ANNALES

DE

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Tome 57, nº 3 (2007), p. 973-1017.

<http://aif.cedram.org/item?id=AIF_2007__57_3_973_0>

Ann. Inst. Fourier, Grenoble
57, 3 (2007) 973-1017

A FINITENESS RESULT FOR THE COMPACTLY SUPPORTED COHOMOLOGY OF RIGID ANALYTIC VARIETIES, II

by Roland HUBER

Abstract. — Let $h : X \to Y$ be a separated morphism of adic spaces of finite type over a non-archimedean field k with Y affinoid and of dimension $\leqslant 1$, let L be a locally closed constructible subset of X and let $g : (X, L) \to Y$ be the morphism of pseudo-adic spaces induced by h. Let A be a noetherian torsion ring with torsion prime to the characteristic of the residue field of the valuation ring of k and let F be a constant A-module of finite type on $(X, L)_{\text{ét}}$. There is a natural class $\mathscr{C}(Y)$ of A-modules on $Y_{\text{ét}}$ generated by the constructible A-modules and the Zariski-constructible A-modules. We show that, for every $n \in \mathbb{N}_0$, the higher direct image sheaf with proper support $R^n g_! F$ is generically constructible, and if h is locally algebraic, $R^n g_! F$ is an element of $\mathscr{C}(Y)$. As an application we obtain a comparison isomorphism for the ℓ-adic cohomology of a separated scheme of finite type over k and its associated adic space.

Résumé. — Soit $h : X \to Y$ un morphisme séparé d'espaces adiques de type fini sur un corps non archimédien k avec Y affinoïde et de dimension $\leqslant 1$. Soit L un sous-ensemble constructible localement fermé dans X et soit $g : (X, L) \to Y$ le morphisme d'espaces pseudo-adiques induit de h. Soit A un anneau noethérien de torsion première à la caractéristique résiduelle de k et soit F un faisceau de A-modules localement constant de type fini sur $(X, L)_{\text{ét}}$. Il y a une classe naturelle $\mathscr{C}(Y)$ des faisceaux de A-modules sur $Y_{\text{ét}}$ engendrée par des faisceaux de A-modules constructibles et des faisceaux de A-modules Zariski-constructibles. Nous montrons que le faisceau image directe à support propre $R^n g_! F$ est génériquement constructible, et si h est localement algébrique, $R^n g_! F$ est un élément de $\mathscr{C}(Y)$. En conséquence, on obtient un théorème de comparaison entre cohomologie ℓ-adique d'un schéma séparé de type fini sur k et de l'espace adique associé.

1. Introduction

Let k be a non-archimedean field, let $h : X \to Y$ be a separated morphism of adic spaces of finite type over $\mathrm{Spa}(k, k^{\circ})$ with Y affinoid and $\dim Y \leqslant 1$,

Keywords: Rigid analytic spaces, adic spaces, compactly supported cohomology.
Math. classification: 14G22, 14F20.

let L be a locally closed constructible subset of X and let $g : (X, L) \to Y$ be the morphism of pseudo-adic spaces induced by h. Let A be a noetherian torsion ring with torsion prime to $\mathrm{char}(k^\circ/k^{\circ\circ})$ and let F be a constant A-module of finite type on $(X, L)_{\mathrm{\acute{e}t}}$. There is a natural class $\mathscr{C}(Y)$ of A-modules on $Y_{\mathrm{\acute{e}t}}$ generated by the constructible A-modules as defined in [9], 2.7 and the Zariski-constructible A-modules. We are interested to know if $R^m g_! F \in \mathscr{C}(Y)$. In [11] is proved that this is fulfilled if $\mathrm{char}(k) = 0$ and $|A| < \infty$. In this paper we will show that without any restriction on the characteristic of k and the cardinality of A the following two statements hold

(I) For every $m \in \mathbb{N}_0$, $R^m g_! F$ is generically constructible on Y, i.e., there exists an open subset U of Y such that the restriction $R^m g_! F|U$ is constructible on U and every $x \in Y$ whose support $\mathrm{supp}(x) = \{c \in \mathscr{O}_Y(Y) \mid c(x) = 0\} \in \mathrm{Spec}\,\mathscr{O}_Y(Y)$ is a generic point of $\mathrm{Spec}\,\mathscr{O}_Y(Y)$ is contained in U.

(II) If h is locally algebraic then, for every $m \in \mathbb{N}_0$, $R^m g_! F \in \mathscr{C}(Y)$.

As a consequence of (II) we will obtain a comparison isomorphism for ℓ-adic cohomology,

$$H^q_c(X, (F_n)_{n\in\mathbb{N}}) \xrightarrow{\sim} H^q_c(X^{\mathrm{ad}}, (F_n^{\mathrm{ad}})_{n\in\mathbb{N}}),$$

where X is a separated scheme of finite type over $\mathrm{Spec}\,k$ (here k is assumed to be algbraically closed) and X^{ad} is its associated adic space over $\mathrm{Spa}(k, k^\circ)$. (For $\mathrm{char}(k) = 0$ this comparison theorem is already proved in [10]).

