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LI COEFFICIENTS FOR AUTOMORPHIC L -FUNCTIONS

by Jeffrey C. LAGARIAS

ABSTRACT. — Xian-Jin Li gave a criterion for the Riemann hypothesis in terms of the positivity of a set of coefficients λ_n ($n = 1, 2, \dots$). We define similar coefficients $\lambda_n(\pi)$ associated to principal automorphic L -functions $L(s, \pi)$ over $GL(N)$. We relate these coefficients to values of Weil's quadratic functional associated to the representation π on a suitable set of test functions. The positivity of the real parts of these coefficients is a necessary and sufficient condition for the Riemann hypothesis for $L(s, \pi)$. Assuming the Riemann hypothesis for $L(s, \pi)$, we show that $\lambda_n(\pi) = \frac{N}{2}n \log n + C_1(\pi)n + O(\sqrt{n} \log n)$, where $C_1(\pi)$ is a real-valued constant. We construct an entire function $F_\pi(z)$ of exponential type that interpolates the generalized Li coefficients at integer values. Assuming the Riemann hypothesis for $L(s, \pi)$, this function on the real axis has a Fourier transform that is a tempered distribution whose support is a countable set in $[-\pi, \pi]$ having 0 as its only limit point.

RÉSUMÉ. — Xian-Jin Li a montré que l'hypothèse de Riemann est équivalente à la positivité d'une certaine suite de réels λ_n ($n = 1, 2, \dots$). De manière similaire, on associe à une fonction automorphe principale $L(s, \pi)$ sur $GL(N)$ une suite de réels $\lambda_n(\pi)$. On établit une relation entre ces coefficients et les valeurs prises par la fonctionnelle quadratique de Weil associée à la représentation π , sur un espace de fonctions tests convenablement choisi. La positivité de la partie réelle de ces coefficients est équivalente à la conjecture de Riemann pour $L(s, \pi)$. En supposant que l'hypothèse de Riemann est satisfaite pour $L(s, \pi)$, on montre que : $\lambda_n(\pi) = \frac{N}{2}(n \log n) + C_1(\pi)n + O(\sqrt{n} \log n)$, où $C_1(\pi)$ est une constante réelle. On construit une fonction entière $F_\pi(z)$, de type exponentielle, qui interpole ces coefficients de Li généralisés en les valeurs entières de la variable. En supposant que l'hypothèse de Riemann est satisfaite pour $L(s, \pi)$, la restriction de cette fonction à l'axe réel admet une transformé de Fourier qui est une distribution tempérée, dont le support est un sous-senséme dénombrable de $[-\pi, \pi]$, ayant le point 0 comme unique point d'accumulation.

Keywords: Automorphic L -function, zeta function.

Math. classification: 11M26, 11M36, 11S40.

1. Introduction

In 1997 Xian-Jin Li [35] derived a necessary and sufficient condition for the Riemann hypothesis in terms of the positivity of the set of coefficients

$$(1.1) \quad \lambda_n := \sum_{\rho}' \left[1 - \left(1 - \frac{1}{\rho} \right)^n \right],$$

in which the sum runs over the nontrivial zeros of the Riemann zeta function, counted with multiplicity, and $'$ indicates that the (conditionally convergent) sum is to be interpreted as $\lim_{T \rightarrow \infty} \sum_{\{\rho: |\rho| \leq T\}}$; we term this $*$ -convergence. The expression (1.1) $*$ -converges for positive and negative integer n , and so defines these coefficients for all integers. One has $\lambda_0 = 0$ and

$$(1.2) \quad \lambda_{-n} = \lambda_n, \text{ for all } n \geq 1.$$

Li's original definition

$$(1.3) \quad \tilde{\lambda}_n := \frac{1}{(n-1)!} \frac{d^n}{ds^n} [s^{n-1} \log \xi(s)] \Big|_{s=1}, \quad n \geq 1,$$

functorially corresponds to $\tilde{\lambda}_n = \lambda_{-n}$ for $n > 0$, and the identity $\tilde{\lambda}_n = \lambda_n$ then holds using (1.2). Li's paper gave more generally a criterion for the Riemann hypothesis to hold for the (completed) Dedekind zeta functions of any algebraic number field K .

These coefficients are expressible in terms of power-series coefficients of functions constructed from the Riemann ξ -function:

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

The coefficients for $n \geq 1$ occur in

$$(1.4) \quad \frac{d}{dz} \log \xi\left(\frac{-z}{1-z}\right) = \frac{-1}{(1-z)^2} \frac{\xi'}{\xi}\left(\frac{-z}{1-z}\right) = \sum_{n=0}^{\infty} \lambda_{n+1} z^n,$$

and those for $n \leq -1$ occur in an analogous formula

$$(1.5) \quad \frac{d}{dz} \log \xi\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2} \frac{\xi'}{\xi}\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} \lambda_{-n-1} z^n.$$

Here $\xi'(\cdot)$ denotes the derivative with respect to s . Our definition (1.1) of the Li coefficients corresponds to an expansion around the point $s = 0$, visible in (1.4), rather than around $s = 1$ as in (1.3) and (1.5).

Closely related coefficients appear in earlier work of J. B. Keiper [34, Sect. 4] in 1992, in an investigation of methods to compute Stieltjes constants to high precision; we describe Keiper's computations further below. Keiper's

coefficients equal $\frac{1}{n}\lambda_{-n}$, in terms of (1.1). Keiper noted that the Riemann hypothesis implies the nonnegativity of the λ_{-n} . We adopt the term Li coefficients because Li’s work shows the nonnegativity of λ_n is equivalent to the Riemann hypothesis and because he generalized them to various other zeta functions.

In [4] E. Bombieri and the author made three observations about these coefficients. The first observation was that a Li criterion can be formulated for very general sets of complex numbers ρ , as follows. Consider any multiset Z of complex numbers ρ satisfying

$$(1.6) \quad \sum_{\rho \in Z} \frac{\operatorname{Re}(\rho)}{(1 + |\rho|)^2} < \infty.$$

If the multiset Z omits the value $\rho = 1$ then the sums

$$(1.7) \quad \operatorname{Re}(\lambda_n(Z)) := \sum_{\rho \in Z} \operatorname{Re} \left(1 - \left(1 - \frac{1}{\rho}\right)^n \right)$$

converge absolutely for all nonpositive integers $n \leq 0$. The positivity condition $\operatorname{Re}(\lambda_n(Z)) \geq 0$ for $n \leq 0$ then implies that all ρ lie in the half-plane $\operatorname{Re}(s) \leq \frac{1}{2}$. If the multiset Z omits the value $\rho = 0$, then the sum (1.7) then converges absolutely for $n \geq 0$, and the positivity condition $\operatorname{Re}(\lambda_n(Z)) \geq 0$ for $n \geq 0$ implies that all ρ lie in the half-plane $\operatorname{Re}(s) \geq \frac{1}{2}$. Combining these criteria, for multisets Z that omit both 0 and 1 the positivity condition $\operatorname{Re}(\lambda_n(Z)) \geq 0$ for all integers n implies that all $\operatorname{Re}(\rho) = \frac{1}{2}$. If the multiset Z is also invariant under the symmetry $\rho \mapsto 1 - \bar{\rho}$, so that $\operatorname{Re}(\lambda_n(Z)) = \operatorname{Re}(\lambda_{-n}(Z))$, it suffices to check this positivity condition $\operatorname{Re}(\lambda_n(Z)) \geq 0$ for $n > 0$ to conclude that all $\operatorname{Re}(\rho) = \frac{1}{2}$. Finally, if Z omits the values 0 and 1 and the sum $\sum_{\rho \in Z} \frac{1}{\rho}$ is $*$ -convergent, then the coefficients $\lambda_n(Z)$ are well-defined for all integers n by the following $*$ -convergent sum:

$$(1.8) \quad \lambda_n(Z) := \sum'_{\rho \in Z} \left(1 - \left(1 - \frac{1}{\rho}\right)^n \right).$$

The second observation in [4, Theorem 2] was that the the “explicit formula” of prime number theory may be used to obtain an arithmetic expression for Li’s coefficients λ_n in (1.1) having the form

$$\lambda_n = S_\infty(n) - S_f(n) + 1,$$

in which $S_\infty(n)$ and $S_f(n)$ correspond to the contributions of the archimedean place and the finite places, respectively, and the last term is a contribution from the pole at $s = 0$ of $\xi(s)/s(s - 1)$. The third observation was

that each positivity condition $\lambda_n \geq 0$ encodes “Weil positivity” of Weil’s quadratic functional for a particular test function $g_n(x)$.

In more recent work K. Maslanka [40] computed λ_n for $1 \leq n \leq 3300$ and empirically studied the growth behavior of the Li coefficients. He observed that in this range they exhibited a smoothly growing dominant asymptotic term with superposed small oscillations exhibiting some internal structure. The dominant term comes from the archimedean prime contribution $S_\infty(n)$ in the arithmetic formula above, and the small oscillations come from the term $S_f(n)$ represented the finite places, and $|S_f(n)| < 20$ over this range. Coffey [8], [9] studied the “arithmetic formula” for the Li coefficients, obtaining many results, including a lower bound for the archimedean place contribution [9, Theorem 2].

The object of this paper is to generalize the Li coefficients to automorphic L -functions, and to determine their asymptotic behavior as $n \rightarrow \infty$, with or without the assumption of the Riemann hypothesis. The automorphic L -functions we treat are principal L -functions over $GL(N)$ for \mathbb{Q} , as given in [10], [29], [31], and [42]. These are (completed) Langlands L -functions attached to irreducible cuspidal unitary automorphic representations appearing in the right action of $GL(n, \mathbb{A}_{\mathbb{Q}})$ on $L^2(GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}_{\mathbb{Q}}))$. The associated generalized Li coefficients will be denoted $\lambda_n(\pi)$.

In §2 we associate to each irreducible cuspidal unitary automorphic representation π an analogue $\xi(s, \pi)$ of the Riemann ξ -function. Our definition of *generalized Li coefficients* $\lambda_n(\pi)$ sets them equal to $\lambda_n(Z)$ for the multiset $Z = Z(\pi)$ of zeros of $\xi(s, \pi)$, and we show that these coefficients $\lambda_n(\pi)$ are well-defined as $*$ -convergent series. The multiset $Z(\pi)$ is invariant under the transformation $\rho \mapsto 1 - \bar{\rho}$, which implies the symmetry

$$(1.9) \quad \lambda_{-n}(\pi) = \overline{\lambda_n(\pi)}.$$

It therefore suffices to study $\lambda_n(\pi)$ for $n \geq 0$. Furthermore the zero sets of the L -functions of a representation π and its contragredient representation π^\vee are related by complex conjugation $Z(\pi^\vee) = \overline{Z(\pi)}$, which implies a second symmetry

$$(1.10) \quad \lambda_n(\pi^\vee) = \overline{\lambda_n(\pi)}.$$

The results of [4] apply to give a Riemann hypothesis criterion for $L(s, \pi)$ in the form: The Riemann hypothesis holds for $L(s, \pi)$ if and only if the real parts $\text{Re}(\lambda_n(\pi))$ are nonnegative for all $n \geq 0$.

In §3 we give a “Weil positivity” interpretation of the generalized Li coefficients. We express the Weil scalar product associated to the representation π for Li’s test functions in terms of $\lambda_n(\pi)$. Here the imaginary parts of the $\lambda_n(\pi)$ appear in the scalar products. The test functions are the same for all representations π .

In §4 we give an arithmetical interpretation of the coefficients in terms of the logarithmic derivative of $\xi(s, \pi)$ expanded about at the point $s = 1$. We express it as

$$(1.11) \quad \lambda_n(\pi) = S_\infty(n, \pi^\vee) - S_f(n, \pi^\vee) + \delta(\pi^\vee),$$

in which the two terms $S_\infty(n, \pi^\vee)$ and $S_f(n, \pi^\vee)$ reflect contributions coming from the Euler product factorization of $\xi(s, \pi^\vee)$ into archimedean places and finite places, respectively. The final term $\delta(\pi) = \delta(\pi^\vee) = 1$ for the trivial representation $\pi = \pi_{\text{triv}}$ over $GL(1)$ and $\delta(\pi) = 0$ otherwise. The contragredient representation appears in (1.11) via the functional equation relating $\xi(s, \pi)$ and $\xi(1 - s, \pi^\vee)$, because our definition (1.1) of the Li coefficients corresponds to values at $s = 0$.

In §5 we obtain an unconditional asymptotic formula for the archimedean contribution $S_\infty(n, \pi)$. This quantity is real-valued, and we show there is a real-valued constant $C_1(\pi)$ such that for all $n \geq 1$,

$$(1.12) \quad S_\infty(n, \pi) = \frac{N}{2} n \log n + C_1(\pi) n + O(1),$$

and the implied constant in the $O(1)$ term depends on π . Here

$$(1.13) \quad C_1(\pi) = \frac{N}{2} (\gamma - 1 - \log(2\pi)) + \frac{1}{2} \log Q(\pi),$$

in which γ is Euler’s constant and $Q(\pi)$ is the conductor of π . In particular $C_1(\pi)$ does not depend on the archimedean parameters $\{\kappa_j(\pi) : 1 \leq j \leq N\}$ of the representation π , and $C_1(\pi) = C_1(\pi^\vee)$.

In §6 we obtain an unconditional estimate for the finite place contribution $S_f(n, \pi)$ in terms of the zeros to a suitable height, by a contour integral estimate. Define the *incomplete Li coefficient* at height T by

$$(1.14) \quad \lambda_n(T, \pi) = \sum_{\substack{\rho \in Z(\pi) \\ |\text{Im}(\rho)| < T}} 1 - \left(1 - \frac{1}{\rho}\right)^n,$$

We show that

$$(1.15) \quad S_f(n, \pi) = \lambda_n(\sqrt{n}, \pi^\vee) + O(\sqrt{n} \log n),$$

where π^\vee is the contragredient representation, and the implied constant in the O -notation depends on π . If the Riemann hypothesis holds for $L(s, \pi)$,

then it holds for $L(s, \pi^\vee)$ by the functional equation, and we obtain

$$\lambda_n(\sqrt{n}, \pi^\vee) = O(\sqrt{n} \log n),$$

so that

$$(1.16) \quad S_f(n, \pi) = O(\sqrt{n} \log n).$$

Furthermore if the Riemann hypothesis holds up to height T , then a bound of shape (1.16) holds for all $n \leq T^2/2(\log T)^2$, with the implied O -constant depending on π .

These results are summarized in the following theorem.

THEOREM 1.1. — *Let π be an irreducible cuspidal unitary automorphic representation for $GL(N)$ over \mathbb{Q} . For $n \geq 1$ there holds*

$$(1.17) \quad \lambda_n(\pi) = \frac{N}{2} n \log n + C_1(\pi)n - \lambda_n(\sqrt{n}, \pi) + O(\sqrt{n} \log n),$$

in which $C_1(\pi)$ is real-valued and the implied constant in the O -notation depends on π . If the Riemann hypothesis holds for $L(s, \pi)$ then the incomplete Li coefficient $\lambda_n(\sqrt{n}, \pi)$ is $O(\sqrt{n} \log n)$, so that for $n \geq 1$,

$$(1.18) \quad \lambda_n(\pi) = \frac{N}{2} n \log n + C_1(\pi)n + O(\sqrt{n} \log n),$$

where the implied constant in the O -notation depends on π .

Theorem 1.1 follows on combining Lemma 4.4, Theorem 5.1 and Theorem 6.1, together with the relation $C_1(\pi) = C_1(\pi^\vee)$. If the Riemann hypothesis does not hold for $L(s, \pi)$ then the incomplete Li coefficient term $\lambda_n(\sqrt{n}, \pi)$ will sometimes be very large, of size exponential in n . This fact was already observed for the Riemann zeta function in [4, Theorem 1 (c)].

In §7 we construct for each π an entire function $F_\pi(z)$ of order one and exponential type that interpolates the Li coefficients at integer values, i.e. $F_\pi(n) = \lambda_n(\pi)$ for all $n \in \mathbb{Z}$. Assuming the Riemann hypothesis for $L(s, \pi)$, this function $F_\pi(z)$ can be chosen to have exponential type at most π . It is then (almost) uniquely characterized by the interpolation property; the only ambiguity concerns zeros at the central critical value $s = \frac{1}{2}$, which require special treatment, see Theorem 7.1. Assuming the Riemann hypothesis we deduce that the function $F_\pi(z)$ has a Fourier transform which is a tempered distribution, whose support is a countable closed set in $[-\pi, \pi]$ having 0 as its only limit point.

