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ON THE PRODUCT OF FUNCTIONS IN BMO AND H¹

by Aline BONAMI, Tadeusz IWANIEC, Peter JONES & Michel ZINSMEISTER

Abstract. — The point-wise product of a function of bounded mean oscillation with a function of the Hardy space H^1 is not locally integrable in general. However, in view of the duality between H^1 and BMO, we are able to give a meaning to the product as a Schwartz distribution. Moreover, this distribution can be written as the sum of an integrable function and a distribution in some adapted Hardy-Orlicz space. When dealing with holomorphic functions in the unit disc, the converse is also valid: every holomorphic of the corresponding Hardy-Orlicz space can be written as a product of a function in the holomorphic Hardy space H^1 and a holomorphic function with boundary values of bounded mean oscillation.

Résumé. — Le produit d'une fonction à oscillation moyenne bornée avec une fonction de l'espace de Hardy H^1 n'est pas intégrable en général. Nous montrons toutefois qu'on peut lui donner un sens en tant que distribution tempérée, ceci grâce à la dualité H^1, BMO. Cette distribution peut de plus s'écrire comme la somme d'une fonction intégrable et d'une distribution appartenant à un espace de Hardy-Orlicz adapté. Lorsqu'on considère un tel produit pour les fonctions holomorphes du disque unité, cet énoncé possède une réciproque : toute fonction holomorphe de l'espace de Hardy-Orlicz considéré peut s'écrire comme un tel produit.

1. Introduction and Overview

The BMO–H^1 Pairing

This paper is largely concerned with the Hardy space $\boldsymbol{H}^1(\mathbb{R}^n)$ and its dual $\boldsymbol{BMO}(\mathbb{R}^n)$, $n \geqslant 2$. An excellent general reference for these spaces is [56].

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Viewing $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$ as a continuous linear functional on $\boldsymbol{H}^1(\mathbb{R}^n)$, see the seminal work of C. Fefferman [18], we shall denote its value at $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ by

$$\int_{\mathbb{R}^n}^{*} \mathfrak{b}(x)\mathfrak{h}(x)\,dx \quad \preccurlyeq \|\mathfrak{b}\|_{BMO} \cdot \|\mathfrak{h}\|_{H^1}\ . \tag{1.1}$$ [1]

There are at least two possible rigorous definitions of (1.1). Denote by $\mathscr{C}^\infty_\bullet(\mathbb{R}^n)$ the space of smooth functions with compact support whose integral mean equals zero. This is a dense subspace of $\boldsymbol{H}^1(\mathbb{R}^n)$. We set out the following definition

$$\int_{\mathbb{R}^n}^{*} \mathfrak{b}(x)\mathfrak{h}(x)\,dx \stackrel{\text{def}}{=\!=} \lim_{j\to\infty}\int_{\mathbb{R}^n} \mathfrak{b}(x)\mathfrak{h}_j(x)\,dx \tag{1.2}$$

where the limit exists for every sequence of functions $\mathfrak{h}_j \in \mathscr{C}^\infty_\bullet(\mathbb{R}^n)$ converging to $\mathfrak{h}$ in the norm topology of $\boldsymbol{H}^1(\mathbb{R}^n)$. An equivalent and very useful way of defining (1.1) is through the almost everywhere approximation of the factor $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}^{*} \mathfrak{b}(x)\mathfrak{h}(x)\,dx \;=\lim_{k\to\infty}\int_{\mathbb{R}^n} \mathfrak{b}_k(x)\mathfrak{h}(x)\,dx\ . \tag{1.3}$$

The limit exists and coincides with that in (1.2) for every sequence $\{\mathfrak{b}_k\} \subset \boldsymbol{L}^\infty(\mathbb{R}^n)$ converging to $\mathfrak{b}$ almost everywhere, provided it is bounded in the space $\boldsymbol{BMO}(\mathbb{R}^n)$. For example,

$$\mathfrak{b}_k(x) = \begin{cases} k & \text{if} \quad k \leqslant \mathfrak{b}(x) \\ \mathfrak{b}(x) & \text{if} \quad -k \leqslant \mathfrak{b}(x) \leqslant \ k \\ -k & \text{if} \quad \mathfrak{b}(x) \leqslant -k \end{cases} \qquad \text{or} \quad \mathfrak{b}_k = \frac{k\,\mathfrak{b}}{k+\mathfrak{b}} \quad k = 1, 2, \ldots$$

One should be warned, however, that the sequence $\{\mathfrak{b}_k\}$ need not converge to $\mathfrak{b}$ in the *BMO*-norm. The celebrated $H^1 - BMO$ inequality gives us the uniform estimate

$$\int_{\mathbb{R}^n} \frac{k\,\mathfrak{b}\cdot\mathfrak{h}}{k+\mathfrak{b}} \quad \preccurlyeq \quad \left\| \frac{k\,\mathfrak{b}}{k+\mathfrak{b}} \right\|_{BMO} \cdot \|\mathfrak{h}\|_{H^1} \quad \leqslant \quad 2\|\mathfrak{b}\|_{BMO} \cdot \|\mathfrak{h}\|_{H^1}\ .$$

An especially interesting case occurs when the point-wise product $\mathfrak{b}\cdot\mathfrak{h}$ is either integrable or a nonnegative function. In view of the equivalence of

[1] Hereafter we propose the following abbreviation $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ for inequalities of the form $|\boldsymbol{A}| \leqslant C\cdot\boldsymbol{B}$, where the constant $C \geqslant 0$ (called implied constant) depends on parameters insignificant to us, such as the dimension n and so forth. One shall easily recognize those parameters from the context.

the limits at (1.2) and (1.3) , we obtain

$$(1.4)\qquad \int_{\mathbb{R}^n}^{*} \mathfrak{b}(x)\mathfrak{h}(x)\,dx = \int_{\mathbb{R}^n} \mathfrak{b}(x)\mathfrak{h}(x)\,dx\,, \quad \text{whenever } \begin{cases} \mathfrak{b}\cdot\mathfrak{h} \in \boldsymbol{L}^1(\mathbb{R}^n) \\ \text{or} \quad \mathfrak{b}\cdot\mathfrak{h} \geqslant 0 \end{cases}$$

by the dominated and monotone convergence theorems, respectively.

Although in general the point-wise product $\mathfrak{b}\cdot\mathfrak{h}$ need not be integrable we are able to give meaning to it as a Schwartz distribution. In what follows, when it is important to emphasize this new meaning, we will use the notation $\mathfrak{b}\times\mathfrak{h} \in \mathscr{D}'(\mathbb{R}^n)$. For the definition of the distribution $\mathfrak{b}\times\mathfrak{h}$, we look at the test functions $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ as multipliers for *BMO* -spaces.(2) First notice the inequality

$$(1.5)\qquad \|\varphi\mathfrak{b}\|_{BMO} \preccurlyeq \|\nabla\varphi\|_\infty\,(\,\|\mathfrak{b}\|_{BMO} + |\mathfrak{b}_{\mathbb{Q}}|\,)$$

where $\mathfrak{b}_{\mathbb{Q}}$ stands for the average of $\mathfrak{b}$ over the unit cube $\mathbb{Q} = [0,1]^n \subset \mathbb{R}^n$. We reserve the following notation,

$$\|\mathfrak{b}\|_{BMO^+} \stackrel{\text{def}}{=\!=} \|\mathfrak{b}\|_{BMO} + |\mathfrak{b}_{\mathbb{Q}}|\,,$$

for the quantity that appears in the right hand side. Now the distribution $\mathfrak{b}\times\mathfrak{h}$ operates on a test function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ by the rule

$$\begin{aligned}\langle \mathfrak{b}\times\mathfrak{h} \,|\, \varphi\rangle \;&\stackrel{\text{def}}{=\!=}\; \int_{\mathbb{R}^n}^{*} [\varphi(x)\mathfrak{b}(x)]\,\mathfrak{h}(x)\,dx \;= \\ &\lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi(x)\mathfrak{b}(x)\mathfrak{h}_j(x)\,dx \;=\; \lim_{k\to\infty}\int_{\mathbb{R}^n} \varphi(x)\mathfrak{b}_k(x)\mathfrak{h}(x)\,dx \\ &\preccurlyeq\; \|\varphi\mathfrak{b}\|_{BMO}\,\|\mathfrak{h}\|_{H^1} \;\preccurlyeq\; \|\nabla\varphi\|_\infty\,.\end{aligned}$$

The latter means precisely that the distribution $\mathfrak{b}\times\mathfrak{h} \in \mathscr{D}'(\mathbb{R}^n)$ has order at most 1. Obviously, the class $\mathscr{C}_0^\infty(\mathbb{R}^n)$ of test functions for the distribution $\mathfrak{b}\times\mathfrak{h} \in \mathscr{D}'(\mathbb{R}^n)$ can be extended to include all multipliers for $\boldsymbol{BMO}(\mathbb{R}^n)$. But we do not pursue this extension here as the need will not arise. It is both illuminating and rewarding to realize, by reasoning as before, that in case when $\mathfrak{b}\cdot\mathfrak{h}$ happens to be locally integrable or nonnegative on some open set $\Omega \subset \mathbb{R}^n$, then $\mathfrak{b}\times\mathfrak{h}$ is a regular distribution on Ω; its action on a test function $\varphi \in \mathscr{C}_0^\infty(\Omega)$ reduces to the integral formula

$$(1.6)\quad \langle \mathfrak{b}\times\mathfrak{h} \,|\, \varphi\rangle \;=\; \int_\Omega \mathfrak{b}(x)\cdot\mathfrak{h}(x)\;\;\varphi(x)\,dx\,, \qquad \text{whenever} \quad \mathfrak{b}\cdot\mathfrak{h} \in \boldsymbol{L}^1_{\text{loc}}(\Omega).$$

(2) A study of multipliers for *BMO* -spaces originated in 1976 by the paper of S. Janson [39] and developed by Y. Gotoh [22, 23], E. Nakai and K. Yabuta [51, 50].

The previous discussion on the product distribution $\mathfrak{b} \times \mathfrak{h}$ can be summarized in the following lemma.

Lemma 1.1. — *For $\mathfrak{b}$ a fixed function in $\boldsymbol{BMO}(\mathbb{R}^n)$, the mapping $\mathfrak{h} \mapsto \mathfrak{b}\,\mathfrak{h}$, which is a priori defined on $\mathscr{C}_\bullet^\infty(\mathbb{R}^n)$ with values in $\mathscr{D}'(\mathbb{R}^n)$, extends continuously into a mapping from $\boldsymbol{H}^1(\mathbb{R}^n)$ into $\mathscr{D}'(\mathbb{R}^n)$, and this new mapping is denoted by $\mathfrak{h} \mapsto \mathfrak{b} \times \mathfrak{h}$. Moreover, for $\mathfrak{b}_k$ tending to $\mathfrak{b}$ as above, the sequence $\mathfrak{b}_k \times \mathfrak{h}$ tends to $\mathfrak{b} \times \mathfrak{h}$ (in the weak topology of $\mathscr{D}'(\mathbb{R}^n)$).*

Remark that adding a constant to $\mathfrak{b}$, which does not change its *BMO* norm, translates into adding a multiple of $\mathfrak{h}$ to the product $\mathfrak{b} \times \mathfrak{h}$. So, we can restrict to functions $\mathfrak{b}$ such that $\mathfrak{b}_{\mathbb{Q}} = 0$, for instance. If $\mathfrak{b}_k$ and $\mathfrak{b}$ satisfy this property, it is sufficient for the conclusion of the last lemma to assume that the sequence $\mathfrak{b}_k$ tends to $\mathfrak{b}$ in the weak* topology of $BMO(\mathbb{R}^n)$. This is a direct consequence of the fact that, while $\varphi \in \mathscr{C}_0^\infty(\Omega)$ is not a multiplier of $\boldsymbol{H}^1(\mathbb{R}^n)$, nevertheless for $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$,

$$\varphi\mathfrak{h} - \left(\int \varphi\mathfrak{h}\,dx\right)\chi_{\mathbb{Q}}$$

is also in $\boldsymbol{H}^1(\mathbb{R}^n)$. One sees already in this elementary case how the product with a function in $\boldsymbol{H}^1(\mathbb{R}^n)$ splits naturally into two parts, the one with cancellations (here the part in $\boldsymbol{H}^1(\mathbb{R}^n)$) and the L^1 part (here the characteristic function).

Weak Jacobian

Recent developments in the geometric function theory [1, 26, 27] and nonlinear elasticity [3, 47, 58, 62] clearly motivated our investigation of the distribution $\mathfrak{b} \times \mathfrak{h}$. These theories are largely concerned with mappings $F = (f^1, f^2, ..., f^n) : \Omega \to \mathbb{R}^n$ (elastic deformations) in the Sobolev class $\boldsymbol{W}^{1,p}(\Omega, \mathbb{R}^n)$, and its Jacobian determinant $\mathcal{J}(x, F) = \det\left[\frac{\partial f^i}{\partial x_j}\right]$. A brief mention of the concept of the weak Jacobian [3] is in order. For $p = n$ one may integrate by parts to obtain

$$\int_\Omega \phi(x)\mathcal{J}(x,F)\,dx = \int_\Omega \phi\, df^1 \wedge df^2 \wedge ... \wedge df^n = -\int_\Omega f^1\, d\phi \wedge df^2 \wedge ... \wedge df^n$$

for all $\phi \in \mathscr{C}_0^\infty(\Omega)$. Now this latter integral gives rise to a distribution of order 1, whenever $|F| \cdot |DF|^{n-1} \in \boldsymbol{L}^1_{\mathrm{loc}}(\Omega)$. By the Sobolev imbedding theorem this is certainly the case when $F \in \boldsymbol{W}^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, with $p = \frac{n^2}{n+1}$.

As is customary, we define the distribution $\Im_F \in \mathscr{D}'(\Omega)$ by the rule

$$(1.7) \qquad \langle \Im_F \mid \phi \rangle = -\int_\Omega f^1 \, d\phi \wedge df^2 \wedge ... \wedge df^n \ , \quad \text{for all } \phi \in \mathscr{C}_0^\infty(\Omega)$$

and call it the weak (or distributional) Jacobian.