The main new ingredient of the proof of (I) is a result on algebraization of finite morphisms of adic spaces (Lemma 7.3).

(II) can be deduced from (I). In the following we sketch this. By virtue of (I) it suffices to show that $R^m g_! F$ is constructible around each $y \in Y(\overline{k})$ (with $\overline{k}$ the algebraic closure of k). For simplicity let us assume that

$$Y := \mathbb{B}^1_k = \mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)\ ,\ \{y\} := \{0\} = V(T) \subseteq Y.$$

The topological space $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$ is a subspace of the valuation spectrum $\mathrm{Spv}\,k\langle T\rangle$ of $k\langle T\rangle$,

$$0 \in Y = \mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle) \subseteq \mathrm{Spv}\,k\langle T\rangle.$$

The element 0 has no proper generalization in $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$ but there is a unique valuation $v \in \mathrm{Spv}\,k\langle T\rangle$ which is a proper generalization of 0 in $\mathrm{Spv}\,k\langle T\rangle$ and extends the valuation $|\ \ |$ of k. One may expect that v can be helpful to study, for a sheaf E on $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)_{\mathrm{\acute{e}t}}$, the behavior of E around 0. Some evidence for this appears in the paper [18]. In the present

paper we use v in order to prove (II).
If $(U_n)_{n\in\mathbb{N}}$ is a fundamental system of neighbourhoods of zero in k then $(U_n\langle T\rangle)_{n\in\mathbb{N}}$ is a fundamental system of neighbourhoods of zero in $k\langle T\rangle$ where $U_n\langle T\rangle = \{\sum a_\ell T^\ell \in k\langle T\rangle \mid a_\ell \in U_n$ for all $\ell\}$. Let $k\langle T\rangle_T$ be the localization of $k\langle T\rangle$ with respect to the multiplicative system $\{1, T, T^2, \ldots\}$. We endow $k\langle T\rangle_T$ with the ring topology such that $(T^n \cdot U_n\langle T\rangle)_{n\in\mathbb{N}}$ is a fundamental system of neighbourhoods of zero. We will show that the topological ring $k\langle T\rangle_T$ is strictly noetherian. Hence we have the affinoid adic space

$$Y' := \mathrm{Spa}\,(k\langle T\rangle_T, k^\circ\langle T\rangle).$$

Since $v(T) \neq 0$, the valuation v of $k\langle T\rangle$ extends uniquely to a valuation w of $k\langle T\rangle_T$. The valuation w is continuous with respect to the topology of $k\langle T\rangle_T$ and $w(b) \leqslant 1$ for all $b \in k^\circ\langle T\rangle$, i.e.,

$$w \in Y'.$$

For an element $a \in k^*$ with $|a| < 1$ put

$$Y'_\varepsilon := \{x \in Y' \mid |a(x)|_x < 1\} \subseteq Y'.$$

This set is independent of a. We have

(1) There is a natural morphism of adic spaces

$$\pi : Y - \{0\} \longrightarrow Y'.$$

π is an open embedding and it extends to a homeomorphism

$$\tau : Y \longrightarrow Y'_\varepsilon$$

with $\tau(0) = w$.

(Remark. The set $Y' - Y'_\varepsilon$ consists of exactly one element).
Since $h : X \to Y$ is locally algebraic, we may assume that the diagram of pseudo-adic spaces

$$\begin{array}{ccc} (X - h^{-1}(0), L - h^{-1}(0)) & & \\ \downarrow{\scriptstyle u} & & \\ Y - \{0\} & \xrightarrow{\ \pi\ } & Y' \end{array}$$

where u is the restriction of g can be extended to a cartesian diagram of pseudo-adic spaces

$$\begin{tikzcd}
(X - h^{-1}(0), L - h^{-1}(0)) \arrow[r, "\pi'"] \arrow[d, "u"'] & (X', L') \arrow[d, "g'"] \\
Y - \{0\} \arrow[r, "\pi"'] & Y'
\end{tikzcd}$$

where the morphism of adic spaces $X' \to Y'$ underlying g' is of finite type and separated and L' is a locally closed constructible subset of X'. Let F' be the constant A-module on $(X', L')_{\text{ét}}$ such that $\pi'^*(F') = F|(X - h^{-1}(0), L - h^{-1}(0))$. Obviously,

(2) $R^m g_! F|Y - \{0\} = \pi^* R^m g'_! F'$.

We will show that (I) holds analogously for g' and F' instead of g and F. Hence as the support of w is the generic point of $\operatorname{Spec} k\langle T\rangle_T$, we obtain that $R^m g'_! F'$ is constructible at w. Then (1) and (2) imply that $R^m g_! F$ is constructible around 0.

Throughout the paper is assumed that, for an affinoid analytic adic X, $\mathscr{O}_X(X)$ is a stictly noetherian Tate ring.

