension and it seems necessary to consider a certain flop of O'Grady's space. So far, we have not succeeded in this direction.

This article consists of three sections. In §1, we review the construction of the Beauville system following [1] to fix our notation and state our Main Theorem in a precise way. In the next section, we review a part of O'Grady's analysis of the singularities of $\overline{M}_S$ and his resolution $\widetilde{M}_S$ as much as necessary for the proof of our result. In the last section, we prove our Main Theorem using the preparations in the previous sections.

1. Settings and statement

Through this article, we fix a generic K3 surface $S \subset \mathbb{P}^4$ of degree 6 with $\mathrm{Pic}(S) = \mathbb{Z}\langle \mathscr{O}_S(1) \rangle$. In this section, we review the construction of the Beauville system in [1] and state our result in a precise manner.

Let $\overline{\mathrm{M}}_S$ be the moduli space of semi-stable torsion free sheaves on S of rank 2, $c_1 = 0$, $c_2 = 4$ and $\mathrm{M}_S \subset \overline{\mathrm{M}}_S$ be the open subset of the points corresponding to the stable sheaves. Then, we can easily see that $\dim \overline{\mathrm{M}}_S = 10$. Moreover we have a natural symplectic form on M_S by the theorem of Mukai [6]. Denote the open dense subset of M_S consisting of the points corresponding to stable vector bundles by M_S^v.

Choose a (smooth) cubic threefold $X \subset \mathbb{P}^4$ containing S. Then there is a quadric Q such that $S = X \cap Q$. Let $\overline{\mathrm{M}}_X$ be the moduli space of semi-stable torsion free sheaves on X of rank 2, $c_1 = 0$, $c_2 = 2$, $c_3 = 0$. Druel [2] showed that $\overline{\mathrm{M}}_X$ is a smooth fivefold and the second Chern class map

$$\mathbf{c}_2 : \overline{\mathrm{M}}_X \to J(X)$$

is just the blowing-up of the intermediate Jacobian $J(X)$ of X with the center $F(X) \subset J(X)$, where $F(X)$ is the Fano surface of lines on X. Denote the exceptional divisor of $\mathbf{c}_2$ by A_X.

Let $\mathrm{M}_X \subset \overline{\mathrm{M}}_X$ be the open subset consisting of the points corresponding to stable sheaves and put $B_X = \overline{\mathrm{M}}_X \backslash \mathrm{M}_X$. Druel also showed that B_X is the strict transform of the hyper-surface $F(X) + F(X) \subset J(X)$. In other word, every strictly semi-stable sheaf $\mathscr{E} \in B_X$ is S-equivalent to $\mathscr{I}_{l_1} \oplus \mathscr{I}_{l_2}$, where l_i $(i = 1, 2)$ is a line on X, and $\mathbf{c}_2(\mathscr{I}_{l_1} \oplus \mathscr{I}_{l_2}) = [l_1] + [l_2]$ ([2], Théorème 3.5). If we denote the open subset of M_X consisting of the points corresponding to vector bundles by M_X^v, we have $\overline{\mathrm{M}}_X \backslash \mathrm{M}_X^v = A_X \cup B_X$. Note that this implies M_X^v can be regarded as an open subset of $J(X)$.

As we chose S generically, for every stable vector bundle $\mathscr{E} \in \mathrm{M}_X^v$, its restriction $\mathscr{E}_{|S}$ to S is also stable (see [1], Proposition 9.4). Thus we have a natural map

$$r_X : \mathrm{M}_X^v \to \mathrm{M}_S.$$

Let $\mathrm{M}_S^\circ \subset \mathrm{M}_S$ be the set of locally free sheaves $\mathscr{F}$ with a resolution

$$0 \longrightarrow \mathscr{O}_Q(-2)^{\oplus 6} \xrightarrow{M_{\mathscr{F}}} \mathscr{O}_Q(-1)^{\oplus 6} \longrightarrow \mathscr{F} \longrightarrow 0$$

where $M_{\mathscr{F}}$ is a skew-symmetric matrix with linear entries. M_S° turns out to be an open set of M_S. Thanks to this resolution, we can define a morphism

$$H : \mathrm{M}_S^\circ \to \Pi = \mathbb{P}H^0(\mathbb{P}^4, \mathscr{I}_S(3)) \cong \mathbb{P}^5$$

sending $\mathscr{F}$ to a cubic defined by the Pfaffian $\mathrm{Pf}(M_{\mathscr{F}})$. Beauville showed in [1] that r_X is injective, the fiber $H^{-1}(X)$ is nothing but $r_X(\mathrm{M}_X^v) \subset \mathrm{M}_S$, and this is actually a Lagrangian sub-manifold for smooth $X \in \Pi$. This implies H is a completely integrable Hamiltonian system over an open subset of Π. We call this system H the *Beauville system*.