With the concept of the product $\mathfrak{b} \times \mathfrak{h} \in \mathscr{D}'(\Omega)$ we may proceed further in this direction. Consider a mapping $F \in \boldsymbol{BMO} \cap \boldsymbol{W}^{1,\,n-1}$. Its first coordinate function $\mathfrak{b} \overset{\text{def}}{=\!=} -f^1$ lies in $\boldsymbol{BMO}$, while the wedge product $\mathfrak{h}(x)\,dx \overset{\text{def}}{=\!=} d\phi \wedge df^2 \wedge ... \wedge df^n$ belongs to the Hardy space $\boldsymbol{H}^1$. Hence

$$(1.8) \quad \langle \Im_F \mid \phi \rangle = \int_\Omega^* [-f^1] \, [d\phi \wedge df^2 \wedge ... \wedge df^n] \ , \quad \text{for all } \phi \in \mathscr{C}_0^\infty(\Omega) \ .$$

Most probably, such an extension of the domain of definition of the Jacobian operator $\Im : \boldsymbol{BMO} \cap \boldsymbol{W}^{1,\,n-1} \to \mathscr{D}'$ will prove useful in full development of the aforementioned theories. However, to go into this in detail would take us too far afield.

Hardy-Orlicz Spaces

Analysis of the relationship between the distribution $\mathfrak{b} \times \mathfrak{h}$ and the pointwise product $\mathfrak{b} \cdot \mathfrak{h}$ brings us to the Hardy-Orlicz spaces. Let us take and use it as a definition the following maximal characterization of $\boldsymbol{H}^1(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$, see for instance [45]. For this we fix a nonnegative $\Phi \in \mathscr{C}_0^\infty(\mathbb{B})$ supported in the unit ball $\mathbb{B} = \{x \in \mathbb{R}^n; |x| < 1\}$ and having integral 1. (3) The one parameter family of the mollifiers

$$\Phi_\varepsilon(x) = \varepsilon^{-n} \Phi\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0$$

gives rise to a maximal operator defined on $\mathscr{D}'(\Omega)$. For a given distribution $f \in \mathscr{D}'(\Omega)$, we may consider smooth functions defined on the level sets $\Omega_\varepsilon = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$,

$$(1.9) \qquad f_\varepsilon(x) = (f * \Phi_\varepsilon)(x) \overset{\text{def}}{=\!=} \langle f \mid \Phi_\varepsilon(x - \cdot) \rangle \ .$$

This latter notation stands for the action of f on the test function $y \mapsto \Phi_\varepsilon(x-y)$ in y-variable. It is well known that $f_\varepsilon \to f$ in $\mathscr{D}'(\Omega)$ as $\varepsilon \to 0$. For a regular distribution $f \in \boldsymbol{L}^1_{\text{loc}}(\Omega)$ the above convolution formula takes the integral form,

$$(1.10) \qquad f_\varepsilon(x) = \int_\Omega f(y)\Phi_\varepsilon(x-y)\, dy \ \to f(x) \, , \qquad \text{as } \varepsilon \text{ goes to zero}$$

(3) For convenience of the subsequent discussion we actually assume that Φ is supported in a cube centered at the origin and contained in $\mathbb{B}$.

whenever $x \in \Omega$ is a Lebesgue point of f.

As a matter of fact such point-wise limits exist almost everywhere for all distributions $f \in \mathscr{D}'(\Omega)$ of order zero (signed Radon measures). The point-wise limit is none other than the Radon-Nikodym derivative of the measure. Call it the regular (or absolutely continuous) part of f,

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = f_{\text{reg}}(x) \quad \text{almost everywhere .} \tag{1.11}$$

The Lebesgue decomposition of measures tells us that $f_{\text{reg}} \in \boldsymbol{L}^1_{\text{loc}}(\Omega)$. If f is a nonnegative distribution, meaning that $\langle f \,|\, \varphi \rangle \geqslant 0$ for all nonnegative test functions $\varphi \in C_0^\infty(\Omega)$, then f is a Borel measure. Thus

$$\lim_{\varepsilon \to 0} f_\varepsilon = f_{\text{reg}} \in \boldsymbol{L}^1_{\text{loc}}(\Omega) \quad \text{when } f \text{ is a nonnegative distribution.} \tag{1.12}$$

Next, the maximal operator $\mathscr{M}$ is defined on $\mathscr{D}'(\Omega)$ by the rule

$$\mathscr{M}f(x) = \sup\{|f_\varepsilon(x)|\,; \quad 0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)\}\,. \tag{1.13}$$

Definition 1.2. — *A distribution $f \in \mathscr{D}'(\Omega)$ is said to belong to the Hardy space $\boldsymbol{H}^1(\Omega)$ if $\mathscr{M}f \in \boldsymbol{L}^1(\Omega)$.*

Naturally, $\boldsymbol{H}^1(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{H^1(\Omega)} = \int_\Omega \mathscr{M}f(x)\,dx. \tag{1.14}$$

An account and subtlety concerning weak convergence in $\boldsymbol{H}^1(\mathbb{R}^n)$ is given in [41] and [14]. Although it is not immediate from this definition, for sufficiently regular domains, the Hardy space $\boldsymbol{H}^1(\Omega)$ consists of restrictions to Ω of functions in $\boldsymbol{H}^1(\mathbb{R}^n)$ [45], see also [9, 43] for Lipschitz domains. Obviously, these are regular distributions; actually we have the inclusion $\boldsymbol{H}^1(\Omega) \subset \boldsymbol{L}^1(\Omega)$.

Next we recall a general concept of Orlicz spaces. Given a sigma-finite measure space (Ω, μ) and given a continuous function $\mathcal{P} : [0, \infty) \to [0, \infty)$ increasing from zero to infinity (but not necessarily convex), the Orlicz space $\boldsymbol{L}^{\mathcal{P}}(\Omega, \mu)$ consists of μ-measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\llbracket f \rrbracket_{\mathcal{P}} = \llbracket f \rrbracket_{L^{\mathcal{P}}(\Omega,\,\mu)} \stackrel{\text{def}}{=\!=} \inf\left\{k > 0\,;\, \int_\Omega \mathcal{P}\left(k^{-1}|\,f|\right)\, d\mu \leqslant 1\right\} < \infty\,. \tag{1.15}$$

In general, the nonlinear functional $\llbracket \;\; \rrbracket_{\mathcal{P}}$ need not satisfy the triangle inequality. However, it does when $\mathcal{P}$ is convex and in this case $\boldsymbol{L}^{\mathcal{P}}(\Omega, \mu)$ becomes a Banach space with respect to the norm $\| \;\; \|_{\mathcal{P}} \stackrel{\text{def}}{=\!=} \llbracket \;\; \rrbracket_{\mathcal{P}}$. In either case $\boldsymbol{L}^{\mathcal{P}}(\Omega, \mu)$ is a complete linear metric space, see [52]. The $\boldsymbol{L}^{\mathcal{P}}$-distance between f and g is given by

$$\operatorname{dist}_{\mathcal{P}}[f,g] \stackrel{\text{def}}{=\!=} \inf\left\{\rho>0\,;\int_\Omega \mathcal{P}\left(\rho^{-1}\,|f-g|\right)\,d\mu \leqslant \rho\right\} < \infty\,. \tag{1.16}$$

Remark 1.3. — It is true that $[\![f-g]\!]_{\mathcal{P}} \leqslant \operatorname{dist}_{\mathcal{P}}[f,g] \leqslant 1$, provided the rightmost inequality holds. Thus $f_j \to f$ in $\boldsymbol{L}^{\mathcal{P}}(\Omega,\mu)$ implies that $[\![f_j - f]\!]_{\mathcal{P}} \to 0$. For the converse implication it is required that the Orlicz function satisfies:

$$\lim_{\varepsilon\to 0}\ \sup_{t>0} \frac{\mathcal{P}(\varepsilon t)}{\mathcal{P}(t)} \;=\; 0\,.$$

This is the case when $\mathcal{P}(t)=\frac{t}{\log(e+t)}$, which we shall repeatedly exploit in the sequel, see for instance the proof of Theorem 1.6.

We are largely concerned with $\Omega=\mathbb{R}^n$ for which it is necessary to work with weighted Orlicz spaces. These weights will be inessential in case of bounded domains. Here are two examples of weighted Orlicz-spaces of interest to us.

The exponential class

$$\operatorname{Exp}\boldsymbol{L} = L^{\Xi}(\mathbb{R}^n,\sigma), \qquad d\sigma = \frac{dx}{(1+|x|)^{2n}}\,, \qquad \Xi(t)=e^t-1.$$

The Zygmund class

$$\boldsymbol{L}^{\wp} = \boldsymbol{L}^{\wp}(\mathbb{R}^n,\mu)\,, \qquad d\mu = \frac{dx}{\log(e+|\,x|\,)}\,, \qquad \wp(t) = \frac{t}{\log(e+t)}\,.$$

Let us explicitly emphasize that $\boldsymbol{L}^{\wp}$, often denoted by $\boldsymbol{L}\log^{-1}\boldsymbol{L}$, is lacking a norm.

The Hardy-Orlicz space $\boldsymbol{H}^{\mathcal{P}}(\Omega,\mu)$ consists of distributions $f\in\mathscr{D}'(\Omega)$ such that $\mathscr{M}f \in \boldsymbol{L}^{\mathcal{P}}(\Omega,\mu)$. We supply $\boldsymbol{H}^{\mathcal{P}}(\Omega,\mu)$ with the nonlinear functional

$$[\![\, f\,]\!]_{H^{\mathcal{P}}} = [\![\, f\,]\!]_{H^{\mathcal{P}}(\Omega,\,\mu)} \stackrel{\text{def}}{=\!=} [\![\, \mathscr{M}f\,]\!]_{L^{\mathcal{P}}(\Omega,\,\mu)} < \infty. \tag{1.17}$$

Thus $\boldsymbol{H}^{\mathcal{P}}(\Omega,\,\mu)$ is a complete linear metric space, a Banach space when $\mathcal{P}$ is convex. These spaces have previously been dealt with by many authors, see [6, 35, 57] and further references given there.

We shall make substantial use of the following weighted Hardy-Orlicz space:

$$\boldsymbol{H}^{\wp} = \boldsymbol{H}^{\wp}(\mathbb{R}^n,\,\mu), \qquad \wp(t)=\frac{t}{\log(e+t)}\,,\quad d\mu = \frac{dx}{\log(e+|x|)}. \tag{1.18}$$

At this point, let us remark that the space $\boldsymbol{H}^1(\mathbb{R}^n)$ is contained in $\boldsymbol{H}^{\wp}(\mathbb{R}^n,\,\mu)$. The two spaces have common “cancellation" properties, such as the following one.

Lemma 1.4. — *A compactly supported integrable function in $\boldsymbol{H}^{\wp}(\mathbb{R}^n,\,\mu)$ has necessarily zero mean.*

Indeed, for such a function f with non zero mean, it is well known and elementary that $\mathscr{M}f(x) \geqslant c|x|^{-n}$, a behavior that is forbidden in $\boldsymbol{H}^{\wp}(\mathbb{R}^n,\,\mu)$.

Next we take on stage a definition of the *BMO* -norm on a domain $\Omega \subset \mathbb{R}^n$, as proposed and developed in [39];

$$\|\mathfrak{b}\|_{BMO(\Omega)} = \sup\left\{ \fint_Q |\,\mathfrak{b} - \mathfrak{b}_Q\,|\,; \quad Q \text{ is a cube in } \Omega \right\}, \qquad \mathfrak{b}_Q = \fint_Q \mathfrak{b}\,.$$

Functions which differ by a constant are indistinguishable in $\boldsymbol{BMO}(\Omega)$. For the space $\boldsymbol{BMO}(\mathbb{R}^n)$ it is sometimes desirable to add $|\mathfrak{b}_{\mathbb{Q}}|$ to this norm, as we have done when defining $\|\,\mathfrak{b}\,\|_{BMO^+}$. For Ω a bounded domain we shall define

$$\|\,\mathfrak{b}\,\|_{BMO^+(\Omega)} \stackrel{\text{def}}{=\!=} \|\,\mathfrak{b}\,\|_{BMO(\Omega)} + \|\,\mathfrak{b}\,\|_{L^1(\Omega)}\,.$$

The well-developed study of the Jacobian determinants is concerned with the *grand Hardy space* $\boldsymbol{H}^{1)}(\Omega)$, see [33, 30, 32]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

Definition 1.5. — *A distribution $f \in \mathscr{D}'(\Omega)$ belongs to $\boldsymbol{H}^{1)}(\Omega)$ if*

$$\|\,f\,\|_{H^{1)}(\Omega)} = \sup_{0<p<1} \left[(1-p) \fint_\Omega |\mathscr{M}f(x)|^p \, dx \right]^{\frac{1}{p}} < \infty. \tag{1.19}$$

We emphasize that $\boldsymbol{L}^1(\Omega) \subset \boldsymbol{H}^{1)}(\Omega)$, where $\boldsymbol{L}^1(\Omega)$ is a Banach space but $\boldsymbol{H}^{1)}(\Omega)$ is not. In this connection it is worth recalling the Hardy-Littlewood maximal function of $f \in \boldsymbol{L}^1(\Omega)$ [(4)]

$$\mathfrak{M}f(x) \stackrel{\text{def}}{=\!=} \sup\left\{ \fint_Q |f(y)|\, dy\,;\ \ x \in Q \subset \Omega \right\}.$$

In general the maximal function $M(x) = \mathfrak{M}f(x)$ is not integrable but it belongs to the Marcinkiewicz class $\boldsymbol{L}^1_{\text{weak}}(\Omega)$, which is understood to mean that

$$\big|\{x \in \Omega;\ M(x) > t\}\big| \leqslant \frac{A}{t}\,, \qquad \text{for some } A > 0 \text{ and all } t > 0.$$

(4) With obvious modification the Hardy-Littlewood maximal operator can be defined on Borel measures.

An elementary computation then reveals that for each $M \in \boldsymbol{L}^1_{\mathrm{weak}}(\Omega)$ we have

$$(1.20) \qquad \left[(1-p) \fint_{\Omega} |M(x)|^p \, dx\right]^{\frac{1}{p}} \leqslant \frac{A}{|\Omega|} \, .$$

For $M = \mathfrak{M} f$, with $f \in \boldsymbol{L}^1(\mathbb{R}^n)$, the growth of its p-norms is reflected in the following equation

$$(1.21) \qquad \lim_{p \to 1} \, (1-p) \int_{\Omega} |M(x)|^p \, dx \; = \; 0 \, .$$

This is definitely false for arbitrary $M \in \boldsymbol{L}^1_{\mathrm{weak}}(\Omega)$, as an inspection of the Dirac mass in place of f shows.

