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SIMULTANEOUS REDUCTION TO NORMAL FORMS OF COMMUTING SINGULAR VECTOR FIELDS WITH LINEAR PARTS HAVING JORDAN BLOCKS

by Masafumi YOSHINO & Todor GRAMCHEV (*)

Abstract. — We study the simultaneous linearizability of d–actions (and the corresponding d-dimensional Lie algebras) defined by commuting singular vector fields in $\mathbb{C}^n$ fixing the origin with nontrivial Jordan blocks in the linear parts. We prove the analytic convergence of the formal linearizing transformations under a certain invariant geometric condition for the spectrum of d vector fields generating a Lie algebra. If the condition fails and if we consider the situation where small denominators occur, then we show the existence of divergent solutions of an overdetermined system of linearized homological equations. In the $\mathbb{C}^\infty$ category, the situation is completely different. We show Sternberg's theorem for a commuting system of $\mathbb{C}^\infty$ vector fields with a Jordan block although they do not satisfy the condition.

Résumé. — Nous étudions la linéarisation simultanée de d–actions (et les algèbres correspondants de Lie d–dimensionelles) definie par des champs de vecteurs singuliers dans $\mathbb{C}^n$ fixant l'origine avec des parties linéaires ayant des blocs de Jordan. Nous montrons la convergence analytique des transformations linéarisantes formelles sous une condition d'invariance géométrique pour le spectre de d-champs de vecteurs qui engendrent une algèbre de Lie. Si la condition n'est pas satisfaite et si il y a des petits diviseurs, nous montrons l'existence de solutions divergentes pour un système sous déterminé d'équations linéarisées homologiques. Dans le cadre de fonctions $\mathbb{C}^\infty$ la situation est complètement différente. Nous montrons le théorème de Sternberg pour une famille commutative de champs de vecteurs qui ne satisfait pas la condition.

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1. Simultaneous normalization

Let $\mathbb{K}$ be $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, and $B = \infty$, $B = \omega$ or $B = k$ for some $k > 0$. Let $\mathcal{G}_B^n$ denote a d–dimensional Lie algebra of germs at $0 \in \mathbb{K}^n$ of C^B vector fields vanishing at 0. Let ρ be a germ of singular infinitesimal $\mathbb{K}^d$–actions of class C^B $(d \geqslant 2)$

$$\rho : \mathbb{K}^d \longrightarrow \mathcal{G}_B^n. \tag{1.1}$$

We denote by $Act^B(\mathbb{K}^d : \mathbb{K}^n)$ the set of germs of singular infinitesimal $\mathbb{K}^d$–actions of class C^B at $0 \in \mathbb{K}^n$. By choosing a basis $e_1, \ldots, e_d \in \mathbb{K}^n$, the infinitesimal action can be identified with a d–tuple of germs at 0 of commuting vector fields $X^j = \rho(e_j)$, $j = 1, \ldots, d$ (cf. [13], [17]). We can define, in view of the commutativity relation, the action

$$(1.2) \qquad \tilde{\rho} : \mathbb{K}^d \times \mathbb{K}^n \longrightarrow \mathbb{K}^n,$$
$$\tilde{\rho}(s; z) = X^1_{s_1} \circ \cdots \circ X^d_{s_d}(z) \quad = \quad X^{\sigma_1}_{s_{\sigma_1}} \circ \cdots X^{\sigma_d}_{s_{\sigma_d}}(z), \quad s = (s_1, \ldots, s_d),$$

for all permutations $\sigma = (\sigma_1, \ldots, \sigma_d)$ of $\{1, \ldots, d\}$, where X^j_t denotes the flow of X^j. We denote by ρ_{lin} the linear action formed by the linear parts of the vector fields defining ρ.

We shall investigate necessary and sufficient conditions for the linearization of ρ, namely, whether there exists a C^B diffeomorphism g preserving 0 such that g conjugates $\tilde{\rho}$ and $\tilde{\rho_{lin}}$

$$\tilde{\rho}(s; g(z)) \quad = \quad g(\widetilde{\rho_{lin}}(s, z)), \qquad (s, z) \in \mathbb{K}^d \times \mathbb{K}^n. \tag{1.3}$$

We recall that in [13] and [24], the linear parts were supposed to be diagonalizable, while in [29] the existence of $n - d$ anlalytic first integrals was required. (See also [1], [15]). Following Katoks' argument in [17], we take a positive integer $m \leqslant n$ such that $\mathbb{K}^n$ is decomposed into a direct sum of m linear subspaces invariant under all $A^\ell = \nabla X_\ell(0)$ $(\ell = 1, \ldots, d)$:

$$(1.4) \qquad \mathbb{K}^n \quad = \quad \mathbb{I}^{s_1} + \cdots + \mathbb{I}^{s_m}, \qquad \dim \mathbb{I}^{s_j} = s_j,\ j = 1, \ldots, m,$$
$$s_1 + \cdots + s_m = n.$$

The matrices $A^1, \ldots, A^d$ can be simultaneously brought in an upper triangular form, and we write again A^ℓ for the matrices,

$$A^\ell \quad = \quad \begin{pmatrix} A_1^\ell & 0_{s_1 \times s_2} & \ldots & 0_{s_1 \times s_m} \\ 0_{s_2 \times s_1} & A_2^\ell & \ldots & 0_{s_2 \times s_m} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{s_m \times s_1} & 0_{s_m \times s_2} & \ldots & A_m^\ell \end{pmatrix}, \quad \ell = 1, \ldots, d. \tag{1.5}$$

If $\mathbb{K}=\mathbb{C}$, the matrix A_j^ℓ is given by

$$A_j^\ell = \begin{pmatrix} \lambda_j^\ell & A_{j,12}^\ell & \dots & A_{j,1s_j}^\ell \\ 0 & \lambda_j^\ell & \dots & A_{j,2s_j}^\ell \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_j^\ell \end{pmatrix}, \ \ell=1,\dots,d,\ j=1,\dots,m, \tag{1.6}$$

with $\lambda_j^\ell, A_{j,\nu\mu}^\ell \in \mathbb{C}$. On the other hand, if $\mathbb{K}=\mathbb{R}$, then we have, for every $1 \leqslant j \leqslant m$ two possibilities: firstly, all A_j^ℓ $(\ell=1,\dots,d)$ are given by (1.6) with $\lambda_j^\ell \in \mathbb{R}$. Secondly, $s_j = 2\tilde{s_j}$ is even and A_j^ℓ is a $\tilde{s_j} \times \tilde{s_j}$ square block matrix given by

$$A_j^\ell = \begin{pmatrix} R_2(\lambda_j^\ell,\mu_j^\ell) & A_{\ell,j}^{12} & \dots & A_{\ell j}^{1\tilde{s_j}} \\ 0 & R_2(\lambda_j^\ell,\mu_j^\ell) & \dots & A_{\ell j}^{2\tilde{s_j}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_2(\lambda_j^\ell,\mu_j^\ell) \end{pmatrix}, \quad \ell=1,\dots,d, \tag{1.7}$$

where

$$R_2(\lambda,\mu) := \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}, \qquad \lambda,\mu\in\mathbb{R}, \tag{1.8}$$

and $A_{\ell j}^{rs}$ are appropriate real matrices.

Following the decomposition (1.6) (respectively, (1.7)) we define $\tilde{\lambda^j}$ by

$$\tilde{\lambda^k} = {}^t(\lambda_1^k,\dots,\lambda_m^k) \in \mathbb{K}^m, \qquad k=1,\dots,d. \tag{1.9}$$

Then we assume

$$\tilde{\lambda^1},\cdots,\tilde{\lambda^d} \ \text{ are linearly independent in } \mathbb{K}^m. \tag{1.10}$$

One can easily see that (1.10) is invariantly defined.

By (1.5) we define

$$\vec{\lambda_j} = {}^t(\lambda_j^1,\cdots,\lambda_j^d) \in \mathbb{K}^d, \qquad j=1,\dots,m, \tag{1.11}$$

and

$$\Lambda_m := \{\vec{\lambda_1},\dots,\vec{\lambda_m}\}. \tag{1.12}$$

We define the cone $\Gamma[\Lambda_m]$ by

$$\Gamma[\Lambda_m] = \left\{ \sum_{j=1}^m t_j \vec{\lambda_j} \in \mathbb{K}^d;\ t_j \geqslant 0, j=1,\dots,m,\ \sum_{j=1}^m t_j \neq 0 \right\}. \tag{1.13}$$

Definition 1.1. — *We say that the* $\mathbb{K}^d$*–action* ρ *is a Poincaré morphism if there exists a basis* $\Lambda_m \subset \mathbb{K}^m$ *such that* $\Gamma[\Lambda_m]$ *is a proper cone in* $\mathbb{K}^m$*, namely it does not contain a straight real line. If the condition is not satisfied, then, we say that the* $\mathbb{K}^d$*–action is in a Siegel domain.*

Note that the definition is invariant under the choice of the basis Λ_m.

Remark 1.2. — As to the alternative definition of a Poincaré morphism we refer to [Definition 6.2.1, [24]].

Next, we introduce the notion of simultaneous resonance. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{K}^m$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{K}^m$, we set $\langle \alpha, \beta \rangle = \sum_{\nu=1}^m \alpha_\nu \beta_\nu$. For a positive integer k we define $\mathbb{Z}_+^m(k) = \{\alpha \in \mathbb{Z}_+^m;\ |\alpha| \geqslant k\}$. Put

$$\omega_j(\alpha) = \sum_{\nu=1}^{d} |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu|, \qquad j = 1, \ldots, m, \tag{1.14}$$

$$\omega(\alpha) = \min\{\omega_1(\alpha), \ldots, \omega_m(\alpha)\}. \tag{1.15}$$

Definition 1.3. — *We say that* Λ_m *is simultaneously nonresonant (or, in short* ρ *is simultaneously nonresonant), if*

$$\omega(\alpha) \neq 0, \qquad \forall \alpha \in \mathbb{Z}_+^m(2). \tag{1.16}$$

If (1.16) does not hold, then we say that Λ_m *is simultaneously resonant.*

Clearly, the simultaneously nonresonant condition (1.16) is invariant under the change of the basis Λ_m. We state the first main result of our paper

Theorem 1.4. — *Let* ρ *be a Poincaré morphism. Then* ρ *is conjugated to a polynomial action by an holomorphic change of variables.*

Remark 1.5. — In case ρ has a semi simple linear part, then Theorem 1.4 is already known. (cf. [Theorem 2.1.4, [24]]).

Example 1.6. — We compare our theorem with the results of Stolovitch [24] and Zung [29]. Let ρ be a $\mathbb{R}^2$–action in $\mathbb{R}^n$, $n \geqslant 4$ with $m = 3$. We choose a basis Λ_2 of $\mathbb{R}^3$ such that

$$\Lambda_2 = \left\{ {}^t(1, 1, \nu), {}^t(0, 1, \mu) \right\}, \qquad \nu, \mu \in \mathbb{R}. \tag{1.17}$$

(cf. [12] for similar and more general reductions of commuting vector fields on the torus).

We will characterize the set of $(\nu, \mu) \in \mathbb{R}^2$ so that the action is a Poincaré morphism, and determine the simultaneous resonances. By (1.13), $\Gamma[\Lambda_2]$ is generated by the set of vectors $\{(1, 0), (1, 1), (\nu, \mu)\}$. Hence the action is a Poincaré morphism if and only if these vectors generate a proper cone, namely (ν,μ) is not in the set $\{(\nu,\mu)\in\mathbb{R}^2;\nu\leqslant\mu\leqslant 0\}$. We note that the interesting case is $\mu<\nu\leqslant 0$, where every generator in (1.17) is in a Siegel domain. Theorem 1.4 can be applied to such a case. In §3 we will show that if the action is not a Poincaré morphism, i.e., $\nu<\mu<0$, then there exist (ν,μ) with the density of continuum such that the linearized overdetermined system of two homological equations has a divergent solution.

Next we will determine (ν,μ) so that a simultaneous resonance exists. If $\eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{Z}_+^3(2)$ is a simultaneous resonance, we have the following set of equations:

$$\begin{aligned}
&(1) \qquad \eta_1+\eta_2+\nu\eta_3=1,\ \eta_2+\mu\eta_3=0,\\
&(2) \qquad \eta_1+\eta_2+\nu\eta_3=1,\ \eta_2+\mu\eta_3=1,\\
&(3) \qquad \eta_1+\eta_2+\nu\eta_3=\nu,\ \eta_2+\mu\eta_3=\mu.
\end{aligned}$$

Elementary computations imply that, in order that one of these equations has a solution η the (ν,μ) satisfies the following:

a) Case $\nu\leqslant\mu\leqslant 0$. The resonance exists iff $(\nu,\mu)\in\mathbb{Q}_-\times\mathbb{Q}_-$, where $\mathbb{Q}_-$ is the set of nonpositive rational numbers. The resonance is given by $(1+(\mu-\nu)k,-\mu k,k)$ and $((\mu-\nu)k,1-k\mu,k)$ where $k\geqslant 1/(1-\nu)$, $k\in\mathbb{Z}_+$, and $((\nu-\mu)(1-k),\mu(1-k),k)$, where $k\geqslant(2-\nu)(1-\nu)$, $k\in\mathbb{Z}_+$.

b) Case $\nu>\mu$ and $\mu\leqslant 0$. The resonance is given by $(0,-\mu/(\nu-\mu),1/(\nu-\mu))$, where $-\mu/(\nu-\mu)\in\mathbb{Z}_+$, $1/(\nu-\mu)\in\mathbb{Z}_+$ and $2\nu-\mu\leqslant 1$.

c) Case $\mu>0$, $\nu\leqslant\mu$. The resonance is given by $(0,0,1/\nu)$, when $\nu=\mu$, $\nu\leqslant 1/2$, $\nu^{-1}\in\mathbb{Z}_+$; $(0,\nu,0)$, when $\nu=\mu\geqslant 2$, $\nu\in\mathbb{Z}_+$; $((\mu-\nu)/\mu,0,1/\mu)$, if otherwise, where $(\mu-\nu)/\mu\in\mathbb{Z}_+$, $1/\mu\in\mathbb{Z}_+$ and $\nu+\mu\leqslant 1$.

d) Case $\nu>\mu$, $\mu\geqslant 0$. The resonance is given by $(\nu-\mu,\mu,0)$, where $\nu-\mu\in\mathbb{Z}_+$, $\mu\in\mathbb{Z}_+$ and $\nu\geqslant 2$.

Let ν be a negative rational number, $\nu=-k_1/k_2$, $k_1,k_2\in\mathbb{Z}_+$, $k_2\neq 0$. Let μ be a rational number and satisfy $\mu<\nu$. Assume that the nonlinear part of X^2 is zero. If the nonlinear part of X^1 consists of the resonant terms of X^2, then we have $[X^1,X^2]=0$. We can easily see that the linearizability of X^1 holds provided $\mu\neq\nu-1/k_2=-(k_1+1)/k_2$.

2. A Poincaré morphism

We start by showing equivalent forms of a Poincaré morphism.