This interpolation function $F_\pi(z)$ appears to be a new object associated to the zeta zeros. We do not know any relation of it to various other functions constructed from the zeta zeros such as Cramer's V -function ([11], [24], [19], [33], [28]) or functions studied by Voros [44], [45].

The results of this paper explain some of the empirical observations of K. Maslanka [40]. The asymptotic formula (1.16) applied to π_{triv} , explains the observed behavior of the small Li coefficients. Since the non-trivial zeros of $\zeta(s)$ are known to lie on the critical line up to height $T \approx 10^9$ we may expect the first 10^{16} Li coefficients will also exhibit similar asymptotic behavior, i.e. the term $|S_f(n)|$ will remain small over this range. Maslanka's computations of $S_f(n)$ allow the possibility that $|S_f(n)|$ is of smaller order of growth than $O(\sqrt{n} \log n)$; if so, this requires further explanation. The approximate formula (1.17) gives no information on the precise spectral nature of the "small oscillations" in $S_f(n)$. Perhaps additional information can be extracted from the Fourier transform of the interpolating function defined in §7.

Earlier computations of J. B. Keiper [34] also apply to the Li coefficients, as noted by K. Maslanka. In our notation Figure 1 of his paper plots the function

$$\bar{\lambda}_n := \lambda_{-n} - \left(\frac{1}{2}n \log n + \frac{1}{2}(\gamma - 1 - \log(2\pi))n \right).$$

over the range $1 \leq n \leq 7000$. In view of Lemma 4.4 and Theorem 5.1 below (for π_{triv}), the functions $\bar{\lambda}_n$ and $S_f(n)$ differ by a bounded quantity; in fact the actual difference appears to be very small. One may therefore interpret Keiper's Figure 1 as essentially picturing $S_f(n)$ over this range; it indicates that $|S_f(n)| < 20$ holds for $1 \leq n \leq 7000$.

Other related work on Li coefficients includes that of F. Brown [5], who determined zero-free regions for Dirichlet and Artin L -functions expressed in terms of the sizes of generalized Li coefficients. P. Freitas [20] relates zero-free regions to the positivity of a different generalization of Li coefficients. There is a probabilistic interpretation of some Li coefficients given in Biane, Pitman and Yor [2, Sec. 2.3]. While this paper was being completed, an asymptotic formula for the Li coefficients λ_n was announced by A. Voros [43], under the Riemann hypothesis. Comparison of his formula with that obtained here in (5.31) led to a simplification of the expression for $C_1(\pi)$ in Theorem 5.1.

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2. Li coefficients for automorphic L -functions

We recall basic facts about principal L -functions $L(s, \pi)$ attached to irreducible cuspidal unitary automorphic representations of $GL(N)$, as in Jacquet [31] and Rudnick and Sarnak [42, Sect. 2], see also Cogdell [10] and Gelbart and Miller [21, Sect. 7.2]. These L -functions are associated to $GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}_{\mathbb{Q}})$, and they are more precisely denoted $L(s, \pi, \rho)$ in which the Langlands L -group ${}^L G = GL(N, \mathbb{C})$ and $\rho : {}^L G \rightarrow GL(N, \mathbb{C})$ is the standard representation.

For the trivial representation π_{triv} of $GL(1)$ we have the completed automorphic L -function $\Lambda(s, \pi_{\text{triv}}) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$. This function has simple poles at $s = 0$ and $s = 1$. Aside from this representation, all other $\Lambda(s, \pi)$ are entire functions.

Each completed automorphic L -function $\Lambda(s, \pi)$ has an Euler product factorization

$$(2.1) \quad \Lambda(s, \pi) := Q(\pi)^{\frac{s}{2}} L_{\infty}(s, \pi) L(s, \pi).$$

Here $Q(\pi)$ is a positive integer called the *conductor* of the representation π , and the *archimedean factor* is

$$(2.2) \quad L_{\infty}(s, \pi) = \prod_{j=1}^N \Gamma_{\mathbb{R}}(s + \kappa_j(\pi)),$$

in which $\kappa_j(\pi)$ are certain constants and

$$(2.3) \quad \Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right).$$

The (*finite*) L -function $L(s, \pi)$ is given by an Euler product over the finite places

$$\begin{aligned} L(s, \pi) &= \prod_p \prod_{j=1}^N (1 - \alpha_{p,j}(\pi) p^{-s})^{-1}, \\ &= \sum_{n=1}^{\infty} a_n(\pi) n^{-s}. \end{aligned}$$

This Euler product and its associated Dirichlet series converges absolutely in a half-plane specified below. The functions $\Lambda(s, \pi)$ satisfy a functional equation

$$(2.4) \quad \Lambda(s, \pi) = \varepsilon(\pi) \Lambda(1 - s, \pi^{\vee}),$$

in which $\varepsilon(\pi)$ is a constant of absolute value one, and π^\vee denotes the contragredient representation. The contragredient representation has L -function

$$(2.5) \quad L(s, \pi^\vee) = \sum_{n=1}^{\infty} \overline{a_n(\pi)} n^{-s}$$

and archimedean factor

$$(2.6) \quad L_\infty(s, \pi^\vee) = L_\infty(s, \pi),$$

and has conductor $Q(\pi^\vee) = Q(\pi)$. The functional equation implies that $\varepsilon(\pi^\vee) = \overline{\varepsilon(\pi)}$. The functions $\Lambda(s, \pi)$ are bounded in vertical strips, with exponential decay as $|\operatorname{Im}(s)| \rightarrow \infty$, with $-B < \operatorname{Re}(s) < B$ for any fixed B .

We define the ξ -function $\xi(s, \pi)$ associated to π by

$$(2.7) \quad \xi(s, \pi) := s^{-e(0, \pi)}(s - 1)^{-e(1, \pi)} \left(\frac{1}{\sqrt{(-1)^{e(\frac{1}{2}, \pi)} \varepsilon(\pi)}} \Lambda(s, \pi) \right),$$

where $e(s_0, \pi)$ denotes the order of a zero or pole of $\Lambda(s, \pi)$ at $s = s_0$, with poles being assigned negative orders. We have $e(0, \pi) = e(1, \pi)$ by the functional equation, and this definition ensures that $\xi(s, \pi)$ is holomorphic and nonzero at $s = 0$ and 1 . In this definition the square roots must be chosen consistently so that

$$\sqrt{(-1)^{e(\frac{1}{2}, \pi)} \varepsilon(\pi)} \cdot \sqrt{(-1)^{e(\frac{1}{2}, \pi)} \varepsilon(\pi^\vee)} = 1.$$

There remains a choice of sign, which can be removed by the requirement that $\xi(\frac{1}{2} + it, \pi) > 0$ hold for small positive t , as justified in the result below. For the trivial representation π_{triv} on $GL(1)$ we have $e(0, \pi_{\text{triv}}) = e(1, \pi_{\text{triv}}) = -1$, and $\xi(s, \pi_{\text{triv}}) = 2\xi(s)$. This convention is forced if we wish to have entire functions in all cases, for we must remove the poles at $s = 0$ and $s = 1$ for the case π_{triv} .

The following theorem collects together analytic facts about automorphic L -functions.

THEOREM 2.1. — *Let π be a irreducible cuspidal unitary automorphic representation of $GL(n)$ over \mathbb{Q} .*

(1) *The ordinary Dirichlet series $L(s, \pi) = \sum_{n=1}^{\infty} a_n(\pi) n^{-s}$ converges absolutely in the half-plane $\operatorname{Re}(s) > 1$. We have for all $n \geq 1$ that*

$$(2.8) \quad |a_n(\pi)| \leq C(\pi) d(n) n^{\frac{N}{2}},$$

for some $C(\pi) > 0$ depending on π , and $d(n)$ is the number of divisors of n .

(2) The archimedean factors $\Gamma_{\mathbb{R}}(s + \kappa_j(\pi))$ in the Euler product $\Lambda(s, \pi)$ satisfy

$$(2.9) \quad \operatorname{Re}(\kappa_j(\pi)) > -\frac{1}{2}.$$

The quantities $\{\kappa_j(\pi) : 1 \leq j \leq N\}$ are permuted under complex conjugation, so that

$$(2.10) \quad L_{\infty}(s, \pi) = \overline{L_{\infty}(\bar{s}, \pi)}.$$

(3) The zeros of $\Lambda(s, \pi)$ all lie in the open critical strip $0 < \operatorname{Re}(s) < 1$. In particular $\Lambda(s, \pi)$ is non-vanishing on the lines $\operatorname{Re}(s) = 0$ and $\operatorname{Re}(s) = 1$.

(4) The counting function $N_{\pi}^{+}(T)$ (resp. $N_{\pi}^{-}(T)$) for zeros of $\Lambda(s, \pi)$ with $0 \leq \operatorname{Im}(\rho_{\pi}) < T$ (resp. $-T \leq \operatorname{Im}(\rho_{\pi}) \leq 0$) each satisfy

$$(2.11) \quad N_{\pi}^{\pm}(T) = \frac{N}{2\pi} T \log T + \frac{1}{2} C_0(\pi) T + O(\log T)$$

as $T \rightarrow \infty$. Here

$$(2.12) \quad C_0(\pi) = \frac{1}{\pi} \log Q(\pi) - \frac{N}{\pi} (1 + \log(2\pi)),$$

in which $Q(\pi)$ is the conductor of π , and the O -constant depends on π .

(5) $\xi(s, \pi)$ satisfies the functional equation

$$(2.13) \quad \xi(s, \pi) = (-1)^k \xi(1 - s, \pi^{\vee}),$$

where $k = e(\frac{1}{2}, \pi) = e(\frac{1}{2}, \pi^{\vee})$ is the order of the zero of $\xi(s, \pi)$ at $s = \frac{1}{2}$. It is real-valued on the critical line $\operatorname{Re}(s) = \frac{1}{2}$, so the multiset $Z(\pi)$ is invariant under the map $\rho \mapsto 1 - \bar{\rho}$.

(6) The function $\xi(s, \pi)$ is an entire function of order one and maximal type. It is bounded in vertical strips $-B < \operatorname{Re}(s) < B$ for any finite B , and has rapid decrease there as $|\operatorname{Im}(s)| \rightarrow \infty$.

Proof. — (1) Those irreducible cuspidal automorphic representations π that arise as subrepresentations of the right regular representation on $L^2(GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}_{\mathbb{Q}}), d\mu_{\text{Haar}})$ are necessarily unitary. We use the bound of Jacquet and Shalika [32, Theorem 5.3], that Rankin-Selberg convolutions $L(s, \pi_1 \times \pi_2)$ of unitary irreducible cuspidal automorphic representations π_1, π_2 have Dirichlet series that converge in $\operatorname{Re}(s) > 1$. We take $\pi_1 = \pi$ and π_2 to be the trivial representation. (See also [29, Sect. 5.11, 5.12].)

Jacquet and Shalika [32, Corollary 2.5] also derive the bound that all unramified primes in the Euler product have $|\alpha_{j,p}| < \sqrt{p}$. All but finitely many primes are unramified, and therefore multiplying out the Euler product yields the bound (2.8). Better bounds are known for $|\alpha_{j,p}|$ which give larger regions of absolute convergence. Rudnick and Sarnak [42, Prop. 5.1]

show for irreducible cuspidal automorphic representations π of $GL(N)$ unramified at p satisfy

$$|\alpha_{j,p}(\pi)| \leq p^{\frac{1}{2} - \frac{1}{N^2+1}}.$$

The *generalized Ramanujan conjecture* formulated in [30] asserts that each $|\alpha_{j,p}| = 1$ at unramified places p . It is known to be true for principal L -functions over $GL(1)$ (Dirichlet L -functions).

(2) The inequality $\operatorname{Re}(s) > -\frac{1}{2}$ is established in Rudnick and Sarnak [42, Eq. (2.5) and Sect. 5.3]. There exist ramified archimedean representations over $GL(2)$ for which some $\operatorname{Re}(\kappa_j(\pi))$ is arbitrarily large. However Luo, Rudnick and Sarnak [39] show that unramified archimedean representations⁽¹⁾ satisfy the stronger bound

$$|\operatorname{Re}(\kappa_j(\pi))| \leq \frac{1}{2} - \frac{1}{N^2+1}.$$

The Ramanujan conjecture at the archimedean places asserts for an unramified representation that all $\operatorname{Re}(\kappa_j(\pi)) = 0$. Its truth would imply that the bound (2.9) could be improved to $\operatorname{Re}(\kappa_j(\pi)) \geq 0$ in the general case.

The symmetry under complex conjugation of the $\kappa_j(\pi)$ holds because the local factors in the Euler product are equivalent to those of the contra-redient (see [22, Theorem 2] and [42, Sect. 2.2]), which gives (2.10).

(3) Iwaniec and Kowalski [29, Theorem 5.42] show non-vanishing of cuspidal automorphic L -functions for $GL(N)$ on the line $\operatorname{Re}(s) = 1$, and also obtain a zero-free region inside the critical strip. The nonvanishing on $\operatorname{Re}(s) = 0$ comes from the functional equation (2.4) for $\Lambda(s, \pi)$.

(4) The asymptotic formula for $N_\pi(T)$ is essentially determined by the archimedean factors in the Euler product for $\Lambda(s, \pi)$. The counting result for the zero density from $-T$ to T appears as [29, Theorem 5.8], in which $Q = Q(\pi)$ is the conductor of π (see [29, Sec. 5.1]). The error term given is $O(\log \mathfrak{q}(\pi, iT))$, with an absolute constant, which involves the *analytic conductor*

$$(2.14) \quad \mathfrak{q}(\pi, s) := Q(\pi) \prod_{j=1}^N (|s + \kappa_j(\pi)| + 3).$$

Since we regard π as fixed, this yields $O(\log T)$, with the O -constant depending on π . To get the bound (4) on the upper and lower critical strip separately, the contour integral proof in [29] must be modified to split into

(1) A representation is *unramified* if it is identically one on the maximal compact subgroup of all the archimedean components of $GL(N)$. Note that a Dirichlet character χ for $GL(1)$ with $\chi(-1) = -1$ corresponds to an automorphic representation for $GL(1)$ that is ramified at the archimedean place, and has $\kappa_1(\chi) = 1$.

two contours with a cut along the real axis, which goes off it in small circles to avoid poles at the trivial zeros. (We omit the details.)

(5) The definition of $\xi(s)$ ensures that the functional equation (2.8) holds up to a sign, which depends on the multiplicity (mod 2) of a zero of $\Lambda(s, \pi)$ at $s = \frac{1}{2}$. Now the functions $\Lambda(s, \pi)$ have the symmetry

$$(2.15) \quad \Lambda(s, \pi) = \overline{\Lambda(\bar{s}, \pi^\vee)}.$$

which follows from (2.10) and (2.5). In consequence (2.7) gives

$$\xi(s, \pi) = (-1)^{e(\frac{1}{2}, \pi)} \overline{\xi(\bar{s}, \pi^\vee)}.$$

We deduce that

$$\xi\left(\frac{1}{2} + it, \pi\right) = (-1)^{e(\frac{1}{2}, \pi)} \xi\left(\frac{1}{2} - it, \pi^\vee\right) = \overline{\xi\left(\frac{1}{2} + it, \pi\right)}.$$

Thus $\xi(s, \pi)$ is real on the critical line. The invariance of the multiset $Z(\pi)$ under $\rho \mapsto 1 - \bar{\rho}$ now follows from the reflection principle.

(6) Godement and Jacquet [23, Theorem 13.8] showed for cuspidal automorphic representations π that $\Lambda(s, \pi)$ is meromorphic and bounded in vertical strips, with finitely many poles, and that the ordinary Dirichlet series $L(s, \pi)$ has a nonempty half-plane of absolute convergence, (See also Jacquet [31, Theorem 6.2]). The assumption that π is irreducible and cuspidal gives that they are entire functions, aside from the trivial representation $\pi_{\text{triv}} \|\cdot\|^s$ of $GL(1)$, whose singularities are simple poles at $s = 0, 1$. It follows that $\xi(s, \pi)$ is an entire function in all cases.

We assert that in the vertical strip $-B \leq \text{Im}(s) \leq B$ the function $\xi(s, \pi)$ satisfies

$$|\xi(s, \pi)| \leq C(\pi, B) e^{-N \frac{\pi}{2} |\text{Im}(s)|}$$

as $|\text{Im}(s)| \rightarrow \infty$. To see this, observe that the exponential decay on vertical lines holds for $\text{Re}(s) \geq 1 + \varepsilon$, coming from exponential decay of the archimedean factor and the bound $|L(s, \pi)| = O(1)$. It then holds for $\text{Re}(s) \leq -\varepsilon$ by the functional equation, and then in between by the Phragmén-Lindelöf principle.