Statement of the Results

Our main result is a detailed form of the decomposition $\mathfrak{b} \times \mathfrak{h}$ as a sum of two terms.(5)

THEOREM 1.6 (Decomposition Theorem). — *To every* $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ *there correspond two bounded linear operators*

$$(1.22) \qquad \mathscr{L}_{\mathfrak{h}} : \boldsymbol{BMO}(\mathbb{R}^n) \longrightarrow \boldsymbol{L}^1(\mathbb{R}^n)$$

$$(1.23) \qquad \mathscr{H}_{\mathfrak{h}} : \boldsymbol{BMO}(\mathbb{R}^n) \longrightarrow \boldsymbol{H}^{\wp}(\mathbb{R}^n, \mu)$$

such that for every $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$ *we have a decomposition*

$$(1.24) \qquad \mathfrak{b} \times \mathfrak{h} = \mathscr{L}_{\mathfrak{h}} \, \mathfrak{b} \; + \; \mathscr{H}_{\mathfrak{h}} \, \mathfrak{b}$$

and the uniform bound

$$(1.25) \qquad \|\mathscr{L}_{\mathfrak{h}} \, \mathfrak{b}\|_{L^1} + \|\mathscr{H}_{\mathfrak{h}} \, \mathfrak{b}\|_{H^{\wp}} \;\preccurlyeq\; \| \, \mathfrak{h} \, \|_{H^1} \, \| \, \mathfrak{b} \, \|_{BMO^+} \, .$$

$\mathscr{L}_{\mathfrak{h}}$ and $\mathscr{H}_{\mathfrak{h}}$ will be referred to as *decomposition operators*. There is clearly not uniqueness of such operators, and we will give different such decompositions. Each of them will have the supplementary property that, for $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$, one has the equality

$$(1.26) \qquad \int_{\mathbb{R}^n}^{*} \mathfrak{b}(x) \mathfrak{h}(x) \, dx = \int_{\mathbb{R}^n} \mathscr{L}_{\mathfrak{h}} \, \mathfrak{b}(x) \, dx \, .$$

(5) This has been announced in earlier unpublished documents, and recently in [32].

So, $\mathscr{H}_{\mathfrak{h}}\,\mathfrak{b}$ can be thought as having zero mean, which is the case when $\mathfrak{b}\cdot\mathfrak{h}$ is integrable (see also Lemma 1.4). In other words, $H_{\mathfrak{h}}\,\mathfrak{b}$ inherits the cancellation properties of $\mathfrak{h}$.

Remark that when $\mathfrak{b}$ is constant, then $\mathfrak{b}\,\mathfrak{h}$ belongs to both spaces, $\boldsymbol{L}^1(\mathbb{R}^n)$ and $\boldsymbol{H}^{\wp}(\mathbb{R}^n,\,\mu)$. So we can choose to fix $\mathscr{L}_{\mathfrak{h}}\mathfrak{h}=\mathfrak{h}$, and restrict to write the decomposition operators on functions $\mathfrak{b}$ such that $\mathfrak{b}_{\mathbf{Q}}=0$. With this choice, the L^1-component enjoys a slightly better estimate,

$$\|\mathscr{L}_{\mathfrak{h}}\,\mathfrak{b}\|_1 \preccurlyeq \|\,\mathfrak{h}\,\|_1\cdot|\mathfrak{b}_{\mathbf{Q}}| + \|\,\mathfrak{h}\,\|_{H^1}\cdot\|\,\mathfrak{b}\,\|_{BMO}\,, \tag{1.27}$$

as well as the other component, which satisfies

$$\|\mathscr{H}_{\mathfrak{h}}\,\mathfrak{b}\|_{H^{\wp}} \preccurlyeq \|\,\mathfrak{h}\,\|_{H^1}\,\|\,\mathfrak{b}\,\|_{BMO}\,. \tag{1.28}$$

The question as to whether such operators can depend linearly on $\mathfrak{h}$ remains open. We believe in the affirmative answer.

Conjecture 1.7. — *There exist bilinear operators*

$$\mathscr{L}:\boldsymbol{BMO}(\mathbb{R}^n)\times\boldsymbol{H}^1(\mathbb{R}^n)\longrightarrow\boldsymbol{L}^1(\mathbb{R}^n)$$

$$\mathscr{H}:\boldsymbol{BMO}(\mathbb{R}^n)\times\boldsymbol{H}^1(\mathbb{R}^n)\longrightarrow\boldsymbol{H}^{\wp}(\mathbb{R}^n,\,\mu)$$

such that for every $\mathfrak{b}\in\boldsymbol{BMO}(\mathbb{R}^n)$ *and* $\mathfrak{h}\in\boldsymbol{H}^1(\mathbb{R}^n)$ *it holds*

$$\mathfrak{b}\times\mathfrak{h}=\mathscr{L}(\mathfrak{b}\,,\mathfrak{h}) + \mathscr{H}(\mathfrak{b}\,,\mathfrak{h})$$

and

$$\|\mathscr{L}(\mathfrak{b}\,,\mathfrak{h})\|_{L^1} + \|\mathscr{H}(\mathfrak{b}\,,\mathfrak{h})\|_{H^{\wp}} \preccurlyeq \|\,\mathfrak{h}\,\|_{H^1}\,\|\,\mathfrak{b}\,\|_{BMO^+}\,.$$

In applications to nonlinear PDEs, the distribution $\mathfrak{b}\times\mathfrak{h}\in\mathscr{D}'(\mathbb{R}^n)$ is used to justify weak continuity properties of the point-wise product $\mathfrak{b}\cdot\mathfrak{h}$. It is therefore important to recover $\mathfrak{b}\cdot\mathfrak{h}$ from the action of the distribution $\mathfrak{b}\times\mathfrak{h}$ on the test functions. An idea that naturally comes to mind is to look at the mollified distributions

$$(\mathfrak{b}\times\mathfrak{h})_\varepsilon=(\mathfrak{b}\times\mathfrak{h})*\Phi_\varepsilon\,,\qquad\text{and let }\varepsilon\to 0.$$

As a consequence of Theorem 1.6, we will see that the limit exists and equals $\mathfrak{b}\cdot\mathfrak{h}$ almost everywhere.

Theorem 1.8. — *Let* $\mathfrak{b}\in\boldsymbol{BMO}(\mathbb{R}^n)$ *and* $\mathfrak{h}\in\boldsymbol{H}^1(\mathbb{R}^n)$. *For almost every* $x\in\mathbb{R}^n$ *it holds*

$$\lim_{\varepsilon\to 0}(\mathfrak{b}\times\mathfrak{h})_\varepsilon(x) = \mathfrak{b}(x)\cdot\mathfrak{h}(x)\,. \tag{1.29}$$

Here is another interesting fact. Suppose that $\mathfrak{b}\cdot\mathfrak{h}$ is nonnegative almost everywhere in an open set $\Omega \subset \mathbb{R}^n$. Then, as we have already mentioned, $\mathfrak{b}\cdot\mathfrak{h}$ lies in $\boldsymbol{L}^1_{\mathrm{loc}}(\Omega) \subset \mathscr{D}'(\Omega)$ and coincides with the distribution $\mathfrak{b}\times\mathfrak{h} \in \mathscr{D}'(\Omega)$. The reader is urged to distinguish between the hypothesis $\mathfrak{b}(x)\cdot\mathfrak{h}(x) \geqslant 0$, for almost every $x \in \Omega$, and that of $\mathfrak{b}\times\mathfrak{h}$ being a nonnegative distribution on Ω. This latter hypothesis precisely means that $\mathfrak{b}\times\mathfrak{h}$ is a Borel measure on Ω (which is practically impossible to verify without understanding the regularity properties of the point-wise product). That in this case the measure $\mathfrak{b}\times\mathfrak{h}$ contains no singular part is not entirely obvious; it is indeed a consequence of the point-wise approximation at (1.29).

Corollary 1.9. — *Let $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$ and $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ and let Ω be an open subset of $\mathbb{R}^n$. The following conditions are equivalent*

i) $\langle\, \mathfrak{b}\times\mathfrak{h} \mid \varphi \rangle \geqslant 0$, *for all nonnegative* $\varphi \in \mathscr{C}_0^\infty(\Omega)$

ii) $\mathfrak{b}(x)\cdot\mathfrak{h}(x) \geqslant 0$, *for almost every* $x \in \Omega$.

In either case the point-wise product $\mathfrak{b}\cdot\mathfrak{h}$ is locally integrable on Ω and, as a distribution, coincides with $\mathfrak{b}\times\mathfrak{h} \in \mathscr{D}'(\Omega)$.

If $\mathfrak{b}\times\mathfrak{h} \in \mathscr{D}'(\mathbb{R}^n)$ is subjected to no restriction concerning the sign, we still observe an improved regularity phenomenon.

Theorem 1.10. — *Let $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$ and $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$. Then the distribution $\mathfrak{b}\times\mathfrak{h} \in \mathscr{D}'(\mathbb{R}^n)$ belongs to the grand Hardy space $\boldsymbol{H}^{1)}_{loc}(\mathbb{R}^n)$, and we have the estimate*

$$\| \mathfrak{b}\times\mathfrak{h} \|_{H^{1)}(\Omega)} \preccurlyeq \| h \|_{H^1(\mathbb{R}^n)} \cdot \| \mathfrak{b} \|_{BMO^+(\mathbb{R}^n)} \,. \tag{1.30}$$

for every bounded open subset $\Omega \subset \mathbb{R}^n$,

In symbols,

$$\boldsymbol{BMO}(\mathbb{R}^n) \times \boldsymbol{H}^1(\mathbb{R}^n) \subset \boldsymbol{L}^1(\mathbb{R}^n) + \boldsymbol{H}^\wp(\mathbb{R}^n) \subset \boldsymbol{H}^{1)}_{\mathrm{loc}}(\mathbb{R}^n)\,. \tag{1.31}$$

Next recall that $\boldsymbol{L}^1_{\mathrm{weak}}(\Omega) \subset \boldsymbol{L}^{1)}(\Omega)$. Hence it is entirely natural to ask whether $\mathfrak{b}\times\mathfrak{h}$ lies in the weak Hardy space $\boldsymbol{H}^1_{\mathrm{weak}}(\Omega)$, meaning that $\mathscr{M}(\mathfrak{b}\times\mathfrak{h})$ belongs to the Marcinkiewicz class $\boldsymbol{L}^1_{\mathrm{weak}}(\Omega)$. While this is obviously the case for $\mathscr{L}_{\mathfrak{h}}\mathfrak{b}$ in (1.24), the $\mathscr{H}_{\mathfrak{h}}\mathfrak{b}$ -component need not be so nice. Nevertheless, $\mathscr{M}(\mathfrak{b}\times\mathfrak{h})$ has properties reminiscent of $\mathfrak{M}f$, where $f \in \boldsymbol{L}^1(\mathbb{R}^n)$. For example, the limit at (1.21) continues to be equal to zero if $M = \mathscr{M}(\mathfrak{b}\times\mathfrak{h}) \notin \boldsymbol{L}^1_{\mathrm{weak}}(\Omega)$. That is:

$$\lim_{p\nearrow 1} (1-p)\int_\Omega |\mathscr{M}(\mathfrak{b}\times\mathfrak{h})|^p = 0\,. \tag{1.32}$$

This additional regularity of $\mathscr{M}(\mathfrak{b} \times \mathfrak{h})$ follows immediately from Theorem 1.6 once we observe a general fact that (1.21) is true for arbitrary $M \in \boldsymbol{L} \log^{-1} \boldsymbol{L}(\Omega)$.

We shall give two proofs of Theorem 1.6, the first one based on *div-curl* atoms, the second one on the atomic decomposition of $\boldsymbol{H}^1(\mathbb{R}^n)$. This last one generalizes in different contexts, such as the dyadic one, or Hardy spaces on the boundary of the complex unit ball.

The first proof is not valid in the case $n = 1$, which is rather special, but another proof, based on complex analysis, is available. For simplicity we shall then skip to the periodic setting and work with the Hardy space $\mathscr{H}^1(\mathbb{D})$ of analytic functions in the unit disk $\mathbb{D} \subset \mathbb{R}^2 \cong \mathbb{C}$ and the associated analytic BMO-space, denoted by $\mathscr{BMO}(\mathbb{D})$, see [20, 63]. Rather unexpectedly, in this context the analogue of Theorem 1.6 is more precise and elegant.

THEOREM 1.11. — *Product of functions in $\mathscr{BMO}(\mathbb{D})$ and $\mathscr{H}^1(\mathbb{D})$ lies in $\mathscr{H}^{\wp}(\mathbb{D})$, with $\wp(t) = \frac{t}{\log(e+t)}$. Moreover, we have the equality*

$$\mathscr{BMO}(\mathbb{D}) \cdot \mathscr{H}^1(\mathbb{D}) = \mathscr{H}^{\wp}(\mathbb{D}). \tag{1.33}$$

In Section 8 we consider the product of functions in $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{BMO}(\Omega)$, for Ω a bounded Lipschitz domain, and make some remarks on the definition of $\boldsymbol{H}^1(\Omega)$, which may be of independent interest, based on the developments in [8, 15, 42, 43, 45, 54].

Epilog

There are several natural reasons for investigating the distribution $\mathfrak{b} \times \mathfrak{h}$. *First*, in PDEs we find various nonlinear differential expressions identified by the theory of compensated compactness, see the seminal work of F. Murat [49] and L. Tartar [59], and the subsequent developments [16, 17, 25]. New and unexpected phenomena concerning higher integrability of the Jacobian determinants and other null Lagrangians have been discovered [46, 29, 33, 28, 21] , and used in the geometric function theory [27, 26, 1], calculus of variations [58, 30], and some areas of applied mathematics, [47, 48, 62]. Recently a viable theory of existence and improved regularity for solutions of PDEs where the uniform ellipticity is lost, has been built out of the distributional *div-curl* products and null Lagrangians [25, 31] . *Second*, these investigations bring us to new classes of functions, distributions and measures [32] , just to mention the grand $\boldsymbol{L}^p$ -spaces [29, 24, 53].