Let $\mathscr{E}$ be a torsion-free sheaf on X. Then we have an exact sequence

$$\mathscr{E}(-2) \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}_{|S} \longrightarrow 0.$$

The correspondence

$$\mathscr{E} \mapsto \mathscr{E}_{|S}$$

induces a map from $\overline{\mathrm{M}}_X$ to $\overline{\mathrm{M}}_S$ if the (semi-)stability is preserved, thanks to the co-representability of the moduli spaces. We actually have a rational map $\bar{r}_X : \overline{\mathrm{M}}_X \dashrightarrow \overline{\mathrm{M}}_S$ in our setting, since $\mathscr{E}_{|S}$ is stable if $\mathscr{E} \in \mathrm{M}_X$ is locally free and stable, as we noticed above. $[\mathscr{E}] \in \overline{\mathrm{M}}_X$ is strictly semi-stable if and only if $\mathscr{E}$ is an extension of $\mathscr{I}_{l_1}$ by $\mathscr{I}_{l_2}$, where l_1 and l_2 are lines on X. Since S contains no line, $\mathscr{E}_{|S}$ must be an extension of $\mathscr{I}_{Z_1}$ by $\mathscr{I}_{Z_2}$, where Z_1 and Z_2 is a 0-dimensional sub-scheme on S of length 2. This is exactly a strictly semi-stable sheaf in $\overline{\mathrm{M}}_S$ ([7], Lemma 1.1.5). A sheaf $\mathscr{E} \in \mathrm{M}_X$ is stable, but not locally free, if and only if there exists an elementary transformation

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X^{\oplus 2} \longrightarrow \iota_* L \longrightarrow 0,$$

where $\iota : C \hookrightarrow X$ is a smooth conic and $L^{\otimes 2} = \mathscr{O}_C(1)$ ([2], Lemme 3.4, Théorèm 3.5). Since $\mathrm{Pic}(S) = \mathbb{Z}\langle \mathscr{O}_S(1)\rangle$, S contains no conic so that this exact sequence restricts to

$$0 \longrightarrow \mathscr{E}_{|S} \longrightarrow \mathscr{O}_S^{\oplus 2} \overset{\Psi}{\longrightarrow} \mathscr{O}_Z \longrightarrow 0$$

where $Z \subset S$ is a 0-dimensional sub-scheme of length 4. For generic choice of Ψ, its kernel turns out to be a stable sheaf on S ([7], p.97).

Thus we have shown the following proposition.

PROPOSITION 1.1. — *There exists a natural rational map*

$$\bar{r}_X : \overline{\mathrm{M}}_X \dashrightarrow \overline{\mathrm{M}}_S$$

which sends stable (resp. strictly semi-stable) sheaves to stable (resp. strictly semi-stable) sheaves where $\bar{r}_X$ is defined. This map is regular and injective on $\mathrm{M}_X^v \cup B_X$.

To get a (partial) compactification of the Beauville system H, we certainly have to take some blow-up of $\overline{\mathrm{M}}_S$. To see this, if we take a representative $\mathscr{F} = \mathscr{I}_{Z_1} \oplus \mathscr{I}_{Z_2}$ of a point in $\overline{\mathrm{M}}_S \backslash \mathrm{M}_S$, there exists a unique line

$l_i \subset \mathbb{P}^4$ which pass through Z_i ($i = 1, 2$). Therefore, a stable torsion free $\mathscr{O}_X$-coherent sheaf $\mathscr{E} = \mathscr{I}_{l_1} \oplus \mathscr{I}_{l_2}$ restricts to $\mathscr{F}$ on S for any cubic three-fold X containing S and l_1, l_2. This means that the images of $\overline{M}_{X_1}$ and $\overline{M}_{X_2}$ under $\bar{r}_X$ intersect at $[\mathscr{E}] \in \overline{M}_S$, where X_1 and X_2 are cubic threefolds containing S, l_1, and l_2, so that we cannot extend H to $\overline{M}_S \backslash M_S$.