PROPOSITION 2.1. — *The action is a Poincaré morphism if and only if each of the following conditions holds*

i) *there exist a positive constant C and an integer k_0 such that*

$$\sum_{k=1}^{d}|\sum_{j=1}^{m}\lambda_j^k\alpha_j| \geqslant C_1|\alpha|, \qquad \forall\alpha\in\mathbb{Z}_+^m(k_0). \tag{2.1}$$

ii) *there exists a nonzero vector $c=(c_1,\dots,c_d)\in\mathbb{C}^d$ if $\mathbb{K}=\mathbb{C}$ (respectively, $c=(c_1,\dots,c_d)\in\mathbb{R}^d$ if $\mathbb{K}=\mathbb{R}$) such that*

$$c_1\tilde{\lambda}^1+\cdots+c_d\tilde{\lambda}^d \text{ is in a Poincaré domain,} \tag{2.2}$$

namely, the convex hull of the set $\{\sum_{j=1}^d c_j\lambda_k^j; k=1,\dots,m\}$ in $\mathbb{C}$ does not contain $0\in\mathbb{C}$ (respectively,

$$\text{the real parts of } c_1\lambda_j^1+\cdots+c_d\lambda_j^d,\ j=1,\dots,m, \text{ are positive.)} \tag{2.3}$$

Proof. — First we show (2.1). Suppose that (2.1) does not hold. Then there exists a sequence $\alpha^\ell\in\mathbb{Z}_+^m$, $\ell\in\mathbb{N}$ such that $|\alpha^\ell|\to\infty$ $(\ell\to\infty)$ and

$$\sum_{k=1}^{d}|\sum_{j=1}^{m}\lambda_j^k\alpha_j^\ell| \leqslant \frac{|\alpha^\ell|}{\ell}, \qquad \ell\in\mathbb{N}. \tag{2.4}$$

By taking a subsequence, if necessary, we may assume that $\alpha^\ell/|\alpha^\ell|\to t^0=(t_1^0,\dots,t_m^0)\in S_{\ell^1}^1\bigcap\mathbb{R}_+^m$ when $\ell\to\infty$, where $S_{\ell^1}^1:=\{x\in\mathbb{K}^m;\ \|x\|_{\ell^1}=\sum_{j=1}^m|x_j|=1\}$ stands for the ℓ^1 unit sphere. By letting $\ell\to\infty$ in (2.4) we get

$$\sum_{k=1}^{d}|\sum_{j=1}^{m}\lambda_j^k t_j^0|=0.$$

It follows that $\sum_{j=1}^{m} t_j^0\vec{\lambda}_j=0$. Let $J\subset\{1,\dots,m\}$ be such that $\sum_{j\in J}t_j^0\vec{\lambda}_j\neq 0$. Such a set J exists by (1.10). It follows that

$$0\neq\sum_{j\in J}t_j^0\vec{\lambda}_j=-\sum_{j\in\{1,\dots,m\}\setminus J}t_j^0\vec{\lambda}_j.$$

Hence $\Gamma[\Lambda_m]$ contains a straight line generated by $\sum_{j\in J}t_j^0\vec{\lambda}_j\neq 0$. This contradicts the assumption that $\Gamma[\Lambda_m]$ is a proper cone.

Conversely, suppose that (2.1) is satisfied. We shall show that $\Gamma[\Lambda_m]$ is proper. Indeed, if otherwise, we can find $t^0=(t_1^0,\dots,t_m^0)\in S_{\ell^1}^1\bigcap\mathbb{R}_+^m\setminus 0$ such that

$$\sum_{j=1}^{m}t_j^0\lambda_j^k=0, \qquad k=1,\dots,d. \tag{2.5}$$

Because the set $\{\alpha/|\alpha|; \alpha \in \mathbb{Z}_+^m(2)\}$ is dense in $S^1_{\ell^1} \bigcap \mathbb{R}^m_+$, there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m$, $\ell \in \mathbb{N}$ such that $|\alpha^\ell| \to \infty$ $(\ell \to \infty)$ and $\lim_{\ell\to\infty} \alpha^\ell/|\alpha^\ell| = t^0$. Therefore, in view of (2.5), we get

$$\lim_{\ell\to\infty} \left(\frac{1}{|\alpha^\ell|} \sum_{k=1}^{d} |\sum_{j=1}^{m} \lambda_j^k \alpha_j^\ell| \right) = 0,$$

which contradicts (2.1).

Next, we shall show ii). Suppose that $\Gamma[\Lambda_m]$ be a proper cone in $\mathbb{K}^d$. Then we can find $c = (c_1, \ldots, c_d) \in \mathbb{C}^d$ such that $\Gamma[\Lambda_m]$ is contained in the real half–space $P_c := \{z \in \mathbb{K}^d,\ \mathrm{Re}(\sum_{k=1}^d c_k z_k) > 0\}$. Therefore

$$0 < \mathrm{Re}(\sum_{k=1}^{d} c_k \sum_{j=1}^{m} t_j \lambda_j^k) = \sum_{j=1}^{m} t_j \mathrm{Re}(\sum_{k=1}^{d} c_k \lambda_j^k) \tag{2.6}$$

for all $t \in \mathbb{R}^m_+ \setminus 0$, which yields $\mathrm{Re}(\sum_{k=1}^d c_k \lambda_j^k) > 0$ for $j = 1, \ldots, m$. We note that, if $\mathbb{K} = \mathbb{R}$, then the use of the real part in the definition of the half–space is superfluous. Finally, we readily see, from (2.2) that, if $\mathbb{K} = \mathbb{C}$ (respectively, (2.3) if $\mathbb{K} = \mathbb{R}$), then the cone $\Gamma[\Lambda_m]$ is contained in P_c. Hence $\Gamma[\Lambda_m]$ is proper. □

Although the following proposition is known, we give an alternative proof for the sake of completeness. (cf. [Lemma 3.1, [25]].)

Proposition 2.2. — *Let the action ρ be a Poincaré morphism. Then we can find a vector field in the corresponding Lie algebra which has the same resonace as the simultaneous resonance of ρ and is in the Poincaré domain.*

Proof. — By ii) of Proposition 2.1 we can find a Poincaré vector field in the Lie algebra as a linear combination of a base corresponding to (2.2). Let c_ν be the numbers in (2.2), and define $\tilde{\lambda}^0 := (\lambda_1^0, \ldots, \lambda_m^0) = \sum_{\nu=1}^d c_\nu \tilde{\lambda}^\nu$. Let S be a similtaneous resonance of ρ. Consider

$$\langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = \sum_{\nu=1}^{d} c_\nu \left(\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu \right).$$

Because $\sum_{\nu=1}^d |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu| \neq 0$ for every $\alpha \in \mathbb{Z}_+^m(2) \setminus S$, it follows that the set $\langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = 0$ in $c = (c_1, \ldots, c_d) \in \mathbb{C}^d$ is a hyperplane if $\alpha \notin S$. It follows that the set

$$\{c = (c_1, \ldots, c_d) \in \mathbb{C}^d; \langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = 0, \exists j, 1 \leqslant j \leqslant m, \exists \alpha \in \mathbb{Z}_+^m(2) \setminus S\}$$

is a countable union of nowhere dense closed set. Therefore we can find $c = (c_1, \dots, c_d)$ for which $\sum_{\nu=1}^{d} c_\nu \tilde{\lambda}^\nu$ satisfies the Poincaré condition and has the resonance S. □

We propose a geometric expression of a Poincaré morphism.

Definition 2.3. — *Let $r > 0$ and g be a Riemannian metric on $\mathbb{R}^n$. We denote by $\langle \cdot, \cdot \rangle_g$ and $\| \cdot \|_g$ the inner product and the norm with respect to g, respectively. We say that $\mathcal{X}_\nu := \sum_{j=1}^{n} X_j^\nu(x)\partial_{x_j}$ $(\nu = 1, \dots, d)$ are simultaneously transversal to the sphere $\|x\|_g = r$ if, the vectors $X^\nu := (X_1^\nu, \dots, X_n^\nu)$ $(\nu = 1, \dots, d)$ satisfy*

$$\sum_{\nu=1}^{d} |\langle X^\nu, x\rangle_g| \neq 0, \quad \forall x, \quad \|x\|_g = r. \tag{2.7}$$

Theorem 2.4. — *Let $r > 0$. Suppose that $\mathcal{B}_\nu := \sum_{j=1}^{n} (A^\nu x)_j \partial_{x_j}$ $(\nu = 1, \dots, d)$ be a commuting system of semi simple linear real vector fields in $\mathbb{R}^n$. Let ρ be the action generated by $\{\mathcal{B}_\nu\}$. We choose a real non-singular matrix P such that $\Lambda^\nu = PA^\nu P^{-1}$ is a block diagonal matrix given by $\Lambda^\nu =$ diag $\{R_2(\xi_1^\nu, \eta_1^\nu), \dots, R_2(\xi_{n_1}^\nu, \eta_{n_1}^\nu), \lambda_{n_1+1}^\nu, \dots, \lambda_n^\nu\}$ for some integer $n_1 \leqslant n$. Let g be a Riemannian metric defined by ${}^tPP$. Then the following conditions are equivalent.*
(a) $\mathcal{B}_\nu$ $(\nu = 1, \dots, d)$ are simultaneously transversal to the sphere $\|x\|_g = r$.
(b) ρ is a Poincaré morphism.
(c) There exist real numbers c_ν $(\nu = 1, \dots, d)$ such that $\sum_{\nu=1}^{d} c_\nu \mathcal{B}_\nu$ is transversal to the sphere $\|x\|_g = r$.

Proof. — We note that $\langle x, y\rangle_g = \langle Px, Py\rangle$ and $\|x\|_g = \|Px\|$. By inserting the relation $X^\nu = A^\nu x = P^{-1}\Lambda^\nu Px$ into (2.7) we can easily see that the simultaneous transversality condition is equivalent to

$$\sum_{\nu=1}^{d} |\langle \Lambda^\nu y, y\rangle| \neq 0, \quad \forall y = (y_1, \dots, y_n), \quad \|y\| = r. \tag{2.8}$$

By definition, (2.8) can be written in

$$\sum_{\nu=1}^{d} \Big| \sum_{j=1}^{n_1} \xi_j^\nu (y_{2j-1}^2 + y_{2j}^2) + \sum_{j=n_1+1}^{n} y_j^2 \lambda_j^\nu \Big| \neq 0, \quad \forall y, \quad \|y\| = r. \tag{2.9}$$

We define $t = (t_1, \dots, t_n)$, $t \in \mathbb{R}_+^n$, $\sum t_j = 1$ by $t_j = (y_{2j-1}^2 + y_{2j}^2)/2$ if $j \leqslant n_1$ and $t_j = y_j^2$ if $j > 2n$. Noting that $\xi_j^\nu (y_{2j-1}^2 + y_{2j}^2) = 2t_j \xi_j^\nu = t_j(\xi_j^\nu + i\eta_j^\nu + \xi_j^\nu - i\eta_j^\nu)$ we see that (2.9) is written in $\sum_{\nu=1}^{d} |\sum_{j=1}^{n} t_j \lambda_j^\nu| \neq 0$ for every $t \in \mathbb{R}_+^n$ and $\sum t_j = 1$. This is equivalent to (b) by definition. Hence we have proved the equivalence of (a) and (b).

By Proposition 2.1 the condition (b) is equivalent to the existence of real numbers c_ν ($\nu = 1, \ldots, d$) such that $\sum_{\nu=1}^{d} c_\nu \mathcal{B}_\nu$ is a Poincaré vector field. By what we have proved in the above ($d = 1$) this is equivalent to say that $\sum_{\nu=1}^{d} c_\nu \mathcal{B}_\nu$ is transversal to the sphere $\|x\|_g = r$. $\square$

Proof of Theorem 1.4. By Proposition 2.2 there exists a Poincaré vector field χ_0 in ρ which is in a Poincaré domain and has the same resonance as ρ. If ρ is not resonant, then we have Theorem 1.4. In case there is a resonance of ρ, then it follows from Lemma 3.2 of [25] that, if χ_0 is normalized, then so is ρ. $\square$

3. Divergent solutions of overdetermined systems of linearized homological equations

We now study the action ρ_{lin} which admits a Jordan block. We assume that the action is formally (simultaneously) linearizable and is not a Poincaré morphism and that the family of linear parts is Diophantine.

Let $\mathbb{C}_2^n\{x\}$ be the set of n vector functions of convergent power series of x without constant and linear terms. We consider

$$L_A v = {}^t(L_1 v, \ldots, L_d v) = f, \quad f := {}^t(f^1, \ldots, f^d) \in (\mathbb{C}_2^n\{x\})^d, \tag{3.1}$$

under the compatibility conditions

$$L_j f^k = L_k f^j, \qquad j, k = 1, \ldots, d, \tag{3.2}$$

where L_j is the Lie derivative of the linear vector field $A_j x \partial_x$

$$L_j v = [A_j x, v] = \langle A_j x, \partial_x \rangle v - A_j v, \quad j = 1, \ldots, d.$$

First we consider a 2–action studied in Example 1.6. We assume that there exists a vector field in the two-dimensional Lie algebra which is not semi simple. In view of Example 1.6 we can choose a base X_1, X_2 with linear parts $A_j \in GL(4; \mathbb{C})$ satisfying $\mathrm{Spec}\,(A_1) = \{1, 1, \nu, \nu\}$ and $\mathrm{Spec}\,(A_2) = \{0, 1, \mu, \mu\}$, respectively, where $\nu \leqslant \mu \leqslant 0$, $(\nu, \mu) \notin \mathbb{Q}^2$, and

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \nu & \varepsilon \\ 0 & 0 & 0 & \nu \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & \varepsilon_0 \varepsilon \\ 0 & 0 & 0 & \mu \end{pmatrix}, \tag{3.3}$$

where $\varepsilon \neq 0$ and $\varepsilon_0 \in \mathbb{C}$. We can make $|\varepsilon| > 0$ arbitrarily small by an appropriate linear change of variables.

Let $\omega(\alpha)$ be defined by (1.15). We say that the simultaneous Diophantine order of $\{\mathrm{Spec}\,(A_1), \mathrm{Spec}\,(A_2)\}$ is τ_0, if, for every $\tau > \tau_0$ there exists $C = C_\tau > 0$ such that

$$\omega(\alpha) \geqslant C|\alpha|^{-\tau}, \quad \forall \alpha \in \mathbb{Z}^4_+(2), \tag{3.4}$$

while, for every $\tau < \tau_0$ there exist $C' > 0$ and a subsequence $\alpha^\ell \in \mathbb{Z}^4_+(2)$ ($\ell = 1, 2, \ldots$) such that

$$\omega(\alpha^\ell) \leqslant C'|\alpha^\ell|^{-\tau}, \qquad \ell \in \mathbb{N}. \tag{3.5}$$

First we note that the conditions (3.4) and (3.5) for $\omega(\alpha)$ are equivalent to the corresponding ones for $\|q\nu\| + \|q\mu\|$ when $q \in \mathbb{N}$, $q \to \infty$, where $\|t\| = \min_{p\in\mathbb{Z}} |p - t|$. Hence the number τ_0 in (3.4) and (3.5) is equal to the speed of the simultaneous approximation of ν and μ, namely $\|q\nu\| + \|q\mu\| \sim Cq^{-\tau_0}$ for some constant $C > 0$ independent of q. Clearly, if (3.4) holds, then we have an upper bound of τ_0. By the result of M. Herman, [16], we have an upper bound $2 + \varepsilon$ for every $\varepsilon > 0$ for almost all ν and μ. On the other hand, Moser showed that there exist Liouville numbers ν and μ such that $\|q\nu\| + \|q\mu\| \geqslant cq^{-\tau}$ for any given $\tau > 2$. (See [Theorem 2, [20]]). This implies that for every $\tau > 2$, there exist Liouville numbers ν and μ such that $\tau_0 \leqslant \tau$. We have another upper bound of τ_0 if either ν or μ is an algebraic number. Indeed, by Roth's theorem, for any given $\tau > 1$ there exists $c > 0$ such that $\|q\nu\| + \|q\mu\| \geqslant cq^{-\tau}$. Hence we have $\tau_0 \leqslant 1$. Finally, by [Corollary 1B, p.27, [22]], if either ν or μ is an irrational number, then we have a lower bound $\tau_0 \geqslant 1/2$.