In the half-plane $\text{Re}(s) \geq 1 + \varepsilon$ we have $|\Lambda(s, \pi)| = O(1)$ because its Dirichlet series representation converges absolutely. It follows that on this half-plane $|\Lambda(s, \pi)| = O(e^{N|s|(\log|s|+1)})$, with the growth rate coming from the archimedean factors, which can be bounded by Stirling’s formula. The functional equation now shows that the same bound holds on the half plane $\text{Re}(s) \leq -\varepsilon$. Since $|\Lambda(s, \pi)|$ is known to be bounded on the vertical strip $-\varepsilon \leq \text{Re}(s) \leq 1 + \varepsilon$, it follows that $|\Lambda(s)| = O(e^{N|s|(\log(|s|+2))})$ for all s . Thus $\Lambda(s)$ is an entire function of order one. It is of maximal type because its growth-rate is faster than $e^{R|s|}$ for any finite R along the positive real axis,

coming from the archimedean factors. The function $|\xi(s, \pi)|$ inherits these properties, since it differs by at most a polynomial factor from $\Lambda(s, \pi)$. \square

We let $Z(\pi)$ denote the multi-set of zeros of $\xi(s, \pi)$ (counted with multiplicity) which is the same as that of $\Lambda(s, \pi)$ except possibly at $s = 0$ and $s = 1$.

LEMMA 2.2. — *For any principal L -function $L(s, \pi)$ for $GL(N)$ the power sums*

$$(2.16) \quad \sigma_n(\pi) := \sum'_{\rho \in Z(\pi)} \frac{1}{\rho^n}, \quad n \geq 1,$$

are absolutely convergent for $n \geq 2$, and are $*$ -convergent for $n = 1$. The real parts of these sums are absolutely convergent for all $n \geq 1$.

Proof. — The absolute convergence for $n \geq 2$ follows easily from the zero-counting bound (2.11).

The $*$ -convergence for $\sigma_1(\pi)$ follows from the asymptotics (2.11) for the zeros. Here the zeros below and above the real axis are paired in increasing order of their imaginary parts (in absolute values) and using the asymptotic formulas in Theorem 2.1(4) we get $*$ -convergence by partial summation. It is important that the remainder term in (2.11) be $O(T^{1-\delta})$ for some $\delta > 0$. (In general the zeros are not symmetric about the real axis. The symmetry $\rho \mapsto 1 - \bar{\rho}$ is of no help in proving $*$ -convergence.) Finally, for any zero $\rho = \beta + i\gamma$ we have

$$\operatorname{Re} \left(\frac{1}{\rho} \right) = \frac{\beta}{\beta^2 + \gamma^2} = O \left(\frac{1}{\gamma^2} \right),$$

which gives absolute convergence for $n = 1$ of the real parts of the terms in the sum (2.16). \square

LEMMA 2.3. — *For all irreducible cuspidal unitary automorphic representations on $GL(N)$ the sums*

$$(2.17) \quad \lambda_n(\pi) := \sum_{\rho \in Z(\pi)} 1 - \left(1 - \frac{1}{\rho} \right)^n$$

are $*$ -convergent for all $n \in \mathbb{Z}$. They are given by

$$(2.18) \quad \lambda_n(\pi) = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \sigma_j(\pi),$$

with $\lambda_0(\pi) = 0$. They satisfy

$$(2.19) \quad \lambda_{-n}(\pi) = \overline{\lambda_n(\pi)} = \lambda_n(\pi^\vee).$$

Proof. — We have

$$\sum_{|\rho| < T} 1 - \left(1 - \frac{1}{\rho}\right)^n = \sum_{|\rho| < T} \frac{n}{\rho} + \sum_{j=2}^n (-1)^{j-1} \binom{n}{j} \left(\sum_{|\rho| < T} \frac{1}{\rho^j}\right).$$

On letting $T \rightarrow \infty$, Lemma 2.2 shows that the first sum on the right $*$ -converges and the second sum on the right converges absolutely. This gives (2.18).

The symmetry $\lambda_{-n}(\pi) = \overline{\lambda_n(\pi)}$ is inherited from the symmetry that if $\rho = \beta + i\gamma$ is a zero of $\xi(s)$, then so is $1 - \bar{\rho} = 1 - \beta + i\gamma$, with the same multiplicity. This holds by the reflection principle since $\xi(s)$ is real on the critical line. That is, the multiset of zeros is invariant under the map $\rho \mapsto 1 - \bar{\rho}$. Now we have

$$1 - \left(1 - \frac{1}{\rho}\right)^{-n} = 1 - \left(\frac{\rho - 1}{\rho}\right)^{-n} = 1 - \left(\frac{-\rho}{1 - \rho}\right)^n = 1 - \overline{\left(1 - \frac{1}{1 - \bar{\rho}}\right)^n}.$$

and this gives $\lambda_{-n}(\pi) = \overline{\lambda_n(\pi)}$.

For the contragredient representation, we have the symmetry $\Lambda(s, \pi) = \overline{\Lambda(\bar{s}, \pi^\vee)}$ in (2.15). This shows that the zero sets of the two completed L -functions are complex conjugate, i.e. $Z(\pi^\vee) = \overline{Z(\pi)}$. This then yields the other relation $\lambda_n(\pi^\vee) = \overline{\lambda_n(\pi)}$. \square

We can now state a general version of Li’s criterion.

THEOREM 2.4. — *Let π be an irreducible cuspidal unitary automorphic representation of $GL(n)$. The following conditions are each equivalent to the Riemann hypothesis for $\xi(s, \pi)$.*

(1) *For all $n \geq 1$,*

$$(2.20) \quad \operatorname{Re}(\lambda_n(\pi)) \geq 0.$$

(2) *For each $\varepsilon > 0$, there is a positive constant $C(\varepsilon)$ such that*

$$(2.21) \quad \operatorname{Re}(\lambda_n(\pi)) \geq -C(\varepsilon)e^{\varepsilon n} \text{ for all } n \geq 1.$$

(3) *The generalized Li coefficients $\lambda_n(\pi)$ satisfy*

$$(2.22) \quad \lim_{n \rightarrow \infty} |\lambda_n(\pi)|^{\frac{1}{n}} \leq 1.$$

Proof. — Theorem 2.1 gives that the multiset $Z(\pi)$ omits the values 0 and 1 and is invariant under the symmetry $\rho \mapsto 1 - \bar{\rho}$. The equivalence of conditions (1) and (2) to the Riemann hypothesis for $Z(\pi)$ follows from the Corollary in Theorem 1 of [4].

It remains to show the equivalence of (3) to the Riemann hypothesis. The $\lambda_n(\pi)$ are identifiable as the power series coefficients around $z = 0$ of

$$(2.23) \quad \frac{d}{dz} \log \xi \left(\frac{-z}{1-z}, \pi \right) = \frac{-1}{(1-z)^2} \frac{\xi'}{\xi} \left(\frac{-z}{1-z}, \pi \right) = \sum_{n=0}^{\infty} \lambda_{n+1}(\pi) z^n.$$

The map $z \mapsto s = -z/(1-z)$ conformally maps the unit disk to the half-plane $\operatorname{Re}(s) \geq \frac{1}{2}$, and the point $z = 0$ corresponds to $s = 1$, a point where $(\xi'/\xi)(s)$ is holomorphic. Assuming the Riemann hypothesis holds for $\xi(s, \pi)$, the function in (2.23) is holomorphic in the unit disk, hence its power series coefficients satisfy (2.22). Conversely, if (2.22) holds, then the function (2.23) is holomorphic in the unit disk, hence $(\xi'/\xi)(s, \pi)$ has no singularity in $\operatorname{Re}(s) > \frac{1}{2}$. The functional equation then shows it has no singularity in $\operatorname{Re}(s) < \frac{1}{2}$ so the Riemann hypothesis holds for $\xi(s, \pi)$. \square

3. Li coefficients and Weil's quadratic functional

A. Weil [46] formulated the “explicit formula” of prime number theory in terms of distributions, using the Fourier transform on the real line. He gave a necessary and sufficient condition for the Riemann hypothesis in terms of the positivity of a quadratic functional on a suitable space of test functions $F(u)$ on the real line, contained in the Hilbert space $L^2(\mathbb{R}, du)$. There is a natural generalization of Weil's quadratic functional associated to the completed L -function $\Lambda(s, \pi)$ of any irreducible cuspidal automorphic representation π of $GL(N)$, which we shall consider here.

We first note that, using an exponential change of variable ($F(u) = e^{u/2} f(e^u)$), the “explicit formula” is expressible in terms of test functions contained in the Hilbert space $L^2(\mathbb{R}_{>0}, dx)$ on the positive real line $\mathbb{R}_{>0}$ in the x -variable, with the Fourier transform replaced by the Mellin transform. This is the framework for the “explicit formula” taken in Bombieri [3] and in [4]. Now we make a second change of variable, using the Mellin transform

$$\hat{f}(s) = \int_0^{\infty} f(x) x^s \frac{dx}{x},$$

to transfer Weil's functional to a quadratic functional on a space of test functions contained inside the Hilbert space $L^2(\frac{1}{2} + i\mathbb{R}, \frac{dt}{2\pi})$; these test functions are the Mellin transforms on the critical line of the test functions on $\mathbb{R}_{>0}$ above. (The Mellin transform extends to an isometry between $L^2(\mathbb{R}_{>0}, dx)$ and $L^2(\frac{1}{2} + i\mathbb{R}, \frac{dt}{2\pi})$.) We treat the “explicit formula” in these coordinates (a viewpoint taken in Burnol [6]). The resulting test functions

in the s -variable have the property of being analytic in some open domain that includes the critical line $\text{Re}(s) = \frac{1}{2}$ in its interior. If the Riemann hypothesis is not assumed, the set of test functions must be further restricted to functions analytic in a region containing the closed critical strip.

We consider as test functions the vector space \mathcal{A} of all functions $F(s)$ holomorphic in the strip $0 < \text{Re}(s) < 1$ which satisfy a uniform growth bound $F(s) = O(1/|s|)$ in the strip outside⁽²⁾ the region $|\text{Im}(s)| \leq 1$, with O -constant depending on the function. The class \mathcal{A} is closed under the action of the involution $\tilde{G}(s) := G(1 - s)$. The functions in \mathcal{A} are completely determined by their values on the critical line $\text{Re}(s) = \frac{1}{2}$ by analytic continuation, and the growth bound ensures that they belong to $L^2(\frac{1}{2} + i\mathbb{R}, \frac{dt}{2\pi})$. Under the inverse Mellin transform they convert to a class of test functions contained in $L^2(\mathbb{R}_{>0}, dx)$ of the type considered in [4]; these functions are smooth away from the point $x = 1$.

Given $F(s), G(s) \in \mathcal{A}$ we define the *Weil scalar product* associated to the automorphic representation π by

$$(3.1) \quad \langle F, G \rangle_{\mathcal{W}(\pi)} := \sum_{\rho \in Z(\pi)} F(\rho) \overline{G(1 - \bar{\rho})}.$$

The sum on the right counts zeros with multiplicity, and it converges absolutely due to the growth bound on F and G for large $|s|$. This scalar product is linear in the first factor and conjugate-linear in the second factor. The multiset of automorphic L -function zeros $Z(\pi)$ (counting multiplicities) is invariant under the involution $\rho \rightarrow 1 - \bar{\rho}$. This yields the Hermitian symmetry

$$\langle F, G \rangle_{\mathcal{W}(\pi)} = \overline{\langle G, F \rangle_{\mathcal{W}(\pi)}}.$$

In the appendix we clarify the relation between this definition and Weil’s definition of the quadratic functional.

The Riemann hypothesis for $\Lambda(s, \pi)$ implies that the Weil scalar product is positive semidefinite on the test function vector space \mathcal{A} . To see this we note that $\rho = 1 - \bar{\rho}$ holds if and only if ρ lies on the critical line $\text{Re}(s) = \frac{1}{2}$. The Riemann hypothesis implies that for all $F(s) \in \mathcal{A}$,

$$\langle F, F \rangle_{\mathcal{W}(\pi)} = \sum_{\rho \in Z(\pi)} F(\rho) \overline{F(1 - \bar{\rho})} = \sum_{\rho \in Z(\pi)} F(\rho) \overline{F(\rho)} = \sum_{\rho \in Z(\pi)} |F(\rho)|^2 \geq 0.$$

The Weil scalar product is not positive definite on the full class \mathcal{A} since $F(s) := \xi(s, \pi) \in \mathcal{A}$ with $\langle F, F \rangle_{\mathcal{W}(\pi)} = 0$. In the converse direction, Weil

⁽²⁾The region $|\text{Im}(s)| \leq 1$ is omitted here to avoid the points $s = 0$ and $s = 1$, because every nonzero element in the Li class \mathcal{L} of test functions defined later necessarily has a pole at one or both of these points.

showed for various L -functions that the positive semidefiniteness of the Weil scalar product on suitable subsets of the test functions in \mathcal{A} . implies the Riemann Hypothesis for $\Lambda(s, \pi)$. A number of criteria of this sort are known, using different collections of test functions; essentially one needs the test function set to be sufficiently large to separate all the zeros $\rho = \rho_\pi$ (counting multiplicities).

We define the *Li class* \mathcal{L} of test functions to be the set of rational functions in the function field $\mathbb{C}(s)$ that vanish at infinity (on the Riemann sphere) and whose polar divisor is contained in the set $\{0, 1\}$. The class \mathcal{L} is closed under addition, multiplication and scalar multiplication, but does not contain the constant functions. A vector space basis for \mathcal{L} consists of $\{s^{-n} : n \geq 1\}$ and $\{(1 - s)^{-n} : n \geq 1\}$.

The special test functions $G_n(s) \in \mathcal{L}$ corresponding to the Li coefficients are

$$(3.2) \quad G_n(s) := 1 - \left(1 - \frac{1}{s}\right)^n \text{ for } n \in \mathbb{Z}.$$

The set of all $G_n(s)$, excluding $G_0(s) \equiv 0$, forms a vector space basis of \mathcal{L} . Indeed the change-of-basis matrix $G_n(s)$ ($n \geq 1$) relating these functions to $\{s^{-n} : n \geq 1\}$ is an upper triangular unipotent matrix, and similarly for $G_{-n}(s)$ ($n \geq 1$) and $\{(1 - s)^{-n} : n \geq 1\}$. The class \mathcal{L} is contained in \mathcal{A} , because the vanishing condition at ∞ implies a bound $F(s) = O(1/|s|)$ uniformly in the region $|\text{Im}(s)| \geq 1$. The class \mathcal{L} has the property that every nonzero member of it has a pole at either $s = 0$ or $s = 1$, or both.

The following result computes the Weil scalar product with respect to this basis.

THEOREM 3.1. — *Let π be an irreducible cuspidal unitary automorphic representation of $GL(N)$, with associated ξ -function $\xi(s, \pi)$ and with Weil scalar product $\langle \cdot, \cdot \rangle_{\mathcal{W}(\pi)}$. For the Li test functions $G_n(s) = 1 - (1 - \frac{1}{s})^n$ there holds*

$$(3.3) \quad \langle G_n, G_m \rangle_{\mathcal{W}(\pi)} = \lambda_n(\pi) + \lambda_{-m}(\pi) - \lambda_{n-m}(\pi).$$

In particular

$$(3.4) \quad \|G_n\|_{\mathcal{W}(\pi)}^2 = \lambda_n(\pi) + \lambda_{-n}(\pi) = 2 \text{Re}(\lambda_n(\pi)).$$

Proof. — By definition

$$\lambda_n(\pi) = \sum'_{\rho \in Z(\pi)} G_n(\rho).$$

Since each function $G_n(s)$ is real on the real axis, the reflection principle gives

$$\overline{G_n(1 - \bar{\rho})} = G_n(1 - \rho).$$

We also have the identity

$$(3.5) \quad G_{-m}(s) = 1 - \left(1 - \frac{1}{s}\right)^{-m} = 1 - \left(1 - \frac{1}{s-1}\right)^m = G_m(1-s)$$

and

$$(3.6) \quad G_n(s)G_{-m}(s) = G_n(s) + G_{-m}(s) - G_{n-m}(s).$$

Combining all these gives

$$\begin{aligned} \langle G_n, G_m \rangle_{\mathcal{W}(\pi)} &= \sum_{\rho \in Z(\pi)} G_n(\rho)G_m(1-\rho) = \sum_{\rho} G_n(\rho)G_m(\rho) \\ &= \sum'_{\rho \in Z(\pi)} (G_n(\rho) + G_{-m}(\rho) - G_{n-m}(\rho)) \\ &= \sum'_{\rho \in Z(\pi)} G_n(\rho) + \sum'_{\rho \in Z(\pi)} G_m(\rho) - \sum'_{\rho \in Z(\pi)} G_{n-m}(\rho) \\ &= \lambda_n(\pi) + \lambda_{-m}(\pi) - \lambda_{n-m}(\pi), \end{aligned}$$

the required result. \square

The Li class \mathcal{L} is large enough to characterize the Riemann hypothesis in terms of the semidefiniteness of the Weil scalar product. Theorem 3.1 shows that semidefiniteness implies that all $\operatorname{Re}(\lambda_n) \geq 0$. The Riemann hypothesis then follows from Theorem 2.4 (2).