Subtle and clever ideas of convergence in these spaces have been adopted from probability and measure theory, biting convergence for instance [7, 4, 5, 62]. Recent investigations of so-called very weak solutions of nonlinear PDEs [30, 24] rely on these new classes of functions.
Third, it seems likely that these methods will shed new light on harmonic analysis with more practical applications.

2. Div-Curl Atoms

A key to our first proof of Theorem 1.6 is the use of div-curl products as generators of $\boldsymbol{H}^1(\mathbb{R}^n)$. We shall draw on the seminal ideas in [10]. Consider vector fields $E \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ with div $E = 0$ and $B \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ with curl $B = 0$, where $1 < p, q < \infty$ is a fixed Hölder conjugate pair, $p+q = pq$. The inner product $E \cdot B$ lies in the Hardy space $\boldsymbol{H}^1(\mathbb{R}^n)$ and we have a uniform bound,

$$\| E \cdot B \|_{H^1} \preccurlyeq \| E \|_p \cdot \| B \|_q . \tag{2.1}$$

We shall, by convenient abuse of previous terminology, continue to call $E \cdot B$ the div-curl atom. In [CLMS] the authors conjecture that for every element $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$, $n \geqslant 2$, the *Jacobian Equation*

$$\mathcal{J}(x, F) = \mathfrak{h} \ , \qquad \text{has a solution} \quad F \in \boldsymbol{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) .$$

In particular, every $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ is a single div-curl atom; $p = n \, , q = \frac{n}{n-1}$. We are inclined to conjecture more specific way of solving this equation:

Conjecture 2.1 (Resolvent of the Jacobian operator). — *There exists a continuous (nonlinear) operator* $\mathscr{F} : \boldsymbol{H}^1(\mathbb{R}^n) \to \boldsymbol{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ *such that*

$$\mathcal{J}(x, \mathscr{F}\mathfrak{h}) = \mathfrak{h} \ , \quad \textit{for every} \quad \mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n) \ , \ n \geqslant 2 .$$

The following result motivates our calling $E \cdot B$ a div-curl atom.

Proposition 2.2 (Div-Curl Decomposition). — *To every* $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ *there correspond vector fields* $E_\nu, B_\nu \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $\nu = 1, 2, \ldots$, *with* div $E_\nu = 0$ *and* curl $B_\nu = 0$, *such that*

$$\mathfrak{h} = \sum_{\nu=1}^{\infty} E_\nu \cdot B_\nu \tag{2.2}$$

$$\| \mathfrak{h} \|_{H^1} \preccurlyeq \sum_{\nu=1}^{\infty} \|E_\nu\|_p \cdot \|B_\nu\|_q \preccurlyeq \| \mathfrak{h} \|_{H^1} \ . \tag{2.3}$$

The proof is based upon the arguments for Theorem III.2 in [10], with one principal modification. First, the decomposition at (2.2) with $E \in \boldsymbol{L}^p(\mathbb{R}^n, \mathbb{R}^n)$ and $B \in \boldsymbol{L}^q(\mathbb{R}^n, \mathbb{R}^n)$, follows in much the same way as demonstrated in [10] for $p = q = 2$. However, by an approximation argument one can easily ensure an additional regularity that the vector fields E and B actually lie in the space $\mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Details are left to the reader.

In what follows we fix, largely for aesthetical reason, the following Hölder conjugate exponents:

$$p = n + 1 \quad \text{and} \quad q = \frac{1}{n} + 1. \tag{2.4}$$

One major advantage of using the expressions $E_\nu \cdot B_\nu$ over the usual H^1-atoms is their product structure. We will be able to apply singular integrals and maximal operators in the spaces $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $L^q(\mathbb{R}^n, \mathbb{R}^n)$, where those operators are bounded.

3. A Few Prerequisites

We note two elementary inequalities

$$ab \leqslant a \log(1 + a) + e^b - 1 \tag{3.1}$$

$$\frac{ab}{\log(e + ab)} \leqslant a + e^b - 1 \tag{3.2}$$

for $a, b \geqslant 0$. Of these, the first is the key to the duality between $\operatorname{Exp} \boldsymbol{L}$ and $\boldsymbol{L} \log \boldsymbol{L}$. Indeed, we have the inequality

$$\| fg \|_{L^1} \leqslant 2 \| f \|_{L \log L} \| g \|_{\operatorname{Exp} L} \tag{3.3}$$

where the Orlicz functions defining $\boldsymbol{L} \log \boldsymbol{L}$ and $\operatorname{Exp} \boldsymbol{L}$ are $t \log(e + t)$ and $e^t - 1$, respectively.

Next we recall the space $\boldsymbol{L} \log^{-1} \boldsymbol{L} = \boldsymbol{L}^\wp$, where $\wp(t) = t \log^{-1}(e + t)$. With the aid of (3.2) we obtain Hölder's inequality

$$\llbracket fg \rrbracket_\wp \leqslant 4 \, \| f \|_{L^1} \| g \|_{\operatorname{Exp} L} . \tag{3.4}$$

Although the functional $\llbracket \quad \rrbracket_\wp$ fails to be subadditive we still have a substitute for the triangle inequality

$$\llbracket f + g \rrbracket_\wp \ \leqslant \ 4 \llbracket f \rrbracket_\wp \ + \ 4 \llbracket g \rrbracket_\wp \ . \tag{3.5}$$

There are no substitutes of subadditivity for infinite sums. In the above inequalities we have understood that the underlying measure is the same for each space. However, weighted exponential classes $\operatorname{Exp} \boldsymbol{L}(\mathbb{R}^n, \sigma)$ are better suited for the study of functions with bounded mean oscillations. In this connection we recall the familiar John-Nirenberg inequalities. Let $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$ be nonconstant and let Q be a cube in $\mathbb{R}^n$. Then

$$\fint_Q \exp\left(\frac{|\mathfrak{b}(x) - \mathfrak{b}_Q|}{\lambda\ \|\mathfrak{b}\|_{BMO}}\right)\, dx \leqslant 2\,, \tag{3.6}$$

where $\lambda > 0$ depends only on the dimension. It is apparent from these inequalities that $\boldsymbol{BMO}(\mathbb{R}^n) \subset \operatorname{Exp}\boldsymbol{L}(\mathbb{R}^n, \sigma)$ for some weights σ, for instance when

$$d\sigma(x) = \frac{dx}{(1+|x|)^{2n}}\ . \tag{3.7}$$

Let us state, without proof, the following lemma.

Lemma 3.1. — *Let const* $\not\equiv \mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$ *and* $\mathfrak{b}_{\mathbb{Q}}$ *denote the integral mean of* $\mathfrak{b}$ *over the unit cube* $\mathbb{Q} \subset \mathbb{R}^n$. *Then*

$$\int_{\mathbb{R}^n} \frac{\left(e^{\frac{|\mathfrak{b}(x) - \mathfrak{b}_{\mathbb{Q}}|}{k}} - 1\right) dx}{(1+|x|)^{2n}} \leqslant 1 \tag{3.8}$$

where $k = C_n \|\mathfrak{b}\|_{BMO}$. *In other words, we have*

$$\|\mathfrak{b} - \mathfrak{b}_{\mathbb{Q}}\|_{\operatorname{Exp}\boldsymbol{L}(\mathbb{R}^n,\sigma)} \preccurlyeq \|\mathfrak{b}\|_{BMO(\mathbb{R}^n)}. \tag{3.9}$$

From now on we shall make use of the abbreviated notation

$$\boldsymbol{L}^{\Xi} = \operatorname{Exp}\boldsymbol{L} = \operatorname{Exp}\boldsymbol{L}(\mathbb{R}^n, \sigma)\ . \tag{3.10}$$

We want to introduce a weight into the space $\boldsymbol{L}^{\wp}(\mathbb{R}^n) = \boldsymbol{L}\log^{-1}\boldsymbol{L}(\mathbb{R}^n)$ so that Hölder's inequality at (3.4) will hold with $\boldsymbol{L}^1 = \boldsymbol{L}^1(\mathbb{R}^n, dx)$. This weight has already been discussed in the introduction, i.e.

$$d\mu(x) = \frac{dx}{\log(e+|x|)}\ . \tag{3.11}$$

The notation for the space $\boldsymbol{L}\log^{-1}\boldsymbol{L}$ with respect to this weight is abbreviated to:

$$\boldsymbol{L}^{\wp} = \boldsymbol{L}^{\wp}(\mathbb{R}^n) = \boldsymbol{L}^{\wp}(\mathbb{R}^n, \mu) = \boldsymbol{L}\log^{-1}\boldsymbol{L}(\mathbb{R}^n). \tag{3.12}$$

The following two Hölder type inequalities will be used in the proofs.

Lemma 3.2. — *Let* $\mathfrak{b} \in \operatorname{Exp}\boldsymbol{L}(\mathbb{R}^n, \sigma) = \boldsymbol{L}^{\Xi}$. *Then for each* $\lambda \in \boldsymbol{L}^1(\mathbb{R}^n)$ *we have* $\lambda \cdot \mathfrak{b} \in \boldsymbol{L}^{\wp}(\mathbb{R}^n, \mu)$ *and*

$$[\![\lambda \cdot \mathfrak{b}]\!]_{\wp} \preccurlyeq \|\lambda\|_1 \cdot \|\mathfrak{b}\|_{\Xi}\ . \tag{3.13}$$

If, moreover, $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n) \subset \operatorname{Exp}\boldsymbol{L}(\mathbb{R}^n, \sigma)$ *then*

$$[\![\lambda \cdot \mathfrak{b}]\!]_{\wp} \preccurlyeq \|\lambda\|_1 \cdot \|\mathfrak{b}\|_{BMO^+}\ . \tag{3.14}$$

Proof. — We will prove (3.13) with the constant $64n^2$. As both sides of (3.13) have the same homogeneity with respect to λ and $\mathfrak{b}$, we may assume that $\|\lambda\|_1 = \|\mathfrak{b}\|_\Xi = \frac{1}{8n}$ and prove the inequality

$$\int_{\mathbb{R}^n} \frac{|\lambda(x)\mathfrak{b}(x)|\ dx}{\log[\,e+|\lambda(x)\mathfrak{b}(x)|\,]\ \log(e+|x|)} \leqslant 1. \tag{3.15}$$

We use the two elementary inequalities:

$$2n\ \log(e+|x|) > \log(e+(1+|x|^{2n}),$$

and, for $a, b > 0$,

$$\log(e+a)\log(e+b) > \frac{1}{2}\log(e+ab).$$

Combining the above inequalities with (3.2), we now estimate the integrand at (3.15)

$$\frac{|\lambda\mathfrak{b}|}{\log(e+|\lambda\mathfrak{b}|)\log(e+|x|)} \leqslant 4n|\lambda| + 4n\frac{e^{|\mathfrak{b}|}-1}{(1+|x|)^{2n}}.$$

It remains to observe that upon integration the right hand side will be bounded by 1, because of our normalization $\|\lambda\|_1 = \|\mathfrak{b}\|_\Xi = \frac{1}{8n}$, the definition of the norm in $\operatorname{Exp}\boldsymbol{L}(\mathbb{R}^n,\sigma)$ and the elementary inequality $8n(e^a-1) \leqslant e^{8na}-1$. This establishes the first inequality of the lemma.

The second inequality is obvious when $\mathfrak{b}$ is constant. To see the general case we apply (3.13) to the function $\mathfrak{b}-\mathfrak{b}_{\mathbb{Q}}$ in place of $\mathfrak{b}$, where $\mathbb{Q}$ is the unit cube, to obtain

$$\begin{aligned}
\|\lambda\mathfrak{b}\|_{\wp} &\leqslant 4\,\|\,(\mathfrak{b}-\mathfrak{b}_{\mathbb{Q}})\ \lambda\,\|_{\wp} + 4\,\|\mathfrak{b}_{\mathbb{Q}}\,\lambda\,\|_{\wp}\\
&\preccurlyeq \|\mathfrak{b}-\mathfrak{b}_{\mathbb{Q}}\|_\Xi\,\|\lambda\|_1 + |\mathfrak{b}_{\mathbb{Q}}|\ \|\lambda\|_1\\
&\preccurlyeq \|\mathfrak{b}\|_{BMO}\,\|\lambda\|_1 + |\mathfrak{b}_{\mathbb{Q}}|\ \|\lambda\|_1\\
&\preccurlyeq \|\lambda\|_1\,\|\mathfrak{b}\|_{BMO^+}
\end{aligned}$$

as desired. □

4. Construction of Operators $\mathscr{L}_{\mathfrak{h}}$ and $\mathscr{H}_{\mathfrak{h}}$ via div-curl atoms

In this section and the following, the dimension n is larger than one. We propose decomposition operators that are defined using a div-curl decomposition. This makes particularly sense in view of Conjecture 2.1.

The familiar Helmholtz decomposition of a vector field $\mathbf{V} \in \boldsymbol{L}^2(\mathbb{R}^n, \mathbb{R}^n)$, also known as Hodge decomposition, asserts that

$$\mathbf{V} = \nabla u + \mathbf{F} \tag{4.1}$$

where $\nabla u \in \boldsymbol{L}^2(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathbf{F}$ is a divergence free vector field in $\boldsymbol{L}^2(\mathbb{R}^n, \mathbb{R}^n)$. Integration by parts shows that $\mathbf{F}$ and ∇u are orthogonal. These orthogonal components of $\mathbf{V}$ are unique and can be expressed explicitly in terms of $\mathbf{V}$ by using Riesz transforms $\mathscr{R} = (\mathscr{R}_1, \dots, \mathscr{R}_n)$ in $\mathbb{R}^n$. Precisely we have

$$-\nabla u = \mathscr{R}(\mathscr{R} \cdot \mathbf{V}) \stackrel{\text{def}}{=\!=} \mathscr{R}\left(\mathscr{R}_1 V^1 + \ldots + \mathscr{R}_n V^n\right) = [\mathscr{R} \otimes \mathscr{R}]\mathbf{V} \tag{4.2}$$

where $(V^1, \dots, V^n)$ are the coordinate functions of $\mathbf{V}$ and the tensor product $\mathscr{R} \otimes \mathscr{R}$ is the matrix of the second order Riesz transforms, $\mathscr{R} \otimes \mathscr{R} = [\mathscr{R}_i \circ \mathscr{R}_j]$. Thus the divergence free component takes the form

$$\mathbf{F} = (\mathscr{I} + \mathscr{R} \otimes \mathscr{R})\mathbf{V} \ , \quad \text{where } \mathscr{I} \text{ denotes the identity operator .} \tag{4.3}$$

We introduce two singular integral operators

$$\mathscr{A} = \mathscr{I} + \mathscr{R} \otimes \mathscr{R} \quad \text{and} \quad \mathscr{B} = -\mathscr{R} \otimes \mathscr{R} \ . \tag{4.4}$$

These are none other than the orthogonal projections of $\boldsymbol{L}^2(\mathbb{R}^n, \mathbb{R}^n)$ onto divergence free and curl free vector fields, respectively.