We say that ν and μ are simultaneously Liouville, if (3.5) holds for every $\tau > 0$.

Let $\sigma \geqslant 1$. We say that a formal power series $f(x) = \sum_\alpha f_\alpha x^\alpha$ is in a Gevrey space $G^\sigma_2(\mathbb{C}^4)$ if $f_\alpha = 0$ for $|\alpha| \leqslant 1$ and, there exist $C > 0$ and $R > 0$ such that $|f_\alpha| \leqslant CR^{|\alpha|}|\alpha|!^{\sigma-1}$, $(\forall \alpha \in \mathbb{Z}^4_+)$.

We consider the following equation

$$L_A v := {}^t(L_1 v, L_2 v) = f, \quad f = {}^t(f^1, f^2) \in (\mathbb{C}^4_2\{x\})^2,\ x \in \mathbb{C}^4, \tag{3.6}$$

where ${}^t(f^1, f^2)$ satisfies the compatibility condition $L_1 f^2 = L_2 f^1$. Then we have:

Theorem 3.1. — *Assume that $\varepsilon_0 \in \mathbb{R} \setminus \{0\}$. Let $1 < \tau_0 < \infty$ be given. Then there exists $E_0 \subset \{(\nu, \mu) \in \mathbb{R}^2; \nu < \mu \leqslant 0\}$ with the density of continuum satisfying $\{(\nu, \mu) \in \mathbb{Q}^2; \nu < \mu \leqslant 0\} \subset E_0$ such that for every $(\nu, \mu) \in E_0$, there exists an $f = {}^t(f^1, f^2) \in (\mathbb{C}^4_2\{x\})^2$ such that $L_1 f^2 = L_2 f^1$ and Eq. (3.6) has a formal power series solution $v \notin \bigcup_{1\leqslant\sigma<2+\tau_0} G^\sigma_2(\mathbb{C}^4)$.*

Furthermore, if the conditions $(\nu,\mu) \notin \mathbb{Q}^2$, (3.4) and $\tau_0 < +\infty$ hold, then (3.6) has a unique solution $v \in \bigcap_{\sigma>2+\tau_0} G_2^\sigma(\mathbb{C}^4)$ for every ${}^t(f^1,f^2) \in (\mathbb{C}_2^4\{x\})^2$ satisfying $L_1 f^2 = L_2 f^1$.

In order to prove Theorem 3.1, we need a function space $\mathcal{G}$ which is a subspace of a set of holomorphic functions in a neighborhood of the origin. First we give the definition in the case $\varepsilon_0 = 1$, i.e., the nilpotent parts of A_1 and A_2 coincide. We define $\mathcal{G}$ by

$$\mathcal{G} := \left\{ f = {}^t(f_1,f_2,f_3,f_4);\ f_j \equiv f_j(x) = \sum_{\alpha\in C_j} f_{\alpha,j}x^\alpha,\ j=1,2,3,4 \right\},$$

where $C_j \subset \mathbb{Z}_+^4(2)$ satisfies the following two conditions.
(1) There exist $c_0 > 0$ and $\tau > \tau_0$ such that for every $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in C_j$, we have $N = \alpha_3+\alpha_4 \neq 0$, $|\alpha| \geqslant 2$, and

$$|\alpha_1 - 1 + (\nu-\mu)N| < N^{-N(\tau+1)}c_0^N, \qquad \forall\alpha \in C_j\ (j=1,2),$$
$$|\alpha_1 + (\nu-\mu)(N-1)| < N^{-N(\tau+1)}c_0^N, \quad \forall\alpha\in C_j\ (j=3,4).$$

(2) The Diophantine condition for $\mathit{Spec}(A_1)$ holds: namely for every $\tau' < \tau_0 < \tau''$, there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 N^{-\tau''} < |\alpha_1+\alpha_2-1+\nu N| < c_2 N^{-\tau'} \quad \text{if } \alpha\in C_j, j=1,2,$$
$$c_1 N^{-\tau''} < |\alpha_1+\alpha_2+\nu N-\nu| < c_2 N^{-\tau'} \quad \text{if } \alpha\in C_j, j=3,4,$$

where $N = \alpha_3+\alpha_4 \neq 0$.

Remark 3.2. — If $\varepsilon_0 \neq 1$, we replace (1) with the following (1)'.

(1)' There exist $c_0 > 0$ and $\tau > \tau_0$ such that for every $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in C_j$, we have $N = \alpha_3+\alpha_4 \neq 0$, and

$$|\varepsilon_0(\alpha_1 - 1 + \alpha_2 + \nu N) - (\alpha_2+\mu N)| < N^{-N(\tau+1)}c_0^N, \quad \text{if } \alpha\in C_1.$$

In the case $\alpha \in C_2$, we replace α_1 and α_2 in the left-hand side of the above inequality with α_1+1 and α_2-1, respectively. Similarly, in the case $\alpha\in C_3$ or $\alpha \in C_4$, we replace α_1 and N in the left-hand side of the above inequality with α_1+1 and $N-1$, respectively.

Remark 3.3. — The space $\mathcal{G}$ is a normed space with the norm $\|f\| := \sum_\alpha |f_\alpha|$, where $|f_\alpha| = \sum_j |f_{\alpha,j}|$ ($f \in \mathcal{G}$). If the conditions (1) and (2) in the definition of $\mathcal{G}$ hold, then the Diophantine condition for $\mathit{Spec}(A_2)$ holds. Hence we have a simultaneous Diophantine condition for $\mathit{Spec}(A_1)$ and $\mathit{Spec}(A_2)$. In the following, we will show that on the support C_j of $\mathcal{G}$, the divergence of the solutions of L_A occurs, with a sharp Gevrey loss equal to $1+\tau$ ($G^1 \to G^{2+\tau}$).

Remark 3.4. — The space $\mathcal{G}$ is not empty for an appropriate choice of ν and μ such that $\nu < \mu < 0$, i.e., the action is not a Poincaré morphism. We first consider the case $\varepsilon_0 = 1$ for the sake of simplicity. If we construct ν and μ so as to satisfy the conditions (1) and (2) for $C_1 = C_2$, then (1) and (2) for $C_3 (= C_4)$ hold if we replace α_1 and N in C_1 with $\alpha_1 + 1$ and $N - 1$, respectively. Hence we will consider C_1.

We can easily construct an irrational number $\nu < 0$ which satisfies (2). In fact, $\alpha_1 + \alpha_2$ and N are given by a continued fraction expansion of ν. Note that α_1 can be taken arbitrarily. Next, by the standard measure theoretic argument, we can show that there exist an irrational number μ with $\nu - \mu < 0$ and the sequence $\{\alpha_1\}$ such that (1) holds. By the construction, we can also choose $\mu < 0$ such that $\nu < \mu < 0$. It follows that the action is not a Poincaré morphism. Moreover, we can easily see that the set of ν and μ satisfying (1) and (2) has the density of continuum.

Next we consider the case $\varepsilon_0 \neq 1$. For the sake of simplicity, we give the sketch of the proof for C_1 in the case $0 < \varepsilon_0 < 1$. The other cases can be treated similarly. First we construct ν so as to satisfy (2). Then the sequence of the integers $k \equiv \alpha_1 + \alpha_2 - 1$ and N are also given. In order to show that there exists μ satisfying $(1)'$, we consider the inequality

$$\left| \frac{\alpha_1 - 1 + (1 - \varepsilon_0^{-1})\alpha_2}{N} - (\varepsilon_0^{-1}\mu - \nu) \right| < N^{-N(\tau+1)-1} c_0^N \varepsilon_0^{-1}.$$

We consider closed intervals of length $2N^{-N(\tau+1)-1} c_0^N \varepsilon_0^{-1}$ with the centers at $\frac{\alpha_1 - 1 + (1-\varepsilon_0^{-1})\alpha_2}{N}$, $(\alpha_1 + \alpha_2 = k + 1)$. Let N and one of these intervals I_N are given. Then we can choose $N' > N$ and $I_{N'}$ such that I_N contains $I_{N'}$. Hence we can construct a sequence of monotone decreasing intervals. By taking a subsequence, if necessary, we see that there exists μ which satisfies $(1)'$. By construction the set of μ has the density of continuum. We remark that we can take $\tilde{\nu} := \varepsilon_0^{-1}\mu - \nu > 0$ if $0 < \varepsilon_0 < 1$. Indeed, since $1 - \varepsilon_0^{-1} < 0$ and $k/N \to -\nu > 0$ as $k, N \to \infty$, it follows that one can take the interval I_N so that I_N is contained in the positive real axis and it is arbitrarily close to the origin. Hence we have $\mu = \varepsilon_0(\tilde{\nu} + \nu) > \varepsilon_0 \nu > \nu$, which implies that the action is not a Poincaré morphism. Similarly, we can show that there exists μ such that the condition does not hold in other cases.

The proof of Theorem 3.1 follows from the following propositions.

Proposition 3.5. — *Assume that $\varepsilon_0 \in \mathbb{R} \setminus \{0\}$. Let $1 < \tau_0 < \infty$. Then there exists $E_0' \subset \{(\nu, \mu) \in \mathbb{R}^2 \setminus \mathbb{Q}^2; \nu < \mu < 0\}$ with the density of continuum such that the following property holds. For every $(\nu, \mu) \in E_0'$ there exist real numbers c_1, c_2 and $k_0 > 0$ such that for every $g \in \mathcal{G}$,*

$g=\sum_{|\alpha|\geqslant k_0} g_\alpha x^\alpha$, there exist $f^j \in \mathcal{G}$ $(j=1,2)$ such that $L_1 f^2 = L_2 f^1$, $g = c_1 f^1 + c_2 f^2$. Moreover, $B := c_1 A_1 + c_2 A_2$ is nonresonant, and ω defined by (1.15) for Spec(B) satisfies (3.5).

PROPOSITION 3.6. — *Assume that $\varepsilon_0 \in \mathbb{R}\setminus\{0\}$ and $1<\tau_0<\infty$. Let $(\nu,\mu)\in E_0'$, where E_0' is given by Proposition 3.5. Let c_1, c_2 and B be as in Proposition 3.5. Then there exists $g\in\mathcal{G}$ such that the homology equation $L_B v = g$ has a unique formal power series solution v which is not contained in $\cup_{1\leqslant\sigma<2+\tau_0} G_2^\sigma(\mathbb{C}^4)$.*

Remark 3.7. — Our divergence results imply in the case of a single holomorphic vector field, that generically vector fields obtained by nonlinear holomorphic perturbations are nonlinearizable (see R. Pérez Marco [19] for more details). We point out that our results generalize those for single vector fields in the presence of nontrivial Jordan blocks (see [15]). As to the case of smooth C^∞ hyperbolic $\mathbb{R}^2$–actions we refer [13].

First we will prove Theorem 3.1, assuming Propositions 3.5 and 3.6.

Proof of Theorem 3.1. — We will prove the former half. Let E_0' be the set given by Proposition 3.5. We define $E_0 := E_0' \cup \{(\nu,\mu)\in\mathbb{Q}^2; \nu<\mu\leqslant 0\}$. By the result of Example 1.6 (a), we know that if $(\nu,\mu)\in\mathbb{Q}^2$, $\nu<\mu\leqslant 0$, then (3.6) has an infinite resonance, $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $\alpha_1 = 1+(\mu-\nu)k$, $\alpha_2=-\mu k$, $\alpha_3+\alpha_4=k$, where $k\geqslant(1-\nu)^{-1}$, $k\in\mathbb{Z}_+$. Hence $v=\sum_\alpha v_\alpha x^\alpha$ is a formal solution of (3.6) with $f=0$ for $v_\alpha\in\mathbb{C}^4$, where the summation with respect to α is taken over the resonances α in the above. Because $|v_\alpha|$ is arbitrary, we take v_α such that $|v_\alpha| = |\alpha|!^{2+\tau_0}$ $(|\alpha|\to\infty)$, which implies $v\notin\bigcup_{1\leqslant\sigma<2+\tau_0} G_2^\sigma(\mathbb{C}^4)$.

Next we study the case $(\nu,\mu)\in E_0'$. By Proposition 3.6 there exists $g\in\mathcal{G}$ such that the unique solution v of $L_B v = g$ with $B=c_1A_1+c_2A_2$ is not contained in $\bigcup_{1\leqslant\sigma<2+\tau_0} G_2^\sigma(\mathbb{C}^4)$. By Proposition 3.5 we can choose $f^j\in\mathcal{G}$ $(j=1,2)$ such that $L_1f^2 = L_2 f^1$ and $g = c_1 f^1 + c_2 f^2$. Because the solution v of $L_A v = f$ is a unique solution of $L_B v = g$, we see that $v\notin\bigcup_{1\leqslant\sigma<2+\tau_0} G_2^\sigma(\mathbb{C}^4)$.

We will prove the latter half. We consider the system of equations $L_j v = f^j (j=1,2)$, where $L_1 f^2 = L_2 f^1$. Let B denote either A_1 or A_2. For the sake of simplicity, we assume that B is put in a Jordan normal form with the diagonal part $B^0 := \mathrm{diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_3\}$ and the off-diagonal element ε_1. We note that, for the equation $L_1 v = f^1$ we have $\lambda_1=\lambda_2=1$, $\lambda_3=\nu$, $\varepsilon_1=\varepsilon$, while for $L_2 v = f^2$ we have $\lambda_1=0,\lambda_2=1$, $\lambda_3=\mu$, $\varepsilon_1=\varepsilon_0\varepsilon$. The homology operator L_B corresponding to B is given by

$$L_B v = \langle B^0 x, \partial_x\rangle v + \varepsilon_1 R[v] - Bv, \qquad v \in \mathbb{C}_2^4\{x\}, \tag{3.7}$$

$$\langle B^0 x, \partial_x\rangle v = \sum_{|\alpha|\geqslant 2} (\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3(\alpha_3+\alpha_4))v_\alpha x^\alpha, \tag{3.8}$$

where $v(x) = \sum_{|\alpha|\geqslant 2} v_\alpha x^\alpha$ and

$$R[v] = \sum_{|\alpha|\geqslant 2} (\alpha_3+1) v_{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4-1)} x^\alpha. \tag{3.9}$$

For $g(x) = {}^t(g_1, g_2, g_3, g_4) \in \mathbb{C}_2^4\{x\}$ we expand $g_k(x)$ in the Taylor series $g_k(x) = \sum_\alpha g_{\alpha;k} x^\alpha$. For nonnegative integers N, α_1 and α_2 we define V_k and G_k by

$$V_k := {}^t\{v_{(\alpha_1,\alpha_2,N-\ell,\ell);k}\}_{\ell=0}^N, \quad G_k := {}^t\{g_{(\alpha_1,\alpha_2,N-\ell,\ell);k}\}_{\ell=0}^N,$$

where $k = 1,2,3,4$. In view of (3.9), the equation $L_B v = g$ is equivalent to

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3 N - \lambda_1)V_1 + \varepsilon_1 \mathcal{M}_N V_1 = G_1, \tag{3.10}$$

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3 N - \lambda_2)V_2 + \varepsilon_1 \mathcal{M}_N V_2 = G_2, \tag{3.11}$$

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3 (N-1))V_3 + \varepsilon_1 \mathcal{M}_N V_3 = G_3 + \varepsilon_1 V_4, \tag{3.12}$$

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3 (N-1))V_4 + \varepsilon_1 \mathcal{M}_N V_4 = G_4, \tag{3.13}$$

where $\mathcal{M}_N$ is given by

$$\mathcal{M}_N = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ N & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & N-1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & N-2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \quad N \geqslant 1, \tag{3.14}$$

and $\mathcal{M}_0 = 0$.