If the Riemann hypothesis holds, then the Weil scalar product is positive definite on \mathcal{L} . For it $F(x) \in \mathcal{L}$ is nonzero, then since it is a rational function it has finitely many zeros, so cannot vanish at all points of $Z(\pi)$. Thus

$$\|F\|_{\mathcal{W}(\pi)}^2 = \sum_{\rho \in Z(\pi)} |F(\rho)|^2 > 0.$$

The vector space \mathcal{L} is then a pre-Hilbert space with this Hermitian scalar product, so it can be completed to a Hilbert space $\mathcal{H}_{\mathcal{L}}(\pi)$.

4. Arithmetic formula for Li coefficients

We relate the Li coefficients to sums of the form $\sum'_{\rho \in Z(\pi)} f(\rho)$ for various functions $f(s)$. In particular we split $\lambda_n(\pi)$ into two parts $S_{\infty}(n, \pi)$ and $S_f(n, \pi)$ given in Lemma 4.4 below.

The Li coefficients are expressible in terms of Laurent series coefficients around the point $s = 1$ of the logarithmic derivative of the ξ -function $\xi(s, \pi)$. These coefficients are given in terms of power sums of zeros.

LEMMA 4.1. — *The power series expansion around $s = 1$ of $(\xi'/\xi)(s, \pi)$ is*

$$(4.1) \quad \frac{\xi'}{\xi}(s + 1, \pi) = \sum_{n=0}^{\infty} (-1)^n \sigma_{n+1}(\pi^\vee) s^n.$$

in which the power sum $\sigma_n(\pi^\vee) = \sum'_{\rho \in Z(\pi^\vee)} \rho^{-n}$.

Proof. — Since $\xi(s, \pi)$ is entire of order one we have the Hadamard product expansion

$$\xi(s, \pi) = e^{A(\pi)s + B(\pi)} \prod_{\rho \in Z(\pi)} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

Taking logarithms yields,

$$\begin{aligned} \log \xi(s, \pi) &= A(\pi)s + B(\pi) + \sum_{\rho \in Z(\pi)} \left(\log\left(1 - \frac{s}{\rho}\right) + \frac{s}{\rho} \right) \\ &= A(\pi)s + B(\pi) - \sum_{\rho \in Z(\pi)} \left(\sum_{n=2}^{\infty} \frac{1}{n} \frac{s^n}{\rho^n} \right) \\ &= A(\pi)s + B(\pi) - \sum_{n=2}^{\infty} \sigma_n(\pi) \frac{s^n}{n}. \end{aligned}$$

One can also deduce by a $*$ -convergent rearrangement that

$$A(\pi) = - \sum'_{\rho \in Z(\pi)} \frac{1}{\rho} = -\sigma_1(\pi).$$

Differentiating yields

$$-\frac{\xi'}{\xi}(s, \pi) = \sum_{n=0}^{\infty} \sigma_{n+1}(\pi) s^n.$$

The functional equation (2.13) gives $\xi'(s, \pi) = (-1)^{k+1} \xi'(1 - s, \pi^\vee)$, from which we obtain

$$\frac{\xi'}{\xi}(s + 1, \pi) = -\frac{\xi'}{\xi}(-s, \pi^\vee) = \sum_{n=0}^{\infty} (-1)^n \sigma_{n+1}(\pi^\vee) s^n,$$

as required. □

We now write

$$(4.2) \quad \frac{\xi'}{\xi}(s+1, \pi) = \left(\frac{1}{2} \log Q(\pi) + \sum_{j=1}^N \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s+1 + \kappa_j(\pi)) \right) + \frac{L'}{L}(s+1, \pi) - \frac{e(0, \pi)}{s+1} - \frac{e(1, \pi)}{s}$$

We define Laurent series coefficients around $s = 1$ of the first two terms on the right, as follows.

DEFINITION 4.2. — The coefficients $\{\tau_n(\pi) : n \geq 1\}$ are defined by

$$(4.3) \quad \sum_{n=0}^{\infty} \tau_n(\pi) s^n := \frac{1}{2} \log Q(\pi) + \sum_{j=1}^N \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s+1 + \kappa_j(\pi))$$

in which we have using (2.3) that

$$(4.4) \quad \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s) = -\frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right).$$

DEFINITION 4.3. — The coefficients $\{\eta_n(\pi) : n \geq 1\}$ are defined by

$$(4.5) \quad \sum_{n=0}^{\infty} \eta_n(\pi) s^n := -\frac{L'}{L}(s+1, \pi) - \frac{\delta(\pi)}{s}$$

where $\delta(\pi) = 1$ if $\pi = \pi_{\text{triv}}$ and $\delta(\pi) = 0$ otherwise. (The minus sign is present to agree with the convention in [4, Eq. (4.3)].)

We note that the coefficients $\tau_n(\pi)$ are real-valued, as a consequence of Theorem 2.1 (2). Also, the coefficients $\eta_n(\pi)$ for $\pi = \pi_{\text{triv}}$ are exactly the coefficients η_n given in [4].

LEMMA 4.4. — *Let π be an irreducible cuspidal automorphic representation on $GL(N)$ over \mathbb{Q} . Then for all $n \geq 1$,*

$$(4.6) \quad \lambda_n(\pi^{\vee}) = S_{\infty}(n, \pi^{\vee}) - S_f(n, \pi^{\vee}) + \delta(\pi),$$

in which

$$(4.7) \quad S_{\infty}(n, \pi^{\vee}) = \sum_{j=1}^n \binom{n}{j} \tau_{j-1}(\pi^{\vee})$$

and

$$(4.8) \quad S_f(n, \pi^{\vee}) = \sum_{j=1}^n \binom{n}{j} \eta_{j-1}(\pi^{\vee}),$$

and $\delta(\pi) = 1$ if $\pi = \pi_{\text{triv}}$ and $\delta(\pi) = 0$ otherwise.

Remark. — Note that $S_\infty(n, \pi) = S_\infty(n, \pi^\vee)$ and $\delta(\pi) = \delta(\pi^\vee)$. Also, since $\lambda_{-n}(\pi) = \overline{\lambda_n(\pi)} = \lambda_n(\pi^\vee)$, we obtain from (4.6) applied to π^\vee that

$$(4.9) \quad \lambda_{-n}(\pi) = S_\infty(n, \pi) - S_f(n, \pi) + \delta(\pi).$$

Proof. — Comparison of Lemma 4.1 for π^\vee and the formula (4.2) yields

$$(4.10) \quad (-1)^j \sigma_{j+1}(\pi) = \tau_j(\pi^\vee) - \eta_j(\pi^\vee) + (-1)^j \delta(\pi^\vee).$$

Substituting this formula into (2.18), and using $-e(0, \pi) = \delta(\pi) = \delta(\pi^\vee)$, we obtain the desired result. □

In the expression (4.6) the term $S_\infty(n, \pi^\vee)$ corresponds to the contribution of the archimedean primes and $S_f(n, \pi^\vee)$ corresponds to the finite primes. There are also extra contributions from singularities at $s = 0$ and $s = 1$. The $s = 1$ contribution cancels against the singularity at $s = 1$ of $(L'/L)(s, \pi)$, but the $s = 0$ singularity remains and contributes the constant $-e(0, \pi)\delta(\pi^\vee)$.

For $\pi_{\text{triv}} = \pi_{\text{triv}}^\vee$ on $GL(1)$ the terms $S_\infty(n, \pi)$ and $S_f(n, \pi)$ form parts of the arithmetic expression for the Li coefficients λ_n given in Theorem 2 of [4]. We have $\mu_1(\pi_{\text{triv}}) = 0$ and

$$(4.11) \quad S_\infty(n, \pi_{\text{triv}}) = - \sum_{j=1}^n (-1)^{j+1} \binom{n}{j} \left(1 - \frac{1}{2^j}\right) \zeta^*(j),$$

in which $\zeta^*(j) = \zeta(j)$ for $j \geq 2$, and $\zeta^*(1) = \log(4\pi) + \gamma$, where γ is Euler's constant. For the other term, writing

$$(4.12) \quad -\frac{\zeta'}{\zeta}(s+1) = \frac{1}{s} + \sum_{j=0}^\infty \eta_j s^j,$$

we have

$$(4.13) \quad S_f(n, \pi_{\text{triv}}) = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \eta_{j-1}.$$

In [4] it is shown that the coefficients η_j are given by

$$\eta_j := \frac{(-1)^j}{j!} \lim_{N \rightarrow \infty} \left(\sum_{m=1}^N \frac{\Lambda(m)(\log m)^j}{m} - \frac{1}{j+1} (\log N)^{j+1} \right),$$

in which $\Lambda(m)$ is the von Mangoldt function.

There exist formal arithmetic expressions for $S_f(n, \pi)$ in the general case that are similar in spirit to (4.13). Logarithmically differentiating the Euler product for $L(s, \pi)$ in the region $\text{Re}(s) > 1$ yields

$$-\frac{L'}{L}(s, \pi) = \sum_{n=1}^\infty \Lambda_\pi(n) n^{-s}$$

in which for $n = p^m$ a prime power,

$$(4.14) \quad \Lambda_\pi(p^m) = \frac{1}{m} \sum_{k=1}^N (\alpha_{k,p}(\pi))^m,$$

and $\Lambda_\pi(n) = 0$ otherwise. Now for all $\pi \neq \pi_{\text{triv}}$ using (4.5) (which is now analytic at $s = 1$) we have

$$\eta_j(\pi) = -\frac{1}{j!} \frac{d^j}{ds^j} \left[\frac{L'}{L}(s, \pi) \right] \Big|_{s=1}.$$

Using (4.14) we obtain, formally,

$$(4.15) \quad \eta_j(\pi) \simeq \frac{(-1)^j}{j!} \sum_{m=1}^\infty \frac{\Lambda_\pi(m)(\log m)^j}{m}$$

The sum on the right at best converges conditionally, when viewed as $\lim_{T \rightarrow \infty} \sum_{n \leq T}$. The conditional convergence is known to hold for L -functions on $GL(1)$. One can show

$$\eta_j(\pi) = \lim_{s \searrow 1} \frac{(-1)^j}{j!} \sum_{m=1}^\infty \frac{\Lambda_\pi(m)(\log m)^j}{m^s}$$

with $s \searrow 1$ along the real axis. Explicit expressions for $\eta_j(\pi)$ for various L -functions and modular forms are derived in X. Li [36], [38], [37].

There exist simplified formulas for the coefficients $\tau_n(\pi)$ analogous to (4.11), which involve values of the Hurwitz zeta function $\zeta(s, z) = \sum_{j=0}^\infty (n+z)^{-s}$ with $\text{Re}(z) > 0$.

LEMMA 4.5. — *For an irreducible cuspidal automorphic representation π on $GL(N)$ and $n \geq 1$, there holds*

$$(4.16) \quad \tau_n(\pi) = \left(-\frac{1}{2}\right)^{n+1} \sum_{j=1}^n \zeta\left(n+1, \frac{\kappa_j(\pi)+1}{2}\right).$$

In addition for $n = 0$ there holds

$$(4.17) \quad \tau_0(\pi) = \frac{1}{2} \log Q(\pi) - \frac{N}{2} \log \pi + \frac{1}{2} \sum_{j=1}^N \frac{\Gamma'}{\Gamma} \left(\frac{\kappa_j(\pi) - 1}{2} \right).$$

where γ is Euler's constant.

Proof. — We recall that $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$, so that

$$(4.18) \quad \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s) = -\frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} \right).$$

The digamma function $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ has the partial fraction expansion

$$(4.19) \quad \psi(s+1) = \frac{\Gamma'}{\Gamma}(s+1) = -\gamma + \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{s+m} \right),$$

in which $\gamma \approx 0.5771$ is Euler's constant. For $\text{Re}(z) > -1$ we define the power series coefficients $\{\tilde{\tau}_n(z) : n \geq 0\}$ by

$$(4.20) \quad \sum_{n=0}^{\infty} \tilde{\tau}_n(z) s^n := \frac{\Gamma'}{\Gamma}(s+1+z).$$

Here

$$\tilde{\tau}_n(z) = \frac{1}{n!} \frac{d^n}{ds^n} \left(\frac{\Gamma'}{\Gamma}(s+1+z) \right) \Big|_{s=0}.$$

We have

$$(4.21) \quad \tilde{\tau}_0(z) = -\gamma + \sum_{m=1}^{\infty} \frac{z}{m(m+z)}$$

and, for $n \geq 1$,

$$(4.22) \quad \tilde{\tau}_n(z) = (-1)^{n+1} \sum_{m=1}^{\infty} \left(\frac{1}{m+z} \right)^{n+1},$$

which follows by differentiating the partial fraction expansion (4.19) at $s = z + 1$. This is essentially a special value of the Hurwitz zeta function, for $n \geq 1$,

$$(4.23) \quad \tilde{\tau}_n(z) = (-1)^{n+1} \zeta(n+1, z+1).$$

Comparing power series coefficients of the definition (4.3) with those of (4.18) yields, for $n \geq 1$,

$$(4.24) \quad \tau_n(\pi) = \frac{1}{2} \sum_{j=1}^N \tilde{\tau}_n \left(\frac{\kappa_j(\pi) - 1}{2} \right) \left(\frac{1}{2} \right)^n,$$

and using (4.23) then yields (4.16).

Finally, the case $n = 0$ follows from (4.3) for $n = 0$ using (4.21). □

Remark. — For the case π_{triv} on $GL(1)$ Lemma 4.5 yields

$$\begin{aligned} \tau_n(\pi_{\text{triv}}) &= \left(-\frac{1}{2}\right)^{n+1} \zeta\left(n+1, \frac{1}{2}\right) \\ &= \left(-\frac{1}{2}\right)^{n+1} \sum_{m=0}^{\infty} \frac{1}{\left(m+\frac{1}{2}\right)^{n+1}} \\ &= (-1)^{n+1} \sum_{m=0}^{\infty} \left(\frac{1}{2m+1}\right)^{n+1} \\ &= (-1)^{n+1} \left(1 - \frac{1}{2^{n+1}}\right) \zeta(n+1). \end{aligned}$$

With further work one can deduce

$$\tau_0(\pi_{\text{triv}}) = -\left(\frac{1}{2}\right) (\log(4\pi) + \gamma),$$

from which one can deduce (4.11).

5. Bounds for $S_{\infty}(n, \pi)$

In this and the next section we consider the order of growth of the terms $S_{\infty}(n, \pi)$ and $S_f(n, \pi)$, respectively. In this section we obtain an unconditional result for $S_{\infty}(n, \pi)$.

THEOREM 5.1. — *For any irreducible cuspidal (unitary) automorphic representation π on $GL(N)$ the quantities $S_{\infty}(n, \pi)$ are real-valued. There is a constant $K(\pi)$ such that for $n \geq K(\pi)$ there holds*

$$(5.1) \quad S_{\infty}(n, \pi) = \left(\frac{N}{2}\right) n \log n + C_1(\pi) n + O(N(K(\pi) + 1)).$$

Here

$$(5.2) \quad C_1(\pi) = \frac{N}{2}(\gamma - 1 - \log(2\pi)) + \frac{1}{2} \log Q(\pi),$$

where γ is Euler's constant. One can take

$$(5.3) \quad K(\pi) = \max_{1 \leq j \leq N} |\kappa_j(\pi)|^2,$$

and the implied constant in the O -notation is absolute.

Note that $C(\pi) = C_1(\pi^{\vee})$, since $Q(\pi) = Q(\pi^{\vee})$, and that $C_1(\pi)$ does not depend on the archimedean parameters $\{\kappa_j(\pi) : 1 \leq j \leq N\}$, though these values appear in the analysis. For the Li coefficients we have

$$(5.4) \quad C_1(\pi_{\text{triv}}) = \frac{1}{2}(\gamma - 1 - \log(2\pi)) \simeq -1.1303307.$$

using $Q(\pi_{\text{triv}}) = 1$.