As Beauville [1] already mentioned, O'Grady's resolved moduli space $\widetilde{M}_S$ is certainly a candidate to compactify H. Let $M_S^v \subset M_S$ be the open subset consisting of stable vector bundles. Then we have obvious inclusions $M_S^\circ \subset M_S^v \subset M_S \subset \overline{M}_S$, and define

$$A_S = (M_S \backslash M_S^v)^-, \quad B_S = (\overline{M}_S \backslash M_S)^-,$$

where the bars stands for the closures in $\overline{M}_S$. Also we define $B_S' \subset B_S$ to be the points corresponding to semi-stable sheaves of the form $\mathscr{I}_Z \oplus \mathscr{I}_Z$ ($Z \in \mathrm{Hilb}^2(S)$). Let $\mu : \widetilde{M}_S \to \overline{M}_S$ be the O'Grady's resolution and $\widetilde{M}_S' = \widetilde{M}_S \backslash \mu^{-1}(A_S \cup B_S')$. Under these notations, we state our result as follows.

THEOREM 1.2 (Main Theorem). — *The Beauville's system $H : M_S^\circ \to \Pi$ extends to a Lagrangian fibration on an open subset in $\widetilde{M}_S'$ whose fiber over $X \in \Pi$ is isomorphic to the complement of a closed subset of codimension not less than 2 in the intermediate Jacobian $J(X)$ of X.*

O'Grady's resolution μ restricted to $\overline{M}_S \backslash B_S'$ is just the blowing-up of $B_S \backslash B_S'$ from the construction (we can actually say that the whole μ is the blowing-up of B_S in $\overline{M}_S$ by a singularity theoretic argument, see [5]). The essential part of our proof of the Main Theorem is the following.

PROPOSITION 1.3. — *Let $B_X' \subset B_X$ be the closed subset consisting of the points corresponding to semi-stable sheaves $\mathscr{I}_l \oplus \mathscr{I}_l$ in $\overline{M}_X$, where l is a line on X, $M_X' = \overline{M}_X \backslash (A_X \cup B_X')$, $M_S' = \overline{M}_S \backslash (A_S \cup B_S')$, and $r_X' : M_X' \to M_S'$ be the restriction of $\bar{r}_X$. Then, r_X' lifts to an embedding $\tilde{r}_X' : M_X' \to \widetilde{M}_S'$.*

M_X' is nothing but $J(X) \backslash (F(X) \cup \mathbf{c}_2(B_X'))$ by Druel's result, where $\mathbf{c}_2 : \overline{M}_X \to J(X)$ is the second Chern class map. $F(X) \subset J(X)$ has the codimension 3. Note that the codimension of $\mathbf{c}_2(B_X') \subset J(X)$ is also 3, since the codimension of $B_X' \subset \overline{M}_X$ is 3 and B_X' maps birationally onto its image under $\mathbf{c}_2$. Therefore, the complement of $M_X' \subset J(X)$ has the codimension 3, which is, of course, not less than 2. Also note that Proposition 1.3 immediately implies that $r_X'(M_X')$ is a Lagrangian sub-manifold of an algebraic symplectic manifold $\widetilde{M}_S'$, because $r_X(M_X^v) \subset M_S^\circ$ is a Lagrangian sub-manifold and $r_X(M_X^v) \subset r_X'(M_X')$ is open.

2. Versal deformation space and local study of O'Grady's blow-up

In this section, we review a part of the local description of O'Grady's resolution. In O'Grady's argument in [7], the following proposition was one of the fundamental tools.

PROPOSITION 2.1 (O'Grady [7], Proposition 1.2.3). — *Let* $(Y, \mathscr{O}_Y(1))$ *be a projective scheme and* $\mathscr{E}$ *is a semi-stable sheaf on* Y. *We define* $G(\mathscr{E}) = \mathrm{Aut}(\mathscr{E})/\mathbb{C}^*$ *and assume* $G(\mathscr{E})$ *is reductive. Then there exists an affine versal deformation space* $(0 \in D(\mathscr{E}))$ *of* $\mathscr{E}$ *with the following properties:*

(i) $D(\mathscr{E})$ *has a natural* $G(\mathscr{E})$-*action fixing the reference point* $0 \in D(\mathscr{E})$.

