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THE DRINFELD MODULAR JACOBIAN $J_1(N)$ HAS CONNECTED FIBERS

by Sreekar M. SHASTRY

Abstract. — We study the integral model of the Drinfeld modular curve $X_1(n)$ for a prime $n \in \mathbb{F}_q[T]$. A function field analogue of the theory of Igusa curves is introduced to describe its reduction mod n. A result describing the universal deformation ring of a pair consisting of a supersingular Drinfeld module and a point of order n in terms of the Hasse invariant of that Drinfeld module is proved. We then apply Jung-Hirzebruch resolution for arithmetic surfaces to produce a regular model of $X_1(n)$ which, after contractions in the special fiber, gives a regular model with geometrically integral fiber over n. Thus the mod n component group of $J_1(n)$ is trivial, i.e. $J_1(n)$ has connected fibers.

Résumé. — Nous étudions le modèle intégral de la courbe modulaire $X_1(n)$ de Drinfeld pour un élément irreductible $n \in \mathbb{F}_q[T]$. Un analogue du corps de fonctions de la théorie des courbes d'Igusa est introduit pour décrire sa réduction mod n. Un résultat décrivant l'anneau universel de déformation d'une paire se composant d'un module de Drinfeld supersingulier et d'un point d'ordre n en termes de l'invariant de Hasse de ce module de Drinfeld est prouvé. Nous appliquons alors la résolution de Jung-Hirzebruch afin que les surfaces arithmétiques produisent un modèle régulier de $X_1(n)$ qui, après des contractions dans la fibre spéciale, donne un modèle régulier tel que la fibre au-dessus de n est géométriquement intègre. Ainsi, la réduction mod n du groupe des composants de $J_1(n)$ est triviale, c'est-à-dire les fibres de $J_1(n)$ sont connexes.

1. Introduction

Let $n \in A := \mathbb{F}_q[T]$ be a monic prime polynomial and $X_1(n)_{/\mathbb{F}_q(T)}$ be the smooth projective curve associated to the moduli problem of classifying pairs consisting of a Drinfeld module over an $\mathbb{F}_q(T)$-scheme and a nowhere vanishing point of its n-torsion. Let $J_1(n)_{/\mathbb{F}_q(T)}$ be its Jacobian. The goal of this paper is to prove

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THEOREM 8.5. — *The closed fiber of the Néron model of $J_1(n)$ over $A_{(n)}$ has trivial geometric component group.*

As is well known (see [2, 9.6/1]), to prove this it suffices to show that there exists a regular proper model of $X_1(n)_{/\mathbb{F}_q(T)}$ over $A_{(n)}$ with geometrically integral special fiber. This is the approach we take.

We will study Drinfeld's integral model of $X_1(n)$ over $A_{(n)}$, defined by means of Drinfeld level structures; it is regular in the stack sense but the coarse modular curve $X_1(n)$ has some singularities. To analyze these singularities we will cover the special fiber by moduli problems specific to characteristic n, namely by function field analogues of the Igusa curves as developed in the case of classical modular curves in [21].

An outline of the paper is as follows: In Section 2 we fix our conventions for Drinfeld modules over schemes and define the relative Frobenius and Verschiebung morphisms of a Drinfeld module. The existence of the Verschiebung depends in an essential way on the canonical generator of the prime ideal (n) provided by the monic polynomial n. Section 3 reviews results (with some proofs) from [5] about the deformation theory of formal modules, especially the crucial fact 3.13(ii) which is implicit in [5, §4]. Following [21], the regularity of the moduli scheme of Drinfeld modules together with a point of order n and an auxiliary rigidifying étale level structure is proved in Section 4. In Section 5 we study the moduli problems specific to characteristic n and prove a result expressing the local ring at a supersingular point in terms of the roots of the Hasse invariant; this result will enable us to give in Section 6 an explicit presentation for the local ring in terms of local rings along the components of the special fiber (where the special fiber has the expected type: two components whose underlying reduced schemes cross transversally at the supersingular points). In Section 7 we identify the unique nonregular point on $X_1(n)$ to be that point corresponding to (ϕ, P) such that ϕ has j-invariant 0 and $P = 0$ is a generator of the kernel of Frobenius. This is a cyclic quotient singularity and we use the Jung-Hirzebruch resolution as developed for arithmetic surfaces in [3] to resolve this singularity in Section 8. Contractions in the special fiber of the Jung-Hirzebruch resolution then yield the sought regular model with integral special fiber.

A remark on the title of this paper: it really means that the Néron model of $J_1(n)$ over $\mathbb{P}^1_{\mathbb{F}_q} - \infty$ has connected fibers. Of course, in this geometric setting, one can form the Néron model over all of $\mathbb{P}^1_{\mathbb{F}_q}$ but the component group at the place ∞ is highly nontrivial; in fact the results of [5] show that it may be described in terms of automorphic forms. See [29, Lemma

12]. However, ∞ is not part of the moduli problem of Drinfeld modules and here plays the role of the archimedean place of $\mathbb{Q}$, whence our title.

Acknowledgments. The problem taken up in this paper was mentioned in the introduction to [3]. In that paper the parallel problem for classical modular curves is solved.

This paper is essentially the author's Michigan Ph.D. thesis. He would like to express his sincere gratitude to his adviser Brian Conrad, under whose direction this work was completed. He is also grateful to Tong Liu for many helpful discussions. Finally, the author would like to mention the manuscripts [23] and [30] which were of great aid while studying the seminal paper [5].

2. Frobenius and Verschiebung

Definition 2.1. — *Let A be any ring. An A-*module scheme *is a pair (G,ϕ) consisting of a commutative group scheme G over an A-scheme S and a map $\phi : A \longrightarrow End_S(G)$ of rings such that the induced map $\phi : A \longrightarrow End_{\mathscr{O}_S}(\omega_{G/S})$ coincides with the composite $A \to H^0(S,\mathscr{O}_S) \to End_{\mathscr{O}_S}(\omega_{G/S})$. Here $\omega_{G/S}$ is the conormal sheaf $\mathscr{I}/\mathscr{I}^2$ where $\mathscr{I}$ is the ideal sheaf along the zero section of G. See* [27, §1].

Let $\mathcal{C}$ be a proper smooth geometrically connected curve over the finite field $\mathbb{F}_q$ of characteristic p, $\infty \in \mathcal{C}$ a closed point, and write $\mathcal{C}-\infty = \operatorname{Spec} A$. Denote by $\operatorname{ord}_\infty(a)$ the valuation at ∞ of $a \in A$.

For an invertible sheaf $\mathscr{E}$ on a scheme S, we write $E := \operatorname{Spec}\operatorname{Sym}\mathscr{E}^\vee$ for the associated line bundle over S, so that $E(U) = \mathscr{E}(U)$ for $U \subset S$ open. We will use the following definition of a Drinfeld module over an A-scheme S (see [22, pp. 6-8] for the equivalence with other definitions). Let $E\{\tau\}$ be the the noncommutative ring $\bigoplus_{i\geqslant 0} E^{\otimes(1-q^i)}(S).\tau^i$, where τ stands for the qth power Frobenius mapping. Multiplication in this ring is given by $\alpha_i\tau^i \cdot \alpha_j\tau^j = (\alpha_i \otimes \alpha_j^{\otimes q^i})\tau^{i+j}$.

Definition 2.2. — *A* Drinfeld A-module of rank r *over an A-scheme S is given by a line bundle E over S together with a ring homomorphism $\phi^E : A \longrightarrow E\{\tau\}$ such that $\phi^E_a = \sum_{i=0}^{m(a)} \alpha_i(a)\tau^i$ where $m(a) := -r\deg(\infty)ord_\infty(a)$, $\alpha_{m(a)}(a)$ is a nowhere vanishing section of $E^{\otimes(1-m(a))}$, and α_0 coincides with the map $\chi : A \longrightarrow H^0(S,\mathscr{O}_S)$ giving the structure of an A-scheme to S.*

A Drinfeld A-module over S is clearly an A-module scheme over S.

Definition 2.3. — *An* isogeny *of Drinfeld modules is a finite homomorphism of A-module schemes.*

Example 2.4. — For a Drinfeld A-module E over S and a nonzero $a \in A$, $\phi_a^E : E \longrightarrow E$ is an isogeny of Drinfeld modules over S.

A Drinfeld module will be denoted simply as $E_{/S}$ with associated invertible sheaf $\mathscr{E}$ and with the choice of A being clear from the context. When S is the spectrum of a local ring R, we sometimes will omit mention of E and just write $\phi_{/R}$ to stand for a ring homomorphism $\phi : A \longrightarrow R\{\tau\}$ as above. In this case, we will also write ϕ_a^* for the additive polynomial with coefficients in R corresponding to ϕ_a.

Suppose that χ has as kernel a maximal ideal $\mathfrak{n}$. In this case we say that S is of *characteristic* $\mathfrak{n}$ and write $S/\mathbb{F}_{\mathfrak{n}}$, where $\mathbb{F}_{\mathfrak{n}} := A/\mathfrak{n}$. The map χ will also be called the *characteristic morphism.*

In the sequel we will only be interested in the case $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}_q}, \infty = (1:0)$, and $r = 2$ so $A = \mathbb{F}_q[T]$ and a Drinfeld module over an A-scheme S is determined by $\phi_T^E = \chi(T) + \alpha_1\tau + \alpha_2\tau^2$ with $\alpha_1 \in E^{\otimes(1-q)}(S), \alpha_2 \in E^{\otimes(1-q^2)}(S)$. Note that as α_2 is nonvanishing, we have $E^{\otimes q^2} \simeq E$.

Fix a monic prime polynomial $n \in \mathbb{F}_q[T]$ of degree d.

Lemma 2.5. — *Given $\phi_{/k}$ where k is a field of characteristic n we have either $\phi_n = (\alpha_d + \cdots + \alpha_{2d}\tau^d)\tau^d$ with $\alpha_d \neq 0$, or else $\phi_n = \alpha_{2d}\tau^{2d}$.*

Proof. — For all $a \in A, \phi_a\phi_n = \phi_n\phi_a$ since $\phi(A)$ is a commutative subring of $k\{\tau\}$. As $\chi(n) = 0$ we have $\phi_n = b\tau^r +$ (higher terms) for some $b \neq 0, r > 0$. Equating the terms of lowest degree in $\phi_a\phi_n = \phi_n\phi_a$ we have $\chi(a)b\tau^r = b\tau^r\chi(a)$ for all $a \in A$. Hence $\chi(a) = \chi(a)^{q^r}$ for all a so $\chi(A) = \mathbb{F}_n \subset k$ is fixed by τ^r. Therefore $d = [\mathbb{F}_n : \mathbb{F}_q]$ divides r. $\square$

Remark 2.6. — Let $\mathfrak{M}^r$ be the category for which an object is a Drinfeld module of rank r over an A-scheme. In [22, 1.5.1], it is shown that $\mathfrak{M}^r$ is a Deligne-Mumford stack and that the natural morphism of stacks $\mathfrak{M}^r \longrightarrow \operatorname{Spec} A$ is smooth of relative dimension $r-1$. In particular, the functor of deformations of a given Drinfeld module over a field k to artin local R-algebras with residue field k (where R is complete noetherian local A-algebra with residue field k) is prorepresented by a power series ring in $r-1$ variables over R (see [26, 2.5.(i)]).

Proposition 2.7. — *Given $E_{/S}$, where S is an A-scheme of characteristic n. Then $\phi_n = (\alpha_d + \cdots + \alpha_{2d}\tau^d)\tau^d$ in $E\{\tau\}$.*

Proof. — We may assume that $S = \operatorname{Spec} R$ is affine and that the underlying bundle of E is trivial. Further, it suffices to check the assertion over the subalgebra of R generated by $\chi(A)$ and the coefficients of ϕ_T so we may take R to be noetherian. Fix a coefficient a_i of τ_i in ϕ_n with $i < d$. Now, $a_i = 0$ if and only if its images in all localizations at a maximal ideal is zero. Thus we may assume that R is a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. As $R/\mathfrak{m}^s$ is artinian for all $s > 0$, $E \otimes_R R/\mathfrak{m}^s$ is uniquely a pullback of $\Phi_{/D}$ where the latter is the universal equicharacteristic deformation of $\phi := E \otimes_R k$, and where $D \simeq k[\![t]\!]$. Hence it suffices to prove the assertion of the proposition for $\Phi_{/D}$. But D is an integral domain so this follows from 2.5 by passing to its fraction field. □

2.8. For a scheme S over $\mathbb{F}_n$, let τ^d be the "absolute q^d-Frobenius" which is by definition the identity on topological spaces and the q^d-th power on structure sheaves. For a Drinfeld module $E_{/S}$, we define the relative Frobenius $\mathrm{F}_{E/S}$ (or just F) by the diagram:

$$\begin{tikzcd}
E \arrow[r, "\mathrm{F}"] \arrow[rr, bend left, "\tau^d"] \arrow[dr] & E^{(n)} \arrow[r, "\mathrm{pr}"] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d] \\
& S \arrow[r, "\tau^d"'] & S
\end{tikzcd}$$

Evidently, $E^{(n)}_{/S}$ is a Drinfeld module.

Before proceeding any further, let us recall that for invertible sheaves $\mathscr{L}$,$\mathscr{M}$ on a scheme S over $\mathbb{F}_p$ (regarded as S-group schemes) we have

$$\mathrm{Hom}_{S-\mathrm{gp}}(\mathscr{L}, \mathscr{M}) = H^0(S, \bigoplus_{n \geqslant 0} \mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{L}^{\otimes(-p^n)})$$

or equivalently: homomorphisms of *S-group schemes* correspond bijectively to *$\mathscr{O}_S$-linear* maps $\mathscr{M}^{-1} \longrightarrow \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes(-p^n)}$. In the case where $\mathscr{L} = \mathscr{M} = \mathscr{E}$ is the invertible sheaf underlying a Drinfeld module E over S, we see that $E\{\tau\}$ is contained in $H^0(S, \oplus_{n \geqslant 0} \mathscr{E}^{\otimes(1-p^n)})$.

Now, by 2.7, we may define the *Verschiebung* V by

$$V := \alpha_d + \cdots + \alpha_{2d}\tau^d$$

(notation as in 2.7). Since $\alpha_{d+j} \in E^{\otimes 1-q^{d+j}}(S).\tau^j$ for $0 \leqslant j \leqslant d$, V corresponds to an $\mathscr{O}_S$-linear map $\mathscr{E}^{-1} \longrightarrow \bigoplus_{j \geqslant 0} (\mathscr{E}^{\otimes q^d})^{\otimes q^j}$ and hence to a homomorphism $E^{(n)} \longrightarrow E$ of S-group schemes.