To begin the proof we introduce the quantities

$$(5.5) \quad T(n, z) := \sum_{j=1}^n \binom{n}{j} \left(\frac{1}{2}\right)^j \tilde{\tau}_{j-1}(z).$$

The formula for $S_\infty(n, \pi)$ in Lemma 4.4 can be expressed in terms of these for various values of z . Indeed we obtain, for $n \geq 1$,

$$(5.6) \quad S_\infty(n, \pi) = \sum_{j=1}^N T\left(n, \frac{\kappa_j(\pi^\vee) - 1}{2}\right) + \binom{n}{1} \left(\frac{1}{2} \log Q(\pi) - \frac{N}{2} \log \pi\right),$$

using (4.24). We proceed to estimate an individual sum $T(n, z)$. The formulas for $\tilde{\tau}_j(z)$ give

$$T(n, z) = T_1(n, z) + T_2(n, z)$$

in which

$$(5.7) \quad T_1(n, z) = \frac{n \Gamma'}{2 \Gamma}(1+z) = \binom{n}{1} \frac{1}{2} \left(-\gamma + \sum_{m=1}^\infty \frac{z}{m(m+z)}\right).$$

and

$$(5.8) \quad T_2(n, z) := \sum_{j=2}^n \binom{n}{j} \frac{1}{2^j} \left((-1)^j \sum_{m=1}^\infty \frac{1}{(m+z)^j}\right).$$

The second sum converges absolutely and can be rearranged as

$$(5.9) \quad \begin{aligned} T_2(n, z) &= \sum_{m=1}^\infty \sum_{j=2}^n \binom{n}{j} \left(-\frac{1}{2}\right)^j \frac{1}{(m+z)^j} \\ &= \sum_{m=1}^\infty \left(\left(1 - \frac{1}{2m+2z}\right)^n - 1 + \frac{n}{2m+2z}\right). \end{aligned}$$

We divide this sum into two parts

$$T_2(n, z) = T_{20}(n, z) + T_{21}(n, z),$$

by splitting the summation range from $m = [1, n]$ and $m = [n + 1, \infty)$ respectively, and treat these in the following two lemmas.

LEMMA 5.2. — *For any complex number z with $\text{Re}(z) \geq -\frac{3}{4}$, the quantity*

$$(5.10) \quad T_{20}(n, z) = \sum_{m=1}^n \left(\left(1 - \frac{1}{2m+2z}\right)^n - 1 + \frac{n}{2m+2z}\right)$$

satisfies, for all $n \geq |z|^2$,

$$(5.11) \quad T_{20}(n, z) = \frac{1}{2}(n \log n) + \left(-\frac{1}{2} \frac{\Gamma'}{\Gamma}(1+z) + \beta_\infty + \frac{1}{\sqrt{e}} - 1 \right) n + O(|z| + 1).$$

in which

$$\beta_\infty = \int_1^\infty e^{-\frac{t}{2}} \frac{dt}{t} \simeq 0.559774.$$

Proof. — The sum (5.10) consists of three subsums, from each of its terms. The middle subsum in (5.10) gives $-n$. For the third subsum,

$$(5.12) \quad \begin{aligned} K(n, z) &= \sum_{m=1}^n \frac{n}{2m+2z} = \frac{n}{2} \sum_{m=1}^n \left(\frac{1}{m} - \frac{z}{m(m+z)} \right) \\ &= \frac{n}{2} \left(\log n + \gamma + O\left(\frac{1}{n}\right) \right) - \frac{n}{2} \left(\frac{\Gamma'}{\Gamma}(1+z) + \gamma + O\left(\frac{|z|+1}{n}\right) \right) \\ &= \frac{1}{2} n \log n - \frac{n}{2} \frac{\Gamma'}{\Gamma}(1+z) + O(|z| + 1), \end{aligned}$$

in which we used the partial fraction formula for the digamma function and a bound for the error in truncating it at the n -th term.

We now consider the first subsum in (5.10), call it

$$(5.13) \quad J(n, z) := \sum_{m=1}^n \left(1 - \frac{1}{2m+2z} \right)^n.$$

Setting $z = x + iy$, for real $t \geq 1$ we have

$$(5.14) \quad \left| 1 - \frac{1}{2t+2z} \right|^2 = 1 + \frac{1 - 4(t+x)}{4((t+x)^2 + y^2)},$$

The condition $\operatorname{Re}(z) \geq -\frac{3}{4}$ now implies that $|1 - \frac{1}{2t+2z}|^2 \leq 1$, so that each term above is $O(1)$. We approximate the sum by an integral, and assert that for $n \geq |z|^2$ there holds

$$(5.15) \quad J(n, z) = \int_1^n \left(1 - \frac{1}{2t+2z} \right)^n dt + O(|z| + 1)$$

with a constant independent of z . To establish this is we show that both sum and integral separately contribute $O(1)$ over the range $1 \leq t \leq n^{2/3}$, and then show on the remaining range $n^{2/3} \leq t \leq n$ that their difference

is $O(1)$. On the initial range we use $1 - x \leq e^{-x}$ for $0 \leq x \leq 1$ to get from (5.14) that, for $t \geq 3$,

$$\begin{aligned} \left| 1 - \frac{1}{2t + 2z} \right|^n &\leq \left(1 + \frac{1 - 4(t + x)}{4((t + x)^2 + y^2)} \right)^{\frac{n}{2}} \\ &\leq \exp \left(\frac{1 - 4(t + x)}{4((t + x)^2 + y^2)} \frac{n}{2} \right) \\ &\leq \exp \left(-\frac{(t + x)}{(t + x)^2 + y^2} \frac{n}{4} \right). \end{aligned}$$

Now the function $f(w) = w/(w^2 + y^2)$ is nonnegative on the positive real axis, increasing to a maximum at $w = y$ and decreasing monotonically thereafter. Applied to the exponential above, we find the the terms for integer $t = m$ with $y \leq m \leq n^{2/3}$ have negative exponent decreasing in absolute value, so the largest term occurs at the top endpoint, and is bounded by $\exp(-cn^{-1/3})$, so the sum over all these terms is $O(1)$. The first y terms each contribute $O(1)$, giving $O(|z| + 1)$ in all. This bounds the sum, and bounding the integral over this range is similar.

On the remaining range $n^{2/3} \leq m \leq 1$ with $m \leq t \leq m + 1$ we have

$$\begin{aligned} \left(1 - \frac{1}{2t + 2z} \right)^n &= \left(1 - \frac{1}{2m + 2z} \right)^n \left(\frac{1 - \frac{1}{2m+2z}}{1 - \frac{1}{2t+2z}} \right)^n \\ &= \left(1 - \frac{1}{2m + 2z} \right)^n \left(1 + O\left(\frac{1}{|2t + 2z|^2} \right) \right)^n \\ &= \left(1 - \frac{1}{2m + 2z} \right)^n \left(1 + O\left(\frac{n}{|2t + 2z|^2} \right) \right), \end{aligned}$$

where we used $|2t + 2z|^2 \geq 2n$. Thus, for $n \geq |z|^2$, the error between the sum and the integral is bounded by an absolute constant times

$$(5.16) \quad \int_{n^{2/3}}^n \left(1 - \frac{1}{2t + 2z} \right)^n \frac{n}{|2t + 2z|^2} dt \leq O(1).$$

This last estimate is obtained by observing that the integrand is maximized (in absolute value) at its top endpoint, where it is $O(1/n)$. We conclude that (5.15) holds.

We now integrate by parts on the right side of (5.15) to obtain

$$J(n, z) = J_1(n, z) - J_2(n, z) + O(|z| + 1),$$

in which

$$(5.17) \quad \begin{aligned} J_1(n, z) &:= t \left(1 - \frac{1}{2t + 2z} \right)^n \Big|_{t=1}^n \\ &= n \left(1 - \frac{1}{2n + 2z} \right)^n - \left(1 - \frac{1}{2 + 2z} \right)^n, \end{aligned}$$

and

$$(5.18) \quad J_2(n, z) = n \int_1^n \frac{2t}{(2t + 2z)^2} \left(1 - \frac{1}{2t + 2z} \right)^{n-1} dt.$$

The first term on the right side is estimated for $n > |z|^2$ by

$$(5.19) \quad \begin{aligned} J_1(n, z) &= n \exp\left(-\frac{n}{2n + 2z}\right) \left(1 + O\left(\frac{|z| + 1}{n}\right) \right) + O(1) \\ &= \frac{1}{\sqrt{e}} n + O(|z| + 1). \end{aligned}$$

For the remaining integral $J_2(n, z)$, we show that $\tilde{J}_2(n, z) := \frac{1}{n} J_2(n, z)$ approaches a limit as $n \rightarrow \infty$, using the fact that $(1 - \frac{1}{u})^u$ approaches e as $\operatorname{Re}(u) \rightarrow \infty$. We rescale it with the variable change $t = nu$ to obtain

$$\begin{aligned} \tilde{J}_2(n, z) &= \int_{\frac{1}{n}}^1 \frac{2nu}{(2nu + z)^2} \left(1 - \frac{1}{2(nu + z)} \right)^{n-1} n du \\ &= \frac{1}{2} \int_{\frac{1}{n}}^1 \frac{u}{(u + \frac{z}{2n})^2} \left[\left(1 - \frac{1}{2(nu + z)} \right)^{2(nu + z)} \right]^{\frac{n-1}{2(nu + z)}} du \\ &= \frac{1}{2} \int_{\frac{1}{n}}^1 \frac{1}{u} e^{-\frac{1}{2u}} du + O\left(\frac{|z| + 1}{n}\right) \\ &= \frac{1}{2} \int_0^1 \frac{1}{u} e^{-\frac{1}{2u}} du + O\left(\frac{|z| + 1}{n}\right). \end{aligned}$$

Now we set

$$\beta_\infty := \int_0^1 e^{-\frac{1}{2u}} \frac{du}{u} = \int_1^\infty e^{-\frac{t}{2}} \frac{dt}{t},$$

and we have obtained

$$J_2(n, z) = \frac{1}{2} \beta_\infty n + O(|z| + 1).$$

Combining all these estimates gives, for $n \geq |z|^2$,

$$\begin{aligned} T_{20}(n, z) &= J_1(n, z) - J_2(n, z) - n + K(n, z) + O(|z| + 1) \\ &= (\sqrt{e} - 1)n + \frac{1}{2}n \log n - \frac{n}{2} \frac{\Gamma'}{\Gamma}(1 + z) - \frac{1}{2}\beta_\infty n + O(|z| + 1) \\ &= \frac{1}{2}n \log n + \left(-\frac{1}{2} \frac{\Gamma'}{\Gamma}(1 + z) - \frac{1}{2}\beta_\infty + \frac{1}{\sqrt{e}} - 1 \right) n + O(|z| + 1), \end{aligned}$$

the desired estimate. □

LEMMA 5.3. — For $\text{Re}(z) \geq -1$ and $n \geq |z| + 2$, the quantity

$$T_{21}(n, z) = \sum_{m=n+1}^\infty \left(\left(1 - \frac{1}{2m+2z} \right)^n - 1 + \frac{n}{2m+2z} \right)$$

satisfies

$$(5.20) \quad T_{21}(n, z) = \left(\frac{1}{2} - \frac{1}{\sqrt{e}} + \frac{1}{2}\alpha_\infty \right) n + O(|z| + 1),$$

in which

$$(5.21) \quad \alpha_\infty = \int_0^1 \left(1 - e^{-\frac{1}{2}t} \right) \frac{dt}{t} \simeq 0.443842.$$

The implied O -constant is absolute.

Proof. — In what follows we assume $n \geq |z| + 2$. We approximate the sum by an integral

$$(5.22) \quad T_{21}(n, z) = \int_n^\infty \left(\left(1 - \frac{1}{2t+2z} \right)^n - 1 + \frac{n}{2t+2z} \right) dt + O(1).$$

To justify the error term, we observe that for $n \leq m \leq t \leq m + 1$ there holds

$$\begin{aligned} \left(1 - \frac{1}{2t+2z} \right)^n &= \left(1 - \frac{1}{2m+2z} + O\left(\frac{1}{|(2m+2z)(2t+2z)|} \right) \right)^n \\ &= \left(1 - \frac{1}{2m+2z} \right)^n \left(1 + O\left(\frac{n}{(t+|z|)^2} \right) \right) \end{aligned}$$

where we used $\text{Re}(z) \geq -1$ and $n \geq 2$. We also use

$$\left| \frac{n}{t+z} - \frac{n}{m+z} \right| \leq \left| \frac{n(m-t)}{(t+z)(m+z)} \right| \leq \frac{n}{(m+|z|-1)^2},$$

Viewing the sum as an integral of a step function, we have bounded the difference between the integrands at t by an absolute constant times $n/(t+|z|-1)^2$, and for $n \geq 3$ we have

$$\int_n^\infty \frac{n}{(t+|z|-1)^2} dt \leq n \int_{n-1}^\infty \frac{1}{t^2} dt = O(1).$$

and (5.22) follows.

Making the change of variable $u = 1/(2t + 2z)$ we obtain

$$(5.23) \quad T_{21}(n, z) = \frac{1}{2} \int_0^{\frac{1}{2n+2z}} [(1-u)^n - 1 + nu] \frac{du}{u^2} + O(1).$$

This integral is a contour integral (since z is complex) but the answer is independent of the contour, since the integrand is analytic. An integration by parts yields $T_{21}(n, z) = K_1(n, z) - K_2(n, z) + O(1)$ in which

$$(5.24) \quad \begin{aligned} K_1(n, z) &= -\frac{1}{2u} [(1-u)^n - 1 + nu] \Big|_{u=0}^{\frac{1}{2n+2z}} \\ &= -(n+z) \left(\left(1 - \frac{1}{2n+2z}\right)^n - 1 + \frac{n}{2n+2z} \right), \end{aligned}$$

and

$$(5.25) \quad \begin{aligned} K_2(n, z) &:= \frac{1}{2} \int_0^{\frac{1}{2n+2z}} [-n(1-u)^{n-1} + n] \frac{du}{-u} \\ &= \frac{n}{2} \int_0^{\frac{1}{2n+2z}} [(1-u)^{n-1} - 1] \frac{du}{u}. \end{aligned}$$

To estimate $K_1(n, z)$ we write $z = x + iy$, with $x \geq -1$, to obtain

$$1 - \frac{1}{2n+2z} = \left(1 - \frac{2n+2x}{4(n+x)^2 + 4y^2}\right) + i \frac{2y}{4(n+x)^2 + 4y^2} = re^{i\theta},$$

in which $r = 1 - \frac{1}{2n} + O(\frac{1}{n^2})$ and $\theta = O(\frac{|z|+1}{n^2})$. It follows that $r^n e^{in\theta} = e^{-\frac{1}{2}} + O(\frac{|z|+1}{n})$, and we obtain

$$(5.26) \quad \begin{aligned} K_1(n, z) &= -n \left[\left(1 - \frac{1}{2n+2z}\right)^n - \frac{1}{2} \right] + O(|z| + 1) \\ &= -n \left[r^n e^{in\theta} - \frac{1}{2} \right] + O(|z| + 1) \\ &= n \left(\frac{1}{2} - \frac{1}{\sqrt{e}} + O\left(\frac{|z|+1}{n}\right) \right) + O(|z| + 1). \end{aligned}$$

Expanding the integral $K_2(n, z)$ in powers of u yields

$$(5.27) \quad \begin{aligned} K_2(n, z) &= \frac{n}{2} \left(\sum_{j=1}^n \int_0^{\frac{1}{2n+2z}} (-1)^j \binom{n}{j} u^{j-1} du \right) \\ &= \frac{n}{2} \left(\sum_{j=1}^n \frac{(-1)^j}{j} \binom{n}{j} \left(\frac{1}{2n+2z}\right)^j \right). \end{aligned}$$

We set

$$\alpha_n(z) = \sum_{j=1}^n \frac{(-1)^j}{j} \binom{n}{j} \left(\frac{1}{2n+2z}\right)^j,$$

and

$$(5.28) \quad \alpha_\infty := \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \frac{1}{j!2^j} = \int_0^{\frac{1}{2}} (1 - e^{-v}) \frac{dv}{v} = \int_0^1 (1 - e^{-\frac{1}{2}t}) \frac{dt}{t}.$$

We assert that for $\operatorname{Re}(z) \geq -1, n \geq 2$,

$$(5.29) \quad |\alpha_n(z) + \alpha_\infty| = O\left(\frac{|z|+1}{n}\right).$$

where the constant in the O -notation is independent of z . Indeed we note that

$$\binom{n}{j} \left(\frac{1}{2n+2z}\right)^j = \frac{1}{j!2^j} \binom{n}{n+z} \binom{n-1}{n+z} \cdots \binom{n-j+1}{n+z},$$

which since $\operatorname{Re}(z) \geq -1$ yields

$$\left| \binom{n}{j} \left(\frac{1}{2n+2z}\right)^j \right| \leq \frac{2}{j!2^j}.$$

and also for $0 \leq j \leq 2 \log n$ that

$$\left| \binom{n}{j} \left(\frac{1}{2n+2z}\right)^j - \frac{1}{j!2^j} \right| = O\left(\frac{|z|+1}{n}\right).$$

Combining these estimates with

$$\left| \alpha_\infty + \sum_{j=1}^{2 \log n} \frac{(-1)^j}{j} \frac{1}{j!2^j} \right| \leq O\left(\frac{1}{n}\right),$$

yields (5.29), so that $K_2(n, z) = -\frac{1}{2}\alpha_\infty n + O(|z|+1)$.