Let $f^j(x) = {}^t(f_1^j(x), \dots, f_4^j(x))$ and let $f_k^j(x) = \sum_\alpha f_{\alpha;k}^j x^\alpha$ $(j = 1,2; k = 1,\dots,4)$ be the Taylor expansion of $f_k^j(x)$. We substitute the Taylor expansions of v and f^j into the equations $L_j v = f^j$. For every $(\alpha_1,\alpha_2) \in \mathbb{Z}_+^2$ and $N \in \mathbb{Z}_+$ such that $\alpha_1 + \alpha_2 + N \geqslant 2$ we compare the coefficients of x^α $(\alpha_3 + \alpha_4 = N)$ with homogeneous degree $\alpha_1 + \alpha_2 + N$. If we set

$$F^j = {}^t(F_1^j, F_2^j, \dots, F_4^j), \quad F_k^j = {}^t\{f^j_{(\alpha_1,\alpha_2,N-r,r);k}\}_{r=0}^N, \tag{3.15}$$

and $V = {}^t(V_1, V_2, \dots, V_4)$, $V_k := {}^t\{v_{(\alpha_1,\alpha_2,N-r,r);k}\}_{r=0}^N$, $H = {}^t(0,0,V_4,0)$, then we can write the system of equations $L_j v = f^j$ $(j = 1,2)$ in the following form

$$\mathcal{A}V = F^1 + \varepsilon H, \quad \mathcal{B}V = F^2 + \varepsilon\varepsilon_0 H, \tag{3.16}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the block diagonal matrices given by

$$\begin{aligned}\mathcal{A} &:= \operatorname{diag}\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} \\ &= \operatorname{diag}\begin{pmatrix} (\alpha_1+\alpha_2+\nu N-1)Id + \varepsilon M_N \\ (\alpha_1+\alpha_2+\nu N-1)Id + \varepsilon M_N \\ (\alpha_1+\alpha_2+\nu N-\nu)Id + \varepsilon M_N \\ (\alpha_1+\alpha_2+\nu N-\nu)Id + \varepsilon M_N \end{pmatrix},\end{aligned} \tag{3.17}$$

$$\begin{aligned}\mathcal{B} &:= \operatorname{diag}\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\} \\ &= \operatorname{diag}\begin{pmatrix} (\alpha_2+\mu N)Id + \varepsilon_0\varepsilon M_N \\ (\alpha_2+\mu N-1)Id + \varepsilon_0\varepsilon M_N \\ (\alpha_2+\mu N-\mu)Id + \varepsilon_0\varepsilon M_N \\ (\alpha_2+\mu N-\mu)Id + \varepsilon_0\varepsilon M_N \end{pmatrix}.\end{aligned} \tag{3.18}$$

We will solve (3.16). Because either ν or μ is an irrational number, we suppose that $\nu \notin \mathbb{Q}$. We will show that for each $k = 1, \ldots, 4$ either $\mathcal{A}_k$ or $\mathcal{B}_k$ is nonsingular. Indeed, suppose that $|\alpha| = \alpha_1 + \alpha_2 + N \geqslant 2$. If $N \neq 0, 1$, then by the irrationality of ν, the matrices $\mathcal{A}_k$ $(k = 1, \ldots, 4)$ are nonsingular. If $N = 0$ or $N = 1$, then by the condition $\alpha_1 + \alpha_2 + N \geqslant 2$, $\mathcal{A}_k$ $(k = 1, \ldots, 4)$ are nonsingular. We can similarly argue if $\mu \notin \mathbb{Q}$.

We will determine V_4. By inductive arguments and $L_1 f^2 = L_2 f^1$ we get

$$\begin{aligned} v_{(\alpha_1,\alpha_2,N-\ell,\ell);4} &= \sum_{r=0}^{\ell} \frac{(-\varepsilon_1)^r}{(\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3(N-1))^{r+1}} \\ &\times \frac{(N-\ell+r)!}{(N-\ell)!} g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4}\end{aligned} \tag{3.19}$$

for $\ell = 0, 1, \ldots, N$, provided $\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3(N-1) \neq 0$. Note that, if $\mathcal{A}_4$ is nonsingular, then (3.19) is valid for $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \nu$, $\varepsilon_1 = \varepsilon$, $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4} = f^1_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4}$, while if $\mathcal{B}_4$ is nonsingular, then (3.19) is valid for $\lambda_1 = 0, \lambda_2 = 1$, $\lambda_3 = \mu$, $\varepsilon_1 = \varepsilon_0\varepsilon$, $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4} = f^2_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4}$. Similar explicit formulas are derived for $v_{\alpha_1,\alpha_2,N-\ell,\ell;k}$, $k = 1, 2$. As to the term $v_{(\alpha_1,\alpha_2,N-\ell,\ell);3}$, there appears the term $\varepsilon_1 V_4^N$ in the right-hand side of (3.12).

By (3.4) we have

$$|\alpha_1 + \alpha_2 + \nu N - \nu| + |\alpha_2 + \mu N - \mu| \geqslant C|\alpha_1 + \alpha_2 + N|^{-\tau} \tag{3.20}$$

for some $C > 0$. It follows that either $|\alpha_1+\alpha_2+\nu N-\nu| \geqslant C|\alpha_1+\alpha_2+N|^{-\tau}/2$ or $|\alpha_2 + \mu N - \mu| \geqslant C|\alpha_1 + \alpha_2 + N|^{-\tau}/2$ holds. Suppose that the former estimate holds. We have the same estimate in case the latter inequality holds. Without loss of generality we may assume that $C < 2$. Let τ be such that $\tau > \tau_0$. Then we have

$$(3.21)\quad \begin{aligned} |\alpha_1+\alpha_2+\nu(N-1)|^{r+1} &\geqslant (C/2)^{r+1}|\alpha_1+\alpha_2+N|^{-\tau(r+1)} \\ &\geqslant (C/2)^{N+1}|\alpha_1+\alpha_2+N|^{-\tau(N+1)}. \end{aligned}$$

Noting that $(N-\ell+r)!/(N-\ell)! \leqslant N!$, we see from (3.19) that if $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4}$ has a G^s estimate, namely, $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4} = O((\alpha_1+\alpha_2+N)!^{s-1})$ modulo exponential factors, then $v_{(\alpha_1,\alpha_2,N-\ell,\ell);4} = O((\alpha_1+\alpha_2+N)!^{s+\tau})$. Especially, if $s=1$, then we have $v_{(\alpha_1,\alpha_2,N-\ell,\ell);4} = O((\alpha_1+\alpha_2+N)!^{\tau+1})$. Similarly, we can easily see that $v_{(\alpha_1,\alpha_2,N-\ell,\ell);j}$ $(j=1,2,4)$ have the estimate $v_{(\alpha_1,\alpha_2,N-\ell,\ell);j} = O((\alpha_1+\alpha_2+N)!^{\tau+1})$.

Next we determine $v_{(\alpha_1,\alpha_2,N-\ell,\ell);3}$ by a similar relation like (3.19). We note that there appears $v_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4}$ in the term $g_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);3}$ of (3.19). By (3.19) we can easily see that $v_{(\alpha_1,\alpha_2,N-\ell+r,\ell-r);4} =$ $O(N^{(\ell-r+1)\tau}(\ell-r)!)$ modulo terms of exponential growth C^N for some $C>0$. Substituting the estimate into (3.19), we see that the right-hand side of (3.19) is estimated by $N^{(\ell-r+1)\tau+(r+1)\tau}(N-\ell+r)!(\ell-r)!/(N-\ell)! = N^{(\ell+2)\tau}N!/(N-\ell)!$ modulo terms of exponential growth. Because $\ell \leqslant N$ and $\tau \geqslant 1$, we see that $v_{(\alpha_1,\alpha_2,N-\ell,\ell);3} = O(N^{N\tau}N!)$ modulo terms of exponential growth. Since $\tau > \tau_0$ is arbitrary, it follows that $v_{(\alpha_1,\alpha_2,N-\ell,\ell);3} = O((\alpha_1+\alpha_2+N)!^{\sigma})$ for every $\sigma > 1+\tau_0$. □

Proof of Proposition 3.5. — Let E_0' be the set of $(\nu,\mu) \in \mathbb{R}^2\backslash\mathbb{Q}^2$, $\nu<\mu\leqslant 0$ such that $\mathcal{G} \neq \emptyset$. The set E_0' has the density of continuum. (cf. Remark 3.4.) We shall show that if $(c_1,c_2) \notin \mathbb{R}^2$ is not contained in the some set E with Lebesgue measure zero, then $B := c_1A_1+c_2A_2$ is nonresonant. Indeed, the eigenvalues of B are given by $c_1, c_1+c_2, c_1\nu+c_2\mu$ with multiplicity. For every $\alpha = (\alpha_1,\ldots,\alpha_4) \in \mathbb{Z}_+^4(2)$, the resonance relations are given by

$$(3.22)\qquad c_1\alpha_1+(c_1+c_2)\alpha_2+(c_1\nu+c_2\mu)(\alpha_3+\alpha_4) = c_1,$$

and the ones with c_1 in the right-hand side replaced by c_1+c_2 and $c_1\nu+c_2\mu$, respectively. Because the argument is similar, we consider the first relation. It follows from (3.22) that

$$c_1(\alpha_1+\alpha_2+\nu(\alpha_3+\alpha_4)-1)+c_2(\alpha_2+\mu(\alpha_3+\alpha_4)) = 0.$$

Because $(\nu,\mu) \notin \mathbb{Q}^2$ and $|\alpha| \geqslant 2$, we can easily see that either $\alpha_1+\alpha_2+\nu(\alpha_3+\alpha_4)-1 \neq 0$ or $\alpha_2+\mu(\alpha_3+\alpha_4) \neq 0$ holds. Hence the set of $(c_1,c_2) \in \mathbb{R}^2$ satisfying (3.22) is a straight line. Therefore the set E of all (c_1,c_2) satisfying a resonance relation for some α has Lebesgue measure zero.

In order to show that $Spec(B)$ satisfies (3.5), let $\tilde{\omega}_j(\alpha)$ and $\tilde{\omega}(\alpha)$ ($\alpha \in \mathbb{Z}_+^4$) be defined by (1.14) and (1.15) for B, respectively. Then we have $\tilde{\omega}(\alpha) \leqslant \tilde{\omega}_j(\alpha) \leqslant \max\{|c_1|, |c_2|\}\omega_j(\alpha)$ for $j = 1, \ldots, 4$ and all $\alpha \in \mathbb{Z}_+^4$. Next, we will estimate $\omega_4(\alpha)$ for $\alpha \in C_4$, where C_4 is given in the definition of $\mathcal{G}$. We note $\omega_4(\alpha) = |\alpha_1 + \alpha_2 + \nu(N-1)| + |\alpha_2 + \mu(N-1)|$ and $|\alpha|/N \to \nu$ ($N, |\alpha| \to \infty$). By the conditions (1) and (2) in the definition of $\mathcal{G}$ we have that for every $\tau' < \tau_0$ there exists $c_0 > 0$ such that $\omega_4(\alpha) \leqslant c_0|\alpha|^{-\tau'}$, when $|\alpha| \to \infty$, $\alpha \in C_4$. This proves (3.5).

Let $(c_1, c_2) \notin E$ and $g \in \mathcal{G}$ be given. We consider

$$L_1 f^2 = L_2 f^1, \qquad c_1 f^1 + c_2 f^2 = g. \tag{3.23}$$

By expanding $f^j(x) = {}^t(f_1^j, f_2^j, f_3^j, f_4^j)$ into the Taylor series we define F^j by (3.15). We similarly define

$$G = {}^t(G_1, G_2, G_3, G_4), \quad G_k = {}^t\{g_{(\alpha_1,\alpha_2,N-r,r);k}\}_{r=0}^N,$$

where $g(x) = {}^t(g_1, g_2, g_3, g_4)$, $g_k(x) = \sum_\alpha g_{\alpha;k}x^\alpha$. We set $H^1 := {}^t(0, 0, F_4^1, 0)$ and $H^2 := {}^t(0, 0, F_4^2, 0)$. We substitute the expansions of f^j and g into (3.23). For every $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and $N \in \mathbb{Z}_+$ such that $\alpha_1 + \alpha_2 + N \geqslant 2$ we compare the coefficients of x^α of homogeneous degree $\alpha_1 + \alpha_2 + N$. Then (3.23) is equivalent to

$$\mathcal{A}F^2 - \mathcal{B}F^1 - \varepsilon H^2 + \varepsilon\varepsilon_0 H^1 = 0, \qquad c_1 F^1 + c_2 F^2 = G, \tag{3.24}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by (3.17) and (3.18).

First we will construct a formal power series solution F^j ($j = 1, 2$) of (3.24). It follows from (3.24) and the definition of H^j that

$$\mathcal{A}_k F_k^2 - \mathcal{B}_k F_k^1 = 0, \quad c_1 F_k^1 + c_2 F_k^2 = G_k, \quad k = 1, 2, 4.$$

We recall that (cf. the proof of Theorem 3.1) either $\mathcal{A}_k$ or $\mathcal{B}_k$ is nonsingular for each $k = 1, 2, \ldots, 4$. Assuming that $\mathcal{A}_k$ is nonsingular we obtain $F_k^2 = \mathcal{A}_k^{-1}\mathcal{B}_k F_k^1$, and hence $c_1 F_k^1 + c_2\mathcal{A}_k^{-1}\mathcal{B}_k F_k^1 = G_k$. It follows that

$$F_k^1 = (c_1 Id + c_2\mathcal{A}_k^{-1}\mathcal{B}_k)^{-1} G_k = (c_1\mathcal{A}_k + c_2\mathcal{B}_k)^{-1}\mathcal{A}_k G_k, \tag{3.25}$$

if $c_1\mathcal{A}_k + c_2\mathcal{B}_k$ is nonsingular. The last condition holds if (c_1, c_2) is not contained in a set of Lebesgue measure zero in $\mathbb{R}^2$, which may depend on α_1, α_2, N. We have similar relations if $\mathcal{B}_k$ is nonsingular.