LEMMA 2.9. — *$\mathrm{F} : E \longrightarrow E^{(n)}$ and $\mathrm{V} : E^{(n)} \longrightarrow E$ are isogenies of Drinfeld modules over S.*

Proof. — There is an isomorphism $E^{(n)} = E \otimes_{\mathscr{O}_S, \tau^d} \mathscr{O}_S \xrightarrow{\sim} E^{\otimes q^d}$ given by $s \otimes a \rightsquigarrow a^{q^d} s^{\otimes q^d}$. By means of this isomorphism we see that if $\phi_a^E = \sum_{j=0}^{m(a)} \alpha_j(a)\tau^j$ then $\phi^{E^{(n)}} : A \longrightarrow E^{\otimes q^d}\{\tau\}$ is given by

$$a \rightsquigarrow \phi_a^{E^{(n)}} = \sum_{j=0}^{m(a)} \alpha_j(a)^{\otimes q^d} \tau^d.$$

That F is an isogeny of Drinfeld modules follows from the equation

$$\mathrm{F} \cdot \phi_a^E = \phi_a^{E^{(n)}} \cdot \mathrm{F} \in E^{(n)}\{\tau\}.$$

Finiteness of V follows from the fact that α_{2d} is nonvanishing. That V is A-equivariant follows from the calculation in $E\{\tau\}$:

$$\phi_T^E \cdot \mathrm{V} \cdot \mathrm{F} = \phi_T^E \cdot \phi_n^E = \phi_n^E \cdot \phi_T^E = \mathrm{V} \cdot \mathrm{F} \cdot \phi_T^E = \mathrm{V} \cdot \phi_T^{E^{(n)}} \cdot \mathrm{F}.$$

Since $\mathrm{F} = \tau^d$ is not a right zero divisor in the noncommutative ring $E\{\tau\}$ we may cancel it to obtain $\phi_T^E \cdot \mathrm{V} = \mathrm{V} \cdot \phi_T^{E^{(n)}}$ as required. □

COROLLARY 2.10. — *We have the commutative diagram*

$$\begin{tikzcd}
E \arrow[rr, "\phi_n^E"] \arrow[dr, "\mathrm{F}"'] & & E \arrow[dr, "\mathrm{F}"] & \\
& E^{(n)} \arrow[ur, "\mathrm{V}"] \arrow[rr, "\phi_n^{E^{(n)}}"'] & & E^{(n)}
\end{tikzcd}$$

and $0 \longrightarrow \ker \mathrm{F} \longrightarrow E[n] \xrightarrow{\mathrm{F}} \ker \mathrm{V} \longrightarrow 0$ *is an exact sequence of A-module schemes.*

Proof. — The commutativity of the diagram is clear. That the sequence is exact follows from the fact that the corresponding map $\mathscr{O}_{\ker \mathrm{V}} \longrightarrow \mathscr{O}_{E[n]}$ is finite locally free (hence faithfully flat) because the top coefficient of V is nonvanishing. □

DEFINITION 2.11. — *A Drinfeld module $\phi_{/k}$ over an algebraically closed field of characteristic n is* supersingular *if $\phi[n](k) = 0$, or equivalently by 2.5, $\phi_n = (unit) \cdot \tau^{2d}$. Otherwise $\phi_{/k}$ is called* ordinary*. A Drinfeld module $E_{/S/\mathbb{F}_n}$ is called* ordinary *if all of its geometric fibers are ordinary.*

Definition 2.12. — *Given $E_{/S/\mathbb{F}_n}$ so that we have $\phi_n^E = (\alpha_d + \cdots + \alpha_{2d}\tau^d) \cdot \tau^d$ we define the* Hasse invariant *to be the coefficient*

$$\alpha_d \in E^{\otimes 1-q^d}(S).$$

Remark 2.13. — Observe that $\mathrm{Tgt(V)} \in \mathrm{Hom}(\mathrm{Lie}E^{(n)}, \mathrm{Lie}E) = \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{E}^{\otimes q^d}, \mathscr{E})$ is given by $\alpha_d \in E^{\otimes 1-q^d}(S)$ so that this definition is completely analogous to the corresponding notion for elliptic curves [21, 12.4.1].

See also [8, §5]. The Hasse invariant is a modular form of weight $q^d - 1$ in the sense of [13, 1.14].

Proposition 2.14. — *The following are equivalent:*
(i) *$E/S/\mathbb{F}_n$ is ordinary.*
(ii) $\ker \mathrm{V}$ *is étale over S.*
(iii) *The Hasse invariant is nonvanishing.*

Proof. — Since $\ker \mathrm{V}$ is finite locally free over S it is étale if and only if all of its geometric fibers are étale and so we can take S to be the spectrum of an algebraically closed field. Now, by the functoriality of the connected-étale sequence for commutative group schemes over S we see that we have a connected-étale sequence in the category of A-module schemes, and hence we have the following diagram with rows and columns exact sequences of A-module schemes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (\ker \mathrm{F})^\circ \arrow[r] \arrow[d] & \ker \mathrm{F} \arrow[r] \arrow[d] & (\ker \mathrm{F})^{\text{ét}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (E[n])^\circ \arrow[r] \arrow[d] & E[n] \arrow[r] \arrow[d] & (E[n])^{\text{ét}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\ker \mathrm{V})^\circ \arrow[r] \arrow[d] & \ker \mathrm{V} \arrow[r] \arrow[d] & (\ker \mathrm{V})^{\text{ét}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(here, and in the sequel, we denote the connected component of a group scheme G by G°). Since $(\ker \mathrm{F})^{\text{ét}} = 0$ we have (i) $\iff$ (ii).

(ii) $\iff$ (iii) follows from the Jacobi criterion. ☐

3. Some facts from the Deformation Theory of Drinfeld modules and formal $\mathcal{O}$-modules

3.1. Let $\mathcal{O} := \widehat{A}_{(n)}, W := \widehat{\mathcal{O}^{\mathrm{nr}}}$, π a uniformizer of W, k an algebraic closure of $\mathbb{F}_n$ (so $W \simeq k[\![\pi]\!]$), and $K := \operatorname{Frac} \mathcal{O}$.

Let $\mathfrak{C}$ be the category of artin local W-algebras with residue field k, $R \in \mathfrak{C}$. A *deformation* of a Drinfeld module $\phi_{/k}$ to R is a pair $(E, i)_{/R}$ where $E_{/R}$ is a Drinfeld module over R and i is an isomorphism $E \otimes_R k \xrightarrow{\sim} \phi$.

3.2. Let $\widetilde{\phi}_{/R}$ be a deformation of $\phi_{/k}$ with $R \in \mathfrak{C}$ and let $\mathfrak{m}$ be the maximal ideal of R. Let f be an automorphism of $\widetilde{\phi}_{/R}$ which reduces to the identity mod $\mathfrak{m}$. Then we claim that f is the identity. To see this, write $f \in (R\{\tau\})^{\times}$ as $f = 1 + \mu$ with $\mu \in \mathfrak{m}\{\tau\}$. We will show $\mu = 0$.

Let us first check that $\mu \in \mathfrak{m}$. We have $(1+\mu) \cdot \widetilde{\phi}_a = \widetilde{\phi}_a \cdot (1+\mu)$ hence $\mu \cdot \widetilde{\phi}_a = \widetilde{\phi}_a \cdot \mu$ for all $a \in A$. Write $\widetilde{\phi}_a = c\tau^r +$ (lower order terms), $\mu = b\tau^s +$ (lower order terms) where we have chosen a such that $\mathfrak{m}^r = 0$. Then we have $c\tau^r b\tau^s = b\tau^s c\tau^r$ with $b \in \mathfrak{m}, c \in R^{\times}$. Hence $b = 0$ and $\mu \in \mathfrak{m}$.

To see that $\mu = 0$, suppose $\mu \equiv 0 \mod \mathfrak{m}^j$. Then for all $a \in A$ we have

$$\widetilde{\phi}_a \cdot (1+\mu) \equiv \widetilde{\phi}_a + \mu\chi(a) \equiv (1+\mu) \cdot \widetilde{\phi}_a \equiv \widetilde{\phi}_a + \mu\widetilde{\phi}_a \mod \mathfrak{m}^{j+1}.$$

Hence $\mu \equiv 0 \mod \mathfrak{m}^{j+1}$ and therefore $\mu = 0 \in R$. It follows that f is the identity.

Alternatively the claim follows from an application of the theory of relative schematic density [15, 11.10.1.(d)] to the prime to $\operatorname{char} k$ torsion of $\widetilde{\phi}_{/R}$, as is done to prove the analogous assertion for abelian schemes.

3.3. Given a Drinfeld module $\phi_{/k}$, let $\mathscr{D}$ be the functor from $\mathfrak{C}$ to sets given by $\mathscr{D}(R) :=$ the set of isomorphism classes of deformations of $\phi_{/k}$ over R. Then $\mathscr{D}$ is prorepresented by $W[\![t]\!]$ (see 2.6), which comes equipped with a universal formal deformation Φ of $\phi_{/k}$. Let m be an ideal of A divisible by at least two primes and suppose m is prime to n. Let $Y := Y(m)_{/A}$ be the (fine) moduli scheme classifying Drinfeld modules with a full level m structure, and write $\mathbb{E}$ for the universal Drinfeld module over Y.

As k is of characteristic n and m is prime to n we may, by the topological invariance of étale morphisms, identify the universal deformation ring $W[\![t]\!]$ with $\widehat{\mathscr{O}}_{Y_W,y}$. Here, $\phi_{/k}$ corresponds to a point $y \in Y(k)$ which we identify with a closed point of Y_W. Then the restriction of $\mathbb{E}$ to $\operatorname{Spec} \widehat{\mathscr{O}}_{Y_W,y}$ provides an algebraization of Φ.

Also recall that the deformation ring of $\phi_{/k}$ coincides with the deformation ring of its divisible formal $\mathcal{O}$-module $\phi[n^{\infty}]$ ([5, 5.4]) and, as k is algebraically closed, there are up to isomorphism only two possibilities for $\phi[n^\infty]$ ([5, 1.7]) namely

$$\phi[n^\infty]_{/k} = \begin{cases} \text{the unique formal } \mathcal{O}\text{-module} & \\ \text{of height one} \times K/\mathcal{O} & \text{“ordinary”} \\ \text{the unique formal } \mathcal{O}\text{-module} & \\ \text{of height two} & \text{“supersingular”} \end{cases}$$

where $K/\mathcal{O}$ refers to the constant formal group $\mathrm{Spf}_k \prod_{K/\mathcal{O}} k$ (notation as in [6, I.4.2]).

3.4. We continue to use the notation $\mathcal{O} = \widehat{A}_{(n)}$ so that $n \in A$ gives rise to a uniformizer $\pi \in \mathcal{O}$. Let R be a complete noetherian local $\mathcal{O}$-algebra with residue field k. By [27, 1.4] there is an equivalence between the category of connected π-divisible groups over R and the category of formal $\mathcal{O}$-modules of finite relative dimension over R such that $[\pi]$ is an isogeny. In particular, if $\Phi_{/R}$ is a deformation of a supersingular $\phi_{/k}$ then by the connected-étale sequence $\Phi[n^\infty]$ is connected and hence corresponds to $G_{/R} = \mathrm{Spf}_R\, R[\![x]\!]$, the latter being a one-dimensional formal $\mathcal{O}$-module of height 2. We will write $P \leftrightarrow x(P)$ for the bijection $\Phi[n^\infty](R) \leftrightarrow G(R) = \mathfrak{m}_R$ where $\mathfrak{m}_R$ is an $\mathcal{O}$-module via the power series defining G. See also [28, 2.2, Prop. 1].

The shape of the universal formal deformation.

The main result of this subsection, 3.13.(ii), is implicit in [5, §4]. We follow in this subsection the exposition of Yasufuku [30].

3.5. Let $\mathcal{O}$ be any ring and consider the functor which assigns to an $\mathcal{O}$-algebra R the set of formal $\mathcal{O}$-module structures on $\mathrm{Spf}_R\, R[\![X]\!]$. It is represented by

$$\Lambda_{\mathcal{O}} := \mathcal{O}[\Gamma_{ij}, \Theta_k(r) \,|\, i,j,k \geqslant 1, r \in \mathcal{O}]/\mathcal{I}$$

where the variable Γ_{ij} corresponds to the coefficient of X^iY^j in the universal formal $\mathcal{O}$-module $F(X,Y) \in \Lambda_{\mathcal{O}}[\![X,Y]\!]$, $\Theta_k(r)$ corresponds to the coefficient of X^k in $[r]_F.X \in \Lambda_{\mathcal{O}}[\![X]\!]$ for $r \in \mathcal{O}$, and $\mathcal{I}$ is the ideal expressing the relations imposed by the formal module axioms.

Lemma 3.6. — *$\Lambda_{\mathcal{O}}$ has a natural gradation such that Γ_{ij} has degree $i+j-1$, $\Theta_k(r)$ has degree $k-1$, and the set of elements of degree zero may be identified with $\mathcal{O} \subset \Lambda_{\mathcal{O}}$.* □

For a positive integer n let $\nu(n)$ be ℓ if $n > 1$ and n is a power of the prime ℓ, $\nu(n) := 1$ otherwise, and let $C_n(X,Y)$ be the polynomial $\frac{1}{\nu(n)}((X+Y)^n - X^n - Y^n) \in \mathbb{Z}[X,Y]$. Note that $C_n(X,Y)$ is primitive. Indeed, suppose that some prime p divided all of the coefficients of $C_n(X,Y)$. If n is a power of p then p^2 divides all of the coefficients of $(X+Y)^n - X^n - Y^n$. Then $m^n \equiv m \mod p^2$ for all positive integers m, which is impossible. If $n = p^a r$ with $r > 1$ prime to p, then $p \Big| \binom{n}{p^a}$, which again is impossible.

Proposition 3.7. — *Let $I \subset \Lambda_{\mathcal{O}}$ be the ideal generated by elements of positive degree and put $\widetilde{\Lambda}_{\mathcal{O}} := \Lambda_{\mathcal{O}}/I^2$; it is again a graded ring. As an $\mathcal{O}$-module the graded piece of degree $n-1$ in $\widetilde{\Lambda}_{\mathcal{O}}$, call it $\widetilde{\Lambda}_{\mathcal{O}}^{n-1}$, is generated by the symbols $\{\gamma_{n-1}, \theta_{n-1}(r)\}_{r\in\mathcal{O}}$ subject to the relations*

$$(r^n - 1)\gamma_{n-1} = \nu(n)\theta_{n-1}(r) \tag{3.7.1}$$

$$\theta_{n-1}(r + r') - \theta_{n-1}(r) - \theta_{n-1}(r') = C_n(r, r')\gamma_{n-1} \tag{3.7.2}$$

$$r\theta_{n-1}(r') + r'^n\theta_{n-1}(r) = \theta_{n-1}(rr') \tag{3.7.3}$$

where γ_{n-1} is defined by $\sum \Gamma_{i,n-i}X^iY^{n-i} = \gamma_{n-1}C_n(X,Y)$ in $\widetilde{\Lambda}_{\mathcal{O}}^{n-1}$ and $\theta_{n-1}(r) :=$ the image of $\Theta_n(r)$ in $\widetilde{\Lambda}_{\mathcal{O}}^{n-1}$.

Proof. — In $\Lambda_{\mathcal{O}}$, if a monomial of total degree $n-1$ in Γ and Θ does not contain a variable of degree $n-1$, then it must be a product of at least two positive-degree variables. These monomials of degree $n-1$ vanish when we mod out by I^2, so as an $\mathcal{O}$-module $\widetilde{\Lambda}_{\mathcal{O}}^{n-1}$ is generated by the images of $\Gamma_{i,n-i}$ and $\Theta_n(r)$; these images satisfy the formal $\mathcal{O}$-module identities.