Combining all the estimates above gives

$$\begin{aligned} T_{21}(n, z) &= K_1(n, z) - K_2(n, z) + O(1) \\ &= \left(\frac{1}{2} - \frac{1}{\sqrt{e}} + \frac{1}{2}\alpha_\infty\right)n + O(|z|+1), \end{aligned}$$

the desired estimate. □

Proof of Theorem 5.1. — Lemma 4.4 gives

$$\begin{aligned}
 S_\infty(n, \pi) &= \sum_{j=1}^n \binom{n}{j} \tau_{j-1}(\pi) \\
 &= \left(\sum_{k=1}^N \sum_{j=1}^n \binom{n}{j} (-1)^j \tilde{\tau}_{j-1} \left(\frac{\kappa_k(\pi) - 1}{2} \right) \left(\frac{1}{2} \right)^j \right) \\
 &\quad + \binom{n}{1} \left(\frac{1}{2} \log Q(\pi) - \frac{N}{2} \log \pi \right) \\
 &= \left(\sum_{k=1}^N T \left(n, \frac{\kappa_k(\pi) - 1}{2} \right) \right) \\
 &\quad + \left(\frac{1}{2} \log Q(\pi) - \frac{N}{2} \log \pi \right) n \\
 &= \left(\frac{1}{2} \log Q(\pi) - \frac{N}{2} \log \pi + \frac{1}{2} \sum_{k=1}^N \frac{\Gamma'}{\Gamma} \left(\frac{\kappa_k(\pi) + 1}{2} \right) \right) n \\
 (5.30) \quad &\quad + \sum_{k=1}^N T_2 \left(n, \frac{\kappa_k(\pi) - 1}{2} \right).
 \end{aligned}$$

where we made use of the decomposition $T(n, z) = T_1(n, z) + T_2(n, z)$ and the formula (5.7) for $T_1(n, z)$.

Now we suppose that $n \geq K(\pi) := \max\{|\kappa_k(\pi)|^2 : 1 \leq k \leq N\}$ and note by Theorem 2.1(2) that each $\operatorname{Re}((\kappa_k(\pi) + 1)/2) > -3/4$, so that we can apply Lemma 5.2 and 5.3. We obtain

$$\begin{aligned}
 T_2 \left(n, \frac{\kappa_k(\pi) - 1}{2} \right) &= \frac{1}{2} n \log n + \left(-\frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\kappa_k(\pi) + 1}{2} \right) + \frac{1}{\sqrt{e}} - 1 \right) n \\
 &\quad + \left(\frac{1}{2} - \frac{1}{\sqrt{e}} + \frac{1}{2} \alpha_\infty - \frac{1}{2} \beta_\infty \right) n + O(K(\pi) + 1) \\
 &= \frac{1}{2} n \log n + \left(-\frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\kappa_k(\pi) + 1}{2} \right) + \frac{1}{2} (\alpha_\infty - \beta_\infty - 1) \right) n \\
 &\quad + O(K(\pi) + 1).
 \end{aligned}$$

Substituting this in the previous formula yields

$$\begin{aligned}
 (5.31) \quad S_\infty(n, \pi) &= \left(\frac{N}{2} \right) n \log n \\
 &\quad + \left(\frac{N}{2} (\alpha_\infty - \beta_\infty - 1 - \log \pi) + \frac{1}{2} \log Q(\pi) \right) n + O(N(K(\pi) + 1)).
 \end{aligned}$$

To complete the proof we must establish the identity⁽³⁾

$$(5.32) \quad \alpha_\infty - \beta_\infty = \gamma - \log 2,$$

where γ is Euler's constant. We start from

$$\alpha_\infty - \beta_\infty = \int_0^{\frac{1}{2}} (1 - e^{-t}) \frac{dt}{t} - \int_{\frac{1}{2}}^\infty e^{-t} \frac{dt}{t},$$

The assertion (5.32) is a special case of an identity valid for each $w > 0$, that

$$(5.33) \quad \int_0^w (1 - e^{-t}) \frac{dt}{t} - \int_w^\infty e^{-t} \frac{dt}{t} = \gamma + \log w,$$

on taking $w = \frac{1}{2}$. The reviewer observes the following proof of (5.33). Denoting the left side of this equation by $S(w)$, an integration by parts (taking $dv = \frac{dt}{t}$) yields

$$(5.34) \quad S(w) = \log w - \int_0^\infty e^{-t} \log t dt = \log w - \Gamma'(1),$$

and the result follows using $\Gamma'(1) = -\gamma$. The special case $w = 1$,

$$(5.35) \quad \gamma = \int_0^1 (1 - e^{-t} - e^{-\frac{1}{t}}) \frac{dt}{t},$$

was established by Barnes [1], by another method. □

Remark. — Computational evidence of Maslanka (private communication) indicates that $S_\infty(n, \pi)$ has a full asymptotic expansion in inverse powers n^{-k} with $k \geq 0$. The further coefficients of such expansions will likely depend on the archimedean parameters $\{\kappa_j(\pi) : 1 \leq j \leq N\}$, unlike the first two coefficients appearing in Theorem 5.1.

6. Bounds for $S_f(n, \pi)$

We obtain a bound for $S_f(n, \pi)$ in terms of the *incomplete Li coefficient to height T* , defined by

$$(6.1) \quad \lambda_n(T, \pi) := \sum_{\substack{\rho \in Z(\pi) \\ |\text{Im}(\rho)| \leq T}} 1 - \left(1 - \frac{1}{\rho}\right)^n,$$

where T is a cutoff parameter.

⁽³⁾This identity was found by comparison of the author's original formula (5.31) with an asymptotic formula for the Li coefficients (under RH) given by Voros [43, Eq. (11)].

THEOREM 6.1. — *For any irreducible cuspidal (unitary) automorphic representation on $GL(N)$ there holds*

$$(6.2) \quad S_f(n, \pi) = \lambda_n(\sqrt{n}, \pi^\vee) + O(\sqrt{n} \log n),$$

in which the implied constant in the O -notation depends on π . If the Riemann hypothesis holds for $L(s, \pi)$ then

$$(6.3) \quad \lambda_n(\sqrt{n}, \pi^\vee) = O(\sqrt{n} \log n).$$

Proof. — We use a contour integral argument. We introduce the kernel function

$$(6.4) \quad k_n(s) := \left(1 + \frac{1}{s}\right)^n - 1 = \sum_{j=1}^n \binom{n}{j} \left(\frac{1}{s}\right)^j,$$

If C_1 is a contour enclosing the point $s = 0$ counterclockwise on a circle of small enough positive radius R , the residue theorem gives

$$(6.5) \quad \frac{1}{2\pi i} \int_{C_1} k_n(s) \left(-\frac{L'}{L}(s+1, \pi)\right) ds = \sum_{j=1}^n \binom{n}{j} \eta_{j-1} = S_f(n, \pi).$$

The residue comes entirely from the singularity at $s = 0$, as no other singularities lie inside the contour.

We deform the contour to the counterclockwise oriented rectangular contour $C_2(n)$ consisting of vertical lines with real part $\text{Re}(s) = \sigma_0$ and $\text{Re}(s) = \sigma_1$ where we will choose $-3 < \sigma_0 < -2$, and $\sigma_1 = 2\sqrt{n}$, and horizontal lines at $\text{Im}(s) = \pm T$, where we will choose $T = \sqrt{n} + \varepsilon_n$, for some $0 < \varepsilon_n < 1$. The integrand has simple poles at the zeros of $L(s, \pi)$, and some of them will now lie inside the contour. The residue theorem gives

$$\begin{aligned} I_2(n) &:= \int_{C_2(n)} k_n(s) \left(-\frac{L'}{L}(s+1, \pi)\right) ds \\ &= S_f(n, \pi) + \sum_{\substack{\rho \in Z(\pi) \\ |\text{Im}(\rho)| < T}} \left(1 + \frac{1}{\rho-1}\right)^n - 1 + I_2^{\text{triv}}(n), \end{aligned}$$

in which $I_2^{\text{triv}}(n)$ evaluates the residues coming from the trivial zeros of $L(s, \pi)$ which will satisfy $-3 \leq \text{Re}(s) < 1/2$. The trivial zeros are associated to the archimedean factors and their location is dictated by the values $\kappa_j(\pi)$, which by Theorem 2.1 satisfy $\text{Re}(\kappa_j(\pi)) > -1/2$, so that all trivial zeros satisfy $\text{Re}(\rho) < \frac{1}{2}$. We have

$$(6.6) \quad \left|1 + \frac{1}{z-1}\right| = \left|\frac{z}{z-1}\right| < 1 \text{ when } \text{Re}(z) < \frac{1}{2},$$

and using this we conclude that the residue at each trivial zero is $O(1)$. Since the trivial zeros fall in N arithmetic progressions with spacing 2, there are $O(N)$ such zeros in the interval $-3 \leq \operatorname{Re}(s) < 1/2$ we conclude that the trivial zero contribution is $I_2(n, \text{triv}) = O(1)$, with O -constant depending on π .

Using the symmetry $\rho \mapsto 1 - \bar{\rho}$ of $Z(\pi)$, which does not change $|\operatorname{Im}(\rho)|$, we can rewrite the sum over $Z(\pi)$ in terms of

$$\left(\frac{1 - \bar{\rho}}{-\bar{\rho}}\right)^n - 1 = \left(\frac{\bar{\rho} - 1}{\bar{\rho}}\right)^n - 1.$$

We observe that $\bar{\rho} \in Z(\pi^\vee)$ so that (6.5) can be rewritten

$$(6.7) \quad I_2(n) = S_f(n, \pi) - \lambda_n(T, \pi^\vee) + O(1).$$

The main part of the theorem will be to establish

$$I_2(n) = O(\sqrt{n}).$$

The theorem will follow from the two assertions that $I_2(n) = O(\sqrt{n})$ and

$$|\lambda_n(\sqrt{n}, \pi^\vee) - \lambda_n(T, \pi^\vee)| = O(\log n).$$

The second assertion follows from the observation that $|T - \sqrt{n}| < 1$, that there are $O(\log n)$ zeros in an interval of length one at this height (by (2.11)), and that for each zero $\rho = \beta + i\gamma$ with $\sqrt{n} \leq |\operatorname{Im}(\rho)| < \sqrt{n} + 1$ there holds

$$\begin{aligned} \left|\left(\frac{\rho - 1}{\rho}\right)^n\right| &= \left|1 - \frac{\beta - i\gamma}{\beta^2 + \gamma^2}\right|^n \\ &\leq \left|\left(1 - \frac{\beta}{\beta^2 + \gamma^2}\right)^2 + \left(\frac{\gamma}{\beta^2 + \gamma^2}\right)^2\right|^{n/2} \\ &\leq \left|1 + \frac{1}{n}\right|^{n/2} \leq 2. \end{aligned}$$

It remains to bound $I_2(n)$. We will use the following estimate for $(L'/L)(s, \pi)$ in a region including the critical strip.

LEMMA 6.2. — For $-4 \leq \operatorname{Re}(s) < 4$ and $s = \sigma + it$ there holds

$$(6.8) \quad \frac{L'}{L}(s, \pi) = \sum_{\{\rho: |\operatorname{Im}(\rho - s)| < 1\}} \frac{1}{s - \rho} + O\left(\log \mathfrak{q}(\pi, s) \frac{L'}{L}(2, \pi)\right),$$

in which $\mathfrak{q}(s, \pi) = Q(\pi) \prod_{j=1}^N (|s + \kappa_j(\pi)| + 3)$ is the analytic conductor of π , and the constant in the O -notation is absolute.

Proof. — The classical proof for Dirichlet L -functions [12, Chap. 15, p. 102] generalizes to give (6.8), following Chapter 5 of [29]. If we know the Ramanujan conjecture holds for $L(s, \pi)$, then we can further control the size of $(L'/L)(2, \pi)$ in the remainder term. \square

We now choose the parameters σ_0 and T appropriately to avoid poles of the integrand. Since the trivial zeros of $L(s, \pi)$ fall in N arithmetic progressions with spacing 2, we may choose σ_0 so that the contour avoids coming within $1/2N$ of any trivial zero. Similarly we can choose $T = \sqrt{n} + \varepsilon_n$ with $0 \leq \varepsilon_n \leq 1$ so that the horizontal lines do not approach closer than $O(\log n)$ to any zero of $L(s, \pi)$. It then follows from Lemma 6.2 that on the horizontal line in the interval $-3 \leq \operatorname{Re}(s) \leq 4$ we have

$$(6.9) \quad \left| \frac{L'}{L}(s+1, \pi) \right| = O((\log T)^2).$$

The Euler product for $L(s, \pi)$ converges absolutely for $\operatorname{Re}(s) > 1$ (by Theorem 2.1(1)) hence the Dirichlet series for $(L'/L)(s, \pi)$ converges absolutely for $\operatorname{Re}(s) > 1$. More precisely if $(L'/L)(s, \pi) = \sum_{m=2}^{\infty} b(m, \pi)m^{-s}$ then for $\sigma > 1$,

$$\left| \frac{L'}{L} \right|(\sigma) = \sum_{m=2}^{\infty} |b(m, \pi)| m^{-\sigma} < \infty.$$

Since the Dirichlet series for $(L'/L)(s, \pi)$ has no constant term, we obtain for $\operatorname{Re}(s) = \sigma > 2$ the bound

$$(6.10) \quad \left| \frac{L'}{L}(s, \pi) \right| \leq \left| \frac{L'}{L}(\sigma, \pi) \right| \leq 2^{-(\sigma-2)} \left| \frac{L'}{L}(2, \pi) \right|.$$

We now consider the integral $I_2(n)$ on the vertical segment (I) having $\operatorname{Re}(s) = \sigma_1 = 2\sqrt{n}$, call it $I_2^{(I)}(n)$. We have

$$\left| \left(1 - \frac{1}{s}\right)^n - 1 \right| \leq \left(1 + \frac{1}{\sigma_1}\right)^n + 1 \leq \left(1 + \frac{1}{2\sqrt{n}}\right)^n \leq \exp\left(\frac{1}{2}\sqrt{n}\right) < 2^{\sqrt{n}}.$$

Now the estimate (6.10) gives on (I) that

$$\left| \frac{L'}{L}(s, \pi) \right| \leq C_0 2^{-2(\sqrt{n}+2)}.$$

so that the integrand is bounded above by $C_1 2^{-\sqrt{n}}$. The length of the contour is $O(n/\log n)$ so we obtain the estimate $|I_2^{(I)}(n)| = o(1)$.