Of course these operators extend continuously to all $L^s(\mathbb{R}^n, \mathbb{R}^n)$, with $1 < s < \infty$. By the definition,

$$\mathscr{A} + \mathscr{B} = \mathscr{I} : \boldsymbol{L}^s(\mathbb{R}^n, \mathbb{R}^n) \to \boldsymbol{L}^s(\mathbb{R}^n, \mathbb{R}^n). \tag{4.5}$$

An important point to emphasize is that $\mathscr{A}$ vanishes on the gradients (the curl free vector fields), while $\mathscr{B}$ acts as identity on the gradients. This observation is immediate from the uniqueness in the Hodge decomposition.

Next we recall Proposition 4.6. Accordingly, every function $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ can be expressed as infinite sum of *div-curl* atoms,

$$\mathfrak{h} = \sum_{\nu=1}^{\infty} E_\nu \cdot B_\nu \ , \qquad \begin{pmatrix} E_\nu \in \mathscr{C}_0^\infty(\mathbb{R}^n) \ , & \operatorname{div} E_\nu = 0 \\ B_\nu \in \mathscr{C}_0^\infty(\mathbb{R}^n) \ , & \operatorname{curl} B_\nu = 0 \end{pmatrix} \tag{4.6}$$

where

$$\sum_{\nu=1}^{\infty} \|E_\nu\|_p \cdot \|B_\nu\|_q \ \approx \ \|\mathfrak{h}\|_{H^1} \ , \quad p = 1 + n \ , \quad q = 1 + \frac{1}{n} \ . \tag{4.7}$$

Now we define the decomposition operators at (1.24) and (1.25) by the rules

$$\mathscr{L}_{\mathfrak{h}}\, \mathfrak{b} \stackrel{\text{def}}{=\!=} \mathfrak{b}_{\mathbb{Q}} \cdot \mathfrak{h} \ + \ \sum_{\nu=1}^{\infty} E_\nu \cdot \mathscr{A}(\mathfrak{b} B_\nu) \tag{4.8}$$

$$(4.9) \qquad \mathscr{H}_{\mathfrak{h}}\,\mathfrak{b} \stackrel{\text{def}}{=} -\mathfrak{b}_{\mathbb{Q}}\cdot\mathfrak{h} + \sum_{\nu=1}^{\infty} E_\nu\cdot\mathscr{B}(\mathfrak{b}B_\nu)$$

for every $\mathfrak{b}\in \boldsymbol{BMO}(\mathbb{R}^n)$. Both series converge in $\mathscr{D}'(\mathbb{R}^n)$ and absolutely almost everywhere, which will easily be seen from the forthcoming estimates. What we want to show is that the partial sums of (4.8) converge in $\boldsymbol{L}^1(\mathbb{R}^n)$, while the partial sums of (4.9) converge in $\boldsymbol{H}^{\wp}(\mathbb{R}^n,\mu)$. Once such convergence is established $\mathscr{L}_{\mathfrak{h}}$ and $\mathscr{H}_{\mathfrak{h}}$ become well defined bounded linear operators on the space $\boldsymbol{BMO}(\mathbb{R}^n)$.

5. Proof of the decomposition theorem through div-curl atoms

We aim to give an L^1-bound for $\mathscr{L}_{\mathfrak{h}}\mathfrak{b}$ and $H^{\wp}$-bound for $\mathscr{H}_{\mathfrak{h}}\mathfrak{b}$, where $\mathfrak{b}$ is an arbitrary function in $\boldsymbol{BMO}(\mathbb{R}^n)$. As we remarked before, we can assume that $\mathfrak{b}_{\mathbb{Q}}=0$ without loss of generality, which we do in the sequel. As a consequence, Formulas (4.8) and (4.9) have no first term.

Let us begin with the easy case.

Estimate of $\mathscr{L}_{\mathfrak{h}}\mathfrak{b}$.

We have already mentioned that $\mathscr{A}$ vanishes on the curl free vector fields. So

$$\mathscr{L}_{\mathfrak{h}}\mathfrak{b} = \sum_{\nu=1}^{\infty} E_\nu\cdot(\mathscr{A}\mathfrak{b}-\mathfrak{b}\mathscr{A})\,B_\nu\ .$$

Here $\mathscr{A}\mathfrak{b}-\mathfrak{b}\mathscr{A}$ denotes the commutator of $\mathscr{A}$ and the operator of multiplication by the function $\mathfrak{b}\in\boldsymbol{BMO}(\mathbb{R}^n)$. At this point we recall the fundamental estimate of R. Coifman, R. Rochberg and G. Weiss [12]

$$(5.1) \qquad \|(\mathscr{A}\mathfrak{b}-\mathfrak{b}\mathscr{A})B_\nu\|_q \preccurlyeq \|\mathfrak{b}\|_{BMO}\,\|B_\nu\|_q\ .$$

We conclude with the estimate of $\mathscr{L}_{\mathfrak{h}}\mathfrak{b}$ claimed at (1.27), by using Hölder's Inequality.

Estimate of $\mathscr{H}_{\mathfrak{h}}\mathfrak{b}$.

Recall that we assumed that $\mathfrak{b}_{\mathbb{Q}}=0$, and Formula (4.9), in view of (4.6), reduces to

$$(5.2) \qquad \mathscr{H}_{\mathfrak{h}}\mathfrak{b} = \sum_{\nu=1}^{\infty} E_\nu\cdot\mathscr{B}(\,\mathfrak{b}B_\nu\,)\ .$$

This can be compared to the div-curl decomposition, except that the curl-free parts, $\mathscr{B}(\,(\mathfrak{b}B_\nu))$ do not satisfy the same estimates in the $\boldsymbol{L}^q$ space.

Some explanation concerning convergence of this series is in order. First, to make our arguments rigorous, we consider finite sums

$$S_k^l = \sum\nolimits^{\sharp} E_\nu \cdot \mathscr{B}(\,(\mathfrak{b}B_\nu\,) \stackrel{\mathrm{def}}{=\!=} \sum_{\nu=k}^{\nu=l} E_\nu \cdot \mathscr{B}(\,(\mathfrak{b}B_\nu\,)\ , \quad \text{for } \ k < l\,,$$

which we shall use to verify the Cauchy condition for convergence of the infinite series with respect to the metric in $\boldsymbol{H}^{\wp}(\mathbb{R}^n)$. In view of Remark 1.3, this is equivalent to showing that

$$\lim_{k\to\infty} \llbracket\, \mathscr{M} S_k^l\, \rrbracket_{\wp} = 0\,.$$

Note that $\mathfrak{b}B_\nu \in \boldsymbol{L}^s(\mathbb{R}^n, \mathbb{R}^n)$ for all $1 < s < \infty$, because $B_\nu \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. The same is valid for its component $\mathscr{B}(\,(\mathfrak{b}B_\nu)$. At this point we need to recall the Hardy-Littlewood maximal operator

$$\mathfrak{M} : L^s(\mathbb{R}^n) \longrightarrow L^s(\mathbb{R}^n), \quad 1 < s < \infty\,.$$

We then refer to the proof of the div-curl lemma given in [10]. Their fundamental estimate written in this context gives the inequality

$$\mathscr{M}\,(E_\nu \cdot \mathscr{B}(\,\mathfrak{b}B_\nu)) \preccurlyeq \big(\,\mathfrak{M}|E_\nu|^n\,\big)^{\frac{1}{n}} \cdot \mathfrak{M}|\mathscr{B}(\mathfrak{b}B_\nu)|\,. \tag{5.3}$$

Let us first examine the term $\mathfrak{M}|\mathscr{B}(\mathfrak{b}B_\nu)|$, for which we have the inequality

$$\mathfrak{M}|\mathscr{B}(\mathfrak{b}B_\nu)| \leqslant \mathfrak{M}|\mathfrak{b}B_\nu| + \mathfrak{M}|\,(\mathscr{A}\mathfrak{b} - \mathfrak{b}\mathscr{A})B_\nu\,|\,.$$

We wish to move the factor $\mathfrak{b}$ in $\mathfrak{M}(\mathfrak{b}B_\nu)$ outside the maximal operator. A device for this procedure is the following commutator:

$$\mathfrak{M}|\mathfrak{b}| - |\mathfrak{b}|\mathfrak{M} : \boldsymbol{L}^s(\mathbb{R}^n) \longrightarrow \boldsymbol{L}^s(\mathbb{R}^n), \quad 1 < s < \infty. \tag{5.4}$$

The inequality now takes the form:

$$\mathfrak{M}|\mathscr{B}(\mathfrak{b}B_\nu)| \leqslant |\mathfrak{b}| \cdot \mathfrak{M}|B_\nu| + \mathfrak{X}_\nu, \tag{5.5}$$

where

$$\mathfrak{X}_\nu = \big(\mathfrak{M}|\mathfrak{b}| - |\mathfrak{b}|\mathfrak{M}\big)|B_\nu| + \mathfrak{M}|(\mathscr{A}\mathfrak{b} - \mathfrak{b}\mathscr{A})B_\nu|\,.$$

A point to make here is that $\|\mathfrak{X}_\nu\|_q$ is controlled by $\|B_\nu\|_q$ and the BMO-norm of $\mathfrak{b}$. Precise bounds are furnished by the inequality due to M. Milman and T. Schonbek [44]. It asserts that

$$\big\|(\mathfrak{M}|\mathfrak{b}| - |\mathfrak{b}|\mathfrak{M})\, f\big\|_s \preccurlyeq \|\mathfrak{b}\|_{BMO} \|f\|_s \qquad \text{for all} \quad 1 < s < \infty. \tag{5.6}$$

Similarly, we can use (5.1) for the commutator $\mathscr{A}\mathfrak{b} - \mathfrak{b}\mathscr{A}$.

These estimates, combined with the usual maximal inequalities, yield

$$\|\mathfrak{X}_\nu\|_q \preccurlyeq \|B_\nu\|_q \cdot \|\mathfrak{b}\|_{BMO}\,. \tag{5.7}$$

We can now return (5.3) to obtain the inequality

$$\mathscr{M}(E_\nu \cdot \mathscr{B}(\mathfrak{b} B_\nu)) \preccurlyeq (\mathfrak{M}|E_\nu|^n)^{\frac{1}{n}} \cdot [\, \mathfrak{X}_\nu + |\mathfrak{b}|\, \mathfrak{M}|B_\nu|\,],$$

which upon substitution into (5.2) yields

$$(\mathscr{M} S_k^l)(x) \preccurlyeq \lambda(x) \cdot |\mathfrak{b}(x)| + A(x) \quad , \tag{5.8}$$

where

$$\lambda = \sum\nolimits^{\sharp} (\mathfrak{M}|E_\nu|^n)^{\frac{1}{n}} \cdot \mathfrak{M}(B_\nu) \tag{5.9}$$

$$A = \sum\nolimits^{\sharp} (\mathfrak{M}|E_\nu|^n)^{\frac{1}{n}} \cdot \mathfrak{X}_\nu\,. \tag{5.10}$$

We approach the critical point of our computation. The goal is to show that λ and A are integrable functions. Note explicitly that direct approach to $L^\wp$-estimates of an infinite series (term by term estimates), or even its finite partial sums such as (5.2), would fail. This failure is due to the lack of countable subadditivity of $\|\ \|_\wp$. Whereas, making the L^1-estimates independently of the number of terms poses no difficulty. It goes as follows:

$$\|\lambda\|_1 \preccurlyeq \sum\nolimits^{\sharp} \|(\mathfrak{M}|E_\nu|^n)^{\frac{1}{n}}\|_p \cdot \|\mathfrak{M}(B_\nu)\|_q \preccurlyeq \sum\nolimits^{\sharp} \|E_\nu\|_p\, \|B_\nu\|_q$$

where we have used the maximal inequalities in the space $\boldsymbol{L}^q(\mathbb{R}^n)$, with $q = \frac{n+1}{n} > 1$; recall that $p = qn$. In much the same way we estimate the L^1-norm of A by (5.7)

$$\|A\|_1 \preccurlyeq \sum\nolimits^{\sharp} \|E_\nu\|_p\, \|\mathfrak{X}_\nu\|_q \preccurlyeq \Big(\sum\nolimits^{\sharp} \|E_\nu\|_p\, \|B_\nu\|_q\Big)\ \|\mathfrak{b}\|_{BMO}\,.$$

We are now in a position to estimate the maximal function of S_k^l. First, by (3.5), we can write

$$\|\mathscr{M} S_k^l)\|_\wp \leqslant 4\,\|\lambda\mathfrak{b}\|_\wp + 4\,\|A\|_\wp\,.$$

Then, with the aid of (3.14), we arrive at the estimate

$$\|S_k^l\|_{H^\wp(\mathbb{R}^n)} \preccurlyeq \|A\|_1 + \|\lambda\|_1 \cdot \|\mathfrak{b}\|_{BMO} \preccurlyeq \Big(\sum\nolimits^{\sharp} \|E_\nu\|_p\, \|B_\nu\|_q\Big)\ \|\mathfrak{b}\|_{BMO}\,. \tag{5.11}$$

Recall that the right hand side stands for $\Big(\sum_{\nu=k}^{\nu=l} \|E_\nu\|_p\, \|B_\nu\|_q\Big)\ \|\mathfrak{b}\|_{BMO}$. This estimate, in view of $\sum_{\nu=1}^{\nu=\infty} \|E_\nu\|_p\, \|B_\nu\|_q \approx \|\mathfrak{h}\|_{H^1} < \infty$, ensures the Cauchy condition for the convergence of the infinite series (5.2) in the metric topology of the space $\boldsymbol{H}^\wp(\mathbb{R}^n)$. Put $k = 1$ and let l go to infinity, to obtain the desired inequality:

$$\|\mathscr{H}_{\mathfrak{h}}\mathfrak{b}\|_{H^\wp} \preccurlyeq \|\mathfrak{h}\|_{H^1} \cdot \|\mathfrak{b}\|_{BMO} \tag{5.12}$$

completing the proof of Theorem 1.