To obtain γ_{n-1}, let $F_n := \sum \widetilde{\Gamma}_{i,n-i}X^iY^{n-i}$ over $\widetilde{\Lambda}_{\mathcal{O}}$. Then we have $F_n(X,Y) = F_n(Y,X)$ and $F_n(X,0) = F_n(0,X) = 0$. Now look at the polynomial degree n part of the identity $F(F(X,Y),Z) = F(X,F(Y,Z))$ where, as before, the universal formal $\mathcal{O}$-module over $\Lambda_{\mathcal{O}}$ is noted F. On the left side, if we consider the contribution from a positive degree coefficient (degree as an element of $\widetilde{\Lambda}_{\mathcal{O}}$) of the outer F, then we cannot have a contribution from a positive degree coefficient of the inner F. Hence, in degree n, the left side is $F_n(X+Y,Z) + F_n(X,Y)$. A similar computation for the right side gives the equality

$$F_n(X+Y,Z) + F_n(X,Y) = F_n(X,Y+Z) + F_n(Y,Z).$$

By [18, 1.6.6] $F_n(X,Y) = \gamma_{n-1}C_n(X,Y)$ for some $\gamma_{n-1} \in \widetilde{\Lambda}_{\mathcal{O}}$. Thus γ_{n-1} has degree $n-1$ in $\Lambda_{\mathcal{O}}$ since $C_n(X,Y)$ has coefficients in $\mathbb{Z}$. It follows that the module generated by the $\{\Gamma_{i,n-i}\}$ is generated by γ_{n-1}.

The relations (3.7.1)-(3.7.3) will follow from looking at the formal group identities, as follows. Write $[r]$ for $[r]_F$ and look at the terms of polynomial degree n in $F([r].X,[r].Y) = [r].F(X,Y)$. We get $F_n(rX,rY) +$

$\theta_{n-1}(r)X^n + \theta_{n-1}(r)Y^n = \theta_{n-1}(r)(X+Y)^n + rF_n(X,Y)$, so

$$\begin{aligned}(r^n - r)\gamma_{n-1}C_n(X,Y) &= \theta_{n-1}(r)((X+Y)^n - X^n - Y^n)\\ &= \theta_{n-1}(r)\nu(n)C_n(X,Y).\end{aligned}$$

This gives the (3.7.1) since $C_n(X,Y) \in \mathbb{Z}[X,Y]$ is primitive. Similarly, comparing T^n-coefficients in the identities $[r+r'].(T) = F([r].T,[r'].T)$ and $[r].([r'].T) = [rr'].T$ finishes the proof of proposition. □

For the rest of this section we take $\mathcal{O}$ to be the ring of integers of a nonarchimedean local field, with uniformizer π, and residue field of size q.

Proposition 3.8. — *For n a power of q there is an isomorphism of $\mathcal{O}$-modules $\widetilde{\Lambda}^{n-1}_{\mathcal{O}} \xrightarrow{\sim} \mathcal{O}$. More precisely, there exists a free generator $u \in \widetilde{\Lambda}^{n-1}_{\mathcal{O}}$ such that $\theta_{n-1}(r) = \frac{r^n - r}{\pi} \cdot u$ for all $r \in \mathcal{O}$ and $\gamma_{n-1} = \frac{p}{\pi} \cdot u$.*

Remark 3.9. — In characteristic p the proposition implies that $\gamma_{n-1} = 0$; this is to be expected since in equal characteristic the underlying formal group of any formal $\mathcal{O}$-module is $\widehat{\mathbb{G}}_a$ [18, 21.1.14].

Proof. — Consider the map of sets $\theta : \mathcal{O} \longrightarrow \widetilde{\Lambda}^{n-1}_{\mathcal{O}}$, $r \rightsquigarrow \theta_{n-1}(r) =: \theta(r)$. We will first show that $\widetilde{\Lambda}^{n-1}_{\mathcal{O}}$ is generated by $\theta(\pi)$ by showing that $M := \widetilde{\Lambda}^{n-1}_{\mathcal{O}}/\mathcal{O}.\theta(\pi)$ vanishes. For $\lambda \in \widetilde{\Lambda}^{n-1}_{\mathcal{O}}$ write $\overline{\lambda}$ for its image in M. For any $r \in \mathcal{O}$ we have $r\theta(\pi) + \pi^n\theta(r) = \theta(r\pi) = \theta(\pi r) = \pi\theta(r) + r^n\theta(\pi)$ by (3.7.3), so in M we must have $\overline{\theta(\pi r)} = \pi^n\overline{\theta(r)} = \pi\overline{\theta(r)}$. Hence $(\pi^n - \pi)\overline{\theta(r)} = 0$ and since $\pi^{n-1} - 1$ is a unit in $\mathcal{O}$ we have

$$\pi\overline{\theta(r)} = \overline{\theta(r\pi)} = 0 \tag{*}$$

for all $r \in \mathcal{O}$. Before proceeding, we note that (3.7.3) gives $\theta(1) = 0$. Now, using (3.7.2) we have

$$\begin{aligned}\theta(p) &= C_n(p-1,1)\gamma_{n-1} + \theta(p-1) + \theta(1)\\ &= C_n(p-1,1)\gamma_{n-1} + C_n(p-2,1)\gamma_{n-1} + \theta(p-2) + \theta(1)\\ &= \cdots\\ &= (C_n(p-1,1) + C_n(p-2,1) + \cdots + C_n(1,1))\gamma_{n-1}\\ &= (p^{n-1} - 1)\gamma_{n-1}\end{aligned}$$

where for the last equation we have used that $\nu(n) = p$ and $\sum_i C_n(p-i,1) = p^{n-1} - 1$ is a telescoping sum. Thus by (*) we have $0 = \overline{\theta(p)} = (p^{n-1}-1)\overline{\gamma_{n-1}}$ so that $\overline{\gamma_{n-1}} = 0$ since the residual characteristic of $\mathcal{O}$ is p and hence $p^{n-1}-1$ is a unit. Using (*) and (3.7.2) we see that $\overline{\theta} : \mathcal{O} \xrightarrow{\theta} \widetilde{\Lambda}^{n-1}_{\mathcal{O}} \xrightarrow{\mathrm{pr}} M$ is an additive map factoring through $\mathcal{O} \longrightarrow \mathcal{O}/\pi$. Now $r^n - r \in (\pi) \subset \mathcal{O}$ for all $r \in \mathcal{O}$ since n is a power of $q = \#\mathcal{O}/\pi$ and therefore by (3.7.3) and (*), $\overline{\theta} : \mathcal{O}/(\pi) \longrightarrow M$ is an $\mathcal{O}/(\pi)$-derivation and hence is zero. On the other hand, since $\overline{\gamma_{n-1}} = 0$, (3.7.1) implies that $\overline{\theta}$ is surjective and hence $M = 0$.

Using the three equations of 3.7 we get a well defined map $\widetilde{\Lambda}^{n-1}_{\mathcal{O}} \longrightarrow \mathcal{O}$ of $\mathcal{O}$-modules by $\gamma_{n-1} \rightsquigarrow \frac{p}{\pi}$ and $\theta(r) \rightsquigarrow \frac{r^n-r}{\pi}$. Since $\theta(\pi)$ is sent to the unit $\pi^{n-1}-1$ this homomorphism is surjective. It is an isomorphism since $\widetilde{\Lambda}^{n-1}_{\mathcal{O}}$ is generated by $\theta(\pi)$. Thus $u := (\pi^{n-1}-1)^{-1}\theta(\pi)$ is sent to 1 via this isomorphism and the proof is complete. □

Proposition 3.10. — *There is an isomorphism* $\mathcal{O}[g_1, g_2, \dots] \xrightarrow{\sim} \Lambda_{\mathcal{O}}$ *of graded $\mathcal{O}$-algebras, where* $\deg g_i = i$.

Proof. — The proof is given in [5, §4]. For our purposes we simply note that the map is defined by sending g_{n-1} to a lift in $\Lambda_{\mathcal{O}}$ of the generator u of $\widetilde{\Lambda}^{n-1}_{\mathcal{O}}$. □

Recall that the formal $\mathcal{O}$-module $\widehat{\mathbb{G}}_a$ over an $\mathcal{O}$-algebra is defined by $F(X,Y) = X+Y$ with the action of $r \in \mathcal{O}$ given by $[r].X := rX$.

Definition 3.11. — *A one-dimensional formal $\mathcal{O}$-module F of height h over a field k over $\mathcal{O}/(\pi)$ is* normal *(with respect to the choice of π) provided that $\mathbb{F}_{q^h} \subset k$ and*

(1) $[\pi]_F.X = X^{q^h}$

(2) $F(X,Y) \in \mathbb{F}_{q^h}[\![X,Y]\!]$ *and* $[r]_F.X \in \mathbb{F}_{q^h}[\![X]\!]$ *for all* $r \in \mathcal{O}$

(3) $F \equiv \widehat{\mathbb{G}}_a$ *(mod poly. deg. q^h).*

Remark 3.12. — Any formal $\mathcal{O}$-module over a separably closed field over $\mathcal{O}/(\pi)$ is isomorphic to a normal one by [5, 1.7].

Proposition 3.13. — *Let G be a formal $\mathcal{O}$-module over $\widehat{\mathcal{O}^{\mathrm{nr}}}/(\pi)$ of height h.*

(i) *The functor of isomorphism classes of deformations of G to complete noetherian local $\widehat{\mathcal{O}^{\mathrm{nr}}}$-algebras with residue field $\widehat{\mathcal{O}^{\mathrm{nr}}}/(\pi)$ is represented by* $\widehat{\mathcal{O}^{\mathrm{nr}}}[\![t_1, \dots, t_{h-1}]\!]$.

(ii) *If G is normal then the universal deformation $\mathbb{G}$ may be taken such that the coefficient of X^{q^j} in $[\pi]_{\mathbb{G}}.X$ is equal to $(\pi^{q^j-1}-1)t_j$ for $j = 1, \dots, h-1$.*

Proof. — Note first that if G is normal then the map $\psi : \mathcal{O}[g_i] \longrightarrow \widehat{\mathcal{O}^{\mathrm{nr}}}/(\pi)$ inducing G sends g_i to 0 for $i < q^h - 1$.

An $\mathcal{O}$-algebra map $f : \mathcal{O}[g_i] \longrightarrow \widehat{\mathcal{O}^{\mathrm{nr}}}[\![t_1, \dots, t_{h-1}]\!]$ is defined by

$$\begin{aligned} f(g_{q^i-1}) &= t_i & & 1 \leqslant j \leqslant h-1 \\ f(g_j) &= 0 & & 1 \leqslant j < q^h - 1 \text{ and } j+1 \neq \text{power of } q \\ f(g_j) &= \text{lift of } \psi(g_j) \text{ to } \widehat{\mathcal{O}^{\mathrm{nr}}} & & \text{otherwise} \end{aligned}$$

Let $\mathbb{G}$ be the resulting formal $\mathcal{O}$-module over $\widehat{\mathcal{O}^{\mathrm{nr}}}[\![t_1, \dots, t_{h-1}]\!]$. Then (i) is proved in [5, 4.2] by reducing to the case that G is normal and showing that $\mathbb{G}$ is the universal deformation.

For (ii), it suffices—by the definition of f—to show that under the map

$$\Lambda_{\mathcal{O}} := \mathcal{O}[\Gamma_{ij}, \Theta_k(r)] \xrightarrow{\sim} \mathcal{O}[g_1, g_2, \dots]$$

$\Theta_{q^j}(\pi)$ is sent to $(\pi^{q^j-1} - 1)g_{q^j-1}$. This follows from 3.8 and the proof of 3.10. $\square$

Remark 3.14. — From part (ii) of the proposition it follows that two deformations G_1, G_2 of a given normal formal $\mathcal{O}$-module G over $\widehat{\mathcal{O}^{\mathrm{nr}}}/(\pi)$ are isomorphic as deformations if and only if the coefficients of $X, X^q, \dots, X^{q^{h-1}}$ in $[\pi]_{G_1}.X$ and $[\pi]_{G_2}.X$ coincide.

Remark 3.15. — Let $\Phi_{/k[\![t]\!]}$ be the universal equicharacteristic deformation of a supersingular Drinfeld module ϕ over an algebraically closed field k of characteristic n. Since $[\pi]_{\Phi[n^\infty]}.X$ coincides with the additive polynomial corresponding to $\Phi_n \in k[\![t]\!]\{\tau\}$, the coefficient of X^{q^d} in $[\pi]_{\Phi[n^\infty]}.X$ is equal to Hasse invariant of Φ. This in turn is equal to the image of $(\pi^{q^d-1} - 1)t$ under $\widehat{\mathcal{O}^{\mathrm{nr}}}[\![t]\!] \longrightarrow k[\![t]\!]$. As $k[\![t]\!]$ is of characteristic n, this image is $-t$ and thus the Hasse invariant "has simple zeros." Cf. [8, 5.6], [21, 12.4.3], and the proof of 5.4 below.

4. $\Gamma_1(n)$-structures

Definition 4.1. — *A $\Gamma_1(n)$-*structure *on a Drinfeld module $E_{/S}$ is a map of (abstract) A-modules $\iota : \mathbb{F}_n \longrightarrow E[n](S)$ such that $G_\iota := \sum_{z\in\mathbb{F}_n}[\iota(z)]$ is an A-submodule scheme of $E[n]$ (and hence $G_\iota \leqslant E[n]$ as relative effective Cartier divisors on E/S).*

4.2. By abuse of language, both ι and $\iota(1) \in E(S)$ will be referred to as a "point of order n." We will often denote $\iota(1)$ by P.

Given $E_{/S}$, let $\mathscr{M}_{E/S}$ be the functor which assigns to an S-scheme T the set of $\Gamma_1(n)$-structures on $E_{/T}$.

Proposition 4.3. — *The functor $\mathscr{M}_{E/S}$ is represented by a closed subscheme of $E[n]$. If n is invertible on S, i.e. if the structure map $S \longrightarrow \operatorname{Spec} A$ factors through $\operatorname{Spec} A[n^{-1}]$, then $\mathscr{M}_{E/S}$ is étale over S.*

Proof. — Clearly, the functor H which sends an S-scheme T to $\mathrm{Hom}_{A\text{-mod}}(\mathbb{F}_n, E(T))$ is represented by $E[n]$, and $\mathscr{M}_{E/S}$ is a subfunctor of H. Let $\alpha^{\mathrm{univ}} : \mathbb{F}_n \longrightarrow E_H(H)$ be the universal homomorphism and put $G_{\alpha^{\mathrm{univ}}} := \sum_{z\in\mathbb{F}_n}[\alpha^{\mathrm{univ}}(z)]$. Then by [21, pp.14-15] there is a closed subscheme $J \subset H$ which is universal for the conditions: (i) $G_{\alpha^{\mathrm{univ}}} \leqslant E_H[n]$ as

Cartier divisors on E_H/H, (ii) $G_{\alpha^{\mathrm{univ}}}$ is a commutative subgroup scheme of $E_H[n]$. Hence $\mathscr{M}_{E/S}$ is a subfunctor of J.