We next consider the integral $I_2(n)$ on the two horizontal segments, call them (II) and (IV). It suffices to bound $I_2^{(II)}(n)$, the treatment for (IV)

being identical. Let $s = \sigma + it$ be a point on (II). We have $T \geq \sqrt{n}$ so that

$$\begin{aligned}
 \left| 1 + \frac{1}{s} \right| &= \left| \left(1 + \frac{\sigma}{\sigma^2 + T^2} \right) + \frac{iT}{\sigma^2 + T^2} \right| \\
 &\leq \left(\left(1 + \frac{\sigma}{\sigma^2 + T^2} \right)^2 + \frac{1}{\sigma^2 + T^2} \right)^{\frac{1}{2}} \\
 (6.11) \qquad &\leq 1 + \frac{\sigma + 1}{\sigma^2 + T^2}.
 \end{aligned}$$

By hypothesis $T^2 \geq n$ so for $-3 \leq \sigma \leq 3$ we have

$$|k_n(s)| \leq \left(1 + \frac{4}{16 + n} \right)^n + 1 \leq e^4 + 1 = O(1)$$

On this interval Lemma 6.2 gives

$$\left| \frac{L'}{L}(s, \pi) \right| = O((\log T)^2) = O((\log n)^2)$$

since we have chosen the ordinate T to stay away from zeros of $L(s, \pi)$. Now we step across the interval (II) towards the right, in segments of length 1, starting from $\sigma = 3$. At the initial point we have $|(L'/L)(s, \pi)| = O(1)$ since we are in the absolute convergence region, and the estimate (6.10) gains a factor of 2 going one unit to the right. In comparison

$$\begin{aligned}
 \left| \frac{k_n(s+1)+1}{k_n(s)+1} \right| &= \left| \frac{1 + \frac{1}{\sigma+1+iT}}{1 + \frac{1}{\sigma+iT}} \right|^n \\
 &\leq \left| 1 + \left(\frac{1}{\sigma+1+iT} - \frac{1}{\sigma+iT} \right) \right|^n \\
 &\leq \left(1 + \frac{1}{|\sigma+1+iT| |\sigma+iT|} \right)^n \\
 (6.12) \qquad &\leq \left(1 + \frac{1}{T^2} \right)^n \leq e.
 \end{aligned}$$

We obtain an upper bound for $|k_n(s)(L'/L)(s, \pi)|$ that decreases geometrically at each step, and after $O(\log n)$ steps it becomes $O(1)$ so we obtain the upper bound

$$|I_2^{(II)}(n)| = O(\sqrt{n} + (\log n)^3) = O(\sqrt{n}).$$

A similar bound holds for $|I_2^{(IV)}(n)|$.

For the remaining vertical segment (III) with $\text{Re}(s) = \sigma_0$, we have that the kernel function $|k_n(s)| = O(1)$ on the line segment (III) by (6.6), and

$|(L'/L)(s, \pi)| = O(\log(|s| + 2))$ using Lemma 6.2. Since the segment (III) has length $O(\sqrt{n})$ we obtain the bound

$$(6.13) \quad \left| I_2^{(III)}(n) \right| = O(\sqrt{n} \log n).$$

Using this bound suffices to establish (6.2), with the remainder term $O(\sqrt{n} \log n)$.

Totalling all these bounds above gives

$$S_f(n, \pi) = \lambda_n(T, \pi^\vee) + O(\sqrt{n} \log n),$$

with $T = \sqrt{n} + \varepsilon_n$.

Now suppose that the Riemann hypothesis holds for $\xi(s, \pi)$. Then we have

$$\left| 1 - \frac{1}{\rho - 1} \right| = \left| \frac{\rho}{\rho - 1} \right| = \left| \frac{-\frac{1}{2} + i\gamma}{\frac{1}{2} + i\gamma} \right| = 1$$

It follows that each zero contributes a term of absolute value at most 2 to the incomplete Li coefficient $\lambda_n(T, \pi)$, and we obtain

$$(6.14) \quad \lambda_n(T, \pi) = O(T \log T + 1),$$

on using the zero density estimate in Theorem 2.1(4). Here the constant in the O -notation depends on π . In particular

$$\lambda_n(\sqrt{n}, \pi^\vee) = \overline{\lambda_n(\sqrt{n}, \pi)} = O(\sqrt{n} \log n),$$

as required. □

7. Interpolation function for Li coefficients

We construct an entire function that interpolates the Li coefficients at integer values.

THEOREM 7.1. — (1) *For any irreducible cuspidal unitary automorphic representation π over $GL(N)$ over \mathbb{Q} , there exists an entire function $F_\pi(z)$ of order one and exponential type having the two properties:*

- (i) *It interpolates the generalized Li coefficients $\lambda_n(\pi)$ at integer values, i.e.*

$$(7.1) \quad F_\pi(n) = \lambda_n(\pi), \quad n \in \mathbb{Z}.$$

- (ii) *It is real-valued on the imaginary axis, with*

$$(7.2) \quad F_\pi(-\bar{z}) = \overline{F_\pi(z)}.$$

(2) If the Riemann hypothesis holds for $L(s, \pi)$, then there exists a unique function $F_\pi(z)$ of exponential type at most π that satisfies (i), and is such that

$$F_\pi^*(z) := F_\pi(z) - e\left(\frac{1}{2}, \pi\right) (1 - \cos \pi z)$$

is of exponential type strictly less than π . Here $e(\frac{1}{2}, \pi)$ is the order of the zero of $\xi(s, \pi)$ at $s = \frac{1}{2}$. On the real axis, $F_\pi(z)$ satisfies the bound

$$(7.3) \quad |F_\pi(x)| \leq C(|x| + 2) \log(|x| + 2),$$

for some constant C depending on π .

Proof. — In this proof we regard π as fixed, with zeros ρ drawn from the multiset $Z(\pi)$.

(1) We define

$$(7.4) \quad \left(1 - \frac{1}{\rho}\right)^z := \exp\left(z \log\left(1 - \frac{1}{\rho}\right)\right),$$

in which we take the principal branch of the logarithm, with argument $-\pi/2 < \text{Im}(\log z) \leq \pi/2$.

We also have the binomial expansion

$$(7.5) \quad \left(1 - \frac{1}{\rho}\right)^z = \sum_{j=0}^{\infty} (-1)^j \binom{z}{j} \frac{1}{\rho^j}.$$

When $|\rho| > 1$ this expansion converges absolutely for all $z \in \mathbb{C}$, since the terms in the series eventually decay geometrically in absolute value.

We now let $R > 0$ and define

$$(7.6) \quad F_R(z) = \sum_{|\rho| \leq R} \left(1 - \left(1 - \frac{1}{\rho}\right)^z\right) + (-z) \left(\sum'_{|\rho| > R} \frac{1}{\rho}\right) + \sum_{j=2}^{\infty} (-1)^j \binom{z}{j} \left(\sum_{|\rho| > R} \frac{1}{\rho^j}\right).$$

Here

$$(7.7) \quad \sum'_{|\rho| > R} \frac{1}{\rho} := \lim_{T \rightarrow \infty} \sum_{R < |\rho| \leq T} \frac{1}{\rho},$$

and this sum is $*$ -convergent by Lemma 2.2. For notational convenience we define the *partial power sums*

$$(7.8) \quad \sigma_j(R, \pi) := \sum_{|\rho| > R} \frac{1}{\rho^j}.$$

These sums converge absolutely for $j \geq 2$ and are $*$ -convergent for $j = 1$, by Lemma 2.2. We can rewrite

$$(7.9) \quad F_R(z) = \sum_{|\rho| \leq R} \left(1 - \frac{1}{\rho}\right)^z - \sigma_1(R, \pi)z + \sum_{j=2}^{\infty} (-1)^j \binom{z}{j} \sigma_j(R, \pi).$$

The zero-counting estimate in Theorem 2.1 (4) implies there are at most $O(\log T)$ zeros in an interval of length 1 at height T , where the O -constant depends on the representation π . Therefore we obtain, for $j \geq 2$

$$(7.10) \quad \begin{aligned} |\sigma_j(R, \pi)| &\leq \sum_{m=R}^{\infty} \#\{m \leq |\rho| < m+1\} \frac{1}{m^j} \\ &\leq \sum_{m=R}^{\infty} C \frac{\log m}{m^j} = O\left((\log R)^2 \frac{1}{R^{j-1}}\right). \end{aligned}$$

where the O -constant depends on π . For the term $j = 1$ we similarly obtain a $*$ -convergence estimate

$$(7.11) \quad |\sigma_1(R, \pi)| = O\left(\frac{(\log R)^2}{R}\right)$$

by partial summation, cancelling the zeros at height T against those at height $-T$. We therefore have, on the disk $|z| \leq \sqrt{R}$, that

$$(7.12) \quad \begin{aligned} \left| \binom{z}{j} \sigma_j(R, \pi) \right| &\leq \left| \binom{z}{j} \right| |\sigma_j(R, \pi)| \\ &\leq C_1 \left(\prod_{k=1}^j \frac{\sqrt{R} + k}{k} \right) (\log R)^2 R^{-j+1} \\ &\leq C_1 \left(\prod_{k=1}^j \frac{1 + k/\sqrt{R}}{k} \right) (\log R)^2 R^{-\frac{j}{2}+1} \\ &\leq C_2 (\log R)^2 R^{-\frac{j}{2}+1}. \end{aligned}$$

On summing over $j \geq 1$ it follows that the terms on the right side of (7.9) converges uniformly on the entire disk $|z| < \sqrt{R}$ to an analytic function $F_R(z)$.

We also obtain from this a bound on the maximum modulus of $F_R(z)$ on $|z| \leq R$. Set

$$(7.13) \quad \beta(\pi) := \sup_{\rho \in Z(\pi)} \left| \log \left(1 - \frac{1}{\rho}\right) \right|,$$

and observe $\beta(\pi)$ is finite because $1 \notin Z(\pi)$, and $|1 - 1/\rho| \rightarrow 0$ as $|\rho| \rightarrow \infty$. then we obtain, for $|z| \leq \sqrt{R} \geq 2$, that

$$\begin{aligned}
 |F_R(z)| &\leq \sum_{|\rho| \leq R} e^{\sqrt{R} \log(1-1/\rho)} + O\left((\log R)^2 \frac{1}{1 - R^{-1/2}}\right) \\
 (7.14) \qquad &= C_3 (R \log R) e^{\beta(\pi)\sqrt{R}}
 \end{aligned}$$

in which the O -constant depends on π .

We assert that all $F_R(z)$ represent the same analytic function. If $R_1 < R_2$ and we consider the functions on the domain $|z| < \sqrt{R_1}$ we can expand the individual terms $(1 - 1/\rho)^z$ with $R_1 < |\rho| \leq R_2$ in $F_{R_2}(z)$ and combine them term-by-term with the binomial expression, using

$$\sigma_j(R_1, \pi) = \sigma_j(R_2, \pi) + \sum_{R_1 < |\rho| \leq R_2} \frac{1}{\rho^j}.$$

Since all sums converge absolutely and uniformly on $|z| < \sqrt{R_1}$, we conclude that $F_{R_2}(z)$ agrees with $F_{R_1}(z)$ there. Letting $R \rightarrow \infty$, we obtain an entire function $F_\pi(z)$ such that

$$F_\pi(z) = F_R(z) \text{ when } |z| \leq \sqrt{R}.$$

The maximum modulus bound (7.13) now establishes that $F_\pi(z)$ is an entire function of order one and exponential type at most $\beta(\pi)$.

Evaluating at integer points $n \in \mathbb{Z}$ and choosing $R = 2n^2$ gives

$$\begin{aligned}
 F_\pi(n) &= \sum_{|\rho| \leq R} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right) - n\sigma_1(R, \pi) + \sum_{j=2}^{\infty} (-1)^j \binom{n}{j} \sigma_j(R, \pi) \\
 (7.15) \qquad &= \sum_{|\rho| \leq R} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right) - n\sigma_1(R, \pi) + \sum_{j=2}^n (-1)^j \binom{n}{j} \sigma_j(R, \pi).
 \end{aligned}$$

The last term is a finite sum, so we can expand all the terms in the first sum $|\rho| < R$ and rearrange to obtain

$$(7.16) \qquad F_\pi(n) = -n\sigma_1(\pi) + \sum_{j=2}^n (-1)^j \binom{n}{j} \sigma_j(\pi) = \lambda_n(\pi)$$

using Lemma 2.3 for the last equality. This proves property (i).

The symmetry property of zeros under $\rho \mapsto 1 - \bar{\rho}$ implies that $F_R(z)$ is real on the imaginary axis, for $|z| < R$. It follows that $F_\pi(z)$ is real on the imaginary axis, and reflection principle then gives the symmetry

$$(7.17) \qquad F_\pi(-x + iy) = \overline{F_\pi(x + iy)} \text{ when } |z| < R.$$

which is property (ii).

We have constructed a specific function $F_\pi(z)$ which has properties (i), (ii). If $\beta(\pi) \geq \pi$, then properties (i), (ii) do not determine the function uniquely because we can add a real multiple of $z \sin \pi z$ to the function while preserving properties (i) and (ii).

(2) Now suppose that the Riemann hypothesis holds for $\xi(s, \pi)$. Then we can write

$$(7.18) \quad \rho = \frac{1}{2} + i\gamma = |\rho|e^{i\varphi_\rho}$$

with $-\pi < \phi_\rho \leq \pi$, and

$$(7.19) \quad \varphi_\rho := \tan^{-1}(2\gamma),$$

so that $\phi_\rho \geq 0$ when $\gamma \geq 0$, and $\varphi_\rho < 0$ otherwise. We have

$$(7.20) \quad 1 - \frac{1}{\rho} = -\frac{\frac{1}{2} - i\gamma}{\frac{1}{2} + i\gamma} = e^{i(-2\varphi_\rho \pm \pi)}$$

with the sign \pm chosen so that the angle falls in $(-\pi, \pi]$. Let us now set

$$(7.21) \quad \varphi_*(\rho) := \begin{cases} -2\varphi_\rho + \pi & \text{if } \varphi_\rho > 0 \leftrightarrow \gamma > 0 \\ -2\varphi_\rho - \pi & \text{if } \varphi_\rho < 0 \leftrightarrow \gamma < 0. \end{cases}$$

The case $\varphi_\rho = 0$, corresponding to $s = \frac{1}{2}$, is problematic, since the sign $\pm\pi$ must have a discontinuity there. For the moment we use the convention that the $+$ sign is chosen. Consequently

$$(7.22) \quad \left| \log \left(1 - \frac{1}{\rho} \right) \right| = |i\varphi_*(\rho)| \leq \pi$$

and equality can occur only if $\varphi_*(\rho) = \pi$, which requires that $\varphi_\rho = 0$ and $\rho = \frac{1}{2}$. It follows that if the Riemann hypothesis holds for $Z(\pi)$ then $\beta(\pi) \leq \pi$, and if there is no zero at $s = \frac{1}{2}$ then $\beta(\pi) < \pi$.

For uniqueness, suppose first that $Z(\pi)$ includes no zero at $s = \frac{1}{2}$. It is well known (from the sampling theorem) that an entire function of order one and exponential type less than π is completely specified by its values at integer points. Indeed, the difference of two such functions would be an entire function of order one and exponential type τ less than π vanishing at all integer points. Since a function of exponential type τ has at most $(2\tau/\pi + o(1))R$ zeros in a disk of radius R , as $R \rightarrow \infty$, it must be identically zero. In this case property (i), together with a growth bound on the function, characterizes the interpolation function uniquely. In the cases where $Z(\pi)$ has zeros at $s = \frac{1}{2}$, we use a convention to handle their contribution

and define $F_\pi(z)$ uniquely. If it has exactly k zeros there, we define their contribution to be

$$f_{\frac{1}{2}}(z) := \frac{k}{2} (e^{-i\pi z} + e^{i\pi z}).$$

This function is real on the imaginary axis, and has $f_{\frac{1}{2}}(z) = f_{\frac{1}{2}}(-z)$, and it corresponds to assigning half of the zeros at $s = \frac{1}{2}$ the argument $-\pi i$ and the other half argument πi . Subtracting this contribution leaves an interpolation function having exponential type strictly less than π , which is then uniquely specified by condition (i).

It remains to bound the size of the interpolation function $F_\pi(z)$ on the real axis. Under the Riemann hypothesis, the bounds on the Li coefficients in §5 and §6 lead one to expect an upper bound of shape $O((|x| + 2) \log(|x| + 2))$, and we now show this holds. We have

$$(7.23) \quad F_\pi(x) = \sum_{\rho} \left(1 - \left(1 - \frac{1}{\rho} \right)^x \right)$$

and we bound each term separately in absolute value. For a zero ρ with $\gamma > 0$ we have

$$\varphi_\rho = \frac{\pi}{2} - \frac{1}{2\gamma} + O\left(\frac{1}{\gamma^2}\right).$$

It follows that for $\gamma > 5|x|$ we have

$$\begin{aligned} 1 - \left(1 - \frac{1}{\rho} \right)^x &= 1 - e^{ix(-2\varphi_\rho + \pi)} \\ &= 1 - \exp\left(2ix \left(\frac{1}{2\gamma} + O\left(\frac{1}{\gamma^2}\right) \right) \right) \\ (7.24) \quad &= i \frac{ix}{\gamma} + O\left(\frac{x}{\gamma^2}\right). \end{aligned}$$

We find a similar expression when $\gamma < 0$.