In case $k = 3$, we obtain $\mathcal{A}_3 F_3^2 - \mathcal{B}_3 F_3^1 = \varepsilon(F_4^2 - \varepsilon_0 F_4^1)$ instead of $\mathcal{A}_k F_k^2 - \mathcal{B}_k F_k^1 = 0$. A simple computation yields that

$$F_3^1 = (c_1\mathcal{A}_3 + c_2\mathcal{B}_3)^{-1}\mathcal{A}_3 G_3 - \varepsilon c_2(c_1\mathcal{A}_3 + c_2\mathcal{B}_3)^{-1}(F_4^2 - \varepsilon_0 F_4^1).$$

By taking the union of all exceptional sets of (c_1, c_2) with α_1, α_2 and N in the set of nonnegative integers such that $\alpha_1 + \alpha_2 + N \geqslant 2$, we see that there exists a unique formal power series solution $f^j(x)$ ($j = 1, 2$) of (3.23), provided (c_1, c_2) is not in an exceptional set of Lebesgue measure zero.

We will show the convergence of $f^j(x)$ ($j = 1, 2$). It is sufficient to prove the convergence of $f^1(x)$ since we may take $c_2 \neq 0$ in view of the choice of c_1 in the above argument. By the condition (2) in the definition of $\mathcal{G}$, we can easily see that L_1^{-1} exists on $\mathcal{G}$, namely $L_1^{-1}L_1 = L_1L_1^{-1} = Id$. Let $g \in \mathcal{G}$. Then it follows from (3.23) and the relation $L_1f^2 = L_2f^1$ that $L_1g = c_1L_1f^1 + c_2L_2f^1$. Hence we have

$$g = c_1f^1 + c_2L_1^{-1}L_2f^1.$$

Now we have

$$L_1^{-1}L_2 = L_2L_1^{-1} = (L_2 - \varepsilon_0L_1)L_1^{-1} + \varepsilon_0Id, \quad \text{on } \mathcal{G}.$$

By definition, we have

$$L_2 - \varepsilon_0L_1 = \langle A_2x, \partial_x\rangle - \varepsilon_0\langle A_1x, \partial_x\rangle + \varepsilon_0A_1 - A_2.$$

Hence, $L_2 - \varepsilon_0L_1$ is semi-simple. By the condition (2) and the proof of the latter half of Theorem 3.1, it follows that the absolute value of the coefficient of x^α of $L_1^{-1}g$ ($g = \sum_\alpha g_\alpha x^\alpha$) is bounded by $N^{(\tau''+1)N}C^N|g_\alpha|$ for some $C > 0$, where $\tau'' > \tau_0$ can be taken arbitrarily close to τ_0. On the other hand, the operator $(L_2 - \varepsilon_0L_1)$ is the one which multiplies the coefficients of x^α with $(\alpha_2 + \mu N - \varepsilon_0(\alpha_1 + \alpha_2 - 1 + \nu N))$ for the first component. We have similar expressions for other components. By the condition $(1)'$, the absolute value of the term is bounded by $N^{-N(\tau+1)}c_0^N$ for some $\tau > \tau_0$. Because $\tau'' > \tau_0$ can be taken arbitrarily close to τ_0, the growth $N^{N(\tau''+1)}C^N$ which comes from L_1^{-1} is absorbed by the term $N^{-N(\tau+1)}c_0^N$. Therefore, the operator $(L_2 - \varepsilon_0L_1)L_1^{-1}$ maps $\mathcal{G}$ to $\mathcal{G}$. By taking k_0 sufficiently large, the norm of $(L_2 - \varepsilon_0L_1)L_1^{-1}$ on the space $\mathcal{G} \cap \{g = \sum_\alpha g_\alpha x^\alpha; |\alpha| > k_0\}$ can be made arbitrarily small.

In view of the construction of c_1 and c_2 we may assume that $c_1 + c_2\varepsilon_0 \neq 0$. Writing

$$g = c_1f^1 + c_2L_1^{-1}L_2f^1 = (c_1Id + c_2\varepsilon_0Id + R)f^1,$$

where $R = (\varepsilon_0L_1 - L_2)L_1^{-1}$, and by noting that R preserves homogeneous polynomials, we see that $(c_1Id + c_2\varepsilon_0Id + R)^{-1}$ exists as a map from $\mathcal{G}$ to $\mathcal{G}$. Therefore we have $f^1 \in \mathcal{G}$. $\square$

Proof of Proposition 3.6. — Because Spec (B) satisfies (3.5) by Proposition 3.5 we may assume, by taking a subsequence if necessary, that $\tilde{\omega}_j(\alpha)$ satisfies (3.5) for $\alpha = \alpha_\ell$. Without loss of generality we may assume that $j = 4$. We consider $\tilde{\omega}_4(\alpha)$. Let $g = {}^t(g_1, \dots, g_4)$, $g_k = \sum_\beta g_{\beta;k}x^\beta$ be the convergent power series defined by $g_{\beta;k} = 0$ for $k = 1, 2, 3$ and all $\beta \in \mathbb{Z}_+^4(2)$;

$g_{(\beta_1,\beta_2,\beta_3,\beta_4);4} = 0$ if $\beta_4 \geqslant 1$; $g_{(\alpha_1,\alpha_2,N,0);4} = 1$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_\ell$, $(\ell = 1, 2, \ldots)$, $N = \alpha_3 + \alpha_4$; $g_{(\beta_1,\beta_2,\beta_3,0);4} = 0$, if otherwise. We want to solve $L_B v = g$. Let λ_j be the eigenvalues of B. By the same argument as in the proof of Theorem 3.1 we have the formula (3.19). Then we have

$$v_{(\alpha_1,\alpha_2,0,N);4} = (-\varepsilon_1)^N (\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3(N-1))^{-N-1} N!, \tag{3.26}$$

for all $\alpha = \alpha^\ell, \ell = 1, 2, \ldots$. By (3.5) we have: for every $\tau' < \tau_0$ we can find a constant $C > 0$ such that

$$|(\lambda_1\alpha_1^\ell + \lambda_2\alpha_2^\ell + \lambda_3(N_\ell - 1))^{-1}| \geqslant C N_\ell^{\tau'}, \quad \forall \ell \in \mathbb{N},$$

where $\alpha^\ell = (\alpha_1^\ell, \alpha_2^\ell, \alpha_3^\ell, \alpha_4^\ell)$, $N_\ell = \alpha_3^\ell + \alpha_4^\ell$. Therefore, by (3.26)

$$|v_{(\alpha_1^\ell,\alpha_2^\ell,0,N_\ell);4}| \geqslant (C|\varepsilon_1|)^{N_\ell} N_\ell^{(N_\ell+1)\tau'} N_\ell!, \quad \ell \in \mathbb{N}, \alpha_1 \in \mathbb{Z}_+(2). \tag{3.27}$$

Because $\varepsilon_1 \neq 0$ and $\tau' < \tau_0$ can be taken arbitrarily close to τ_0, (3.27), Stirling's formula and the inequality $N! \geqslant C^N N^N, \forall N \in \mathbb{Z}_+$ yield the assertion. □

Example 3.8. — We give an example of a formal Gevrey linearization. (cf. Theorem 3.1.) We consider

$$L_\Lambda u = R(x+u), \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\tau & -1 \\ 0 & 0 & -\tau \end{pmatrix}, \tag{3.28}$$

where $\tau > 0$ is an irrational number. For $C \gg 1$, let f be an analytic function $f(x_1, x_2) = \sum_\alpha f_\alpha x_1^{\alpha_1} x_2^{\alpha_2}$, where the summation with respect to α is taken for $\alpha \in \mathbb{Z}_+^2(2)$ such that $1 < \alpha_1 - \tau\alpha_2 < C$. We define $R(x) = {}^t(0, x_3 f(x_1, x_2), 0)$. We shall show that the unique solution of (3.28) is in G^2. Indeed, we may look for the solution u of (3.28) in the form $u = {}^t(0, x_3 w(x), 0)$. We can easily see that w satisfies

$$\begin{aligned} \mathcal{L}w &\equiv (x_1\partial_{x_1} - \tau x_2\partial_{x_2} - \tau x_3\partial_{x_3} + x_3\partial_{x_2})w \\ &= f(x_1, x_2 + x_3 w) \equiv g(x). \end{aligned} \tag{3.29}$$

We substitue the expansion $w(x) = \sum_\alpha w_\alpha x^\alpha$ into (3.29). We can easily see that the summation in the expansion of $w(x)$ can be taken for α such that $\alpha_1 - \tau(\alpha_2 + \alpha_3) > 1$. Indeed, by the definition the support of $g(x)$ satisfies $\alpha_1 - \tau(\alpha_2 + \alpha_3) > 1$ if the support of $w(x)$ satisfies the same condition. On the other hand, by simple computations the support of $\mathcal{L}^{-1}w$ satisfies $\alpha_1 - \tau(\alpha_2 + \alpha_3) > 1$ if the support of w satisfies the condition. From these properties we can show the assertion for the homogeneous part 2 of w because there appears no term form w in g. Inductively, we can prove the assertion. If we expand $g(x) = \sum_\alpha g_\alpha x^\alpha$, then by the same calculations as in (3.19) we obtain

$$w_{(\alpha_1, N-\ell, \ell)} = \sum_{r=0}^{\ell} \frac{g_{(\alpha_1, N-\ell+r, \ell-r)}}{(\alpha_1 - \tau N)^{r+1}} \frac{(N-\ell+r)!}{(N-\ell)!}, \quad \ell = 0, 1, \dots, N. \tag{3.30}$$

If we can show that $g_{(\alpha_1, N-\ell+r, \ell-r)} = O((\ell - r)!)$ modulo terms of order K^{α_1+N} $(K > 0)$, then we can easily see that $w_{(\alpha_1, N-\ell, \ell)} = O(\ell!)$. This proves that the solution u of (3.29) is in G^2.

If $\alpha_1 + N = 2$, then no term from the expansion of w appears in $g_{(\alpha_1, N-\ell+r, \ell-r)}$ in (3.29). Hence, by the analyticity assumption of f, we obtain the desired estimate for w_α with $\alpha_1 + N = 2$, $\alpha_2 + \alpha_3 = N$. Suppose that we have $w_{(\alpha_1, N-\ell, \ell)} = O(\ell!)$ up to $\alpha_1 + N < \nu$ for some $\nu > 2$. Then by the definition of $g(x)$ and simple computations of the substitution of a Gevrey power series into an analytic function, we see that $g_{(\alpha_1, N-\ell, \ell)} = O(\ell!)$. Hence, by the inductive argument we obtain the desired estimate, $w_{(\alpha_1, N-\ell, \ell)} = O(\ell!)$, $\alpha_1 + N = \nu$.

We will briefly mention the general case of d–actions. We suppose that there exist j, $1 \leqslant j \leqslant m$ and ℓ_0, $1 \leqslant \ell_0 \leqslant d$ such that $A_j^{\ell_0}$ in (1.6) admits only one dimensional eigenspace, i.e., the geometric multiplicity of λ_j^ℓ is one. For a positive integer r we define the $r-$ square nilpotent matrix N_r by

$$N_r = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0\dots & 0 & 1 \\ 0 & 0 & 0\dots & 0 & 0 \end{pmatrix}. \tag{3.31}$$

By assumption we have

$$A_j^{\ell_0} = \lambda_j^{\ell_0} Id + \varepsilon N_{s_j}, \quad \varepsilon \neq 0. \tag{3.32}$$

By the explicit description of the centralizers of matrices (cf. [14]) all other matrices have the following form

$$A_j^\ell = \lambda_j^\ell Id + \sum_{k=1}^{s_j - 1} \varepsilon_k^{\ell j} (N_{s_j})^k \quad \varepsilon_k^{\ell j} \in \mathbb{C},\ k = 1, \dots, s_j - 1. \tag{3.33}$$

We have

Theorem 3.9. — *Assume (3.32). Then there exist $\varepsilon_k^{\ell_j}$ in (3.33), λ_j^ℓ, $(\ell = 1, 2, \dots, d; j = 1, 2, \dots, n)$ with the density of continuum such that the followings hold;*

(i) The simultaneously nonresonant condition (1.16) holds, and there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^n(2)$, $\ell \in \mathbb{N}$ *and a positive number* $c_0 > 0$ *such that* $|\alpha^\ell| \to \infty$ $(\ell \to \infty)$ *and* $0 < \omega(\alpha^\ell) \leqslant c_0$, $\ell \in \mathbb{N}$.
(ii) There exists an $f := {}^t(f^1, f^2, \ldots, f^d) \in (\mathbb{C}_2^\sigma\{x\})^d$ *satisfying (3.2) such that* $v := L_A^{-1} f$ *satisfies* $v \notin \bigcup_{1 \leqslant \sigma < 2} G_2^\sigma(\mathbb{C}^n)$.

4. Sternberg's theorem for commuting vector fields

The results in section 2 imply that the simultaneous linearization of a Poincaré morphism with a Jordan block is reduced essentially to the Poincaré–Dulac theorem for a single vector field in an analytic category. On the other hand, in view of the results in section 3, the reduction seems impossible if the action is not a Poincaré morphism.

In this section we shall illustrate that the situtation is completely different in a smooth category. We consider two commuting vector field in $\mathbb{R}^4$ which are in a Siegel domain and only one of the two has a linear part with nontrivial Jordan block. Obviously, the action is not a Poincaré morphism. We will show that they are simultaneously linearizable in C^k for every $k \geqslant 1$.