Let $\beta : \mathbb{F}_n \longrightarrow E_J[n]$ be the universal object over J, with associated divisor $D := \sum[\beta(z)]$. Consider the diagram

$$\begin{tikzcd} E \arrow[d] & D \arrow[l, hook'] \arrow[dl] \\ J & \end{tikzcd}$$

Make the base extension $D \longrightarrow J$ to get a section Δ

$$\begin{tikzcd} E_D \arrow[d] & D \times_J D \arrow[l, hook'] \arrow[dl, shift right] \\ D \arrow[ur, shift right, "\Delta"'] & \end{tikzcd}$$

Then $\mathscr{M}_{E/S} \times_J D$ is represented by the closed subscheme of D which is universal for $[\phi_t(\Delta)] \leqslant D_D$ (that this condition is representable is [21, 1.3.4]). This is the condition that D_D be A-invariant; note that the cotangent condition in the definition of an A-module scheme is automatically satisfied since it is satisfied on E and hence on any closed subgroup scheme on which A acts, so we need only check A-invariance to represent the condition of being an A-module subscheme.

Now, as D is finite locally free over $J \subset E[n]$, $\mathscr{M}_{E/S}$ is represented by a closed subscheme of J, as on [21, p. 14].

To see that $\mathscr{M}_{E/S}$ is étale over S if n is invertible on S, we argue as follows. Define $Q = E[n] - \{0\}$ so that Q is étale. Let $g : \mathscr{M}_{E/S} \longrightarrow Q$ be the map $\iota \rightsquigarrow \iota(1)$. By descent theory (see [17, Exp. VIII, 5.4]), g is an isomorphism if and only if it is after an fpqc base change $T \longrightarrow S$. Take T to be the finite étale cover $E[n] \longrightarrow S$ so that E possesses a full level n structure over T, i.e. an isomorphism $\mathbb{F}_n^{\oplus 2} \xrightarrow{\sim} E[n]$.

By means of the level structure, we see that the map g has the effect of sending an injection $\iota : \mathbb{F}_n \hookrightarrow \mathbb{F}_n^{\oplus 2}$ to $\iota(1) \in \mathbb{F}_n^{\oplus 2} - 0$. Hence g is an isomorphism over T, as claimed. $\square$

4.4. Let m be an ideal of A divisible by at least two primes, so that there exists a fine moduli space $Y(m)$ for the functor of isomorphism classes of Drinfeld modules over A-schemes with full level m structure. Now assume that m is prime to n and let $\mathbb{E}$ be the universal Drinfeld module over $Y(m)$. Then $Y_1(n,m) := \mathscr{M}_{\mathbb{E}/Y(m)}$ is a fine moduli space for the functor of isomorphism classes of triples consisting of a Drinfeld module over an A-scheme, a point of order n, and a full level m structure. We remark that if we restricted ourselves to $A[n^{-1}]$-schemes, then the moduli problem for Drinfeld modules with a point of order n would be representable.

To ease notation, put $\mathcal{Y}_1 := Y_1(n,m)$, $Y := Y(m)$. By 4.3, the obvious map $\pi : \mathcal{Y}_1 \longrightarrow Y$ is a finite map of A-schemes and its restriction over $A[n^{-1}]$ is étale.

Regularity.

Proposition 4.5. — *$\mathcal{Y}_1$ is a regular surface and $\pi : \mathcal{Y}_1 \longrightarrow Y$ is flat.*

Proof. — This is Deligne's homogeneity principle [21, p. 130 ff.]. The argument must be modified in several places to make it work in the function field setting, and various aspects of the proof will be needed later on. Details are given for the reader's convenience.

We may work over $A_{(n)}$ by 4.3 and by the regularity of $Y(m)$ [5, Cor. p. 577]. Define

$$U := \{y \in Y \mid \forall x \to y, \mathscr{O}_{\mathcal{Y}_1,x} \text{ is regular and is flat over } \mathscr{O}_{Y,y}\}.$$

Now π is proper, hence closed, and the nonregular locus in $\mathcal{Y}_1$ is closed since $\mathcal{Y}_1$ is excellent [14, 7.8.6(iii)] as is the nonflat locus [25, 24.3], so that U is the complement of the union of two closed sets, hence open. Further, U contains the generic fiber of Y since π is generically étale. Thus, to show $U = Y$ it is enough to show that U contains every closed point of $Y_{\mathbb{F}_n}$ (since the closed points are dense in the latter). We say $y \in Y_{\mathbb{F}_n}$ is ordinary or supersingular according as the corresponding Drinfeld module $\phi_{/k(y)}$ is ordinary or supersingular.

Let $y_0 \in Y_{\mathbb{F}_n}$ be a closed point. Grant for now the following assertions:

(i) $y_0 \in U$ if and only if $\mathscr{M}_{\Phi/W[\![t]\!]}$ is regular and is flat over $W[\![t]\!]$. Here, as in 3.3, Φ is the universal deformation over $W[\![t]\!]$ of $\phi \otimes_{k(y_0)} \overline{k(y_0)}$.

(ii) If y_0 is a supersingular point then $\mathscr{M}_{\Phi/W[\![t]\!]}$ is regular and is flat over $W[\![t]\!]$.

For arbitrary $E_{/S}$, it is clear that $\mathscr{M}_{E/S}$ depends only on $E[n^\infty]$, i.e. given E, E' over S and an isomorphism $E[n^\infty] \xrightarrow{\sim} E'[n^\infty]$, we have an isomorphism $\mathscr{M}_{E/S} \xrightarrow{\sim} \mathscr{M}_{E'/S}$. Now, as $\mathscr{M}_{\Phi/W[\![t]\!]}$ depends only on $\Phi[n^\infty]$, hence only on $\phi[n^\infty]$, which in turn depends only on whether ϕ is ordinary or supersingular, we see that U contains all ordinary points or none, and likewise contains all supersingular points or none.

As there exist only finitely many supersingular points [12, 4.2], and $Y_{\mathbb{F}_n}$ is smooth (2.6), every neighborhood of a supersingular point contains ordinary points. Since U is open, it therefore suffices to check that U contains a supersingular point. This is so by (ii).

Proof of (i). Recall that if $A \longrightarrow B$ is a finite local homomorphism of noetherian local rings, then B is flat over $A \iff \widehat{B}$ is flat over $\widehat{A} \iff B^{\mathrm{sh}}$ is flat over A^{sh} [EGA IV$_4$, 18.8.10]. Also, B is regular $\iff \widehat{B}$ is regular $\iff B^{\mathrm{sh}}$ is regular [16, 18.8.3]. Hence, we see that $\mathscr{O}_{\mathcal{Y}_1,x}$ is regular and is flat over $\mathscr{O}_{Y,y} \iff \widehat{\mathscr{O}^{\mathrm{sh}}_{\mathcal{Y}_1,x}}$ is regular and is flat over $\widehat{\mathscr{O}^{\mathrm{sh}}_{Y,y}}$.

Next, for any scheme X of finite type over A, there are natural bijections between $X(k)$, the set of pairs $(x_0, k(x_0) \hookrightarrow k)$ where x_0 a closed point of X, and the set of closed points of $x \in X_W$ with residue field k. Here k is an algebraic closure of $\mathbb{F}_n$. Under this bijection, we have $\widehat{\mathscr{O}}_{X_W,x} \simeq \widehat{\mathscr{O}^{\mathrm{sh}}_{X,x_0}}$; indeed, both are initial objects in the category of complete noetherian local $\mathscr{O}_{X,x_0}$-algebras with residue field $k(x)$.

Fix $y_0 \in Y$ and $y \in Y_W$ over it. Then, by the preceding, y is in U if and only if for all $x \in \mathcal{Y}_{1/W}$ over y, $\widehat{\mathscr{O}}_{\mathcal{Y}_{1/W},x}$ is regular and is flat over $\widehat{\mathscr{O}}_{Y_W,y}$. As $\mathcal{Y}_{1/W} \longrightarrow Y_W$ is finite, we have

$$\coprod_{x \to y} \operatorname{Spec} \widehat{\mathscr{O}}_{\mathcal{Y}_{1/W},x} = \mathcal{Y}_{1/W} \times_{Y_W} \operatorname{Spec} \widehat{\mathscr{O}}_{Y_W,y}.$$

The latter scheme is isomorphic to $\mathscr{M}_{\Phi/W[\![t]\!]}$ by means of the identification of $\mathbb{E}_{/\widehat{\mathscr{O}}_{Y_W,y}}$ with $\Phi_{/W[\![t]\!]}$. This proves (i).

Proof of (ii). Suppose $\phi_{/k}$ is supersingular, and let $\operatorname{Spec} D$ denote the finite $W[\![t]\!]$-scheme $\mathscr{M}_{\Phi/W[\![t]\!]}$. Then $\mathscr{M}_{\Phi/W[\![t]\!]}(k)$ is a singleton so that D is a complete local ring.

The map $\operatorname{Spec} D \longrightarrow \operatorname{Spec} W[\![t]\!]$ has dense image since it is finite étale and surjective upon inverting n. Being finite it is closed and hence surjective. Therefore $\dim D \geqslant 2$. Hence to show D is regular it suffices to show $\mathfrak{m}_D$ is generated by two elements. We will use the following fact: Let D be a complete local W-algebra with residue field k, $f, g \in \mathfrak{m}_D$. Then $\mathfrak{m} = (f,g)$ if and only if for every artin local W-algebra R and every $\xi : D \longrightarrow R$ such that $\xi(f) = \xi(g) = 0$, ξ factors through k. [*Proof.* To prove sufficiency (necessity being clear), let $\overline{D} := D/(f,g), \overline{\mathfrak{m}} = \mathfrak{m}/(f,g)$. Let ξ_i be the projection $D \longrightarrow \overline{D}/\overline{\mathfrak{m}}^i$. By assumption, each of these factors through k and as $\overline{D}$ is complete we see that the resulting map $\xi : D \longrightarrow \overline{D}$ factors through $k = D/\mathfrak{m}$. Hence $\mathfrak{m} = (f,g)$.]

Returning to the proof of (ii), let $\Psi := \Phi \otimes_{W[\![t]\!]} D$ and α be the universal $\Gamma_1(n)$-structure on Ψ, so that $(\Psi_{/D}, \alpha, \mathrm{id})$ is the universal object for the functor which sends $R \in \mathfrak{C}$ to the set of isomorphism classes of deformations of $(\phi, 0)$ over R. Such a deformation is by definition a triple $(E, \beta, i)_{/R}$ where $(E, i)_{/R}$ is a deformation of $\phi_{/k}$ and i (necessarily) carries $\beta \otimes k$ to the unique point of order n on $\phi_{/k}$, namely 0.

Put $P := \alpha(1) \in \Psi[n](D) \subset \Psi[n^\infty](D)$. Because $\phi_{/k}$ is supersingular, $\Psi[n^\infty]$ is a formal $\mathcal{O}$-module and $\Psi[n^\infty](D) = \mathfrak{m}_D$ with $\mathcal{O}$-module structure from $\Psi[n^\infty]$ as discussed in 3.4. Let $x(P) \in \mathfrak{m}_D$ be the image of P under the above identifications.

Write $t \in D$ for the image of t under $W[\![t]\!] \longrightarrow D$. Then we will use the above claim to show that $\mathfrak{m}_D = (x(P), t)$. So let $\xi : D \longrightarrow R$ be such that $\xi(t) = \xi(x(P)) = 0$, and let (E, β, i) be the pullback of $(\Psi, \alpha, \mathrm{id})$. Then what we want to show is that (E, β, i) comes from $(\phi, 0, \mathrm{id})$ by extension of scalars $k \longrightarrow R$.

It is enough to show that E and β are constant since otherwise we would have distinct isomorphisms $E \xrightarrow{\sim} \phi \otimes_k R$, contradicting 3.2.

Now, as $\xi(x(P)) = 0 \in R$ it is clear that β is the zero map so that $\sum_{\mathbb{F}_n}[0] = \operatorname{Spec} R[z]/z^{q^d}$ is an A-submodule scheme of $E[n]$. In other words, z^{q^d} divides $\phi_n^*(z)$ so that ϕ_n^* has no linear term and $\chi(n) = \partial \circ \phi_n^E = 0$ where $\chi : A \longrightarrow R$ is the characteristic morphism. But E is induced from $W[\![t]\!]$ via the composite $W[\![t]\!] \longrightarrow D \longrightarrow R$, so $\chi(n) = 0$ implies that the uniformizer of W is sent to zero in R and since t is sent to zero by assumption, we see that $W[\![t]\!] \longrightarrow R$ factors through k. Thus E is constant, as required.[1]

Finally, the map $W[\![t]\!] \longrightarrow D$ is flat since it is a finite map between regular rings of the same dimension [1, V.3.6]. $\square$

Remark 4.6. — Let z be a point of $\mathcal{Y}_{1/k}$ such that the corresponding pair (ϕ, P) has $P = 0$, and write $\widehat{\mathscr{O}}_z := \mathscr{O}_{\mathcal{Y}_{1/W}, z}$. The above proof provides a surjection $W[\![x, t]\!] \twoheadrightarrow \widehat{\mathscr{O}}_z$ the kernel of which is a principal ideal with generator that is part of a regular system of parameters of $W[\![x, t]\!]$.

5. Igusa Curves

Definition 5.1. — *Given $E_{/S/\mathbb{F}_n}$ we define functors from (schemes/S) to (sets) by*

$$\begin{aligned} \mathscr{F}_{E/S} : T &\rightsquigarrow \{\iota : \mathbb{F}_n \longrightarrow E[n](T) \mid \textstyle\sum[\iota(z)] = \ker \mathrm{F}\}, \\ \mathscr{V}_{E/S} : T &\rightsquigarrow \{\iota : \mathbb{F}_n \longrightarrow E^{(n)}[n](T) \mid \textstyle\sum[\iota(z)] = \ker \mathrm{V}\}. \end{aligned}$$

Elements of $\mathscr{V}_{E/S}(T)$ are called Igusa structures *on $E_{/T}$.*

[1] The argument in this paragraph is easier than the corresponding argument in [21]; the very definition of Drinfeld module gives that n (and hence the uniformizer of W) is sent to zero in R, whereas for elliptic curves this is proved by a more difficult argument in [21, 5.3.3,5.3.4].

The functor $\mathscr{F}_{E/S}$ (resp. $\mathscr{V}_{E/S}$) is represented by a closed subscheme of $E[n]$ (resp. $E^{(n)}[n]$) by [21, 1.3.5, 1.6.5]. In the case of $\mathscr{F}_{E/S}$ we will need the following explicit description.

Proposition 5.2. — *$\mathscr{F}_{E/S}$ is equal to* $\operatorname{Spec}(\operatorname{Sym}_{\mathscr{O}_S}\mathscr{E}^\vee)/(\mathscr{E}^\vee)^{\otimes(q^d-1)}$ *as closed subschemes of E.*

Proof. — Choose $m \in A$ prime to n. Then $T := E[m] \longrightarrow S$ is finite étale and has a section, hence descent data with respect to this cover is effective; we may assume without loss that E is endowed with a full level m structure over S. Let $Y := Y(m)_{/\mathbb{F}_n}$ and $\mathbb{E}$ be the universal Drinfeld module over Y. It suffices to prove the assertion for $\mathbb{E}$. As $\mathbb{E}$ carries an étale level structure over Y it has nonvanishing sections and therefore the underlying line bundle of $\mathbb{E}$ is isomorphic to the trivial bundle over Y so that we may identify $\ker \mathrm{F}$ with $\operatorname{Spec}\mathscr{O}_Y[X]/X^{q^d}$. Write $\mathscr{F}$ for $\mathscr{F}_{\mathbb{E}/Y}$.