Let us prove (1.26). By continuity of the linear functional associated to $\mathfrak{b}$,

$$\int_{\mathbb{R}^n}^{*} \mathfrak{b}(x)\mathfrak{h}(x)\,dx \;= \sum \int_{\mathbb{R}^n} \mathfrak{b}(x)E_\nu(x)\cdot B_\nu(x)\,dx.$$

The integral of a div-curl atom is zero, so that

$$\int_{\mathbb{R}^n} \mathfrak{b}E_\nu \cdot B_\nu\,dx = \int_{\mathbb{R}^n} E_\nu \cdot \mathscr{A}(bB_\nu)\,dx,$$

from which we conclude.

6. Proof of the decomposition theorem through classical atoms

For Q a cube in $\mathbb{R}^n$, recall that a Q-atom is a bounded function a that vanishes outside Q, has mean zero and satisfies the inequality $\|a\|_\infty \leqslant |Q|^{-1}$. The atomic decomposition goes as follows (see [56] for instance):

Proposition 6.1 (Atomic Decomposition). — *To every $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ there correspond scalars λ_ν, cubes Q_ν and Q_ν-atoms a_ν, $\nu = 1, 2, \ldots$, such that*

$$\mathfrak{h} = \sum_{\nu=1}^{\infty} \lambda_\nu\, a_\nu \tag{6.1}$$

$$\sum_{\nu=1}^{\infty} \|\lambda_\nu\| \;\approx\; \|\,\mathfrak{h}\,\|_{H^1} \;\;. \tag{6.2}$$

For $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)$ given as above, let us define new decomposition operators as follows. For $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$ with $\mathfrak{b}_{\mathbb{Q}} = 0$, we define

$$\mathscr{L}_{\mathfrak{h}}\,\mathfrak{b} \;\stackrel{\text{def}}{=}\; \sum_{\nu=1}^{\infty} \lambda_\nu(\mathfrak{b} - \mathfrak{b}_{Q_\nu})a_\nu \tag{6.3}$$

$$\mathscr{H}_{\mathfrak{h}}\,\mathfrak{b} \;\stackrel{\text{def}}{=}\; \sum_{\nu=1}^{\infty} \lambda_\nu \mathfrak{b}_{Q_\nu} a_\nu \;. \tag{6.4}$$

Let us prove Theorem 1.6 for these decomposition operators. As in the previous section, we want to show that the partial sums of (6.3) converge in $\boldsymbol{L}^1(\mathbb{R}^n)$, while the partial sums of (6.4) converge in $\boldsymbol{H}^{\wp}(\mathbb{R}^n, \mu)$. Once such convergence is established $\mathscr{L}_{\mathfrak{h}}$ and $\mathscr{H}_{\mathfrak{h}}$ become well defined bounded linear operators on the space $\boldsymbol{BMO}(\mathbb{R}^n)$.

The first assertion follows immediately from the fact that

$$\|\mathfrak{b} - \mathfrak{b}_{Q_\nu}\|_{L^1(Q_\nu)} \leqslant \|\mathfrak{b}\|_{BMO}|Q_\nu| \,,$$

which, combined with (6.2), implies the normal convergence of the series appearing in (6.3).

Let us concentrate on (6.4). We have the inequality

$$|\mathscr{M}(\mathfrak{b}_{Q_\nu} a_\nu)| \leqslant |\mathfrak{b} - \mathfrak{b}_{Q_\nu}|\,|\mathscr{M}(a_\nu)| + |\mathfrak{b}||\mathscr{M}(a_\nu)| \,. \tag{6.5}$$

It is sufficient to prove the two following L^1 inequalities for Q-atoms a:

$$\|(\mathfrak{b} - \mathfrak{b}_Q)\,\mathscr{M}(a)\|_1 \preccurlyeq \|\mathfrak{b}\|_{BMO} \tag{6.6}$$

$$\|\mathscr{M}(a)\|_1 \preccurlyeq 1 \,, \tag{6.7}$$

to be able to proceed as in Section 5. A linear change of variables allows us to assume that Q is the unit cube $\mathbb{Q}$. Then both inequalities are classical and may be found in [56]). For a $\mathbb{Q}$-atom a one has the inequality

$$\mathscr{M} a(x) \preccurlyeq \frac{1}{(1+|x|)^{n+1}} \,,$$

while

$$\int \frac{|\mathfrak{b}(x) - \mathfrak{b}_{\mathbb{Q}}|}{(1+|x|)^{n+1}}\, dx \preccurlyeq \|\mathfrak{b}\|_{BMO} \,.$$

The conclusion follows at once.

Again, we prove (1.26) as in the previous section. By continuity of the linear functional associated to $\mathfrak{b}$,

$$\int^{*}_{\mathbb{R}^n} \mathfrak{b}(x)\mathfrak{h}(x)\, dx = \sum \int_{\mathbb{R}^n} \mathfrak{b}(x) a_\nu(x)\, dx \,.$$

The integral of an atom is zero, so that

$$\int_{\mathbb{R}^n} \mathfrak{b} a_\nu \, dx = \int_{\mathbb{R}^n} (\mathfrak{b} - \mathfrak{b}_{Q_\nu}) a_\nu \, dx \,,$$

from which we conclude.

7. Proof of Theorem 1.8

Recall that the space $\mathscr{C}^\infty_\bullet(\mathbb{R}^n)$ of smooth functions with compact support whose integral mean equals zero is dense in $\boldsymbol{H}^1(\mathbb{R}^n)$. We fix a sequence $\{\mathfrak{h}_j\} \subset \mathscr{C}^\infty_\bullet(\mathbb{R}^n)$ converging to a given function $\mathfrak{h}$ in $\boldsymbol{H}^1(\mathbb{R}^n)$. We also fix a function $\mathfrak{b} \in \boldsymbol{BMO}(\mathbb{R}^n)$. Our proof is based upon the following observation. There exists a subsequence, again denoted by $\{\mathfrak{h}_j\}$, such that

$$\mathscr{M}[\,\mathfrak{b} \times (\mathfrak{h} - \mathfrak{h}_j)\,] \longrightarrow 0 \,, \quad \text{almost everywhere} \,. \tag{7.1}$$

To see this we appeal to the decomposition in Theorem 1.6. Accordingly,

$$\mathscr{M}[\,\mathfrak{b} \times (\mathfrak{h} - \mathfrak{h}_j)\,] \;\leqslant\; \mathscr{M}[\,\mathscr{L}_{\mathfrak{h}-\mathfrak{h}_j}\,\mathfrak{b}\,] \;+\; \mathscr{M}[\,\mathscr{H}_{\mathfrak{h}-\mathfrak{h}_j}\,\mathfrak{b}\,]\,.$$

It suffices to show that each term in the right hand side converges to zero almost everywhere. For the first term we argue by using the inequality (1.25),

$$\|\,\mathscr{L}_{\mathfrak{h}-\mathfrak{h}_j}\,\mathfrak{b}\,\|_{L^1} \;\;\preccurlyeq\;\; \|\,\mathfrak{h} - \mathfrak{h}_j\,\|_{H^1}\,\|\,\mathfrak{b}\,\|_{BMO^+} \;\;\longrightarrow\;\; 0\,.$$

Hence $\mathscr{M}[\,\mathscr{L}_{\mathfrak{h}-\mathfrak{h}_j}\,\mathfrak{b}\,] \leqslant \mathfrak{M}[\,\mathscr{L}_{\mathfrak{h}-\mathfrak{h}_j}\,\mathfrak{b}\,] \to 0$ in $\boldsymbol{L}^1_{\mathrm{weak}}(\mathbb{R}^n)$, thus almost everywhere for a suitable subsequence. Similarly, for the second term, we have

$$\|\mathscr{M}[\,\mathscr{H}_{\mathfrak{h}-\mathfrak{h}_j}\,\mathfrak{b}\,]\,\|_{L^{\wp}} \;\;\preccurlyeq\;\; \|\,\mathfrak{h} - \mathfrak{h}_j\,\|_{H^1}\,\|\,\mathfrak{b}\,\|_{BMO^+} \;\;\longrightarrow\;\; 0\,.$$

Passing to a subsequence we conclude with 7.1.

We now define a set $\mathbb{E} \subset \mathbb{R}^n$ of full measure by requiring that every $x \in \mathbb{E}$ is a Lebesgue point of $\mathfrak{b}$ and, in addition,

$$\mathscr{M}[\,\mathfrak{b} \times (\mathfrak{h} - \mathfrak{h}_j)\,] \;+\; |\,\mathfrak{b}\cdot(\mathfrak{h} - \mathfrak{h}_j)\,| \to 0 \qquad \text{on } \mathbb{E}. \tag{7.2}$$

We shall show that

$$\lim_{\varepsilon \to 0}\,(\mathfrak{b} \times \mathfrak{h})_\varepsilon(x) \;=\; \mathfrak{b}(x)\cdot\;.h(x) \qquad \text{for} \quad x \in \mathbb{E}. \tag{7.3}$$

From now on the computation takes place at a given point $x \in \mathbb{E}$. We begin with a telescoping decomposition

$$(\mathfrak{b} \times \mathfrak{h})_\varepsilon \,-\, \mathfrak{b}\cdot\mathfrak{h} \;=\; [\,\mathfrak{b} \times (\mathfrak{h} - \mathfrak{h}_j)\,]_\varepsilon \,+\, [\,(\mathfrak{b} \times \mathfrak{h}_j)_\varepsilon \,-\, \mathfrak{b}\cdot\mathfrak{h}_j\,] \,+\, \mathfrak{b}\cdot(\mathfrak{h}_j - \mathfrak{h})\,.$$

Hence

$$|\,(\mathfrak{b}\times\mathfrak{h})_\varepsilon - \mathfrak{b}\cdot\mathfrak{h}\,| \;\leqslant\; \mathscr{M}[\,\mathfrak{b}\times(\mathfrak{h}-\mathfrak{h}_j)\,] + |\,(\mathfrak{b}\times\mathfrak{h}_j)_\varepsilon - \mathfrak{b}\cdot\mathfrak{h}_j\,| + |\,\mathfrak{b}\cdot(\mathfrak{h}_j - \mathfrak{h})\,|\,.$$

We choose j sufficiently large so that the first and the last terms are small. With j fixed the middle term goes to zero as $\varepsilon \to 0$, because x was a Lebesgue point of $\mathfrak{b}$. This completes the proof.

8. The case of dimension 1: classical Hardy spaces

We include this section for different reasons. The first one is that the “div-curl method" does not work, obviously, in dimension one. Secondly, we have more accurate results, as well as a converse statement to Theorem 1.6, which may be seen as a factorization theorem for analytic functions. Also the proof is particularly simple and conceptual. We consider the case of periodic functions for simplicity. A last point to mention is the fact that the decomposition operators that we propose depend now linearly of $\mathfrak{h}$ instead of $\mathfrak{b}$. This deserves to be mentioned in view of Conjecture 1.7.

We first recall the definition of Hardy spaces of analytic functions in the unit disk $\mathbb{D} \subset \mathbb{R}^2 \cong \mathbb{C}$.

DEFINITION 8.1. — *The space $\mathscr{H}^p(\mathbb{D})$, $p > 0$, consists of analytic functions $F : \mathbb{D} \to \mathbb{C}$ such that*

$$\|F\|_{\mathscr{H}^p(\mathbb{D})} \stackrel{\text{def}}{=\!=} \sup_{0<r<1} \left(\int_0^{2\pi} |F(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} < \infty . \tag{8.1}$$

If $p \geqslant 1$ this formula defines a norm and $\mathscr{H}^p(\mathbb{D})$ becomes a Banach space. For $0 < p < 1$, $\mathscr{H}^p(\mathbb{D})$ is a complete linear metric space with respect to the distance $\operatorname{dist}_{H^p}[F, G] \stackrel{\text{def}}{=\!=} \|F - G\|_p^p$. A fundamental theorem of Hardy and Littlewood asserts that an analytic function $F : \mathbb{D} \to \mathbb{C}$ is in $\mathscr{H}^p(\mathbb{D})$ if and only if its non-tangential maximal function $F^* \in \boldsymbol{L}^p(\partial\mathbb{D})$, where for every $\xi \in \partial\mathbb{D}$ we define

$$F^*(\xi) \stackrel{\text{def}}{=\!=} \sup_{z \in \Gamma(\xi)} |F(z)| , \qquad \text{and} \quad \Gamma(\xi) = \{z \in \mathbb{D};\ |\xi - z| < 2(1 - |z|)\} .$$

We shall also need the so-called analytic BMO-space, defined and denoted by

$$\mathscr{BMO}(\mathbb{D}) = \mathscr{H}^1(\mathbb{D}) \cap \boldsymbol{BMO}(\partial\mathbb{D}) . \tag{8.2}$$

Perhaps this definition needs some explanation. It is known, see [20], that every function $F \in \mathscr{H}^p(\mathbb{D})$ has non-tangential limit $f(\xi) = \lim_{\Gamma(\xi) \ni z \to \xi} F(z)$ almost everywhere on $\partial\mathbb{D}$. The function f is usually called the boundary value of F, and, by abuse of notation, we will write $F(\xi)$ instead of $f(\xi)$ most of the time. Furthermore, if $p \geqslant 1$, then we recover F from its boundary values by Poisson integral, denoted by $F = \mathcal{P}f$. In other words, elements of the Hardy space $\mathscr{H}^p(\mathbb{D})$, with $p \geqslant 1$, may be viewed as functions on $\partial\mathbb{D}$ in $\boldsymbol{L}^p(\partial\mathbb{D})$. The same is valid for $\mathscr{BMO}(\mathbb{D})$, whose elements can as well be seen as analytic functions inside the unit disc, or as functions at the boundary.