(ii) *The Zariski tangent space to* $0 \in D(\mathscr{E})$ *is* $G(\mathscr{E})$-*equivariantly identified with* $\mathrm{Ext}^1_{\mathscr{O}_Y}(\mathscr{E}, \mathscr{E})$.

(iii) *If* $\overline{\mathrm{M}}(\mathscr{E})$ *is a connected component of the moduli space of* $\mathscr{O}(1)$-*semi-stable sheaves on* Y *containing the point* $[\mathscr{E}]$, *we have an isomorphism of germs*

$$(0 \in D(\mathscr{E})/\!\!/ G(\mathscr{E})) \overset{\sim}{\longrightarrow} ([\mathscr{E}] \in \overline{\mathrm{M}}(\mathscr{E})).$$

O'Grady observed that the deformation space $D(\mathscr{E})$ above is given by an étale slice of the Quot scheme that we use to construct the moduli space $\overline{\mathrm{M}}(\mathscr{E})$. The assumption that $G(\mathscr{E})$ is reductive is equivalent to the closedness of the orbit of the point corresponding to $\mathscr{E}$ in the Quot scheme. This also means that every point of $D(\mathscr{E})$ is represented by a semi-stable sheaf on Y (for detail, see [7], §1.2).

Now we apply the proposition to our situation. We keep the notations in §1. A point in $B_X \backslash B'_X$ is represented by the polystable sheaf $\mathscr{E} = \mathscr{I}_{l_1} \oplus \mathscr{I}_{l_2}$ where l_1 and l_2 are different lines on X. Since $G(\mathscr{E}) = \mathbb{C}^*$, we can apply the proposition for $\mathscr{E}$. If we restrict $\mathscr{E}$ to S, we get $\mathscr{E}_{|S} = \mathscr{I}_{Z_1} \oplus \mathscr{I}_{Z_2}$, where $Z_i = S \cap l_i$ $(i = 1, 2)$. In particular, we have $G(\mathscr{E}_{|S}) = \mathbb{C}^*$ and we can also apply the proposition for $\mathscr{E}_{|S}$.

Now let us put $E_{X,ij} = \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_i}, \mathscr{I}_{l_j})$, $E_{S,ij} = \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{I}_{Z_i}, \mathscr{I}_{Z_j})$. Then we have

(2.1)
$$\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) = E_{X,11} \oplus E_{X,12} \oplus E_{X,21} \oplus E_{X,22},$$
$$\mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S}) = E_{S,11} \oplus E_{S,12} \oplus E_{S,21} \oplus E_{S,22}.$$

By [7], Lemma 1.4.16, both of the $\mathbb{C}^*$-actions on $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$ and on $\mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S})$ are given by

$$(e_{11}, e_{12}, e_{21}, e_{22}) \mapsto (e_{11}, \lambda e_{12}, \lambda^{-1} e_{21}, e_{22})$$

with respect to the decompositions (2.1). Therefore the natural map

$$\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S})$$

defined by restriction to S is $\mathbb{C}^*$-equivariant so that we have a $\mathbb{C}^*$-equivariant morphism

$$\rho : D(\mathscr{E}) \to D(\mathscr{E}_{|S})$$

such that the GIT quotient of ρ by $\mathbb{C}^*$ is nothing but the restriction map $\bar{r}_X$ near $[\mathscr{E}] \in \overline{\mathrm{M}}_X$.

Next we determine the local models of $D(\mathscr{E})$ and $D(\mathscr{E}_{|S})$, where the description of $D(\mathscr{E}_{|S})$ is given by O'Grady [7] and that of $D(\mathscr{E})$ is given by Druel [2] along with the line of argument of O'Grady. We first treat $D(\mathscr{E})$.