Put $Z := \operatorname{Spec}\mathscr{O}_Y[X]/X^{q^d-1}$. We claim that the closed immersion defined by $\mathscr{F} \hookrightarrow \ker \mathrm{F}$, $\iota \rightsquigarrow \iota(1)$ factors through $Z \hookrightarrow \ker \mathrm{F}$. To see this, consider $\iota(1) \in (\ker \mathrm{F})(B)$, for an $\mathscr{O}_S$-algebra B. Since $\sum[\iota(z)] = \ker \mathrm{F}$ we have $\prod_{z\in\mathbb{F}_n}(u-\iota(z)) = u^{q^d}$ in the polynomial ring $B[u]$ and hence $\prod_{z\in\mathbb{F}_n^\times}(u-\iota(z)) = u^{q^d-1}$. Evaluating at $u=\iota(1)$ we see that $\iota(1)^{q^d-1}=0$.

Thus we have a closed immersion $g : \mathscr{F} \hookrightarrow Z$. To show g is an isomorphism it suffices to show it is an isomorphism after making the fpqc base change $Y_k \to Y$, where $k := \overline{\mathbb{F}}_n$. As $g \otimes k$ is globally defined, it is enough to show it is an isomorphism locally. So fix a closed point $y \in Y_k$ corresponding to the Drinfeld module $\phi_{/k}$. By 3.12, we may assume without loss that $\phi[n^\infty]^\circ$ is a normal formal $\mathcal{O}$-module.

Clearly $\mathbb{E}[n^\infty]/\widehat{\mathscr{O}}_{Y_k,y}$ is the universal equicharacteristic deformation of the divisible formal $\mathcal{O}$-module $\phi[n^\infty]_{/k}$ and we may identify $\widehat{\mathscr{O}}_{Y_k,y}$ with $k[\![t]\!]$ as before.

Now, over $k[\![t]\!]$ we have $\mathscr{F} \hookrightarrow Z \hookrightarrow F := \mathbb{E}[n^\infty]^\circ$ and $F_{/k[\![t]\!]}$ is the universal equicharacteristic deformation of the formal $\mathcal{O}$-module $\phi[n^\infty]^\circ$. Fix an identification of $\mathcal{O}$ with $\mathbb{F}_n[\![\pi]\!]$.

There is a formal $\mathcal{O}$-module G over $k[\![t]\!]$ such that

$$[\pi]_G.X = [\pi]_F.X,$$
$$[\zeta]_G.X = \zeta X \text{ for } \zeta \in \mathbb{F}_n \subset \mathbb{F}_n[\![\pi]\!]$$

and an isomorphism $F \xrightarrow{\sim} G$ of deformations by 3.14 (here X is the coordinate of the formal group $\widehat{\mathbb{G}}_a/k[\![t]\!]$ on which $\mathcal{O}$ acts via $[\pi]_F, [\pi]_G$— recall that any formal $\mathcal{O}$-module has as underlying formal group the formal additive group since we are in characteristic p [18, 21.1.14]).

As $\mathscr{F}$ and Z are closed subschemes of the formal scheme F to show $\mathscr{F} \hookrightarrow Z$ is an isomorphism it suffices to show that the induced map $\mathscr{F}(B) \to Z(B)$ is an isomorphism for B an artinian $k[\![t]\!]$-algebra.

The isomorphism $F \xrightarrow{\sim} G$ preserves the (q^d-1)th roots of zero, i.e. preserves Z, and in G we see that an element $\iota \in \mathscr{F}(B)$, i.e. a generator of $\ker \mathrm{F} \subset E[n]$, satisfies

$$\begin{aligned} X^{q^d-1} &= \prod_{z \in \mathbb{F}_n^\times} (X - \iota(z)) = \prod_{\zeta \in \mathbb{F}_n^\times \subset \mathbb{F}_n[\![\pi]\!]} (X - [\zeta]_G . \iota(1)) \\ &= \prod_{\zeta} (X - \zeta \iota(1)) = X^{q^d-1} - \iota(1)^{q^d-1}. \end{aligned}$$

Hence giving $\iota \in \mathscr{F}(B)$ is the same as giving a (q^d-1)th root of zero in B, or in other words, an element of $Z(B)$. □

Lemma 5.3. — *If $E_{/S}$ is ordinary then $\mathscr{V}_{E/S}$ is étale over S.*

Proof. — In this case, $\ker \mathrm{V}_{E/S}$ is an étale A-module scheme over S. We have

$$\mathscr{V}_{E/S}(\,\cdot\,) = \mathrm{Isom}_{A-\text{mod.sch.}}(\underline{\mathbb{F}_n}, \ker \mathrm{V})(\,\cdot\,)$$

and the latter sheaf is obviously (formally) étale. □

Write $Y := Y(m)_{\mathbb{F}_n}$, let $\mathbb{E}$ be the universal Drinfeld module over Y, and let $\mathscr{V} = \mathscr{V}_{\mathbb{E}/Y}$.

Proposition 5.4. — *$\mathscr{V}_{\mathbb{E}/Y}$ is regular and is flat over Y. In particular, it is smooth over $\mathbb{F}_n$.*

Proof. — $\mathscr{V}$ is étale over Y^{ord} by the lemma and so we are reduced to studying what happens near a supersingular point.

Let y_0 be a supersingular point of Y and $x_0 \in \mathscr{V}$ the unique point over it. Let x, y be points of $\mathscr{V}_k, Y_k$ over x_0, y_0 where $k = \overline{\mathbb{F}}_n$. As in the proof of 4.5, $\mathscr{O}_{\mathscr{V},x_0}$ is regular and is flat over $\mathscr{O}_{Y,y_0}$ if and only if $\widehat{\mathscr{O}}_{\mathscr{V}_k,x}$ is regular and is flat over $\widehat{\mathscr{O}}_{Y_k,y}$.

Now $\widehat{\mathscr{O}}_{Y_k,y} \simeq k[\![t]\!]$ is regular and of dimension one and therefore flatness of the finite ring extension $\widehat{\mathscr{O}}_{Y_k,y} \to \widehat{\mathscr{O}}_{\mathscr{V}_k,x}$ follows if the latter is regular and of dimension one.

Let $\phi_{/k}$ be the supersingular Drinfeld module corresponding to $y \in Y_k$ so that the universal equicharacteristic deformation $\Phi/k[\![t]\!]$ of ϕ is identified with $\mathbb{E}/\widehat{\mathscr{O}}_{Y_k,y}$, as usual. We have $\mathscr{V}_k \otimes_{\mathscr{O}_{Y_k,y}} \widehat{\mathscr{O}}_{Y_k,y} = \mathscr{V}_{\Phi/k[\![t]\!]} =: \operatorname{Spec} D$ for D a finite (hence complete) local (since $\phi_{/k}$ is supersingular) $\widehat{\mathscr{O}}_{Y_k,y}$-algebra.

We claim that if E is a Drinfeld module over a separably closed field K of characteristic n then there exists an Igusa structure on E, so that

$\mathscr{V}_{E/K}(K) \neq \varnothing$. Indeed, if E is ordinary then $\ker \mathrm{V}$ is isomorphic to $\underline{\mathbb{F}_n}$ and this isomorphism provides the Igusa structure. On the other hand, if E is supersingular then 0 is a generator of $\ker \mathrm{V}$.

Let us show that $\operatorname{Spec} D \longrightarrow \operatorname{Spec} k[\![t]\!]$ is surjective. It is finite, hence closed. On the other hand, the image contains the dense open $\operatorname{Spec} k((t))$ since there exist Igusa structures on $\Phi/k((t))_{\mathrm{sep}}$. Thus $\dim D \geqslant 1$ and therefore to show D is regular it suffices to show that $\mathfrak{m}_D$ is principal.

Let $\Psi_D := \Phi \otimes_{k[\![t]\!]} D$; it carries the universal Igusa structure, call it α. Then $(\Psi, \alpha)_{/D}$ prorepresents the functor which assigns to an artin local k-algebra R with residue field k the set of isomorphism classes of triples $(E, \beta, i : E \otimes_R k \xrightarrow{\sim} \phi)$ where E is a deformation of ϕ over R, β is an Igusa structure on E, and i is an isomorphism which carries β to the unique Igusa structure on ϕ.

Let $P := \alpha(1) \in (\ker \mathrm{V})(D) \subset E^{(n)}[n](D)$ and x be the coordinate of the formal $\mathcal{O}$-module $\Psi^{(n)}[n^\infty]_{/D}$ We will show that $\mathfrak{m}_D = (x(P))$.

As in the proof of 4.5 it is enough to show that for all $\xi : D \longrightarrow R$ such that $\xi(x(P)) = 0$, the induced triple $\xi_*(\Psi, \alpha) =: (E, \beta, i)_{/R}$ is constant. Fix such an R and ξ. As before, it will suffice to show that E and β are constant. We may assume that $\phi[n^\infty]$ is normal by 3.12.

By definition of Ψ, to show E is constant, it is enough to show that under the composite $k[\![t]\!] \longrightarrow D \xrightarrow{\xi} R$ the element t is sent to $0 \in R$.

Write $G_{/R}$ for the formal $\mathcal{O}$-module $E[n^\infty]$. Since $\xi(x(P)) = 0$ we have that $\ker(\mathrm{V} : E^{(n)} \longrightarrow E)$ is generated by $0 \in R$, hence β is constant. Since the equations defining the Cartier divisors $\ker \mathrm{V}$ and $\sum_{\mathbb{F}_n}[0]$ differ by a unit multiple, we obtain the equations

$$\alpha_d X + \cdots + \alpha_{2d} X^{q^d} = (\mathrm{unit}) \cdot X^{q^d}$$

But then, by 3.15, the image of $-t$ in R is $\alpha_d = 0$ so that E is a constant deformation, as required. □

Note that since the k-algebra D is a one-dimensional regular local ring with parameter $x(P)$, we may identify D with $k[\![x(P)]\!]$.

Theorem 5.5. — *Let k be an algebraically closed field of characteristic n and suppose given a supersingular $\phi_{/k}$ such that ϕ_T is monic in τ. Let $\Phi/k[\![t]\!]$ be the universal equicharacteristic deformation of ϕ. Then there is an isomorphism of $k[\![t]\!]$-schemes*

$$\mathscr{V}_{\Phi/k[\![t]\!]} \simeq \operatorname{Spec} k[\![t^{1/(q^d-1)}]\!].$$

Proof. — We use the notations D, $(\Psi,\alpha)_{/D}, P=\alpha(1)$ as in the previous proof. As before, $\phi[n^\infty], \phi^{(n)}[n^\infty]$ may be taken to be normal formal $\mathcal{O}$-modules. Let F, G be the formal $\mathcal{O}$-modules $\Psi[n^\infty], \Psi^{(n)}[n^\infty]$ over D. Write

$$[\pi]_F.X = \alpha_d X^{q^d} + \cdots + \alpha_{2d} X^{q^{2d}}$$
$$[\pi]_G.X = \alpha_d^{q^d} X^{q^d} + \cdots + \alpha_{2d}^{q^d} X^{q^{2d}}$$

Identify $\mathcal{O}$ with $\mathbb{F}_n[\![\pi]\!]$ and define a formal $\mathcal{O}$-module G' over D by

$$[\zeta]_{G'}.X := [\zeta]_G.X \text{ for } \zeta \in \mathbb{F}_n \subset \mathbb{F}_n[\![\pi]\!]$$
$$[\pi]_{G'}.X := \alpha_d^{q^d} X^{q^d} + X^{q^{2d}} = (-t)^{q^d} X^{q^d} + X^{q^{2d}}$$

where $-t \in D$ comes from the $k[\![t]\!]$-algebra structure on D. Since $\phi_n^{(n)} = \tau^{2d}$ we see that G' is also a deformation of $\phi^{(n)}[n^\infty]$. By 3.14 there is an isomorphism of formal $\mathcal{O}$-modules $\eta : G \xrightarrow{\sim} G'$.

Now observe that D is regular local hence a domain and $P \in D$ is nonzero (because G is not a constant deformation by the proof of 5.4). Writing $x(P) \in (\ker \mathrm{V})(D) \subset G(D) = \mathfrak{m}_D$ as in 3.4, we have

$$\begin{aligned} 0 &= \eta([\pi]_G.P) \\ &= [\pi]_{G'}.Q \qquad (\text{where } Q := \eta(P) \neq 0) \\ &= (-t)^{q^d} Q^{q^d} + Q^{q^{2d}} \\ &= (-tQ + Q^{q^d})^{q^d} \end{aligned}$$

and hence $Q = t^{1/(q^d-1)}$.

Thus η induces an isomorphism $D = k[\![x(P)]\!] \xrightarrow{\sim} k[\![t^{1/(q^d-1)}]\!]$ over $k[\![t]\!]$. $\square$

Remark 5.6. — By 3.15, the Hasse invariant of $\Phi/k[\![t]\!]$ is given by $-t$. Since k is algebraically closed, 5.5 says that in a formal neighborhood of a supersingular point, giving a generator of the kernel of Verschiebung is the same as giving a (q^d-1)th root of the Hasse invariant. This is analogous to the result [21, 12.8.2] which identifies the moduli problem of prescribing a generator of the kernel of the Verschiebung of an arbirary elliptic curve with the moduli problem of prescribing a $(p-1)$th root of the Hasse invariant of that elliptic curve.

COROLLARY 5.7. — *The Igusa curve $\mathscr{V}_{\mathbb{E}/Y}$ is an étale $\mathbb{F}_n^\times$-torsor over Y^{ord}, fully ramified over the supersingular points. Moreover, it is geometrically connected.*

Proof. — The first assertion is clear; the second is proved precisely as in [21, p. 364]. □

6. Reduction mod n

The goal of this section is to determine the structure of the local ring at a supersingular point (of characteristic n) of $X_1(n)$ in terms of the local rings along the irreducible components passing through that point. This will be accomplished by applying the Crossings Theorem [21, 13.1.3].

For this section we fix the notations: $\mathcal{O} := \widehat{A}_{(n)}, W := \widehat{\mathcal{O}^{\mathrm{nr}}}, \pi \in W$ the uniformizer, $Y = Y(m)_{/A}$, $\mathbb{E}$ the universal Drinfeld module over Y, $\mathcal{Y}_1 = Y_1(n,m)_{/A}$, $\pi : \mathcal{Y}_1 \longrightarrow Y$ is the natural map, k a field such that $\mathbb{F}_n \subset k \subset \overline{\mathbb{F}}_n$ and all supersingular points of Y are k-rational (so that $k \supset \mathbb{F}_{n^2}$ by [8, 5.3], where we abusively write $\mathbb{F}_{n^2} := \mathbb{F}_{q^{2d}}$), $\mathscr{F} = \mathscr{F}_{\mathbb{E}/Y_{\mathbb{F}_n}} \otimes k, \mathscr{V} = \mathscr{V}_{\mathbb{E}/Y_{\mathbb{F}_n}} \otimes k$, z a supersingular point of $\mathcal{Y}_1$, and ϕ the Drinfeld module corresponding to z.