Now, the connection between $\mathscr{H}^1(\mathbb{D})$, viewed as a space of functions on $\partial\mathbb{D}$, and the real Hardy space $\boldsymbol{H}^1(\partial\mathbb{D})$ (also generated by the atoms and the constants) is rather simple :

$$\boldsymbol{H}^1(\partial\mathbb{D}) = \{u + iv ;\quad u, v \in \Re e\, \mathscr{H}^1(\mathbb{D})\} .$$

It turns out, see [20], that $\boldsymbol{H}^1(\partial\mathbb{D})$ consists precisely of those $f \in \boldsymbol{L}^1(\partial\mathbb{D})$ for which the non-tangential maximal function $(\mathcal{P}f)^*$ of the Poisson extension $\mathcal{P}f : \mathbb{D} \to \mathbb{C}$ lies in $\boldsymbol{L}^1(\partial\mathbb{D})$. As a corollary, the Hilbert transform

H is a bounded operator of $\boldsymbol{H}^1(\partial\mathbb{D})$ into itself. We then have

$$\mathscr{H}^1(\mathbb{D})_{|\partial\mathbb{D}} = \{ u + i\,Hu ; \; u \in \boldsymbol{H}^1(\partial\mathbb{D}) \} .$$

We now want to introduce in a similar fashion the Hardy-Orlicz space $\mathscr{H}^{\wp}(\mathbb{D})$, with $\wp(t) = \frac{t}{\log(\mathrm{e+t})}$. By definition, it consists of analytic functions $F : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} \frac{|F(re^{i\theta})|\, d\theta}{\log(e + |F(re^{i\theta})|)} < \infty .$$

Thus we may define

$$\|F\|_{\wp} \stackrel{\mathrm{def}}{=\!=} \sup_{0<r<1} [\![F_r]\!]_{L^{\wp}(\partial\mathbb{D})} , \quad F_r(\xi) = F(r\xi) , \text{ for } \xi \in \partial\mathbb{D} .$$

This space contains $\mathscr{H}^1(\mathbb{D})$, and is contained in $\mathscr{H}^p(\mathbb{D})$, for $p < 1$. As for $\mathscr{H}^p(\mathbb{D})$ spaces, we have the following, which will be useful later,

Proposition 8.2. — *An analytic function $F : \mathbb{D} \to \mathbb{C}$ belongs to $\mathscr{H}^{\wp}(\mathbb{D})$ if and only if its non-tangential maximal function F^* lies in the Orlicz space $\boldsymbol{L}^{\wp}(\partial\mathbb{D})$.*

Proof. — The "if" part is obvious. To prove the converse, we mimic the well known arguments used for $\mathscr{H}^p(\mathbb{D})$ spaces. First of all, every function $F \in \mathscr{H}^{\wp}(\mathbb{D})$ belongs to $\mathscr{H}^{\frac{1}{2}}(\mathbb{D})$ and thus admits a decomposition $F = B\,G$, where B is a Blaschke product and G is non-vanishing in $\mathbb{D}$. The argument used on page 56 in [20] works *mutatis mutandis* to show that $G \in \mathscr{H}^{\wp}(\mathbb{D})$, because of the convexity of the function $s \longrightarrow e^s \log^{-1}(e + e^s)$. As a consequence, $\sqrt{G}$ belongs to the space $\boldsymbol{L}^2\log^{-2}\boldsymbol{L}(\partial\mathbb{D})$. By an easy variant of Hardy-Littlewood maximal inequality,the same is valid for $\sqrt{G}^{\,*}$. To conclude, we have a point-wise inequality between the non-tangential maximal functions $F^* \leqslant G^* = [\sqrt{G}^{\,*}]^2$. The reader may wish to recall that $F^* \preccurlyeq \mathfrak{M}f$ on $\partial\mathbb{D}$, where f is the boundary value of $F \in \mathscr{H}^{\wp}(\mathbb{D})$ and $\mathfrak{M}$ stands for the Hardy-Littlewood maximal operator on $\partial\mathbb{D}$, see [20]. □

Let us go back to products of functions in $\mathscr{H}^1(\mathbb{D})$ and $\mathscr{BMO}(\mathbb{D})$, which are contained in all $\mathscr{H}^p(\mathbb{D})$ spaces for $p < 1$. We are now ready to prove Theorem 1.11, which has already been stated in the introduction and is repeated here for convenience.

Theorem 1.11. — *The product of $G \in \mathscr{BMO}(\mathbb{D})$ and $H \in \mathscr{H}^1(\mathbb{D})$ belongs to $\mathscr{H}^{\wp}(\mathbb{D})$. Moreover, every function in $\mathscr{H}^{\wp}(\mathbb{D})$ can be written as such a product. In other words,*

$$\mathscr{H}^1(\mathbb{D}) \cdot \mathscr{BMO}(\mathbb{D}) = \mathscr{H}^{\wp}(\mathbb{D}).$$

Proof. — Let us prove the first assertion. We only need to bound the integrals

$$I_r \;=\; \int_0^{2\pi} \frac{|G(re^{i\theta})|\cdot|H(re^{i\theta})|\;d\theta}{\log\left[\,e\,+\;|G(re^{i\theta})|\cdot|H(re^{i\theta})|\right|\,]}\ ,$$

independently of $0 < r < 1$. This is given by (3.14), using the fact that G_r have norms in $\mathscr{BMO}(\mathbb{D})$ that are uniformly bounded.

For the converse, let $F \in \mathscr{H}^{\wp}(\mathbb{D})$. We want to find G and H such that $F = GH$. By Proposition 8.2, we know that its non-tangential maximal function F^{*} is such that $F^{*}\log^{-1}(e+F^{*}) \in \boldsymbol{L}^1(\partial\mathbb{D})$. At this point we need a BMO-majorant of $\log(e+F^{*})$.

Lemma 8.3. — *There exists $G \in \mathscr{BMO}(\mathbb{D})$ such that $\log(e+F^{*}) \leqslant |G|$ on $\partial\mathbb{D}$.*

To see this, we factor F as before, $F = B\cdot F_0$, where B is a Blaschke product and F_0 does not vanish in $\mathbb{D}$. Then $F^{*} \leqslant GF_0^{*} = [\sqrt{F_0}\ ^{*}]^2 \preccurlyeq [\mathfrak{M}\sqrt{F_0}\,]^2$. Thus

$$\log(e+F^{*}) \leqslant\; C + 2\log\left(\mathfrak{M}\sqrt{F_0}\right) \;\stackrel{\text{def}}{=\!=}\; b \in \boldsymbol{BMO}(\partial\mathbb{D})$$

by a famous theorem of Coifman-Rochberg-Weiss [CRW]. Define G as the Poisson integral of $b + i\,Hb$. Then G is in $\mathscr{BMO}(\mathbb{D})$ and has the required properties.

To conclude for the proof of Theorem 1.11, we need only show that $H = F/G$ belongs to $\mathscr{H}^1(\mathbb{D})$. We already know that H belongs to $\mathscr{H}^p(\mathbb{D})$ for $p<1$, since the function G is bounded below. The last stage consists in proving that the boundary values of H are given by an integrable function on $\partial\mathbb{D}$ (recall that the boundary values of a non zero function in $\mathscr{H}^p(\mathbb{D})$ cannot vanish almost everywhere). This last fact is obvious:

$$\int_0^{2\pi} |H(e^{i\theta})|\;d\theta = \int_0^{2\pi} \frac{|F(e^{i\theta})|}{|G(e^{i\theta})|}\;d\theta \leqslant \int_0^{2\pi} \frac{F^{*}(e^{i\theta})}{\log[e+\;F^{*}(e^{i\theta})]}\;d\theta < \infty,$$

which completes the proof of Theorem 1.11. □

Let us turn to the so-called "real" Hardy spaces. We define $\mathscr{H}^{\wp}(\partial\mathbb{D})$ as the space of distributions f on $\partial\mathbb{D}$ such that $(Pf)^{*} \in \boldsymbol{L}^{\wp}(\partial\mathbb{D})$. An equivalent definition, as for $\mathscr{H}^p$ spaces, is the following:

$$\boldsymbol{H}^{\wp}(\partial\mathbb{D}) = \;\{u+iv\,;\;\; u,v \in \Re e\,\mathscr{H}^{\wp}(\mathbb{D})\,\}\,. \tag{8.3}$$

Here, in the right hand side, the space $\mathscr{H}^{\wp}(\mathbb{D})$ is identified with the space of corresponding boundary values in the sense of distributions.The fact that $\mathscr{H}^{\wp}(\mathbb{D})$ is contained in $\mathscr{H}^{\wp}(\partial\mathbb{D})$ follows from proposition 8.2. Let us just sketch the proof of the converse. It is sufficient to consider a real distribution f, such that $\mathcal{P}f = \Re\mathfrak{h}$, with $\mathfrak{h} \in \mathscr{H}^p(\mathbb{D})$. To conclude, it is sufficient to recall that the area functions of f and $\mathfrak{h}$ coincide, and that the area function may be equivalently used in the definition of $\mathscr{H}^\wp(\partial\mathbb{D})$ instead of the non tangential maximal function. We refer to [20] for this.

THEOREM 8.4. — *The product of functions in $\boldsymbol{H}^1(\partial\mathbb{D})$and $\boldsymbol{BMO}(\partial\mathbb{D})$ is a distribution in $\boldsymbol{L}^1(\partial\mathbb{D}) + \boldsymbol{H}^\wp(\partial\mathbb{D})$. In other words,*

$$\boldsymbol{H}^1(\partial\mathbb{D}) \times \boldsymbol{BMO}(\partial\mathbb{D}) \subset \boldsymbol{L}^1(\partial\mathbb{D}) \ + \ \boldsymbol{H}^\wp(\partial\mathbb{D}) \,.$$

Before embarking into the proof, the statement has to be given some explanation, as in $\mathbb{R}^n$. Namely, we must give a meaning to the product of $\mathfrak{h} \in \boldsymbol{H}^1(\partial\mathbb{D})$ and $\mathfrak{b} \in \boldsymbol{BMO}(\partial\mathbb{D})$ as a distribution, since one cannot in general multiply distributions. It follows from a result of Stegenga [St] that $\phi\mathfrak{b} \in \boldsymbol{BMO}(\partial\mathbb{D})$ for every test function $\phi \in \mathscr{C}^\infty(\partial\mathbb{D})$. We may, therefore, define the distribution $\mathfrak{b} \times \mathfrak{h} \in \mathscr{D}'(\partial\mathbb{D})$ by the rule,

$$\langle \mathfrak{h} \times \mathfrak{b} \,,\, \phi\rangle \overset{\text{def}}{=\!=} \langle \mathfrak{h} \,, \phi\mathfrak{b}\rangle_{H^1-BMO} \,.$$

Proof. — By Theorem 1.11 we know that $(\mathfrak{h} \ + \ i\,H\mathfrak{h}) \cdot (\mathfrak{b} \ + \ i\,H\mathfrak{b}) \in \mathscr{H}^\wp(\mathbb{D})$ and thus the imaginary part of this function, $\mathfrak{h} \cdot H\mathfrak{b} \ + \ \mathfrak{b} \cdot H\mathfrak{h} \in \boldsymbol{H}^\wp(\partial\mathbb{D})$. To conclude with the decomposition of $\mathfrak{b} \times \mathfrak{h}$ we recall the well known fact (following from the $H^1 - BMO$ duality) that every $\mathfrak{b} \in \boldsymbol{BMO}(\partial\mathbb{D})$ can be expressed as $\mathfrak{b} = \mathfrak{b}_1 + H\mathfrak{b}_2$, where both functions $\mathfrak{b}_1$ and $\mathfrak{b}_2$ lie in $\boldsymbol{L}^\infty(\partial\mathbb{D})$ [40, 19]. Hence we obtain the desired decomposition

$$\mathfrak{h} \times \mathfrak{b} = \alpha + \beta \,, \qquad \text{where} \quad \begin{cases} \alpha = \mathfrak{h}\,\mathfrak{b}_1 - \mathfrak{b}_2\,H\mathfrak{h} \in \boldsymbol{L}^1(\partial\mathbb{D}) \\ \beta = \mathfrak{h}\,H\mathfrak{b}_2 + \mathfrak{b}_2\,H\mathfrak{h} \in \boldsymbol{H}^\wp(\partial\mathbb{D}) \end{cases}$$

by what we have just seen with $\mathfrak{b}_2$ in place of $\mathfrak{b}$. □

Remark that we could also have used the method of Section 6. Here we have a more explicit decomposition, which depends linearly on $\mathfrak{h}$ instead of $\mathfrak{b}$. Moreover, we have a converse, even if it is not as neat as for analytic functions.