LEMMA 2.2 (Druel [2], Lemme 4.3). — For any two lines l_1 and l_2 on a cubic threefold X, $\mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{I}_{l_1}, \mathscr{I}_{l_2})$ vanishes if $i \neq 1$. For $i = 1$,

$$\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}, \mathscr{I}_{l_2}) = \begin{cases} \mathbb{C} & (l_1 \neq l_2), \\ \mathbb{C}^2 & (l_1 = l_2). \end{cases}$$

COROLLARY 2.3. — $D(\mathscr{E})$ is smooth, i.e., isomorphic to an open neighborhood of $0 \in \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$.

We note that we can deduce the smoothness of $\overline{\mathrm{M}}_X$ along $B_X \backslash B'_X$ using this corollary: $E_{X,11} \oplus E_{X,22}$ is the tangent direction of B_X and $(E_{X,12} \oplus E_{X,21})/\!\!/\mathbb{C}^* \cong \mathbb{C}$ is the normal direction to B_X ([2], Théorème 4.6).

On the other hand, $D(\mathscr{E}_{|S})$ is not smooth so that $\overline{\mathrm{M}}_S$ is singular along B_S. But the normal cone to $D(\mathscr{E}_{|S})$ can be described in terms of the Yoneda square. This fact was one of the key observations of [7]. The Yoneda product is just the composition morphism

$$\mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S}) \times \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S}) \xrightarrow{\cup} \mathrm{Ext}^2_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S})$$

and define the Yoneda square $\Upsilon : \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S}) \to \mathrm{Ext}^2_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S})_0$ by

$$e \mapsto \Upsilon(e) = e \cup e,$$

where $\mathrm{Ext}^2_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S})_0$ is the kernel of the trace map on $\mathrm{Ext}^2_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S})$. Let $\mathscr{E}_{|S} \in B_S \backslash B'_S$, i.e., $\mathscr{E}_{|S} = \mathscr{I}_{Z_1} \oplus \mathscr{I}_{Z_2}$ with $Z_1 \neq Z_2$. Taking Serre duality into account, we have

$$\mathrm{Ext}^2_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S}) = \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{I}_{Z_1}, \mathscr{I}_{Z_1})^\vee \oplus \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{I}_{Z_2}, \mathscr{I}_{Z_2})^\vee \cong \mathbb{C}^2$$

where the last isomorphism is given by the trace map. Since the Yoneda pairing on Ext^1 is skew symmetric, the Yoneda square is given by

$$\Upsilon(e) = (e_{12} \cup e_{21}, e_{21} \cup e_{12}), \text{ where } e = e_{11} + e_{12} + e_{21} + e_{22}$$

with respect to the decomposition (2.1). We define $\Psi : \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S}) \to \mathbb{C}$ by $e \mapsto \mathrm{Tr}(e_{12} \cup e_{21})$. We also define $\overline{\Psi} : E_{S,12} \oplus E_{S,21} \to \mathbb{C}$ by the same correspondence $(e_{12}, e_{21}) \mapsto \mathrm{Tr}(e_{12} \cup e_{21})$. Note that $\dim E_{S,11} = \dim E_{S,22} = 4$, $\dim E_{S,12} = \dim E_{S,21} = 2$ by Riemann-Roch and Serre duality.

PROPOSITION 2.4 (O'Grady [7], §1.4). — *Notations as above. Take $\mathscr{E}_{|S} \in B_S \setminus B'_S$ and let $\Sigma_S \subset D(\mathscr{E}_{|S})$ be the locus of sheaves of the form $\mathscr{I}_{Z_1} \oplus \mathscr{I}_{Z_2}$ where $Z_1, Z_2 \in \mathrm{Hilb}^2(S)$. We assume $Z_1 \neq Z_2$ for every point of Σ_S by shrinking $D(\mathscr{E}_{|S})$ if necessary. Then,*

(i) *The tangent cone $C_0 D(\mathscr{E}_{|S})$ in $\mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{E}_{|S}, \mathscr{E}_{|S})$ is given by*
$$C_0 D(\mathscr{E}_{|S}) \cong \Upsilon^{-1}(0,0) = \Psi^{-1}(0).$$

(ii) *Σ_S is smooth and $\mathbb{C}^*$-invariant. The tangent space of Σ_S is identified with $E_{S,11} \oplus E_{S,22}$.*

(iii) *The normal cone $C_{\Sigma_S} D(\mathscr{E}_{|S})$ is a locally trivial family of the affine cone over smooth quadric in $\mathbb{P}^3$ given by*
$$\overline{\Psi}^{-1}(0) \subset E_{S,12} \oplus E_{S,21}.$$

Remark 2.5. — Our Σ_S in the proposition above corresponds to Σ_Q° in O'Grady's notation (or rather, its restriction $\mathscr{W} = \mathscr{V} \cap \Sigma_Q^\circ$ to the étale slice $\mathscr{V}$), [7], p.53, p61.