Let $X(y)$ and $Y(y)$ be commuting C^∞ vector fields with the common singular point at the origin $0 \in R^4$. Suppose that $\nabla X(0) = A$, $\nabla Y(0) = B$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\nu & 0 \\ 0 & 0 & 0 & -\nu \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mu & \varepsilon \\ 0 & 0 & 0 & -\mu \end{pmatrix}, \tag{4.1}$$

where $\varepsilon \neq 0$. We assume that the action is not a Poincaré morphism, namely, (cf. Example 1.6)

$$\nu > \mu > 0,\ \nu \in \mathbb{R} \setminus \mathbb{Q}. \tag{4.2}$$

We also note that the irrationality of ν implies that the pair (X, Y) is nonresonant. Then we have

THEOREM 4.1. — *Suppose that (4.1) and (4.2) are verified. Let* $m \geqslant 1$ *be an integer. Then there exists a* C^m*– change of the variables* $y = u(x) = x + v(x)$, $v(0) = 0$, $\nabla v(0) = 0$ *in some neighborhood of the origin which transforms both* X *and* Y *to their linear parts.*

We need to prepare lemmas in order to prove our theorem. By Sternberg's theorem we may assume that X is linear, i.e. $Xv(y) = \nabla v(y)Ay$. Let $R(y) = {}^t(R_1(y), R_2(y), R_3(y), R_4(y))$ be the nonlinear part of Y

$$Yf(y) = \nabla f(y)(By + R(y)). \tag{4.3}$$

Suppose that the change of variables $y = u(x) = x + v(x)$, $v(0) = 0$, $\nabla v(0) = 0$ linearizes both X and Y. Then $v(x)$ satisfies the system of homology equations

$$\nabla v(x), Bx - Bv = R(x + v(x)), \tag{4.4}$$

and

$$\nabla v(x)Ax - Av = 0. \tag{4.5}$$

We write $x = (x_1, x_2, x'')$ and $z = (z_1, z')$. Let $c_1 > 0$ and $0 < c_2 \leqslant 1$ be constants. Then we define

$$\Omega = \{x' = (x_2, x_3, x_4) = (x_2, x'') \in \mathbb{R}^3; |x_2| < c_1,\ |x''| < c_2\}, \tag{4.6}$$

$$\Omega_1 = \{x_1 \in \mathbb{R}; |x_1| < 1\} \times \Omega. \tag{4.7}$$

Then we have

Lemma 4.2. — *Let $k = \infty$ or $k \geqslant 1$ be an integer. Let L be given by*

$$L = \sum_{j=1}^{2} x_j \partial_{x_j} - \nu \sum_{k=3}^{4} x_k \partial_{x_k}.$$

Then the C^k solution of

$$Lf(x) - f(x) \quad = \quad 0, \quad x = (x_1, x_2, x_3, x_4) \in \Omega_1, \tag{4.8}$$

(respectively,

$$Lw(x) + \nu w(x) \quad = \quad 0 \quad x = (x_1, x_2, x_3, x_4) \in \Omega_1) \tag{4.9}$$

is given by

$$f(x) \quad = \quad x_1 \varphi_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \textit{for } \pm x_1 > 0, \tag{4.10}$$

or

$$f(x) \quad = \quad x_2 \varphi_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \textit{for } \pm x_1 > 0, \tag{4.11}$$

(respectively, by

$$w(x) \quad = \quad |x_1|^{-\nu} \psi_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad \textit{for } \pm x_1 > 0\ ), \tag{4.12}$$

where $\varphi_\pm(z) \in C^k(\Omega)$ (respectively $\psi_\pm(z) \in C^k(\Omega)$.)

Proof. — Let L be the operator given in the lemma. We want to solve (4.8) and (4.9). First we solve (4.8) in the region $x_1 > 0$. If we set $f(x) = x_1\varphi(x)$ (resp. $f(x) = x_2\psi(x)$), then we have that

$$L\varphi(x) = 0, \qquad (\textit{resp. } L\psi(x) = 0). \tag{4.13}$$

By the theorem in page 61 of [2], the solutions of (4.13) are given by the first integral of the corresponding characteristic equation. For the sake of simplicity, we consider the equation $L\varphi(x) = 0$. The characteristic equation is given by

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = -\frac{dx_3}{\nu x_3} = -\frac{dx_4}{\nu x_4}. \tag{4.14}$$

If we integrate (4.14) by taking x_1 as an independent variable, then we obtain

$$x_2 = x_1 x_2^0,\ x_3 = x_1^{-\nu} x_3^0,\ x_4 = x_1^{-\nu} x_4^0, \tag{4.15}$$

where x_2^0, x_3^0, x_4^0 are certain constants. It follows that the first integral $\varphi_+(x)$ is given by

$$\varphi_+(x) \equiv \tilde{\varphi}_+\left(\frac{x_2}{x_1}, x_3x_1^\nu, x_4x_1^\nu\right) = \tilde{\varphi}_+(x_2^0, x_3^0, x_4^0), \tag{4.16}$$

for some differentiable function $\tilde{\varphi}_+$. Hence, the general solution of (4.8) in $x_1 > 0$ is given by $f(x) = x_1\varphi_+(x)$ (resp. $f(x) = x_2\varphi_+(x)$ for possibly different φ_+).

In case $x_1 < 0$ we make the same argument by replacing x_1 with $-x_1$. We see that there exists $\varphi_-(x)$ such that $f(x) = x_1\varphi_-(x)$ (resp. $f(x) = x_2\varphi_-(x)$ for possibly different φ_-.)

Next we consider the equation (4.9). We set $w(x) = |x_1|^{-\nu}\psi(x)$. For the sake of simplicity we consider the case $x_1 > 0$. The case $x_1 < 0$ can be treated similarly if we replace x_1 with $-x_1$. We can easily see that ψ satisfies $L\psi = 0$. Hence it follows from the above argument that

$$w(x) = x_1^{-\nu}\psi_+(x) = x_1^{-\nu}\tilde{\psi}_+\left(\frac{x_2}{x_1}, x_3x_1^\nu, x_4x_1^\nu\right). \tag{4.17}$$

□

By the commutativity we see that every component of $v = R(x) = (R_1, \ldots, R_4)$ satisfies either (4.8) or (4.9). Hence, by Lemma 4.1 we have, for $\pm x_1 > 0$,

$$R_j(x) \quad = \quad x_j\Psi_\pm^j(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad j = 1, 2, \tag{4.18}$$

$$R_j(x) \quad = \quad |x_1|^{-\nu}\Psi_\pm^j(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu),\ j = 3, 4 \tag{4.19}$$

for some functions $\Psi^j_\pm$. In the following we will cut off $R_j(x)$ with a smooth function being identically equal to 1 in some neighborhood of the origin and with support contained in a small neighborhhood of the origin, which we give in the proof of Theorem 4.1. For the sake of simplicity, we denote the modified $R_j(x)$ with the same letter. We set

$$z_1 = x_2/x_1,\ z_2 = x_3|x_1|^\nu,\ z_3 = x_4|x_1|^\nu. \tag{4.20}$$

For every $x_1 \neq 0$, we define $\Psi^j_\pm(z)$ by (4.18) and (4.19), namely, for $\pm x_1 > 0$,

$$\Psi^j_\pm(z) = x_j^{-1} R_j(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3), \quad j = 1, 2, \tag{4.21}$$

$$\Psi^j_\pm(z) = |x_1|^\nu R_j(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3), \quad j = 3, 4. \tag{4.22}$$

We can easily see that $\Psi^j_\pm \in C^\infty(\mathbb{R}^3_z)$ $(j = 1, 2, 3, 4)$.

By (4.5) and simple computations we see that every component of $v(x) = (v_1(x), \dots, v_4(x))$ satisfies either (4.8) or (4.9). It follows from Lemma 4.1 that every component of v has an expression

$$v_j(x) = x_j \varphi^j_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad j = 1, 2, \tag{4.23}$$

and

$$v_j(x) = |x_1|^{-\nu} \varphi^j_\pm(\frac{x_2}{x_1}, x_3|x_1|^\nu, x_4|x_1|^\nu), \quad j = 3, 4, \tag{4.24}$$

for some $\varphi^j_\pm$ with $\pm x_1 > 0$.

We substitute the transformation (4.20) and (4.23), (4.24) into (4.4), and we rewrite (4.4) as an equation of z for the unknown functions $\varphi^j_\pm(z)$ with a parameter x_1. Recalling that $v_j = x_j \varphi^j_\pm$ and $v_j = |x_1|^{-\nu} \varphi^j_\pm$ we obtain

$$x_2 \partial_{x_2} v_1 = x_1 z_1 \partial_{z_1} \varphi^1_\pm(z), \quad x_3 \partial_{x_3} v_1 = x_1 z_2 \partial_{z_2} \varphi^1_\pm(z), \tag{4.25}$$

$$x_4 \partial_{x_4} v_1 = x_1 z_3 \partial_{z_3} \varphi^1_\pm(z), \quad x_4 \partial_{x_3} v_1 = x_1 z_3 \partial_{z_2} \varphi^1_\pm(z), \tag{4.26}$$

and we have similar relations for $v_2 = x_2 \varphi^2_\pm(x)$ and $v_j = |x_1|^{-\nu} \varphi^j_\pm(x)$. In fact we have

$$\langle \nabla v_1(x), Bx \rangle = x_1 \mathcal{L} \varphi^1_\pm(z), \quad \text{for } \pm x_1 > 0, \tag{4.27}$$

$$\langle \nabla v_2(x), Bx \rangle - v_2(x) = x_2 \mathcal{L} \varphi^2_\pm(z), \quad \text{for } \pm x_1 > 0, \tag{4.28}$$

$$\langle \nabla v_j(x), Bx \rangle = |x_1|^{-\nu} \mathcal{L} \varphi^j_\pm(z), \quad \text{for } \pm x_1 > 0,\ j = 3, 4, \tag{4.29}$$

where

$$\mathcal{L} f(z) = z_1 \partial_{z_1} f(z) - (\mu z_2 - \varepsilon z_3) \partial_{z_2} f(z) - \mu z_3 \partial_{z_3} f(z). \tag{4.30}$$

We define $\varphi_\pm(z) = {}^t(\varphi^1_\pm(z), \varphi^2_\pm(z), \varphi^3_\pm(z), \varphi^4_\pm(z))$.

LEMMA 4.3. — *We have the expression*

$$R_j(x+v(x)) = x_j E^j_\pm(z,\varphi_\pm(z)), \quad \textit{for } \pm x_1 > 0,\ j=1,2, \tag{4.31}$$

where $E^j_\pm(z,w)$ *is given by*

$$\begin{aligned} & E^j_\pm(z,w) \\ = \quad & (1+w_j)\Psi^j_\pm\left(z_1\frac{1+w_2}{1+w_1},(z_2+w_3)|1+w_1|^\nu,(z_3+w_4)|1+w_1|^\nu\right) \end{aligned} \tag{4.32}$$

and

$$R_j(x+v(x)) = |x_1|^{-\nu}E^j_\pm(z,\varphi_\pm(z)) \quad \textit{for } \pm x_1>0,\ j=3,4, \tag{4.33}$$

with

$$\begin{aligned} & E^j_\pm(z,w) = |1+w_1|^{-\nu} \\ \times \quad & \Psi^j_\pm\left(z_1\frac{1+w_2}{1+w_1},(z_2+w_3)|1+w_1|^\nu,(z_3+w_4)|1+w_1|^\nu\right). \end{aligned} \tag{4.34}$$

Proof. — We have

$$\begin{aligned} \frac{x_2+v_2(x)}{x_1+v_1(x)} &= \frac{x_2(1+\varphi^2_\pm(z))}{x_1(1+\varphi^1_\pm(z))} \\ &= \frac{x_2}{x_1}\frac{1+\varphi^2_\pm(z)}{1+\varphi^1_\pm(z)} = z_1\frac{1+\varphi^2_\pm(z)}{1+\varphi^1_\pm(z))}. \end{aligned} \tag{4.35}$$

$$\begin{aligned} & (x_3+v_3(x))|x_1+v_1|^\nu \\ =\ & (x_3+|x_1|^{-\nu}\varphi^3_\pm(z))|x_1|^\nu|1+\varphi^1_\pm(z)|^\nu \\ =\ & (x_3|x_1|^\nu+\varphi^3_\pm(z))|1+\varphi^1_\pm(z)|^\nu \\ =\ & (z_2+\varphi^3_\pm(z))|1+\varphi^1_\pm(z)|^\nu. \end{aligned} \tag{4.36}$$

$$\begin{aligned} & (x_4+v_4(x))|x_1+v_1|^\nu \\ =\ & (x_4+|x_1|^{-\nu}\varphi^4_\pm(z))|x_1|^\nu|1+\varphi^1_\pm(z)|^\nu \\ =\ & (x_4|x_1|^\nu+\varphi^4_\pm(z))|1+\varphi^1_\pm(z)|^\nu \\ =\ & (z_3+\varphi^4_\pm(z))|1+\varphi^1_\pm(z)|^\nu. \end{aligned} \tag{4.37}$$

Hence, if $j=1,2$, we get

$$\begin{aligned} & R_j(x+v(x)) = (x_j+v_j(x)) \\ \times\quad & \Psi^j_\pm(\frac{x_2+v_2(x)}{x_1+v_1(x)},(x_3+v_3(x))|x_1+v_1|^\nu,(x_4+v_4)|x_1+v_1(x)|^\nu) \\ =\quad & x_j(1+\varphi^j_\pm) \\ \times\quad & \Psi^j_\pm\left(z_1\frac{1+\varphi^2_\pm}{1+\varphi^1_\pm},(z_2+\varphi^3_\pm)|1+\varphi^1_\pm|^\nu,(z_3+\varphi^4_\pm)|1+\varphi^1_\pm|^\nu\right), \end{aligned} \tag{4.38}$$

which yields (4.31). Similarly, we can readily prove (4.33). □

Now we are ready to write explicitly the reduction of the overdetermined system for v: $(X_A - A)v = 0$, $(X_b - B)v = R(x + v(x))$ into a 4×4 system of equations for $\varphi_\pm(z)$ in $z \in \Omega$ with a parameter x_1. Then the new system of semilinear homological equations for $\varphi_\pm$ is written as follows

$$(4.39) \qquad \begin{aligned} (\mathcal{L} - \tilde{B})(\varphi_\pm) &= E_\pm(z, \varphi_\pm(z)), \\ E_\pm(z, w) &= (E_\pm^1(z, w), \ldots, E_\pm^4(z, w)), \end{aligned}$$

where $E_\pm^j(z, w)$ are given by (4.32) and (4.34) and

$$(4.40) \qquad \tilde{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \varepsilon \\ 0 & 0 & 0 & -\mu \end{pmatrix}.$$

We prepare a lemma.

Lemma 4.4. — *Let $\nu > 0$ be an irrational number. Let $f(x)$ and $w(x)$ be smooth solutions of (4.8) and (4.9) in Ω_1, respectively satisfying that*

$$(4.41) \qquad f(0) = w(0) = 0,$$

$$(4.42) \qquad \nabla f(0) = \nabla w(0) = 0.$$

We cut off $f(x)$ and $w(x)$ with a smooth function being identically equal to 1 in some neighborhood of the origin and with support contained in a small neighborhhood of the origin. For the sake of simplicity we denote the modified functions with the same letter. Let $\varphi_\pm(z)$ and $\psi_\pm(z)$ be defined by (4.10), (4.11) and (4.12), respectively by the same way as (4.21) and (4.22). Then, for every $\alpha \in \mathbb{Z}_+^3$, we have

$$(4.43) \qquad \partial_z^\alpha \Theta(z_1, 0) = 0, \qquad \forall z = (z_1, 0) \in \Omega,$$

with $\Theta = \varphi_\pm$ and $\Theta = \psi_\pm$.