PROPOSITION 8.5. — *Any distribution in $\boldsymbol{L}^1(\partial\mathbb{D}) + \boldsymbol{H}^\wp(\partial\mathbb{D})$ can be written as a sum of no more than two distributions in $\boldsymbol{H}^1(\partial\mathbb{D}) \cdot \boldsymbol{BMO}(\partial\mathbb{D})$.*

Proof. — Using the alternative definition of $\boldsymbol{H}^\wp(\partial\mathbb{D})$ through real parts of holomorphic functions in $\boldsymbol{H}^\wp(\mathbb{D})$ as well as Theorem 1.11, we can consider a function that may be written as $f + \mathfrak{b}_1\mathfrak{h}_1 + \overline{\mathfrak{b}_2\mathfrak{h}_2}$, with $f \in \boldsymbol{L}^1(\partial\mathbb{D})$ with $\mathfrak{b}_1$, $\mathfrak{b}_2$ in $\mathscr{BMO}(\mathbb{D})$ and $\mathfrak{h}_1$, $\mathfrak{h}_2$ in $\mathscr{H}^1(\mathbb{D})$. We claim that $f + \mathfrak{b}_1\mathfrak{h}_1$ can be written as $\mathfrak{b}\mathfrak{h}$, with $\mathfrak{b}_2$ in $\boldsymbol{BMO}(\partial\mathbb{D})$ and $\mathfrak{h}$ in $\mathscr{H}^1(\mathbb{D})$. Indeed, put $g = |f| + |\mathfrak{h}_1| + e$. Then $\log g$ is integrable and there is a function $\mathfrak{h} \in \mathscr{H}^1(\mathbb{D})$ such that $|\mathfrak{h}|_{\partial\mathbb{D}} = g$: it suffices to take $F = \exp(u + iv)$, where u is the Poisson integral of $\log g$ and v is the harmonic conjugate of u. To conclude, it remains to remark that $\frac{f}{\mathfrak{h}} + \mathfrak{b}_1 \frac{\mathfrak{h}_1}{\mathfrak{h}}$ is in $\boldsymbol{BMO}(\partial\mathbb{D})$. The first term is clearly bounded, while the second term is in $\boldsymbol{BMO}(\mathbb{D})$ as the product of a function in $\mathscr{BMO}(\mathbb{D})$ and a function in $\mathscr{H}^\infty(\mathbb{D})$. This concludes for the proof. □

There remains still an interesting question which we leave unanswered:
Question. *Is $\boldsymbol{H}^1(\partial\mathbb{D}) \cdot \boldsymbol{BMO}(\partial\mathbb{D})$ a vector space?*

9. Variants on domains in $\mathbb{R}^n$

We first generalize Theorem 1.6 in the context of bounded Lipschitz domains. We claim that, again, the product $\mathfrak{b} \times \mathfrak{h}$ can be defined in the distribution sense for $\mathfrak{b} \in \boldsymbol{BMO}(\Omega)$ and $\mathfrak{h} \in \boldsymbol{H}^1(\Omega)$. Recall that this BMO space is larger than the dual space of $\boldsymbol{H}^1(\Omega)$, which coincides with the subspace of $\boldsymbol{BMO}(\mathbb{R}^n)$ consisting in functions that vanish outside Ω (see [8]). This allows us to define, for $\varphi \in \mathscr{C}_0^\infty(\Omega)$,

$$\begin{aligned}\langle \mathfrak{b} \times \mathfrak{h} \,|\, \varphi \rangle \quad &\stackrel{\text{def}}{=\!=} \int_\Omega^* [\varphi(x)\mathfrak{b}(x)]\, \mathfrak{h}(x)\, dx \;=\; \int_{\mathbb{R}^n}^* [\varphi(x)\mathfrak{b}(x)]\, \tilde{\mathfrak{h}}(x)\, dx \\ &\preccurlyeq \; \|\, \varphi\mathfrak{b} \,\|_{BMO} \,\|\, \mathfrak{h} \,\|_{H^1(\Omega)} \;\preccurlyeq\; \|\, \nabla\varphi \,\|_\infty \,.\end{aligned}$$

Here $\tilde{\mathfrak{h}}$ is the extension of $\mathfrak{h}$ in $\boldsymbol{H}^1(\mathbb{R}^n)$ given in the next proposition.

Proposition 9.1 (Extension). — *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. There exist linear operators $\mathscr{H}_\Omega : \boldsymbol{H}^1(\Omega) \to \boldsymbol{H}^1_{\mathscr{F}}(\mathbb{R}^n)$ and $\mathscr{B}_\Omega : \boldsymbol{BMO}(\Omega) \to \boldsymbol{BMO}(\mathbb{R}^n)$ such that*

$$\begin{cases} \mathscr{H}_\Omega \mathfrak{h} \;=\; \mathfrak{h} \quad \text{on } \Omega \\ \|\mathscr{H}_\Omega \mathfrak{h}\,\|_{H^1_{\mathscr{F}}(\mathbb{R}^n)} \preccurlyeq \|\mathfrak{h}\,\|_{H^1_{\mathscr{F}}(\Omega)} \end{cases} \quad \text{and} \quad \begin{cases} \mathscr{B}_\Omega \mathfrak{b} \;=\; \mathfrak{b} \quad \text{on } \Omega \\ \|\mathscr{B}_\Omega \mathfrak{b}\,\|_{BMO(\mathbb{R}^n)} \preccurlyeq \|\mathfrak{b}\,\|_{BMO(\Omega)} \end{cases} .$$

For BMO -extension see [39], while an up-to-date connected account of H^1 -extensions appears in [45].

Using these extensions, we get

Corollary 9.2. — *Theorem 1.6 holds with $\mathbb{R}^n$ replaced by any bounded Lipschitz domain.*

As a final remark, we discuss some maximal operators best suited to PDEs and, in particular, to the use of Jacobian determinants or div-curl atoms as generators of $\boldsymbol{H}^1(\Omega)$. There is quite an extensive literature concerning definitions of the Hardy spaces in a domain $\Omega \subset \mathbb{R}^n$, [9, 8, 14, 15, 35, 42, 43, 45]. Let us briefly outline the general concept of a maximal function of a distribution $f \in \mathscr{D}'(\Omega)$. Suppose that for each point $x \in \Omega$, we are given a class $\mathscr{F}_x$ of test functions $\varphi \in \mathscr{C}_0^\infty(Q)$, where $Q \subset \Omega$ are cubes containing x. Denote by $\mathscr{F} = \bigcup_{x\in\Omega} \mathscr{F}_x$. The maximal function $\mathscr{M}_{\mathscr{F}} f$ of a distribution $f \in \mathscr{D}'(\Omega)$ is defined by the rule

$$\mathscr{M}_{\mathscr{F}} f(x) = \sup\left\{\langle f, \varphi\rangle \ ; \quad \varphi \in \mathscr{F}_x \right\}. \tag{9.1}$$

Note that Definition 1.2 is dealt with the test functions of the form $\varphi(y) = \Phi_\varepsilon(x-y)\,, 0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$.

Definition 9.3. — *A distribution $f \in \mathscr{D}'(\Omega)$ is in the Hardy space $\boldsymbol{H}^1_{\mathscr{F}}(\Omega)$ if*

$$\| f \|_{H^1_{\mathscr{F}}(\Omega)} = \int_\Omega \mathscr{M}_{\mathscr{F}} f(x)\, dx \ < \ \infty\,. \tag{9.2}$$

It is perhaps worth considering the class $\mathscr{F}_x$ of **all** test functions such that

$$\|\nabla\varphi\|_\infty \leqslant \frac{1}{(\operatorname{diam} Q)^{n+1}}\,, \quad x \in Q \subset \Omega\,, \quad \varphi \in \mathscr{C}_0^\infty(Q)\,. \tag{9.3}$$

This seemingly modest generalization is a great convenience to PDEs. It is easily seen that the restriction of $f \in \boldsymbol{H}^1_{\mathscr{F}}(\mathbb{R}^n)$ to any domain $\Omega \subset \mathbb{R}^n$ lies in $\boldsymbol{H}^1_{\mathscr{F}}(\Omega)$. We also have the inclusion $\boldsymbol{H}^1_{\mathscr{F}}(\Omega) \subset \boldsymbol{H}^1(\Omega)$, since the class $\mathscr{F}_x$ contains all test functions of the form $\varphi(y) = \Phi_\varepsilon(x-y)$, with $0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$. See footnote 3.

Proceeding further in our attempts to generalize the maximal operator, we still weaken regularity of the test functions in the class $\mathscr{F}_x$. Instead of the gradient condition at (9.3) we shall impose only Hölder's condition

$$\|\varphi\|_{C^\gamma(Q)} \stackrel{\text{def}}{=\!=} \sup_{a,\, b \in Q} \frac{|\varphi(a)-\varphi(b)|}{|a-b|^\gamma} \leqslant \frac{1}{(\operatorname{diam} Q)^{n+\gamma}}\,. \tag{9.4}$$

From now on, our class $\mathscr{F}_x$ consists of test functions $\varphi \in \mathscr{C}_0^\infty(Q)$ that verify (9.4), where $Q \subset \Omega$ can be arbitrary cube containing $x \in \Omega$, whereas the exponent $0 < \gamma \leqslant 1$ is fixed.

Our detailed account will be confined to the case in which the underlying domain Ω is either a cube or the entire space $\mathbb{R}^n$ for simplification.

We are going to prove the following

THEOREM 9.4. — *Let Ω be a cube or the entire space $\mathbb{R}^n$. The following Hardy spaces are the same,*

$$\boldsymbol{H}^1_{\mathscr{F}}(\Omega) \;=\; \boldsymbol{H}^1(\Omega) \;=\; \boldsymbol{H}^1(\mathbb{R}^n)_{|\,\Omega}\,. \tag{9.5}$$

Moreover, for each $\mathfrak{h} \in \boldsymbol{H}^1(\mathbb{R}^n)_{|\,\Omega}$, it holds:

$$\|\mathfrak{h}\,\|_{H^1_{\mathscr{F}}(\Omega)} \;\approx\; \|\mathfrak{h}\,\|_{H^1(\Omega)}\,. \tag{9.6}$$

A generalization of the fundamental lemma of [10] on div-curl atoms supplies the key to the proof of Theorem 9.4.

LEMMA 9.5. — *Given a div-curl couple $E \in \boldsymbol{L}^p(Q,\mathbb{R}^n)$ and $B \in \boldsymbol{L}^q(Q,\mathbb{R}^n)$ in a cube $Q \subset \mathbb{R}^n$, where $p+q = p\cdot q$ for some $p,q > 1$. Then for each test function $\varphi \in \mathscr{C}_0^\infty(Q)$ we have the inequality:*

$$\int_Q [\,E(x)\cdot B(x)\,]\,\varphi(x)\,dx$$

$$\preccurlyeq\; (\operatorname{diam} Q)^{\,n+\gamma}\,\|\varphi\,\|_{C^\gamma(Q)} \left(\fint_Q |E|^{p'}\right)^{\frac{1}{p'}} \left(\fint_Q |B|^{q'}\right)^{\frac{1}{q'}} \tag{9.7}$$

with the exponents satisfying:

$$\frac{1}{p'} + \frac{1}{q'} = 1 + \frac{\gamma}{n}\quad, \quad \textit{where} \quad \begin{cases} 1 \leqslant p' = \frac{n\,p}{n+\gamma p-\gamma} < p \\ 1 \leqslant q' = \frac{n\,q}{n+\gamma q-\gamma} < q \end{cases}. \tag{9.8}$$

Proof. — As a first step we extend E and B as a *div-curl* couple in the entire space $\mathbb{R}^n$, again denoted by E and B. We shall also need to keep track of how their norms increase. Precisely,

$$\|E\,\|_{L^{p'}(\mathbb{R}^n)} \;\preccurlyeq\; \|E\,\|_{L^{p'}(Q)} \qquad \text{and} \qquad \|B\,\|_{L^{p'}(\mathbb{R}^n)} \;\preccurlyeq\; \|B\,\|_{L^{p'}(Q)} \tag{9.9}$$

for all $1 \leqslant p' \leqslant p$ and $1 \leqslant q' \leqslant q$. Such extensions are straightforward once we view the vector fields E and B as exact and coexact differential forms on the cube Q, by virtue of Poincaré Lemma. In this view E and B are the first order differentials of certain Sobolev functions in Q. These functions can routinely be extended to $\mathbb{R}^n$ with uniform bound of their Sobolev norms. We now appeal to the Hodge decomposition operators $\mathscr{A}$ and $\mathscr{B}$ at (4.4) and (4.5). Accordingly,

$$\varphi E = \mathscr{A}(\varphi E) \;+\; \mathscr{B}(\varphi E)$$

where we shall take into account that $\mathscr{A}(\varphi E)$ is divergence free (thus orthogonal to B), whereas the operator $\mathscr{B}$ vanishes on E. Hence, by

Hölder's inequality

$$\begin{aligned}\int \varphi\,[B\cdot E] &= 0 + \int B\cdot(\mathscr{B}\varphi - \varphi\mathscr{B})E \\ &\preccurlyeq \|B\|_{L^{q'}(\mathbb{R}^n)} \cdot \|(\mathscr{B}\varphi-\varphi\mathscr{B})E\|_{L^{\frac{np}{n-\gamma}}(\mathbb{R}^n)}. \end{aligned} \tag{9.10}$$

Here the commutator of the singular integral operator $\mathscr{B}$ with the multiplication by φ is controlled by means of the fractional integral,

$$(\mathscr{B}\varphi-\varphi\mathscr{B})E(x) \preccurlyeq \int \frac{|\varphi(x)-\varphi(y)|}{|x-y|^n}\,|E(y)|\,dy \leqslant \|\varphi\|_{C^\gamma(\mathbb{R}^n)} \int \frac{|E(y)|\,dy}{|x-y|^{n-\gamma}}$$

Hardy-Littlewood-Sobolev inequality yields

$$\|(\mathscr{B}\varphi-\varphi\mathscr{B})E\|_{L^{\frac{np}{n-\gamma}}(\mathbb{R}^n)} \preccurlyeq \|\varphi\|_{C^\gamma(\mathbb{R}^n)} \cdot \|E\|_{L^{p'}(\mathbb{R}^n)}, \quad \text{as } \frac{n-\gamma}{np} = \frac{1}{p'} - \frac{\gamma}{n}.$$

This gives (9.7) when substituted into (9.10), the last step being justified by (9.9) and (9.8). This completes the proof of Lemma 9.5. □

To conclude for the proof of the theorem it is sufficient to use the Div-Curl Decomposition (see Proposition 2.2) when Ω is the entire space, and the extension of a function to the entire space otherwise (see Proposition 9.1), which allows to restrict to div-curl atoms $E\cdot B$. For these last ones, the key inequality (9.7) is used to obtain that

$$\mathscr{M}_{\mathscr{F}}(E\cdot B) \preccurlyeq \mathfrak{M}_{p'}(E)\cdot \mathfrak{M}_{q'}(B). \tag{9.11}$$

Here we remind that the Hardy-Littlewood maximal operator

$$(\mathfrak{M}_s h)(x) = \sup_{x\in Q\subset\Omega} \left(\fint_Q |h|^s \right)^{\frac{1}{s}}$$

is bounded in all spaces $\boldsymbol{L}^r(\Omega)$, with $r > s$. Thus by Hölder's inequality we obtain

$$\begin{aligned}\textstyle\int_\Omega \mathscr{M}_{\mathscr{F}}(E\cdot B) &\preccurlyeq \|\mathfrak{M}_{p'}(E)\|_{L^p(\Omega)} \cdot \|\mathfrak{M}_{q'}(B)\|_{L^q(\Omega)} \\ &\preccurlyeq \|E\|_{L^p(\Omega)} \cdot \|B\|_{L^q(\Omega)} \end{aligned} \tag{9.12}$$

which allows to conclude for the theorem.

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