Now we can describe the local structure of O'Grady's resolution $\mu : \widetilde{\mathrm{M}}_S \to \overline{\mathrm{M}}_S$ near a point $[\mathscr{E}_{|S}] \in B_S \setminus B'_S$.

PROPOSITION 2.6. — *Let $\nu_S : \widetilde{D}(\mathscr{E}_{|S}) \to D(\mathscr{E}_{|S})$ be the blowing-up of $\Sigma_S \subset D(\mathscr{E}_{|S})$. Then its GIT quotient $\bar{\nu}_S : \widetilde{D}(\mathscr{E}_{|S}) /\!\!/ \mathbb{C}^* \to D(\mathscr{E}_{|S}) /\!\!/ \mathbb{C}^*$ is identified with the blowing-up $\mu : \widetilde{\mathrm{M}}_S \to \overline{\mathrm{M}}_S$ along B_S near the point $[\mathscr{E}_{|S}]$ (the GIT quotient of $\widetilde{D}(\mathscr{E}_{|S})$ is linearized by $\mathscr{O}(-F_S)$ where F_S is the exceptional divisor of ν_S).*

The key ingredient of this proposition is Kirwan's blow-up of the GIT quotient [4]. Here we only refer to [7], in particular §§1.1, 1.2, 1.8, for the details. We analyze the set of stable and semi-stable points of $\widetilde{D}(\mathscr{E}_{|S})$. Since the points in $D(\mathscr{E}_{|S}) \setminus \Sigma_S$ are represented by stable sheaves, we know that every point in $\widetilde{D}(\mathscr{E}_{|S}) \setminus F_S$ is $\mathbb{C}^*$-stable. Take a point $x \in \Sigma_S$ and consider its fiber $F_x = \nu_S^{-1}(x) \subset F_S$. Then, by the Hilbert–Mumford criterion of stability, we have

$$F_x^s = F_x^{ss} = \mathbb{P}\{(e_{12}, e_{21}) \in E_{S,12} \oplus E_{S,21} \mid e_{12} \cup e_{21} = 0, e_{12} \neq 0, e_{21} \neq 0\},$$

and the stabilizer of each point of F_x^s is $\mathbb{Z}/2\mathbb{Z}$ (see [7], Lemma 1.6.1, Claim 1.8.8). Since $F_S^s \subset \widetilde{D}(\mathscr{E}_{|S})^s$ is a smooth Cartier divisor and $\widetilde{D}(\mathscr{E}_{|S}) \setminus F_S$

is smooth, $\widetilde{D}(\mathscr{E}_{|S})^s$ is smooth. As the locus of the points with non-trivial stabilizer is the smooth divisor F_S^s, Luna's étale slice theorem implies that $\widetilde{D}(\mathscr{E}_{|S})/\!/\mathbb{C}^*$ is smooth and $F_S/\!/\mathbb{C}^*$ is a smooth divisor. In this way, O'Grady showed that μ is a resolution (at least near the point $[\mathscr{E}_{|S}]$ from the argument above).

3. Proof of the main result

In this section, we prove Proposition 1.3, keeping the description in the previous section in mind, and complete the proof of Theorem 1.2.