Proof. — Because ν is an irrational number we can easily see, from (4.8) and (4.9) that every $f(x)$ and $w(x)$ satisfying (4.41) and (4.42) are flat at the origin, namely all derivatives $\partial_x^\alpha f(x)$, $\partial_x^\alpha w(x)$ ($\alpha \in \mathbb{N}^4$) vanish at the origin $x = 0$. Let $\Theta(z) = \varphi_\pm(z)$, and set $f(x) = x_1 \varphi_\pm(x_2/x_1, x_3|x_1|^\nu, x_4|x_1|^\nu)$, $x_1 \neq 0$. Then we have

$$(4.44) \qquad \begin{aligned} \partial_x^{\alpha'}\left(x_1^{-1} f(x)\right) &= \partial_x^{\alpha'} \varphi_\pm(x_2/x_1, x_3|x_1|^\nu, x_4|x_1|^\nu) \\ &= x_1^{-\alpha_2} |x_1|^{\nu(\alpha_3+\alpha_4)} \partial_{z_1}^{\alpha_2} \partial_{z_2}^{\alpha_3} \partial_{z_3}^{\alpha_4} \varphi_\pm(z)\Big|_{z_1 = x_2/x_1, z_2 = x_3|x_1|^\nu, z_3 = x_4|x_1|^\nu}. \end{aligned}$$

We let x tend to zero so as to satisfy $x_2/x_1 = z_1$, $z_2 = x_3|x_1|^\nu = 0$ and $z_3 = x_4|x_1|^\nu = 0$. Then we have

$$\begin{aligned} (4.45) \qquad & \partial_{z_1}^{\alpha_2}\partial_{z_2}^{\alpha_3}\partial_{z_3}^{\alpha_4}\varphi_\pm(z_1,0,0) \\ = \; & \lim_{x\to 0} x_1^{\alpha_2}|x_1|^{-\nu(\alpha_3+\alpha_4)}\partial_x^{\alpha'}\left(x_1^{-1}f(x_1,x_2,0,0)\right) = 0, \end{aligned}$$

because $f(x)$ is flat at the origin. The other cases will be proved similarly. □

Remark. — Let $\varphi_\pm(z) \in C^k(\Omega)$ be given. Assume that (4.43) is satisfied for $\Theta = \varphi_\pm$ up to some finite $|\alpha|$. Then the function $f(x)$ defined by (4.10) gives a finitely smooth solution of (4.8) if ν is an irrational number. Indeed, the finite smoothness at $x_1 = 0$ follows from the argument of Lemma 4.4.

In order to solve (4.39) we introduce a function space. Let $N \geqslant 1$ and $k \leqslant N$ be integers. Let $0 < c_2' < c_2 \leqslant 1$ be a constant. Then we define

$$(4.46) \qquad \|V\|_{k;N} = \sup_{z\in\mathbb{R}^3, 0<|z'|\leqslant c_2'} \sum_{|\alpha|\leqslant k} |z'|^{|\alpha|}\left|\partial_z^\alpha\left(|z'|^{-N}V(z)\right)\right|,$$

$$|V(z)| \;=\; \Big(\sum_{j=1}^4 |V_j(z)|^2\Big)^{1/2}, \quad V(z) = (V_1(z), V_2(z), V_3(z), V_4(z)).$$

The set of all C^k functions $V(z)$ such that $\|V\|_{k,N} < \infty$ is a Banach space $B_{k;N}$ with the norm $\|\cdot\|_{k;N}$. Then we have

Lemma 4.5. —

i) *For any integers $k \geqslant 0$ and $0 \leqslant \ell \leqslant N$, there exists a constant $C_{k,N} > 0$ such that*

$$(4.47) \qquad \|u\|_{k;\ell} \leqslant C_{k,N}\|u\|_{k;N}, \quad \forall u \in B_{k;N}.$$

ii) *For every $f, g \in B_{k;N}$ we have $fg \in B_{k;N}$ and there exists a constant $C_{k,N} > 0$ such that*

$$(4.48) \qquad \|fg\|_{k;N} \leqslant C_{k,N}\|f\|_{k;N}\|g\|_{k;N}, \quad \forall f, g \in B_{k;N}.$$

Proof. — Because $|z'| \leqslant 1$, we have, for $|\alpha| \leqslant k$

$$\begin{aligned} & |z'|^{|\alpha|}\partial^\alpha(|z'|^{-\ell}u(z)) = |z'|^{|\alpha|}\partial^\alpha(|z'|^{N-\ell}|z'|^{-N}u(z)) \\ = \; & |z'|^{|\alpha|}\sum_{\beta+\gamma=\alpha}\binom{\alpha}{\beta}\partial^\beta|z'|^{N-\ell}\partial^\gamma(|z'|^{-N}u(z)) \\ \leqslant \; & C_1 \sup|z'|^{|\gamma|}|\partial^\gamma(|z'|^{-N}u(z))| \end{aligned}$$

for some $C_1 > 0$. This proves i).

In order to prove ii) we have, for $|\alpha| \leqslant k$

$$
\begin{aligned}
(4.49) \qquad & |z'|^{|\alpha|}|\partial^\alpha(|z'|^{-N}fg)| \\
\leqslant & \sum_{\beta+\gamma=\alpha}\binom{\alpha}{\beta}|z'|^{|\beta|}|\partial^\beta(|z'|^{-N}f)||z'|^{|\gamma|}|\partial^\gamma g| \\
\leqslant & C_2\|f\|_{k;N}\|g\|_{k,0} \leqslant C_3\|f\|_{k;N}\|g\|_{k,N}.
\end{aligned}
$$

Here $C_2 > 0$ and $C_3 > 0$ are constants. This proves ii) . □

We define the operator Q by

$$
\begin{aligned}
(4.50) \qquad QV &= -\int_0^\infty e^{-t\tilde{B}}V(e^{tC}z)dt, \\
V &= (V_1, \dots, V_4) = (\varphi_\pm^1, \varphi_\pm^2, \varphi_\pm^3, \varphi_\pm^4),
\end{aligned}
$$

where

$$
(4.51) \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mu & \varepsilon \\ 0 & 0 & -\mu \end{pmatrix}.
$$

We can easily see that $U = QV$ gives the solution of $(\mathcal{L} - \tilde{B})U = V$. Then we have

LEMMA 4.6. — *Let the integers k and N satisfy that $0 \leqslant 2k < N - \mu$ and $\mu(k + 1 - N) + k < 0$. Then there exists $C_{k,N}(\Omega) > 0$ such that*

$$
(4.52) \qquad \|QV\|_{k;N} \leqslant C_{k,N}(\Omega)\,\|V\|_{k;N}\,, \quad \forall V \in B_{k;N}.
$$

Proof. — First we note that

$$
\begin{aligned}
(4.53) \qquad e^{tC}z &= (e^t z_1, e^{-\mu t}z_2 + e^{-\mu t}\varepsilon t z_3, e^{-\mu t}z_3), \\
(4.54) \qquad e^{-t\tilde{B}}V &= (V_1, V_2, e^{\mu t}(V_3 - \varepsilon t V_4), e^{\mu t}V_4).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(4.55) \qquad V(e^{tC}z) &= V(e^t z_1, e^{-\mu t}(z_2 + \varepsilon t z_3), e^{-\mu t}z_3) \\
&= e^{-\mu N t}((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2}\tilde{V}(e^{tC}z),
\end{aligned}
$$

where $\tilde{V}(\zeta) = V(\zeta)/|\zeta'|^N$. It follows that the right-hand side integral of (4.50) converges, because the growing term $e^{\mu t}$ in e^{-tB} can be absorbed by

$e^{-\mu Nt}$, $(\mu > 0)$. First we estimate $\|QV\|_{0;N}$. By (4.54) and (4.55) we have

$$
\begin{aligned}
(4.56)\qquad & \|QV\|_{0;N} \\
= & \sup_{z\in\mathbb{R}^3, 0<|z'|\leqslant c_2'} \left(\frac{1}{|z'|^N}\int_0^\infty \left|e^{-t\tilde{B}}V(e^{tC}z)\right| dt\right) \\
\leqslant & \sup_{z\in\mathbb{R}^3, 0<|z'|\leqslant c_2'} \left(\frac{1}{|z'|^N}\int_0^\infty (1+|\varepsilon|t)e^{\mu t}\left|V(e^{tC}z)\right| dt\right) \\
\leqslant & \sup\left(\frac{1}{|z'|^N}\int_0^\infty (1+|\varepsilon|t)e^{\mu(1-N)t}\right. \\
& \left.\times ((z_2+|\varepsilon|tz_3)^2+z_3^2)^{N/2}\left|\tilde{V}(e^{tC}z)\right| dt\right).
\end{aligned}
$$

On the other hand we note that

$$
\begin{aligned}
(4.57)\quad |z'|^{-N}((z_2+|\varepsilon|tz_3)^2+z_3^2)^{N/2} & \leqslant |z'|^{-N}(|z'|+|\varepsilon|t|z_3|)^N \\
& \leqslant (1+|\varepsilon|t)^N.
\end{aligned}
$$

In order to estimate $\tilde{V}(e^{tC}z)$ we note the following inequality

$$
(4.58)\qquad e^{-\mu t}(|z_2+\varepsilon tz_3|^2+z_3^2)^{1/2} \leqslant |z'|(1+|\varepsilon|t)e^{-\mu t} \leqslant |z'| \leqslant c_2',
$$

because we have $|\varepsilon| < \mu$. It follows that

$$
(4.59)\qquad |\tilde{V}(e^{tC}z)| \leqslant \sup_{z\in\mathbb{R}^3, 0<|z'|\leqslant c_2'} |\tilde{V}(z)|.
$$

Hence the right-hand side of (4.56) is estimated in the following way

$$
(4.60)\quad \leqslant \sup_{z\in\mathbb{R}^3, 0<|z'|\leqslant c_2'} |\tilde{V}(z)| \int_0^\infty (1+|\varepsilon|t)^{N+1}e^{\mu(1-N)t}dt \leqslant C\|V\|_{0;N}
$$

for some $C > 0$ independent of V. It follows that $\|QV\|_{0;N} \leqslant C\|V\|_{0;N}$ for some $C > 0$.

Next we will estimate the derivative $|z'|^{|\alpha|}\partial_z^\alpha(|z'|^{-N}QV)$. By Leibnitz rule it is sufficient to estimate the term $|z'|^{|\alpha|}\partial^\gamma|z'|^{-N}\partial^{\alpha-\gamma}(QV)$, where $\alpha \geqslant \gamma$. By simple computations, we have $|z'|^{|\alpha|}\partial^\gamma|z'|^{-N} \leqslant C_1|z'|^{-N+|\alpha|-|\gamma|}$ for some $C_1 > 0$ independent of z'. On the other hand, we have

$$
\begin{aligned}
\partial^{\alpha-\gamma}(QV) &= -\partial^{\alpha-\gamma}\int_0^\infty e^{-t\tilde{B}}((z_2+\varepsilon tz_3)^2+z_3^2)^{N/2}e^{-\mu Nt}\tilde{V}(e^{tC}z)dt \\
&= -\sum_{\beta\leqslant\alpha-\gamma}\binom{\alpha-\gamma}{\beta} \\
&\quad \times \int e^{-t\tilde{B}-\mu Nt}\partial_z^\beta((z_2+\varepsilon tz_3)^2+z_3^2)^{N/2}\partial^{\alpha-\gamma-\beta}\tilde{V}(e^{tC}z)dt.
\end{aligned}
$$

We can easily see

$$\left|\partial_z^{\beta}((z_2+\varepsilon t z_3)^2+z_3^2)^{N/2}\right| \leqslant C_2(1+|\varepsilon|t)^N|z'|^{N-|\beta|} \tag{4.61}$$

for some $C_2>0$. If we set $\alpha-\beta-\gamma=\delta$, $\delta=(\delta_1,\delta_2,\delta_3)$, then we have

$$\partial^{\alpha-\beta-\gamma}\tilde{V}(e^{tC}z)=e^{t\delta_1-\mu(\delta_2+\delta_3)t}(\partial_1^{\delta_1}\partial_2^{\delta_2}(\varepsilon t\partial_2+\partial_3)^{\delta_3}\tilde{V})(e^{tC}z). \tag{4.62}$$

It follows that

$$\begin{aligned}
&|z'|^{|\alpha|}\partial^{\gamma}|z'|^{-N}|\partial^{\alpha-\gamma}(QV)| \\
\leqslant\ & C_3|z'|^{-N+|\alpha|-|\gamma|}\sum_{\beta}\int_0^{\infty} e^{\mu t-\mu N t} \\
&\times|\partial_z^{\beta}((z_2+\varepsilon t z_3)^2+z_3^2)^{N/2}||\partial^{\alpha-\gamma-\beta}\tilde{V}(e^{tC}z)|dt \\
\leqslant\ & C_3|z'|^{-N+|\alpha|-|\gamma|}\int_0^{\infty} e^{\mu t-\mu N t}(1+|\varepsilon|t)^{N+1} \\
&\times\sum_{\beta}|z'|^{N-|\beta|}|\partial^{\alpha-\gamma-\beta}\tilde{V}(e^{tC}z)|dt \\
\leqslant\ & C_4\int_0^{\infty}\sum_{|\xi|=|\alpha-\beta-\gamma|\leqslant k}|z'|^{|\xi|}|(\partial_z^{\xi}\tilde{V})(e^{tC}z)| \\
&\times(1+|\varepsilon|t)^{N+1+|\xi|}e^{\mu t-\mu N t+|\alpha|t}dt.
\end{aligned} \tag{4.63}$$

In order to estimate $|z'|^{|\xi|}|(\partial_z^{\xi}\tilde{V})(e^{tC}z)|$, we set $\zeta=e^{tC}z$. Then we have

$$\begin{aligned}
&|z'|^{|\xi|}|(\partial_z^{\xi}\tilde{V})(e^{tC}z)|=|(e^{-tC}\zeta)'|^{|\xi|}|(\partial^{\xi}\tilde{V})(\zeta)| \\
\leqslant\ & e^{\mu|\xi|t}|(\partial^{\xi}\tilde{V})(\zeta)|((\zeta_2+\varepsilon t\zeta_3)^2+\zeta_3^2)^{|\xi|/2} \\
\leqslant\ & e^{\mu k t}(1+|\varepsilon|t)^k|\zeta'|^{|\xi|}|(\partial^{\xi}\tilde{V})(\zeta)| \\
\leqslant\ & \|V\|_{k;N}e^{\mu k t}(1+|\varepsilon|t)^k.
\end{aligned} \tag{4.64}$$

By assumption we have $(1+k-N)\mu+|\alpha|\leqslant(1+k-N)\mu+k<0$. Hence the right-hand side integral in (4.63) converges. Therefore we see that the right-hand side of (4.63) can be estimated by $C_5\|V\|_{k;N}$. □

Proof of Theorem 4.1. — Let the integers k and N satisfy that $0\leqslant 2k<N-\mu$ and $n\geqslant 2$, $\mu(k+1-N)+k<0$. By setting $\varphi_{\pm}=QV$, (4.39) is equivalent to

$$V \quad = \quad E_{\pm}(z,QV). \tag{4.65}$$

We define the sequence $V_{\pm}^j$ $(j=0,1,2,\ldots)$ by

$$V_{\pm}^0=E_{\pm}(z,0),\quad V_{\pm}^1=E_{\pm}(z,QV_{\pm}^0)-E_{\pm}(z,0), \tag{4.66}$$

and

$$V_\pm^{j+1} = E_\pm(z, Q(V_\pm^0 + \cdots + V_\pm^j)) - E_\pm(z, Q(V_\pm^0 + \cdots + V_\pm^{j-1})), \quad j = 1, 2, \ldots \tag{4.67}$$

We will show the convergence of $\sum_{j=0}^\infty V_\pm^j =: V_\pm$. By the definition and Lemma 4.3 we have $V_\pm^0 = E_\pm(z, 0) = \Psi_\pm(z)$. Next we have

$$\begin{aligned} V_\pm^1 &= E_\pm(z, QV_\pm^0) - E_\pm(z, 0) \\ &= QV_\pm^0 \int_0^1 \nabla_w E_\pm(z, \tau QV_\pm^0) d\tau. \end{aligned} \tag{4.68}$$

Let $\varepsilon' > 0$ be a small constant chosen later, and suppose that

$$\|\Psi_\pm\|_{k;N} < \varepsilon', \quad \|\nabla\Psi_\pm\|_{k;N} < \varepsilon'. \tag{4.69}$$

Then, by Lemma 4.6 and the definition of $V_\pm^0$ we have

$$\|\tau QV_\pm^0\|_{k;N} \leqslant c_1 \|V_\pm^0\|_{k;N} = c_1 \|\Psi_\pm\|_{k;N} < c_1 \varepsilon' \tag{4.70}$$

for some $c_1 > 0$ independent of $\Psi_\pm$. Here we recall from (4.66) that $V_\pm^0 = E_\pm(z, 0)$ and $E_\pm(z, 0) = \Psi_\pm(z)$ by (4.32) and (4.34).