We now divide the sum (7.23) into parts $|\rho| < |x| + 2$ and $|\rho| \geq |x| + 2$. The former sum contributes $O((|x| + 2) \log(|x| + 2))$ using the bounds on the number of zeros in Theorem 2.1 (4). On the remaining range we pair the zeros with increasing positive γ against those with increasing negative γ , and find that their imaginary parts cancel out by Theorem 2.1 (4) to an error $O((x \log(|\gamma| + 2))/\gamma^2)$ on each zero. Now summing over the $O(\log T)$ zeros in an interval of length one at height T , for $|x| + 2 \leq T \leq \infty$ gives a total contribution of

$$O\left(x(\log|x| + 2)^2 \frac{1}{|x| + 2}\right) = O((\log|x| + 2)^2).$$

The bound on the sum (7.23) is then

$$|F_\pi(x)| \leq C(|x| + 2) \log(|x| + 2), \quad -\infty < x < \infty,$$

as asserted. \square

We next study the Fourier transform of $F_\pi(x)$ on the real line, regarded as a distribution. We suppose the Riemann hypothesis holds for π , and consider the unique function $F_\pi(z)$ given in Theorem 7.1 (2). The growth bound in (7.3) implies that its Fourier transform is well-defined as a tempered distribution. Viewing (7.23) term-by-term we obtain, a representation for this Fourier transform formally as a sum of delta functions

$$\hat{F}_\pi(\eta) = \sum_{\rho \in Z(\pi)} (\delta_0(\eta) - \delta_{\varphi^*(\rho)}(\eta)),$$

in which $\delta_\alpha(\eta)$ denotes a delta function centered at $\eta = \alpha$. Note that the individual terms in the sum must be grouped as indicated to define a continuous linear functional on test functions in the Schwartz space $\mathcal{S}(\mathbb{R})$. One deduces that the support of this tempered distribution is real and lies in $[-\pi, \pi]$, and is a discrete set, determined by the delta functions, except at the non-isolated limit point $\eta = 0$.

Remarks. — (1) One can carry out a similar procedure to interpolate the archimedean contributions $S_\infty(n, \pi)$, for $n \geq 1$, with an interpolation function $F_{\pi, \infty}(z)$ that is an entire function of exponential type. One must modify the proof to subtract off the contribution of the term linear in z from the other terms in the formula of Lemma 4.4, to get a convergent formula. The resulting interpolation function is, formally,

$$(7.25) \quad F_{\pi, \infty}(z) = C_3(\pi) + C_4(\pi)z + \sum_{j=1}^N \sum_{n=1}^{\infty} \left(1 - \frac{2}{2n + \kappa_j(\pi) - 1} z - \left(1 - \frac{2}{2n + \kappa_j(\pi) - 1} \right)^z \right)$$

for a certain constants $C_3(\pi), C_4(\pi)$. Here we have

$$\left| 1 - \frac{2}{2n + \kappa_j(\pi) - 1} \right| < 1 \text{ for all } n \geq 1.$$

The interpolation function $F_{\pi, \infty}(z)$ will be an entire function of exponential type. It has polynomial growth on the positive real axis (bounded by $O((|x| + 2) \log(|x| + 2))$ for positive real x), but increases exponentially on the negative real axis. Consequently its Fourier transform can be interpreted as a distribution, but not as a tempered distribution.

(2) One can also obtain an entire function $F_{\pi,f}(z)$ of exponential type interpolating $S_f(n, \pi)$, given as

$$F_{\pi,f}(z) := F_{\pi}(z) - F_{\pi,\infty}(z) - \delta(\pi).$$

Again, even assuming the Riemann hypothesis, its Fourier transform must be interpreted as a distribution and not a tempered distribution.

(3) Assume the Riemann hypothesis holds for $L(s, \pi)$, and suppose that $\pi = \pi^\vee$ is self-dual. Then (7.20) yields

$$(7.26) \quad \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right) + \left(1 - \left(1 - \frac{1}{1-\rho}\right)^n\right) = 2 - 2 \cos n\varphi^*(\rho).$$

Note that $\varphi_\rho \rightarrow \pm \frac{\pi}{2}$ as $\gamma \rightarrow \pm\infty$, and the $\pm\pi$ term is chosen of opposite sign, whence $-2\varphi_\rho \pm \pi \rightarrow 0$ as $|\rho| \rightarrow \infty$.

8. Concluding remarks

(1) Instead of the ξ -function treated in §2, one may alternatively consider generalized Li coefficients associated to the function $\xi^+(s, \pi)$ obtained by removing any zeros at $s = \frac{1}{2}$ from $\xi(s, \pi)$. Here

$$\xi^+(s, \pi) := \left(s - \frac{1}{2}\right)^{-e(\frac{1}{2}, \pi)} \xi(s, \pi),$$

and we let $\lambda_n^+(\pi)$ denote the associated Li coefficients. Use of this function has three positive features. First, it allows us to unambiguously obtain the functional equation in the form

$$\xi^+(s, \pi) = \xi^+(1 - s, \pi)$$

with the sign convention that $\xi^+(\frac{1}{2}) > 0$. Second, assuming RH, the interpolating function $F_\pi^+(z)$ for the generalized Li coefficients associated to $\xi^+(s, \pi)$ in §7 will be an entire function of order 1 and type strictly less than π . In consequence it is uniquely determined by the values $\{F_\pi^+(n, \pi) := \lambda_n^+(\pi), n \in \mathbb{Z}\}$. Third, in terms of the Weil scalar product associated to a hypothetical Hilbert-Polya operator, with eigenvalues $\lambda = s^2 - \frac{1}{4}$, removal of the zeros at $s = \frac{1}{2}$ would correspond to taking the orthogonal complement of the eigenspace with eigenvalue $\lambda = 0$. It is hoped that zeros at $s = \frac{1}{2}$ will have some arithmetic-geometric meaning, as in the Birch-Swinnerton Dyer conjecture, and there may well be an arithmetico-geometric way to directly characterize this eigenspace.

(2) There are analogues of the Li coefficients for automorphic L -functions in the function field case. We first note that the ξ -function for the trivial representation over a function field K in one variable over a finite field \mathbb{F}_q can be taken to be

$$\xi(s, \pi_{triv, K}) = q^{-s}(1 - q^{-s})(1 - q^{1-s})Z_K(s),$$

in which $Z_K(s)$ is the (completed) function field zeta function, and is a polynomial in $w = q^{-s}$. For a rational function field $K = \mathbb{F}_q(T)$ we have $\xi(s, \pi_{triv, K}) = 1$. All other automorphic L -functions are polynomials in the variable w . The transformation $s = -z/(1 - z)$ is still used to define the Li coefficients in the function field case. The definition (3.1) for the “Weil scalar product” applies, and the positivity of the Li coefficients is interpretable in terms of this scalar product applied to the Li test functions $G_n(s)$. The asymptotics of the Li coefficients in the function field case are different from those in the number field case, since there are no archimedean places. The main term in their asymptotics is Cn rather than the term $Cn \log n$ occurring in the number field case.

To interpret the Li coefficients in terms of a function field “explicit formula,” one must use a version in terms of the s -variable, noting that L -functions are singly periodic with period $2\pi i / \log q$. The standard function field “explicit formula” is generally stated in terms of the variable w , and it has an algebraic geometry interpretation, related to intersection theory. The allowed test functions in this formula (related to divisors) are Laurent polynomials in $w = q^{-s}$ with integer coefficients, cf. Haran [26]. The Li test functions $G_n(s)$ are not functions of q^{-s} , and must be q -periodized using

$$(8.1) \quad P_q(G_n)(s) := \sum_{n \in \mathbb{Z}} G_n \left(s + \frac{2\pi i n}{\log q} \right).$$

to be viewed as function field test functions. This sum (8.1) is conditionally convergent and is regularized as a limit as $T \rightarrow \infty$ of $\sum_{-T < n < T}$. It would be interesting to see if this framework is compatible with a framework of Deninger [13], [16, Sect. 4]. Another feature of the function field case is that some function field L -functions have multiple zeros on the critical line at positions other than

$$s \equiv \frac{1}{2} \pmod{\frac{2\pi i}{\log q}}.$$

(3) Under the Riemann hypothesis, the “explicit formula” decomposition of the Li coefficients into $S_\infty(n, \pi)$ and $S_f(n, \pi)$ reveals that the dominant contribution to the asymptotics of the Li coefficients comes from the archimedean terms, which correspond in (7.25) to the “trivial zeros” of $L(s, \pi)$.

This contrasts with the definition (1.3) of the Li coefficients (1.3), which is a sum over the non-trivial zeros of $L(s, \pi)$, and does not include the “trivial zeros”. In effect certain asymptotics of the nontrivial zeros are described in terms of the “trivial zeros.”

Appendix A. "Explicit formula" and Weil's quadratic functional

For simplicity we treat here only the case of the trivial representation π_{triv} on $GL(1)$. In that case, there are two versions of the “explicit formula” of prime number theory, the trace form and the covariance form, described below. For all other automorphic representations π there is essentially only one form of the “explicit formula.”

Consider first a vector space \mathcal{A}_δ of test functions $f : \mathbb{R}_{>0} \rightarrow \mathbb{C}$, defined as follows. Associated to a test function is its Mellin transform

$$\hat{f}(s) := \int_0^\infty f(x)x^s \frac{dx}{x}.$$

The vector space \mathcal{A}_δ consists of those functions $f(x)$ whose Mellin transform is analytic in the strip $\frac{1}{2} - \delta < \text{Re}(s) < \frac{1}{2} + \delta$, and extends continuously to the boundary of the strip. We put a metric on test functions given by the uniform norm on the closed strip,

$$d(f_1, f_2) := \sup_{s \in S_\delta} |f(s) - g(s)|.$$

With this topology \mathcal{A}_δ is a complete metric space. We define the involution

$$(A.1) \quad \tilde{f}(x) := \frac{1}{x} f\left(\frac{1}{x}\right),$$

whose effect on Mellin transforms is:

$$(A.2) \quad \hat{\tilde{f}}(s) = \hat{f}(1-s).$$

We define the space $\hat{\mathcal{A}}_\delta$ to be the set of Mellin transforms of the elements of \mathcal{A}_δ , regarded as analytic functions in the specified strip. Note that the space \mathcal{A} in §3 is given by

$$\mathcal{A} = \cup_{\delta < 1/2} \hat{\mathcal{A}}_\delta.$$

We define the *Weil distribution functional*

$$(A.3) \quad W[f] := \sum'_{\{\rho: \xi(\rho)=0\}} \hat{f}(\rho),$$

in which $'$ means that the (possibly conditionally convergent) sum is interpreted as $\lim_{T \rightarrow \infty} \sum_{|\rho| \leq T}$. Under suitable conditions $W[f]$ is a continuous linear functional on the allowed set of test functions \mathcal{A}_δ , for example when $\delta > \frac{1}{2}$. (If the Riemann hypothesis is assumed one can take any $\delta > 0$.) We then define the *trace functional*

$$(A.4) \quad T[f] := \hat{f}(0) - W[f] + \hat{f}(1).$$

The quantity $T[f]$ is sometimes called the “spectral side” of the “explicit formula” of prime number theory.

The “explicit formula” in *trace form* is an formula for $T[f]$ for a suitable set of test functions f , taking the shape

$$(A.5) \quad T[f] = \sum_{\nu} W_{\nu}(f),$$

in which $W_{\nu}(f)$ is a contribution associated to each (non-archimedean or archimedean) place ν of the given field K . The right hand side of (A.5) is sometimes called the “arithmetic-geometric” side of the explicit formula. We shall not be concerned with the exact form of the arithmetic-geometric side here, the individual terms $W_{\nu}(f)$ of which can be expressed in various interesting ways, see Burnol [6], [7], Haran [25] [27]. For general automorphic representations π the terms $W_{\nu}(f)$ represent contributions from individual terms in the Euler product factorization of $\xi(s, \pi)$. A version of the “explicit formula” in the trace formulation is given in Patterson [41, Sect. 3.6]. In order to get an unconditional result one must use a test function space \mathcal{A}_δ with $\delta > \frac{1}{2}$. A version of the “explicit formula” for principal automorphic L -functions over $GL(n)$ appears in Rudick and Sarnak [42, Prop. 2.1].

One hope is that the “explicit formula” (A.5) might be interpretable as a Lefschetz trace formula coming from a dynamical system acting on an (unknown) “geometric” object, in which $T[f]$ is to be viewed as a distributional trace with test functions f living on cohomology groups of the geometric object, and the arithmetic side giving data arising from fixed points or periodic orbits on the geometric object. This viewpoint is taken in Deninger [14], [15], [17], [18].

Weil originally formulated the explicit formula in the *covariance form* given by

$$(A.6) \quad W[f] = - \sum_{\nu} W_{\nu}(f) + W_0(f) + W_1(f).$$

This is a rearrangement of the terms in the “trace form” equality, in which we define

$$W_0(f) := \hat{f}(0) = \int_0^\infty \frac{1}{x} f(x) \frac{dx}{x},$$

and

$$W_1(f) := \hat{f}(1) = \int_0^\infty f(x) \frac{dx}{x}.$$

Attached to this quantity Weil introduced the *scalar product*

$$(A.7) \quad \langle f, g \rangle_{\mathcal{W}} := W[f * \tilde{g}] = \sum_{\rho} \hat{f}(\rho) \overline{\hat{g}(1 - \bar{\rho})} = \sum_{\rho} \hat{f}(\rho) \hat{\hat{g}}(1 - \rho).$$

Weil’s criterion for the Riemann hypothesis is that this scalar product be positive semidefinite on a suitable space of test functions, which could be taken to be \mathcal{A}_δ for any $\delta > \frac{1}{2}$, for example. This scalar product will be indefinite if the Riemann hypothesis does not hold. We may transport this scalar product forward to the Mellin-transformed space $\hat{\mathcal{A}}$ as

$$\langle \hat{f}, \hat{g} \rangle_{\mathcal{W}} := \langle f, g \rangle_{\mathcal{W}}.$$

The Weil functional $W[f]$ has a natural interpretation in the function field case, see Haran [26].

One feature of the covariance form (A.6) of the “explicit formula” compared to the trace form is that one can make sense of the functional $W[f]$ on a larger set of test functions than those allowed in the trace form, permitting test functions that have singularities at $s = 0$ and $s = 1$. However the right hand side of (A.6) must then be redefined as a limit as $T \rightarrow \infty$, with each local term computed using cutoffs at $x = 1/T$ near zero and at $x = T$ near ∞ . This was done in the computations in [4]. This is a “regularization” of the right side of (A.6) because the two terms $W_0(f)$ and $W_1(f)$ diverge as the cutoff parameter $T \rightarrow \infty$. Note that these two terms are present only for the trivial representation π_{triv} on $\text{GL}(1)$, and are absent from the analogous formula for all other cuspidal automorphic representations π . In these other cases the analogous formulas have $T[f] = -W[f]$, so the “trace form” and “covariance form” essentially coincide.

The vector space \mathcal{L} of Li test functions makes sense for this extended covariance form of the “explicit formula” with a cutoff parameter, and not for the trace form. As noted in §3, it consists exclusively of rational functions which have poles either at $s = 0$ or $s = 1$, or both. In consequence the trace function $T[f]$ is undefined for every Li test function. For example, if $g_n(x)$ denotes the test function on $\mathbb{R}_{>0}$ whose Mellin transform

$$G_n(s) \hat{g}_n(s) = \int_0^\infty g_n(x) x^{s-1} dx,$$

gave the Li test functions $G_n(s)$ in (3.2) then $\hat{g}_n(0)$ is infinite, for each $n \geq 1$. In [4, Lemma 2] the functions $g_n(x)$ were explicitly determined, as

$$g_n(x) = \begin{cases} P_n(\log x) & \text{if } 0 < x < 1, \\ n/2 & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$$

in which

$$P_n(x) := \sum_{j=1}^n \binom{n}{j} \frac{x^{j-1}}{(j-1)!}.$$

We note that the polynomial $P_n(x)$ is essentially a Laguerre polynomial $L_n^\alpha(x)$ with $\alpha = 1$, namely $P_n(x) = L_{n-1}^1(-x)$; this fact was also observed by Coffey [9, Appendix E]. The Weil scalar product of g_n with other functions remains well-defined in the covariance form of the “explicit formula” when a cutoff version is used. In terms of the cutoff version of the definition, the unbounded contribution from $\hat{g}_n(0)$ above as $T \rightarrow \infty$ is offset by a corresponding divergence coming from the finite primes in the “explicit formula.” The Weil distribution functional $W[f]$ is well-defined for all $f \in \mathcal{L}$, as is the Weil scalar product, whether or not the Riemann hypothesis holds.

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