The proof of Proposition 1.3. Let $\Sigma_X \subset D(\mathscr{E})$ the locus of points represented by sheaves of the form $\mathscr{I}_{l_1} \oplus \mathscr{I}_{l_2}$ and take the blowing-up $\nu_X : \widetilde{D}(\mathscr{E}) \to D(\mathscr{E})$ along Σ_X. Then we have a commutative diagram of $\mathbb{C}^*$-equivariant morphisms

$$
\begin{array}{ccc}
\widetilde{D}(\mathscr{E}) & \xrightarrow{\tilde{\rho}} & \widetilde{D}(\mathscr{E}_{|S}) \\
\nu_X \downarrow & & \downarrow \nu_S \\
D(\mathscr{E}) & \xrightarrow{\rho} & D(\mathscr{E}_{|S}).
\end{array}
$$

Again by Kirwan's theorem, the GIT quotient $\widetilde{D}(\mathscr{E})/\!/\mathbb{C}^*$ is isomorphic to the blowing-up of $D(\mathscr{E})/\!/\mathbb{C}^*$ along $\Sigma_X/\!/\mathbb{C}^*$. But $\Sigma_X/\!/\mathbb{C}^*$ is a divisor which is identified with $B_X \subset \overline{M}_X$, so the morphism $\widetilde{D}(\mathscr{E})/\!/\mathbb{C}^* \to D(\mathscr{E})/\!/\mathbb{C}^*$ induced by ν_X has to be an isomorphism. This implies that $\bar{r}'_X : \mathrm{M}'_X \to \mathrm{M}'_S$ lifts to a morphism $\bar{r}'_X : \mathrm{M}'_X \to \widetilde{\mathrm{M}}'_S$. Since $\bar{r}'_X$ is obviously injective, it is enough to show that the differential of $\bar{r}'_X$ is injective at the points of $B_X \backslash B'_X$ to show that $\bar{r}'_X$ is an embedding.

LEMMA 3.1. — l_1, l_2 be (not necessarily distinct) lines on a cubic threefold X. Then
$$
\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}, \mathscr{I}_{l_2}(-2)) = 0.
$$

Proof. — Since we have $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}, \mathscr{I}_{l_2}(-2)) \cong \mathrm{Ext}^2_{\mathscr{O}_X}(\mathscr{I}_{l_2}, \mathscr{I}_{l_1})^\vee$ by Serre duality, the lemma follows from Lemma 2.2.   □

LEMMA 3.2. — Let X be a cubic threefold containing a K3 surface S and l_1, l_2 (not necessarily distinct) lines on X. Assume l_1 and l_2 are not contained in S. Then the restriction map
$$
\gamma : \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}, \mathscr{I}_{l_2}) \to \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{I}_{l_1}\mathscr{O}_S, \mathscr{I}_{l_2}\mathscr{O}_S)
$$
is injective.

Proof. — Note that we have an injective map

$$\mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{I}_{l_1}\mathscr{O}_S, \mathscr{I}_{l_2}\mathscr{O}_S) \rightarrow \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}\mathscr{O}_S, \mathscr{I}_{l_2}\mathscr{O}_S).$$

Consider the diagram
(3.1)

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}, \mathscr{I}_{l_2}) & \xrightarrow{\quad\quad\quad\quad\alpha\quad\quad\quad\quad} & \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}, \mathscr{I}_{l_2}\mathscr{O}_S) \\
{\scriptstyle\gamma}\downarrow & & \Vert \\
\mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{I}_{l_1}\mathscr{O}_S, \mathscr{I}_{l_2}\mathscr{O}_S) \longrightarrow & \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}\mathscr{O}_S, \mathscr{I}_{l_2}\mathscr{O}_S) \xrightarrow{\;\beta\;} & \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{l_1}, \mathscr{I}_{l_2}\mathscr{O}_S).
\end{array}$$

For an extension class

$$\eta : 0 \longrightarrow \mathscr{I}_{l_2} \xrightarrow{\;\varphi\;} \mathscr{E} \longrightarrow \mathscr{I}_{l_1} \longrightarrow 0,$$

$\alpha(\eta)$ is given by

$$0 \longrightarrow \mathscr{I}_{l_2}\mathscr{O}_S \longrightarrow \mathscr{E}/\varphi(\mathscr{I}_{l_2}(-2)) \longrightarrow \mathscr{I}_{l_1} \longrightarrow 0.$$