In order to estimate $\|\nabla_w E_\pm(\cdot, \tau QV_\pm^0)\|_{k;N}$, we set $w = (w_1, \ldots, w_4) = \tau QV_\pm^0$ and

$$\zeta = (\zeta_1, \zeta') = \left(z_1 \frac{1 + w_2}{1 + w_1}, (z_2 + w_3)|1 + w_1|^\nu, (z_3 + w_4)|1 + w_1|^\nu \right).$$

The differentiation $\partial_z^\alpha(\nabla_w E_\pm(z, \tau QV_\pm^0))$ consists of terms which are product of $\partial^\beta \nabla\Psi_\pm(\zeta)$ $(\alpha \geqslant \beta)$ and the differentiations of w. First, the product of differentiations of w is bounded by a constant in view of (4.70). On the other hand, in order to estimate $|\partial_\zeta^\beta \nabla\Psi_\pm(\zeta)|$, we note

$$\begin{aligned} |\partial_\zeta^\beta \nabla\Psi_\pm(\zeta)| &= |\partial_\zeta^\beta \left(|\zeta'|^N |\zeta'|^{-N} \nabla\Psi_\pm(\zeta) \right)| \\ &\leqslant C_0 \sum_{\gamma \leqslant \beta} |\partial_\zeta^\gamma |\zeta'|^N| |\partial_\zeta^{\beta-\gamma}(|\zeta'|^{-N} \nabla\Psi_\pm(\zeta))| \end{aligned}$$

for some constant $C_0 > 0$. Because $N \geqslant 2k \geqslant 2|\beta| \geqslant 2|\gamma|$ and $|\zeta'| \leqslant 1$, we have $|\partial_\zeta^\gamma |\zeta'|^N| \leqslant C_1 |\zeta'|^{N-|\gamma|} \leqslant C_1 |\zeta'|^{|\beta|-|\gamma|}$ for some $C_1 > 0$. It follows from the definition of the norm that $|\partial_\zeta^\beta \nabla\Psi_\pm(\zeta)| \leqslant C_2 \|\nabla\Psi_\pm\|_{k;N}$ for some $C_2 > 0$. Hence, if $\varepsilon' > 0$ is sufficiently small, then we obtain, by the definition of $E_\pm(z, w)$ in (4.39), (4.32) and (4.34),

$$\|\nabla_w E_\pm(\cdot, \tau QV_\pm^0)\|_{k;N} \leqslant c_2 \|\nabla\Psi_\pm\|_{k;N} < c_2 \varepsilon', \tag{4.71}$$

for some $c_2 > 0$ independent of ε' and $\Psi_\pm$.

It follows from (4.68) that

$$\|V_\pm^1\|_{k;N} \leqslant \|QV_\pm^0\|_{k;N}\int_0^1 \|\nabla_w E_\pm(z,\tau QV_\pm^0)\|_{k;N}d\tau \leqslant c_1c_2\varepsilon'^2.$$

In order to show the general case, we assume that $\|V_\pm^j\|_{k;N} \leqslant c_1^jc_2^j\varepsilon'^{j+1}$ for $j=0,1,2,\dots,k$. Then we have

$$\|\sum_{j=0}^k V_\pm^j\|_{k;N} \leqslant \sum_{j=0}^k c_1^jc_2^j\varepsilon'^{j+1} \leqslant \frac{\varepsilon'}{1-c_1c_2\varepsilon'}. \tag{4.72}$$

By the definition we have

$$\begin{aligned} V_\pm^{k+1} &= E_\pm(z,Q(V_\pm^0+\cdots+V_\pm^k)) - E_\pm(z,Q(V_\pm^0+\cdots+V_\pm^{k-1})) \\ &= QV_\pm^k\int_0^1 \nabla_w E_\pm(z,Q(V_\pm^0+\cdots+V_\pm^{k-1})+\tau QV_\pm^k)d\tau. \end{aligned} \tag{4.73}$$

By the apriori estimate (4.72) and the boundedness of Q, the substitution in the right-hand side of (4.73) is well–defined. Moreover, by the same argument as in the proof of (4.71) we see that

$$\|\nabla_w E_\pm(z,Q(V_\pm^0+\cdots+V_\pm^{k-1})+\tau QV_\pm^k)\|_{k;N} \leqslant c_2\varepsilon'.$$

It follows from (4.73) that

$$\|V_\pm^{k+1}\|_{k;N} \leqslant \|QV_\pm^k\|_{k;N}c_2\varepsilon'\int_0^1 d\tau \leqslant c_1^{k+1}c_2^{k+1}\varepsilon'^{k+2}.$$

Hence we have the estimate of $V_\pm^j$ for $j=k+1$. It follows that the series $V_\pm := \sum_{j=0}^\infty V_\pm^j$ converges in $B_{k;N}$ and $V_\pm$ is a solution of (4.65). We note that, by (4.72) $V_\pm$ satisfies the estimate $\|V_\pm\|_{k;N} \leqslant \varepsilon'(1-c_1c_2\varepsilon')^{-1}$, and $V_\pm$ is divisable by $|z'|^2$.

Next we verify the smallness assumption (4.69) uniformly with respect to $x_1 \neq 0$ in some neighborhood of $x_1=0$. Because the argument is similar we consider the condition $\|\Psi_\pm\|_{k;N} < \varepsilon'$. In view of the definition of $\Psi_\pm$ in (4.21) and (4.22), we shall estimate

$$x_j^{-1}R_j(x_1,x_1z_1,|x_1|^{-\nu}z_2,|x_1|^{-\nu}z_3), \qquad j=1,2$$

and

$$|x_1|^\nu R_j(x_1,x_1z_1,|x_1|^{-\nu}z_2,|x_1|^{-\nu}z_3), \qquad j=3,4$$

with $x_1\neq 0$ close to 0. Because the argument is similar, we consider the case $j=1$. We have, for $|\alpha|\leqslant k$

$$\begin{aligned} &|z'|^{|\alpha|}\left|\partial_z^\alpha(|z'|^{-N}\Psi_\pm^1(z))\right| \\ &= x_1^{-1}|z'|^{|\alpha|}\left|\partial_z^\alpha(|z'|^{-N}R_1(x_1,x_1z_1,|x_1|^{-\nu}z_2,|x_1|^{-\nu}z_3))\right|. \end{aligned} \tag{4.74}$$

By (4.43) we have that, for every positive integer p, the term

$$R_1(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3)|z'|^{-p}$$

is smooth at $z = 0$. Because

$$|z'|^p = (|x_1|^\nu |x_1|^{-\nu} |z'|)^p = (|x_1|^\nu |x''|)^p, \quad x'' = (x_3, x_4),$$

and $|x''|$ is bounded by the support condition of R_j, the negative power $|z'|^{-N}$ in the right-hand side of (4.74) is absorbed by $|z'|^p$ if p is sufficiently large. On the other hand, if the differentiation ∂_z^α is applied to $R_1(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3)$, then the negative power of $|x_1|$ appears. These terms are also uniformly bounded when $x_1 \to 0$, because there appears positive power of $|x_1|$ from $|z'|^p$. Because all derivatives of $R(x)$ at the origin vanish, we see that the right-hand side of (4.74) can be made arbtrarily small if we cut off $R_(x)$ in a sufficiently small neighborhood of the origin. This proves that we have (4.69).

We set $\varphi_\pm = QV_\pm \in B_{k;N}$, and $\varphi_\pm(z) = (\varphi_\pm^1(z), \varphi_\pm^2(z), \varphi_\pm^3(z), \varphi_\pm^4(z))$. The function $\varphi_\pm$ is a solution of (4.39). Then we define $v^j(x)$ $(j = 1, 2, 3, 4)$ by (4.23) and (4.24). For a given integer m, we can easily see that $v^j(x)$ is a C^m function if we take k and N in $B_{k;N}$ sufficiently large. If we rewrite (4.39) with the variable x, then we see that v is a solution of (4.4), where the nonlinear part R is modified by a cutoff function. In order to show that v is a solution of the original (4.4) we will show the apriori estimate of v. Indeed, if $|x+v| < \varepsilon''$ for sufficiently small ε'', then v is a solution of (4.4). By Lemma 4.6 and the uniform estimate of $V_\pm$ in x_1 we know that $\varphi_\pm^1(z)$ is uniformly bounded in z and x_1. It follows that $v_1(x) = x_1 \varphi_\pm^1$ is arbitrarily small if x_1 is sufficiently small. Similarly we can show that $v_2(x) = x_2 \varphi^2$ is small by the estimate of $V_\pm$. On the other hand, we have $x_3 + v_3(x) = x_3 + |x_1|^{-\nu} \varphi_\pm^3(z)$. Because $\varphi_\pm^3$ is divisable by $|z'|^2$ and $|z'| = |x_1|^\nu |x''|$, by Lemma 4.4 we see that $|x_3 + v_3(x)| < \varepsilon''$ uniformly in x_1. Similarly we can show the same estimate for $x_4 + v_4$. Therefore we see that v is a solution of (4.4). □

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BIBLIOGRAPHY

[1] M. Abate, "Diagonalization of nondiagonalizable discrete holomorphic dynamical systems", *Amer. J. Math.* **122** (2000), p. 757-781.

[2] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, New York, Heidelberg, Berlin ed., Springer, 1983.

[3] A. D. Bruno, "The analytic form of differential equations", *Tr. Mosk. Mat. O-va* (1971), no. 25, p. 119-262, and **26**, 199–239 (1972) (in Russian); see also Trans. Mosc. Math. Soc. **25**, 131-288 (1971) and **26**, 199-239 (1972).

[4] A. D. Bruno & S. Walcher, "Symmetries and convergence of normalizing transformations", *J. Math. Anal. Appl.* **183** (1994), p. 571-576.

[5] T. Carletti, "Exponentially long time stability for non-linearizable analytic germs of $(\mathbb{C}^n, 0)$", *Ann Inst. Fourier (Grenoble)* **54** (2004), no. 4, p. 989-1004.

[6] T. Carletti & S. Marmi, "Linearization of analytic and non-analytic germs of diffeomorphisms of $(\mathbb{C}, 0)$", *Bull. Soc. Math. France* **128** (2000), p. 69-85.

[7] K. T. Chen, "Diffeomorphisms: $C^\infty-$realizations of formal properties", *Amer. J. Math.* **87** (1965), p. 140-157.

[8] G. Cicogna & G. Gaeta, *Symmetry and perturbation theory in nonlinear dynamics*, Berlin ed., New Series m: Monographs, vol. 57, Springer–Verlag, 1999.

[9] G. Cicogna & S. Walcher, "Convergence of normal form transformations: the role of symmetries. Symmetry and perturbation theory", *Acta Math. Appl.* **70** (2002), p. 95-111.

[10] R. De La Llave, *A tutorial on KAM theory.*, to appear in Proc. of 1999 Summer Research Institute on Smooth Ergodic Theory and Applications, Univ. of Washington, Seattle, July 26–August 13 1999.

[11] D. DeLatte & T. Gramchev, "Biholomorphic maps with linear parts having Jordan blocks: Linearization and resonance type phenomena", *Math. Physics Electronic Journal* **8** (2002), no. paper n. 2, p. 1-27.

[12] D. Dickinson, T. Gramchev & M. Yoshino, "Perturbations of vector fields on tori: resonant normal forms and Diophantine phenomena", *Proc. Edinb. Math. Soc. (2)* **45** (2002), no. 3, p. 731-159.

[13] F. Dumortier & R. Roussarie, "Smooth linearization of germs of R^2-actions and holomorphic vector fields", *Ann. Inst. Fourier (Grenoble)* **30** (1980), no. 1, p. 31-64.

[14] F. R. Gantmacher, *The theory of matrices*, vol. 1-2, Chelsea Publishing Co., New York, 1959.

[15] T. Gramchev, "On the linearization of holomorphic vector fields in the Siegel Domain with linear parts having nontrivial Jordan blocks", *World Scientific* (2003), p. 106-115, S. Abenda, G. Gaeta and S. Walcher eds, Symmetry and perturbation theory, Cala Gonone, 16–22 May 2002.

[16] M. Herman, "Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations", *Publ. Math. I.H.É.S.* **49** (1979), p. 5-233.

[17] A. Katok & S. Katok, "Higher cohomology for Abelian groups of toral automorphisms", *Ergodic Theory Dyn. Syst.* **15** (1995), no. 3, p. 569-592.

[18] P. R. Marco, "Non linearizable holomorphic dynamics having an uncountable number of symmetries", *Inv. Math.* **119** (1995), p. 67-127.

[19] ———, "Total convergence or small divergence in small divisors", *Commun. Math. Phys.* **223** (2001), no. 3, p. 451-464.

[20] J. Moser, "On commuting circle mappings and simultaneous Diophantine approximations", *Mathematische Zeitschrift* **205** (1990), p. 105-121.

[21] R. Rousssarie, *Modèles locaux de champs et de formes*, vol. 30, Astérisque, 1975.

[22] W. Schmidt, *Modèles locaux de champs et de formes*, berlin, heidelberg, new york ed., Lect. Notes in Mathematics, vol. 785, Springer Verlag, 1980.

[23] S. Sternberg, "The structure of local homeomorphisms II, III", *Amer. J. Math.* (1958), p. 623-632, **80**, 623-632 and **81**, 578–604.

[24] L. Stolovitch, "Singular complete integrability", *Publ. Math. I.H.E.S.* **91** (2000), p. 134-210.

[25] ———, "Normalisation holomorphe d'algèbres de type Cartan de champs de vecteurs holomorphes singuliers", *Ann. Math.* **161** (2005), p. 589-612.

[26] S. Walcher, "On convergent normal form transformations in presence of symmetries", *J. Math. Anal. Appl.* **244** (2000), p. 17-26.

[27] J.-C. Yoccoz, "A remark on Siegel's theorem for nondiagonalizable linear part", *Astérisque* **231** (1995), p. 3-88, manuscript, 1978, see also Théorème de Siegel, nombres de Bruno et polynômes quadratiques.

[28] M. Yoshino, "Simultaneous normal forms of commuting maps and vector fields", *World Scientific, Singapore* (1999), p. 287-294, A. Degasperis, G. Gaeta eds., Symmetry and perturbation theory SPT 98, Rome 16–22 December 1998.

[29] N. T. Zung, "Convergence versus integrability in Poincaré-Dulac normal form.", *Math. Res. Lett.* **9** (2002), no. 2-3, p. 217-228.

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