Proposition 6.1. — *If z is a supersingular point of $\mathcal{Y}_{1/k}$ then $\widehat{\mathscr{O}}_{\mathcal{Y}_{1/k},z} \simeq k[\![x,y]\!]/fv$ where the complete local rings along the irreducible components are $\widehat{\mathscr{O}}_{\mathscr{F},z} \simeq k[\![x,y]\!]/f$ and $\widehat{\mathscr{O}}_{\mathscr{V},z} \simeq k[\![x,y]\!]/v$, respectively.*

Proof. — We have the map from $\mathscr{F}$ (resp. $\mathscr{V}$) to $\mathcal{Y}_{1/\mathbb{F}_n}$ defined by $(\phi,\iota) \rightsquigarrow (\phi,\iota)$ (resp. $(\phi,\iota) \rightsquigarrow (\phi^{(n)},\iota)$, where we have suppressed the auxiliary étale level m structure from the notation. Going through the list of hypotheses of the Crossings Theorem:

(1) Y is smooth over k as m is prime to n.

(2) π is finite flat by 4.5.

(3) If $k = \overline{\mathbb{F}}_n$ then by 4.6 we have $\widehat{\mathscr{O}}_{\mathcal{Y}_1,z} \otimes k = k[\![x(P),t]\!]/(\text{one equation})$. If k is a finite extension of $\mathbb{F}_n$, let $\mathcal{O}'$ be the unique unramified extension of $\mathcal{O}$ with k as its residue field[(2)], let $\Phi_0/k[\![t]\!]$ be the universal equicharacteristic deformation of ϕ, and let $\Phi/\mathcal{O}'[\![t]\!]$ be an algebraization of the universal formal deformation of ϕ to artin local $\mathcal{O}'$-algebras with residue field k (see 2.6).

Then our local scheme $\mathscr{M}_{\Phi_0/k[\![t]\!]}$ is equal to $\mathscr{M}_{\Phi/\mathcal{O}'[\![t]\!]} \otimes k$ which is in turn equal to

$$\operatorname{Spec}\left(\begin{array}{c}\text{two-dimensional complete regular local}\\ \mathcal{O}'\text{-algebra with residue field } k\end{array}\right) \otimes k$$

[(2)] $\mathcal{O}'$ plays the role of the Witt ring $W(k)$ in our characteristic p setting.

But such an $\mathcal{O}'$-algebra is of the form $\mathcal{O}'[\![x, y]\!]/f$ for some $f \in \mathfrak{m} - \mathfrak{m}^2$. Now tensoring with k, we see that our local ring has the form

$$k[\![x, y]\!]/(\text{one equation}).$$

(4) It is clear that for each supersingular point of Y there exists a unique k-rational closed point of $\mathscr{F}$ (resp. $\mathscr{V}$) lying over it.

(5) Finite flatness of $\mathscr{F} \longrightarrow \mathcal{Y}_{1/k}$ follows from its explicit description; for $\mathscr{V} \longrightarrow \mathcal{Y}_{1/k}$ flatness follows from regularity. We check finiteness in (7).

(6) We have that $\mathscr{F}_{\text{red}}$ (resp. $\mathscr{V}_{\text{red}}$) is a smooth curve over k by 5.2 (resp. 5.7).

(7) The maps from $\mathscr{F}$ and $\mathscr{V}$ to $\mathcal{Y}_{1/k}$ are maps of finite Y_k-schemes and are therefore finite. By the moduli interpretation these maps are monic (in the sense of category theory) and hence are closed immersions.

(8) Consider the map of Y_k^{ord}-schemes: $\mathscr{F}^{\text{ord}} \coprod \mathscr{V}^{\text{ord}} \longrightarrow \mathcal{Y}_{1/k}^{\text{ord}}$. To check it is an isomorphism we may assume that $k = \overline{\mathbb{F}}_n$. From (7) it suffices to check surjectivity; let z be a k-point of $\mathcal{Y}_{1/k}^{\text{ord}}$ corresponding to (ϕ, ι). If ι is zero, then by the sequence of 2.10 we have that $\sum_{a \in \mathbb{F}_n} [\iota(a)]$ is $\ker \mathrm{F}$ which is in the image of $\mathscr{F} \longrightarrow \mathcal{Y}_{1/k}$. If ι is nonzero, then z is the image of the point of $\mathscr{V}$ corresponding to the pair $(\phi^{(1/n)}, \iota)$. Here $\phi^{(1/n)}$ is the Drinfeld module over k defined as follows: write $\phi_T = \chi(T) + \alpha_1 \tau + \alpha_2 \tau^2$. Then $\phi_T^{(1/n)} := \chi(T) + \alpha_1^{1/q^d} \tau + \alpha_2^{1/q^d} \tau^2$.

Thus we may apply the Crossings Theorem to conclude the proof of the proposition. □

By 4.6 we see that if $z \in \mathcal{Y}_{1/W}(k)$ has point of order n equal to zero then we have a surjection $W[\![x, t]\!] \longrightarrow \widehat{\mathscr{O}}_{\mathcal{Y}_{1/W}, z}$ with kernel a principal ideal generated by an element g that is part of a regular system of parameters of $W[\![x, t]\!]$ and such that $g \equiv fv \mod \pi$. Note that now x and t have deformation theoretic interpretations as in 4.5.

Proposition 6.2. — *Suppose $z \in \mathcal{Y}_{1/W}(k)$ corresponds to a pair consisting of a Drinfeld module with point of order n equal to zero. Then for $q^d > 2$ we have*

$$g \mod \pi = \begin{cases} x^{q^d - 1} & z \text{ ordinary}, \\ x^{q^d - 1} v & z \text{ supersingular}. \end{cases}$$

Proof. — Since $\mathscr{F}$ and $\mathscr{V}$ only meet at supersingular points, the explicit description of $\mathscr{F}$ (5.2) gives $g \equiv x^{q^d - 1} \mod \pi$ when z is ordinary.

Suppose z is supersingular. Let us first show that t is not a formal parameter of $\widehat{\mathscr{O}}_{\mathscr{V}, z} = k[\![x, t]\!]/v$. By 5.5 there is an isomorphism of $k[\![t]\!]$-algebras

$k[\![x,t]\!]/v \simeq k[\![t^{1/(q^d-1)}]\!]$; t is obviously not a formal parameter of the latter ring since $q^d > 2$.

On the other hand, $k[\![x,t]\!]/v$ is formally smooth so that $v(x,t) = ax + bt + \cdots$ with not both a, b equal to zero; t is a formal parameter of this ring if and only if $k[\![x]\!]/v(x,0) \simeq k$, which is to say $v(x,0) = ax + \cdots$ with $a \neq 0$. Hence $a = 0$ and $k[\![t]\!]/v(0,t) \simeq k$ so that $\{x, v\}$ is a regular system of parameters of $k[\![x,t]\!]$, and $g \equiv x^{q^d-1}v \mod \pi$, as required. □

Remark 6.3. — If $q^d = 2$, i.e. $A = \mathbb{F}_2[T]$ and $n = T$ or $T-1$, then $J_1(n)$ has trivial component group by [9, 5.11](3) since in this case $X_1(n) = X_0(n)$.

Corollary 6.4. — *$\mathcal{Y}^{\mathrm{red}}_{1/k}$ has smooth irreducible components that are all geometrically irreducible and cross transversally at supersingular points. Furthermore, the supersingular points are the only nonsmooth points on $\mathcal{Y}^{\mathrm{red}}_{1/k}$.*

7. Nonregular points on the coarse modular curve

This section is devoted to the determination of the nonregular points on the coarse modular curve.

7.1. Let us recall some facts from [8, §5] on Drinfeld $\mathbb{F}_q[T]$-modules over an algebraically closed field k: (1) $\mathrm{Aut}(\phi)$ is either $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$ according as $j(\phi) \neq 0$ or $j(\phi) = 0$; we may identify $\mathrm{Aut}(\phi)$ with a subgroup of $k^\times \subset (k\{\tau\})^\times$. (2) If k is of characteristic n then $j = 0$ is supersingular if and only if $d = \deg n$ is odd and moreover

$$\#\{\text{supersingular } j\text{-invariants in } k\} = \begin{cases} q\left(\dfrac{q^{d-1}-1}{q^2-1}\right) + 1 & d \text{ odd}, \\[2ex] \dfrac{q^d - 1}{q^2-1} & d \text{ even}. \end{cases}$$

7.2. We define $Y_1(n)_{/A}$ to be the quotient of $Y_1(n,m)$ by the natural action of $GL_2(A/m)$. It is the coarse moduli space for pairs consisting of a Drinfeld module and a point of order n. There is a finite(4) map

(3) which says: The component group at n of $J_0(n)$ is cyclic of order $\frac{q^d-1}{q^2-1}$ or $\frac{q^d-1}{q-1}$ according as d is even or odd.

(4) In [5, p. 587] (see also [23, 4.2.3]) it is shown that $j : Y(nm) \longrightarrow \mathbb{A}^1_A$ is a finite map for m prime to n. Clearly $j : Y(nm) \longrightarrow \mathbb{A}^1_A$ factors through $j : Y_1(n) \longrightarrow \mathbb{A}^1_A$, $Y_1(n)$ is a quotient of $Y(nm)$ by a finite group action, and everything in sight is noetherian so that $j : Y_1(n) \longrightarrow \mathbb{A}^1_A$ is also a finite map.

$j : Y_1(n) \longrightarrow Y(1) = \mathbb{A}^1_A = \operatorname{Spec} A[j]$ which sends a pair $(\phi, \iota)_{/R}$ to $j(\phi) \in R$. Here, if ϕ is a Drinfeld module over R then the j-invariant is defined by

$$\phi_T = \chi(T) + \beta\tau + \alpha\tau^2 \rightsquigarrow (j \rightsquigarrow \frac{\beta^{q+1}}{\alpha}) \in \mathbb{A}^1_A(R).$$

As $\mathcal{Y}_{1/A}$ is regular (4.5), the quotient $Y_1(n)$ is normal, and being finite over $\mathbb{A}^1_A$, we may define $X_1(n)_{/A}$ by "normalizing near ∞" as in [21, 8.6.3].

By standard arguments (use [21, Theorem, p. 508] and the fact [5, §9] that Drinfeld's compactification $X(I)$ of the moduli space of Drinfeld modules with full level I structure is smooth over $A[I^{-1}]$) $X_1(n)$ is smooth over $A[n^{-1}]$ and hence any nonregular point of $X_1(n)$ is a closed point in the special fiber of $X_1(n)_{/A_{(n)}}$.

Again we write $\mathcal{O} = \widehat{A}_{(n)}, W = \widehat{\mathcal{O}^{\mathrm{nr}}}, k = \overline{\mathbb{F}}_n$. For $y \in Y_1(n)_{/W}$ a closed (k-rational) point and $z \in \mathcal{Y}_{1/W}$ a point over y we have isomorphisms (see [3, 3.1.1], [4, I.8.2.1.])

$$\widehat{\mathscr{O}}_z := \widehat{\mathscr{O}}_{\mathcal{Y}_{1/W},z} \simeq \mathscr{R}_z,$$
$$\widehat{\mathscr{O}}_{Y_1(n)_{/W},y} \simeq \widehat{\mathscr{O}}_z^{\mathrm{Aut}(z)},$$

where $\mathscr{R}_z$ is the deformation ring of the pair $(\phi, P)_{/k}$ corresponding to z and $\mathrm{Aut}(z)$ is the automorphism group of (ϕ, P). Observe that if $P \in \phi(k)$ is nonzero then $\mathrm{Aut}(z)$ is trivial: any automorphism is given by a scalar $u \in k^\times$ and if $uP = P$, then $u = 1$. By 4.5 we conclude that $Y_1(n)_{/W}$ is regular at points along $\mathscr{V}^{\mathrm{ord}}$. So suppose that z corresponds to $(\phi, 0)_{/k}$, i.e. z is a closed point of $\mathscr{F} \hookrightarrow \mathcal{Y}_{1/k}$.

In order to determine when $\widehat{\mathscr{O}}_{Y_1(n)_{/W},y}$ is regular we will use the following theorem. See [3, 2.3.9].

Theorem 7.3 (Serre). — *Let R be a regular noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let G be a finite subgroup of* $\mathrm{Aut}(R)$ *and write R^G for the local ring of G-invariants. Assume*

(i) *The characteristic of k is prime to $\#G$,*

(ii) *G acts trivially on k, and*

(iii) *R is a finitely generated R^G-module.*

Then R^G is regular if and only if the image of G in $\mathrm{Aut}(\mathfrak{m}/\mathfrak{m}^2)$ *is generated by pseudo-reflections. (For a finite dimensional vector space V over a field, an element $\sigma \in GL(V)$ is called a pseudo-reflection if* $\mathrm{rank}(1-\sigma) \leqslant 1$.*)*

In particular, if R as above is moreover two-dimensional and G is cyclic, the subring R^G is regular if and only if the generator of G has a nonzero fixed point in $\operatorname{Cot} R := \mathfrak{m}/\mathfrak{m}^2$.

The action of $\mathrm{Aut}(z)$ on $\widehat{\mathscr{O}}_{\mathcal{Y}_{1/W},z}$ is the abstract one coming from the identification of $\widehat{\mathscr{O}}_{\mathcal{Y}_{1/W},z}$ with $\mathscr{R}_z$; in terms of moduli, $g \in \mathrm{Aut}(z)$ acts on the left by $(E,\beta,i) \rightsquigarrow (E,\beta,g\circ i)$.

Proposition 7.4. — *There is a unique nonregular point on $Y_1(n)_{/W}$, namely the point z corresponding to $(\phi,0)$ with $j(\phi)=0$.*

Proof. — Immediate from Serre's theorem and the following lemma. □

Lemma 7.5. — (i) *$\mathrm{Cot}\,\mathscr{R}_z \simeq k.x \oplus k.t$ compatibly with the action of* $\mathrm{Aut}\,\phi$.

(ii) *There is a canonical isomorphism $k.x \simeq \mathrm{Cot}_0\phi$. Hence $u \in \mathrm{Aut}\,\phi \subset k^\times$ acts as the scalar u on the line $k.x$.*

(iii) *The element $u \in \mathrm{Aut}\,\phi \subset k^\times$ acts as u^{1-q} on $k.t$.*

Proof. — (i) Let $\mathscr{D}$ be the functor which assigns to an artin local k-algebra R with residue field k the set of isomorphism classes of deformations $(E,\beta,i)_{/R}$ of $(\phi,0)$. Then $\mathscr{D}(k[\varepsilon])$ is the equicharacteristic tangent space of $\mathscr{R}_z$ so $\mathrm{Cot}\,\mathscr{R}_z = \mathscr{D}(k[\varepsilon])^\vee$.