β sends an extension class

$$0 \longrightarrow \mathscr{I}_{l_2}\mathscr{O}_S \longrightarrow \mathscr{F} \xrightarrow{\;\psi\;} \mathscr{I}_{l_1}\mathscr{O}_S \longrightarrow 0$$

to the class

$$0 \longrightarrow \mathscr{I}_{l_2}\mathscr{O}_S \longrightarrow \tilde{\mathscr{F}} \xrightarrow{\;pr_2\;} \mathscr{I}_{l_1} \longrightarrow 0,$$

defined by

$$\tilde{\mathscr{F}} = \{(m, f) \in \mathscr{F} \oplus \mathscr{I}_{l_1} \mid \psi(m) = \pi(f)\}$$

where $\pi : \mathscr{I}_{l_1} \longrightarrow \mathscr{I}_{l_1}\mathscr{O}_S$ is the natural surjection. By these explicit formulae, we can easily check that the diagram (3.1) is commutative. To show the injectivity of γ, it is enough to prove that α is injective. But the long exact sequence for Ext's and Lemma 3.1 imply that α is actually injective. $\square$

Since the $\mathbb{C}^*$-actions on $E_{X,11} \oplus E_{X,22}$ and $E_{S,11} \oplus E_{S,22}$ are trivial, the differential of the map $d(\bar{r}'_{X|B_X}) : T_{[\mathscr{E}]}B_X \rightarrow T_{[\mathscr{E}|_S]}B_S$ is injective by Lemma 3.2. The injectivity of $E_{X,12} \oplus E_{X,21} \rightarrow E_{S,12} \oplus E_{S,21}$ implies that any non-zero tangent vector either tangent to the orbit direction in $\nu_X^{-1}([\mathscr{E}])$ or normal direction to the exceptional divisor F_X of $\nu_X : \widetilde{D}(\mathscr{E}) \rightarrow D(\mathscr{E})$ has non-zero image under $d\tilde{\rho}$. This implies the injectivity of the differential of $\bar{r}'_X = \tilde{\rho}/\!\!/\mathbb{C}^*$ on the normal space to $F_X/\!\!/\mathbb{C}^*$. This completes the proof of Proposition 1.3.

Conclusion of the proof of Theorem 1.2. Let U be the open subset of Π consisting of smooth cubic threefolds containing S. Then, we have a family of intermediate Jacobians $\mathcal{J} = \mathcal{J}(\mathscr{X}/U) \to U$, where $\mathscr{X} \to U$ is the tautological family. There exists a closed subset $Z \subset \mathcal{J}$ of codimension not less than 2 such that any fiber of $\mathcal{J}\backslash Z$ over $X \in U$ is isomorphic to M'_X. Proposition 1.3 implies that we have a morphism $r : \mathcal{J}\backslash Z \to \widetilde{\mathrm{M}}'_S$ which smoothly embeds every fiber of $\mathcal{J}\backslash Z \to U$. By Beauville's construction, we already know that r is a birational morphism onto its image.

CLAIM. — *Fix a point $p \in \widetilde{\mathrm{M}}'_S$. Then, the locus of cubics $X \in U$ such that $\tilde{r}'_X(\mathrm{M}'_X) \ni p$ is discrete.*

Proof. — If not, there exists a one-dimensional family $\{X_t\}_{t \in \Delta}$ of cubic threefolds containing S such that $p \in \tilde{r}'_{X_t}(\mathrm{M}'_{X_t})$ for every $t \in \Delta$. We can assume that the corresponding infinitesimal deformation class $\theta \in H^0(\mathrm{M}'_{X_0}, N_{r'_{X_0}(\mathrm{M}'_{X_0})/\widetilde{M}'_S})$ is non-zero and vanishes at p. Since $r'_{X_0}(\mathrm{M}'_{X_0})$ is a Lagrangian sub-manifold in the symplectic manifold $\widetilde{M}'_S$, we have $N_{r'_{X_0}(\mathrm{M}'_{X_0})/\widetilde{M}'_S} \cong \Omega^1_{\mathrm{M}'_{X_0}}$ so that we can regard θ as a holomorphic 1-form on M'_{X_0}. However M'_{X_0} can be regarded as an open subset of an Abelian variety $J(X_0)$ and the codimension of the complement is not less than 2. This implies that the 1-form θ extends to a non-zero holomorphic 1-form on $J(X_0)$ which vanishes at a point. This is absurd. $\square$

The claim implies that r contracts no curve. Therefore r must be an isomorphism to its image, after shrinking U if necessary, by Zariski's main theorem and this completes the proof of Theorem 1.2.

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