Let $\mathscr{D}_0$ be the subfunctor which classifies those $(E,\beta,i)_{/R}$ such that $E \simeq \phi\otimes_k R$ is the constant deformation. Let $\mathscr{D}_1$ be the subfunctor of $\mathscr{D}$ which classifies those $(E,\beta,i)_{/R}$ such that β is the zero map. Clearly $\mathscr{D}_0(k[\varepsilon])^\vee = \mathrm{Cot}\,k[\![x]\!] = k.x$ and $\mathscr{D}_1(k[\varepsilon])^\vee = \mathrm{Cot}\,k[\![t]\!] = k.t$. As $\mathscr{D}_0(k[\varepsilon]) \cap \mathscr{D}_1(k[\varepsilon]) = 0$ by the proof of 4.5, (i) follows.

(ii) By $\mathrm{Cot}_0\phi$ we mean the cotangent space to the underlying $\mathbb{G}_{a/k} = \mathrm{Spec}\,k[X]$ with A-module structure coming from ϕ. We have a natural map

$$\begin{aligned} \mathscr{D}_0(k[\varepsilon]) &\longrightarrow \mathrm{Tgt}_0(\phi[n]^\circ \otimes_k k[\varepsilon]) \\ (\phi\otimes_k k[\varepsilon],\beta,i) &\rightsquigarrow \beta(1) \end{aligned}$$

which is an isomorphism being a nonzero linear map between one-dimensional vector spaces over k. Now

$$\mathrm{Tgt}_0(\phi[n]^\circ \otimes k[\varepsilon]) = \mathrm{Hom}_{k[\varepsilon]\text{-alg}}(\mathscr{O}_{\phi[n]^\circ}\otimes k[\varepsilon], k[\varepsilon]) = \mathrm{Tgt}_0\phi$$

where the last equality follows since $\phi[n]^\circ = \mathrm{Spec}\,k[X]/X^{q^a}$ with $a=d$ or $2d$ ($\mathrm{char} k = n$) and $q^a \geqslant 2$. This proves (ii).

(iii) This will be proved with the aid of the Kodaira-Spencer isomorphism of [10]. Namely, we will show the Kodaira-Spencer map is $\mathrm{Aut}\,\phi$-equivariant and then compute the $\mathrm{Aut}\,\phi$ action on its target. This amounts to unraveling the definitions. The reader may want to skip ahead to 7.6.

Most of this proof does not require the assumption $A=\mathbb{F}_q[T]$, and we will drop this assumption until further notice. We need some notation. Let $m \subset A$ be prime to n and divisible by at least two primes, $Y := Y(m)_{/A[m^{-1}]}$,

$y \in Y$ a closed point, $y^* : \mathscr{O}_Y \longrightarrow k := k(y)$ the corresponding ring map, $\phi_{/k}$ the corresponding Drinfeld module, and $\chi : A \longrightarrow k$ its characteristic morphism.

Let $\mathscr{T}$ be the tangent sheaf of $Y/A[m^{-1}]$. Then, as usual, we identify

$$\mathscr{T}(y) := \mathscr{T}_y/\mathfrak{m}_y\mathscr{T}_y = \mathrm{Der}_{A[m^{-1}]}(\mathscr{O}_Y, k) = \mathrm{Hom}^y_{A[m^{-1}]-\mathrm{alg}}(\mathscr{O}_Y, k[\varepsilon])$$

with the bijection being given by $X \leftrightarrow (b \rightsquigarrow y^*(b) + \varepsilon X(b))$ where X is a derivation and Hom^y consists of those maps reducing to y^* mod ε.

Since the universal Drinfeld module over Y has trivial bundle it is given by a ring homomorphism $\Phi : A \longrightarrow \mathscr{O}_Y\{\tau\}$. Fix $X \in \mathscr{T}(y)$. We obtain a deformation $\widetilde{\phi}^{\eta_X}$ of ϕ over $k[\varepsilon]$ by means of the composite

$$A \xrightarrow{\ \Phi\ } \mathscr{O}_Y\{\tau\} \xrightarrow{y^* + \varepsilon X} k[\varepsilon]\{\tau\}$$

so that $\widetilde{\phi}^{\eta_X} = \phi + \varepsilon\eta_X$ where η_X is defined[(5)] as follows: let $X(\phi)$ be $a \rightsquigarrow X(\phi)(a) := \sum X(\alpha_i(a))\tau^i$ if $\phi_a = \sum_{i\geqslant 0}\alpha_i(a)\tau^i$. We then write $\eta_X(a) := X(\phi)(a)$.

Then $(\widetilde{\phi}^{\eta_X}, \mathrm{id})_{/k[\varepsilon]}$ is a "standard" deformation of ϕ (in the sense that the special fiber isomorphism is the identity with respect to the projection $k[\varepsilon] \to k$). Recall that $u \in \mathrm{Aut}\,\phi$ acts on $\mathscr{D}_1(k[\varepsilon])$ on the left by $(\widetilde{\phi}, i) \rightsquigarrow (\widetilde{\phi}, u \circ i)$. Now any deformation over $k[\varepsilon]$ is isomorphic to a standard deformation; we claim that the $\mathrm{Aut}\,\phi$ action on standard deformations is given by $u.(\widetilde{\phi}^{\eta_X}, \mathrm{id}) \simeq (\widetilde{\phi}^{u\eta_X u^{-1}}, \mathrm{id})$. To see this compute in $k[\varepsilon]\{\tau\}$: $u(\phi + \varepsilon\eta_X) = (\phi + \varepsilon u\eta_X u^{-1})u$. Thus the action of $\mathrm{Aut}\,\phi$ on $\mathscr{T}_y$ is given by $X \rightsquigarrow u * X$ where $u\eta_X u^{-1} =: \eta_{u*X}$.

Now, the $\mathscr{O}_Y$-linear Kodaira-Spencer isomorphism [10, 6.11]

$$KS : \mathscr{T} \xrightarrow{\ \sim\ } \mathscr{H}om_{\mathscr{O}_Y}(H_1^*(\Phi), H_2^*(\Phi))$$

is defined as the composite

$$X \rightsquigarrow KS(X) : H_1^*(\Phi) \longrightarrow H^*_{\mathrm{dR}}(\Phi) \xrightarrow{\nabla_X} H^*_{\mathrm{dR}}(\Phi) \xrightarrow{\mathrm{pr}} H_2^*(\Phi).$$

See [10] and [11] for the definitions and basic properties of these groups. For our purposes, we only need that

- $H_i^*(\Phi), H^*_{\mathrm{dR}}(\Phi)$ are by definition subquotients of the module $D(\Phi)$ where the latter is defined to be the A-bimodule of $\mathbb{F}_q$-linear derivations of A into

$$N(\Phi) := \left\{ \begin{array}{c} \mathbb{F}_q\text{-linear morphisms of } Y\text{-group schemes} \\ a : \Phi \longrightarrow \mathbb{G}_{a/Y} \text{ such that } \mathrm{Lie}(a) = 0 \end{array} \right\},$$

(5) η_X is most naturally regarded as an element of $D_r(\phi)$, the module of reduced derivations. See [10, 3.8,6.12].

- $D(\Phi), H_i^*(\Phi), H_{\mathrm{dR}}^*$ are compatible with base change [11, 4.5],
- $H_{\mathrm{dR}}(\Phi)$ splits as $H_1^*(\Phi) \oplus H_2^*(\Phi)$,
- for $\phi = \Phi \otimes_{\mathscr{O}_Y} k$ there is an $\operatorname{Aut}\phi$-equivariant isomorphism from $H_1^*(\phi)$ to the one-dimensional vector space $k.(\phi - \chi)$,
- the Gauss-Manin connection acts as $\nabla_X(\sum \nu_i \tau^i) := \sum X(\nu_i)\tau^i$.

The space $H_2^*(\phi)$ is naturally isomorphic to the module of strictly-reduced derivations, which we will identify explicitly below in the case $A = \mathbb{F}_q[T]$. By definition, $\operatorname{Aut}\phi$ acts on $D(\phi)$ and its subquotients $H_i^*(\phi)$ by multiplication on the right.

We make $\operatorname{Hom}_k(H_1^*(\phi), H_2^*(\phi))$ into a *left* $\operatorname{Aut}\phi$-module by defining, for $u \in \operatorname{Aut}\phi$, $(u * h)(z) := h(z.u).u^{-1}$. Now $\mathscr{D}_1(k[\varepsilon]) = \mathscr{T}(y)$ is a left $\operatorname{Aut}\phi$-module, and to show KS is $\operatorname{Aut}\phi$-equivariant it suffices to check that

$$u * KS(X)(\phi - \chi) = \eta_{u*X} = u\eta_X u^{-1}.$$

We compute:

$$\begin{aligned} u * KS(X)(\phi - \chi) &= (KS(X)((\phi - \chi).u)).u^{-1} \\ &= KS(X)(u.(\phi - \chi)).u^{-1} \text{ since } u \text{ commutes with } \phi \\ &= u.(KS(X)(\phi - \chi)).u^{-1} \text{ since } KS(X) \text{ is } k\text{-linear} \\ &= u\eta_X u^{-1} \end{aligned}$$

as required.

It is clear that u acts on $H_1^*(\phi)$ as the scalar $u \in k^\times$. Henceforth we take $A = \mathbb{F}_q[T]$, so $H_2^*(\phi)$ is canonically isomorphic to the module of strictly reduced derivations, where the latter by definition consists of those $\varrho \in D(\phi) \simeq \operatorname{Der}_{\mathbb{F}_q}(A, \tau.k\{\tau\})$ ([10, 3.2,3.3]) such that $\deg_\tau \varrho_a < 2(-\operatorname{ord}_\infty(a))$ for all nonzero $a \in A$ or equivalently some nonzero a by [10, 3.9]. Taking $a = T$ we see that $H_2^*(\phi)$ is the one-dimensional vector space spanned by τ. As the action is on the right and $\tau u = u^q\tau$, $u \in \operatorname{Aut}\phi$ acts as the scalar u^q, and hence acts on the *left* on $\operatorname{Hom}_k(H_1^*(\phi), H_2^*(\phi))$ as the scalar u^{1-q}. Therefore $\operatorname{Aut}\phi$ acts[6] by the same scalar u^{1-q} on $\operatorname{Cot} k[\![t]\!] = k.t$ and (iii) is proved. □

The foregoing settles the determination of the nonregular points on $Y_1(n)_{/W}$. That those are all of the nonregular points is guaranteed by

Proposition 7.6. — *$X_1(n)_{/A_{(n)}}$ is regular along its cusps.*

Proof. — In this proof, all modular curves will be over $A_{(n)}$. Write $\operatorname{Spec} A_{(n)} = \{\eta, s\}$ with η the generic point and s the closed point. The

(6) N.B.: the action on cotangent spaces is *not* the contragredient representation but rather the transpose action on the *right*.

map $X_1(n) \longrightarrow X_0(n)$ is the quotient by the natural faithful action of a group of order prime to p, viz. $\mathbb{F}_n^\times/\mathbb{F}_q^\times$. We will apply the following form of Abhyankar's Lemma: Let R be a regular local ring with fraction field K, K'/K a finite Galois extension with group G, R' the integral closure of R in K'. Let ϖ be part of a regular system of parameters of R and suppose that $R.\varpi$ is the unique height one prime ramified in R, and that this ramification is tame. Then R' is regular. See [7, Appendix A1.11].

We will use the following facts:

(a) $X_0(n)$ is regular along its cusps [9, p. 17].

(b) The cusps 0 and ∞ have distinct specializations in $X_0(n)_s$ [9, p. 13].

Let $z' \in X_1(n)$ be a cuspidal closed point and let $z \in X_0(n)$ be its image, so that $R := \mathscr{O}_{X_0(n),z}$ is a regular two-dimensional local ring. Write $R' := \mathscr{O}_{X_1(n),z}$ and let $\mathfrak{p} \subset R$ be a height-one prime corresponding to the point $\zeta \in X_0(n)$. Then by (b) and [24, Ch. 8, 3.4.b.] ζ is either the generic point of an irreducible component of $X_0(n)_s$ or else is a uniquely determined cuspidal closed point of $X_0(n)_\eta$. Suppose the latter. By [20, 2.3] $X_1(n)_\eta \longrightarrow X_0(n)_\eta$ is unramified along the cusps so that in this case $\mathfrak{p}$ is unramified in R'.

Now suppose ζ is the generic point of an irreducible component $C \subset X_0(n)_s$. If C is the component classifying pairs consisting of a Drinfeld module and a cyclic étale subgroup scheme of its n-torsion, then $\mathfrak{p}$ is unramified in R' since the component of $X_1(n)_s$ above C is $\mathscr{V}$ which is étale over C. We conclude that the only height-one prime of R which is ramified in R' is the one corresponding to the generic point ζ of the component $C' \subset X_0(n)_s$ classifying connected cyclic subgroups. Now R is regular local hence a UFD so that the height-one prime $\mathfrak{p}$ is principal with generator ϖ, say. We claim that ϖ is part of a regular system of parameters of R: this follows since R/ϖ is itself regular, being the local ring of the smooth curve $C \simeq \mathbb{P}^1_{\mathbb{F}_n}$ [9, 5.3] at ∞. We have assembled all of the facts needed to apply Abhyankar's Lemma. $\square$

8. Resolution

Write $\mathcal{O} = \widehat{A}_{(n)}, W = \widehat{\mathcal{O}^{\mathrm{nr}}}, k = \overline{\mathbb{F}}_n$, π for a uniformizer of W, and $\operatorname{Spec} W = \{\eta, s\}$. Let $C_{\mathrm{F}}, C_{\mathrm{V}} \subset X_1(n)_s$ be the components corresponding to $\mathscr{F}$,$\mathscr{V}$ in $X_1(n,m)_s$

Lemma 8.1. — $\operatorname{mult} C_{\mathrm{F}} = \frac{q^d-1}{q-1}$ *and* $\operatorname{mult} C_{\mathrm{V}} = 1$.

Proof. — $\mathscr{F}$ and $\mathscr{V}$ have multiplicities q^d-1 and 1 respectively by 6.2. Let ξ', ξ be the generic points of $\mathscr{F}$, C_{F} respectively, $v_{\xi'}, v_\xi$ the corresponding discrete valuations on the function fields of $X_1(n,m)_{/W}, X_1(n)_{/W}$. Then we have $\operatorname{mult}\mathscr{F} = v_{\xi'}(\pi) = e_{\xi'/\xi}\cdot \operatorname{mult} C_{\mathrm{F}}$. But the local rings at the respective generic points are related by $\widehat{\mathscr{O}}_\xi = \widehat{\mathscr{O}}_{\xi'}^{\mathrm{Aut}(\xi)}$ and $\mathrm{Aut}(\xi) = \mathbb{F}_q^\times$ so, by [21, p. 509], $e_{\xi'/\xi} = \#\mathbb{F}_q^\times$.

The case of C_{V} is similar. $\square$

By [3, 2.3.4, p. 341] we may linearize the Aut-action to obtain

$$(*)\qquad \widehat{\mathscr{O}}_z \simeq \mathscr{R}_z \simeq \begin{cases} W[\![x_0,t_0]\!]/(x_0^{q^d-1}t_0-\pi) & \text{supersingular} \\ W[\![x_0,t_0]\!]/(x_0^{q^d-1}-\pi) & \text{ordinary} \end{cases}$$

with the $\mathrm{Aut}(z)$-action on $\operatorname{Cot}\mathscr{R}_z$ exactly as in 7.5. Note that x_0 and t_0 do not have deformation theoretic interpretations as the coordinate of the point of order n and the deformation parameter of the underlying Drinfeld module, respectively. But we no longer need these interpretations.

Definition 8.2. — *Let X' be a normal curve over a the spectrum S of a discrete valuation ring. Assume that S is excellent or that X' has smooth generic fiber. Let $s\in S$ be a closed point of residual characteristic $p \geqslant 0$ and let $x' \in X'_s$ be a closed point such that X'_s has at most two (geometric) analytic branches at x'. Say that x' is a* tame cyclic quotient singularity *if there is an integer $N>1$ prime to p, a unit $r\in(\mathbb{Z}/N\mathbb{Z})^\times$, and integers $m'_1>0$, $m'_2 \geqslant 0$ satisfying $m'_1 \equiv -rm'_2 \mod N$ such that $\widehat{\mathscr{O}^{\mathrm{sh}}_{X',x'}}$ is isomorphic to the subalgebra of $\boldsymbol{\mu}_N(k(s)_{\mathrm{sep}})$-invariants in $\widehat{\mathscr{O}^{\mathrm{sh}}_{S,s}}[\![t'_1,t'_2]\!]/(t'^{m'_1}_1 t'^{m'_2}_2 - \pi_s)$ under the action $t'_1 \rightsquigarrow \zeta t'_1, t'_2 \rightsquigarrow \zeta^r t'_2$.*

8.3. In order to employ the Jung-Hirzebruch resolution of [3], we require a presentation of the singularity on $X_1(n)$ as a tame cyclic quotient singularity.

By (*) and [3, 2.3.4, p. 345] we have

$$\widehat{\mathscr{O}}_z^{\mathbb{F}_q^\times} \simeq \begin{cases} W[\![x,t]\!]/(x^{\frac{q^d-1}{q-1}}t-\pi) & \text{supersingular} \\ W[\![x,t]\!]/(x^{\frac{q^d-1}{q-1}}-\pi) & \text{ordinary.} \end{cases}$$

We will apply the following special case of results from [3].

Theorem 8.4. — *With the notations and assumptions of 8.2, suppose that $x'\in X'_s$ has a tame cyclic quotient singularity with parameters N and r, where we represent $r\in(\mathbb{Z}/N\mathbb{Z})^\times$ by the unique positive integer less than N, and assume that $k(s)$ is separably closed. Consider the Jung-Hirzebruch* *continued fraction expansion*

$$\frac{N}{r} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \frac{1}{b_\lambda}}}$$

with integers $b_j \geqslant 2$ *for all* j.

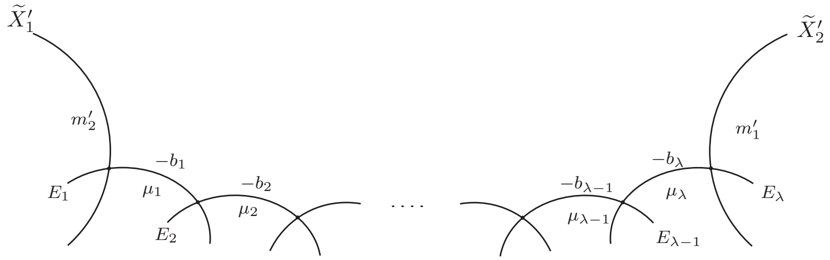


Figure 8.1. Fiber of the minimal regular resolution of x' *over* $k(x')_{\mathrm{sep}}$.

Then the minimal regular resolution of X' *along* x' *has fiber over* $k(x')_{\mathrm{sep}}$ *whose underlying reduced curve looks like the chain of* E_j*'s shown in Figure 1, where:*

(a) all intersections are transverse with $E_j \simeq \mathbb{P}^1_{k(x')_{\mathrm{sep}}}$;

(b) $E_j.E_j = -b_j < -1$ *for all* j;

(c) E_1 *is transverse to the strict transform* $\widetilde{X}'_1$ *of the global algebraic irreducible component* X'_1 *through* x' *with multiplicity* m'_2, *and likewise for* E_λ *and the component* $\widetilde{X}'_2$ *with multiplicity* m'_1 *in the case of two analytic branches. (The case* $X'_2 = X'_1$ *can happen, and there is no* $\widetilde{X}'_1$ *in the case of one analytic branch (i.e. when* $m'_2 = 0$*).)*

(d) Write μ_j *for the multiplicity of* E_j *in the fiber of* X'^{reg} *over* $k(x')_{\mathrm{sep}}$. *If* $r > 1$ *then* $\lambda > 1$ *and the* μ_j*'s are the unique solution to the equation*

$$\begin{pmatrix} b_1 & -1 & & & & \\ -1 & b_2 & -1 & & \text{\Large 0} & \\ & -1 & b_3 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & \text{\Large 0} & & -1 & b_{\lambda-1} & -1 \\ & & & & -1 & b_\lambda \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_{\lambda-1} \\ \mu_\lambda \end{pmatrix} = \begin{pmatrix} m'_2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ m'_1 \end{pmatrix}$$

To apply this we split into two cases corresponding to the different presentations in 8.3.

($j = 0$ supersingular, d odd) We have $\widehat{\mathscr{O}_z}^{\mathrm{Aut}(z)} = (\widehat{\mathscr{O}}_z^{\mathbb{F}_q^\times})^{\overline{\mathrm{Aut}(z)}}$; here $\overline{\mathrm{Aut}(z)} := \mathrm{Aut}(z)/\mathbb{F}_q^\times \simeq \boldsymbol{\mu}_{q+1}(\overline{\mathbb{F}}_n)$, where we identify $\overline{\mathrm{Aut}(z)}$ with $\boldsymbol{\mu}_{q+1}$ via $\mathrm{Aut}(z) = \mathbb{F}_{q^2}^\times \ni u \rightsquigarrow u^{1-q} =: \zeta \in \overline{\mathrm{Aut}(z)}$, so that $\zeta \in \overline{\mathrm{Aut}(z)}$ acts on $\mathrm{Cot}\, \widehat{\mathscr{O}}_z^{\mathbb{F}_q^\times}$ by $x \rightsquigarrow \zeta^{-1}x, t \rightsquigarrow \zeta t$.

We take $t_1' = t, t_2' = x, m_2' = \frac{q^d-1}{q-1}, m_1' = 1, N = q+1, r = -1 = q \in (\mathbb{Z}/N\mathbb{Z})^\times$ in the notation of 8.2,8.4.

The congruence in Definition 8.2 is satisfied: $1 \equiv -(q)\frac{q^d-1}{q-1} \equiv -(-1)\left(\frac{-2}{-2}\right)$ mod $q+1$ since d is odd.

By induction, we obtain the continued fraction expansion

$$\frac{N}{r} = \frac{q+1}{q} = 2 - \cfrac{1}{2 - \cfrac{1}{\cdots - \frac{1}{2}}}$$

such that $\lambda = q$ in the notation of 8.4.

Given α and β consider the equation

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{\lambda-1} \\ \mu_\lambda \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix}$$

Now the matrix on the left is the Cartan matrix A_q; we find its inverse in [19, p. 69] and compute that

$$\mu_i = \alpha + \frac{i}{q+1}(\beta - \alpha). \tag{8.4.1}$$

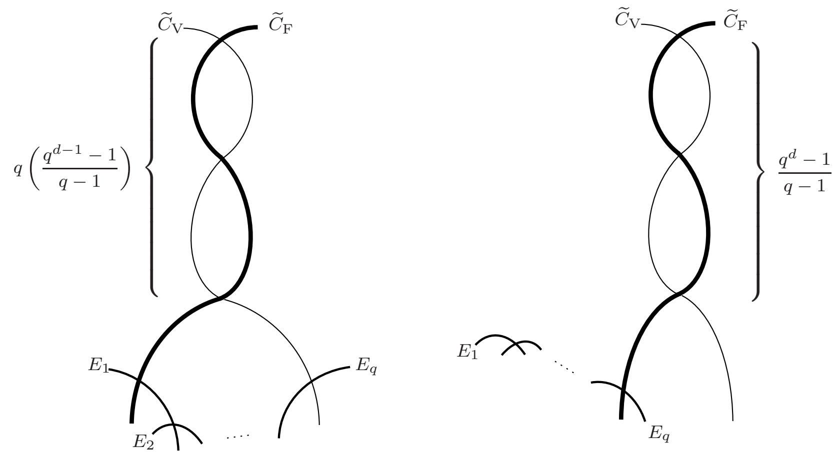
In our case, $\alpha = \frac{q^d-1}{q-1}, \beta = 1$ so that in particular $\mu_1 = \frac{1}{q+1}\left(q\left(\frac{q^d-1}{q-1}\right)+1\right)$.

The special fiber of the minimal resolution $\widetilde{X_1(n)}$ is shown in Figure 2(a); we have

$$\widetilde{C}_{\mathrm{F}} + \sum_1^q \mu_i E_i + \widetilde{C}_{\mathrm{V}} \equiv 0.$$

Intersecting with $\widetilde{C}_{\mathrm{F}}^{\mathrm{red}}$ we obtain

$$m_{\mathrm{F}}\widetilde{C}_{\mathrm{F}}^{\mathrm{red}}.\widetilde{C}_{\mathrm{F}}^{\mathrm{red}} + \mu_1 + \widetilde{C}_{\mathrm{V}}.\widetilde{C}_{\mathrm{F}}^{\mathrm{red}} = 0$$



(a) d odd (b) d even

Figure 8.2. Special fiber of the minimal regular resolution of $X_1(n)$.

where $m_{\mathrm{F}} := \operatorname{mult}(\widetilde{C}_{\mathrm{F}})$.
But $\widetilde{C}_{\mathrm{V}}.\widetilde{C}_{\mathrm{F}}^{\mathrm{red}} = \#\{\text{supersingular geometric points}\} - 1$ so

$$\begin{aligned}
\widetilde{C}_{\mathrm{V}}.\widetilde{C}_{\mathrm{F}}^{\mathrm{red}} + \mu_1 &= q\left(\frac{q^{d-1}-1}{q^2-1}\right) + \frac{1}{q+1}\left(\frac{q(q^d-1)}{q-1}+1\right)\\
&= q\left(\frac{q^{d-1}-1}{q^2-1}\right) + \frac{q^{d+1}-1}{(q-1)(q+1)}\\
&= \frac{q^{d+1}+q^d-q-1}{q^2-1}\\
&= \frac{(q+1)(q^d-1)}{q^2-1}\\
&= \frac{q^d-1}{q-1} = m_{\mathrm{F}}
\end{aligned}$$

and hence $\widetilde{C}_{\mathrm{F}}^{\mathrm{red}}$ is a -1-curve. Contract it. Now we use [3, 2.1.2] which implies that if X is a regular fibered surface over W, $X' = \mathrm{Bl}_P(X) \longrightarrow X$ is the contraction of an exceptional curve, C and D are effective divisors supported in the special fiber of X and passing through a closed point P, and $C', D' \subset X'$ are their strict transforms then $C.D = C'.D' + \operatorname{mult} C \operatorname{mult} D$. Hence in the blow down the image of E_1 is a -1-curve and so is contractible. As all of the E_i's have self intersection -2, we may continue this process until we obtain the sought regular model of $X_1(n)$ with integral special fiber.

($j = 0$ ordinary, d even) In this case We have $\widehat{\mathscr{O}_z}^{\mathrm{Aut}(z)} = (\widehat{\mathscr{O}_z}^{\mathbb{F}_q^\times})^{\overline{\mathrm{Aut}(z)}}$ where we identify $\overline{\mathrm{Aut}(z)}$ with $\boldsymbol{\mu}_{q+1}$ via $\mathrm{Aut}(z) = \mathbb{F}_{q^2}^\times \ni u \rightsquigarrow u^{q-1} =: \zeta \in \overline{\mathrm{Aut}(z)}$, so that $\zeta \in \overline{\mathrm{Aut}(z)}$ acts on $\mathrm{Cot}\, \widehat{\mathscr{O}_z}^{\mathbb{F}_q^\times}$ by $x \rightsquigarrow \zeta x, t \rightsquigarrow \zeta^{-1} t$.

In the notation of 8.2 we take $t_1' = x, t_2' = t, m_1' = \frac{q^d-1}{q-1}, m_2' = 0, N = q+1$, and $r = -1 = q \in (\mathbb{Z}/N\mathbb{Z})^\times$. The required congruence is satisfied: $\frac{q^d-1}{q-1} \equiv (-q)(0) \mod q+1$ since d is even.

The continued fraction is as before, so by 8.4.1 (with $\alpha = 0, \beta = \frac{q^d-1}{q-1}$) we have $\mu_q = \frac{q}{q+1} \cdot \frac{q^d-1}{q-1}$. Intersecting the special fiber of $\widetilde{X_1(n)}$ with $\widetilde{C}_{\mathrm{F}}^{\mathrm{red}}$ again, we obtain

$$m_{\mathrm{F}} \widetilde{C}_{\mathrm{F}}^{\mathrm{red}} . \widetilde{C}_{\mathrm{F}}^{\mathrm{red}} + \mu_q + \widetilde{C}_{\mathrm{V}} . \widetilde{C}_{\mathrm{F}}^{\mathrm{red}} = 0.$$

Noticing that $\widetilde{C}_{\mathrm{V}} . \widetilde{C}_{\mathrm{F}}^{\mathrm{red}} = \#\{\text{supersingular geometric points}\}$ as the point $j = 0$ on C_{F} is away from the crossing we compute

$$\begin{aligned} \#\{\text{supersingular geometric points}\} + \mu_q &= \frac{q^d-1}{q^2-1} + \frac{q}{q+1} \cdot \frac{q^d-1}{q-1} \\ &= \frac{q^d-1}{q-1} = m_{\mathrm{F}} \end{aligned}$$

so again $\widetilde{C}_{\mathrm{F}}^{\mathrm{red}}$ is a -1-curve. Now proceed as before.

As explained in the introduction, we have proven the

Theorem 8.5. — *The Drinfeld modular Jacobian $J_1(n)$ has connected fibers.*

Remark 8.6. — Let $H \subset \mathbb{F}_n^\times / \mathbb{F}_q^\times$ be a subgroup of order h. One would like to study the intermediate curves $X_H(n) := X_1(n)/H$ as in [3]; there they prove in the case of the classical modular curves that the natural map $J_H(p) \to J_0(p)$ induces an injection on mod p component groups. This is achieved by a study of the geometry of the minimal regular resolution of $X_H(p)$.

In our case, in order to carry out an analogous analysis, we would need to find a Jung-Hirzebruch continued fraction expansion for $\frac{q+1}{[-h]}$ where by $[-h]$ we mean the unique positive integer less than $q+1$ and congruent to $-h \mod q+1$. This seems to be an intractable problem. The numerator $q+1$ arises as the cardinality of $\mathrm{Aut}\, \phi/\mathbb{F}_q^\times$ for ϕ with $j = 0$; in the elliptic curve case there are only finitely many automorphism groups to check, so a case by case analysis is possible.

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