I thank S. Bosch for his reference to [19] in Remark 2.9.

2. Some strictly noetherian Tate rings

For a topological ring A, we call a subring of A which is open in A and whose subspace topology is adic a ring of definition of A, and we call A a Tate ring if it has a ring of definition and a topologically nilpotent unit ([7]).

Let A be a topological ring and let f be an element of A. Let A_f be the localization of the ring A with respect to the multiplicative system $\{1, f, f^2, \ldots\}$ and let $\rho : A \to A_f$ be the natural mapping. There is a ring topology on A_f such that $\{\rho(f^n U) \mid n \in \mathbb{N}$ and U a neighbourhood of 0 in $A\}$ is a fundamental system of neighbourhoods of zero. The ring A_f equipped with this topology is denoted by A^f. The mapping $\rho : A \to A^f$ is a universal ring homomorphism from A to a topological ring which maps f to a unit and is open.
If f' is an element of A such that we have an equality $V(f) = V(f')$ of subsets of $\operatorname{Spec} A$ then we have an equality of topological rings $A^f = A^{f'}$.

If A is hausdorff (complete, resp.) and the ideal $\ker(\rho) = \{a \in A \mid f^n a = 0$ for some $n \in \mathbb{N}\}$ is annihilated by some f^m then A^f is hausdorff (complete, resp.). If there exists a unit s of A such that sf is topolgically nilpotent in A then A^f has a topologically nilpotent unit (e.g., $\rho(sf)$) and, for every ring of definition A_0 of A, $\rho(A_0)$ is a ring of definition of A^f. Therefore, if A is a Tate ring then A^f is a Tate ring.

The aim of this section is to show that if A is a strictly noetherian Tate ring then A^f is strictly noetherian, too.

2.1. — Let A be a topological ring which has a fundamental system of neighbourhoods of 0 consisting of additively closed subsets of A. We put, for every subset U of A,

$$\begin{aligned} A\langle X_1, \ldots, X_n\rangle &= \{\textstyle\sum a_\nu X^\nu \in A[[X_1, \ldots, X_n]] \mid (a_\nu)_{\nu\in\mathbb{N}_0^n} \text{ is a} \\ &\qquad \text{zero sequence in } A\} \\ U\langle X_1, \ldots, X_n\rangle &= \{\textstyle\sum a_\nu X^\nu \in A\langle X_1, \ldots, X_n\rangle \mid a_\nu \in U \text{ for all } \nu \in \mathbb{N}_0^n\} \\ U[X_1, \ldots, X_n] &= \{\textstyle\sum a_\nu X^\nu \in A[X_1, \ldots, X_n] \mid a_\nu \in U \text{ for all } \nu \in \mathbb{N}_0^n\}. \end{aligned}$$

In the following we endow $A[X_1, \ldots, X_n]$ and $A\langle X_1, \ldots, X_n\rangle$ with the ring topology such that if $\mathscr{U}$ is the set of all neighbourhoods of 0 in A then $\{U[X_1, \ldots, X_n] \mid U \in \mathscr{U}\}$ and $\{U\langle X_1, \ldots, X_n\rangle \mid U \in \mathscr{U}\}$ are fundamental systems of neighbourhoods of 0 in $A[X_1, \ldots, X_n]$ and $A\langle X_1, \ldots, X_n\rangle$.
A complete Tate ring A is called strictly noetherian if, for every $n \in \mathbb{N}$, the ring $A\langle X_1, \ldots, X_n\rangle$ is noetherian.

Remark 2.2. — (i) Let A be as in (2.1). Then $(A[T]^T)^\wedge = A\langle T\rangle^T$.

(ii) For a ring B and an element s of B let (B, s) indicate that the ring B is endowed with the sB-adic topology. Then $(B[T], sT)^\wedge = ((B, s)\langle T\rangle, sT)$.

We say that a topological ring A satisfies (N) if every ideal I of A is finitely generated and the natural A-module topology of I ([8], §2) agrees with the subspace topology of I in A.

Remark 2.3. — (i) Let A be a topological ring and let I be a finitely generated ideal of A. The natural A-module topology of I agrees with the subspace topology of I in A if and only if for some (and then for any) finite system of generators $a_1, \ldots, a_n$ of I the mapping

$$(*) \qquad A^n \to I,\ (x_1, \ldots, x_n) \mapsto x_1a_1 + \ldots + x_na_n$$

is open where A^n is equipped with the product topology and I is equipped with the subspace topology of A.

(ii) Let A be a Tate ring that is hausdorff and unequal $\{0\}$. Let $\| \quad \|$ be a norm of A, i.e. $\| \quad \|$ is a mapping $A \to \mathbb{R}_{\geqslant 0}$ such that
  (a) $\|0\| = 0,\ \|1\| = 1$
  (b) $\|x + y\| \leqslant \max(\|x\|, \|y\|)$ for all $x, y \in A$
  (c) $\|xy\| \leqslant \|x\| \cdot \|y\|$ for all $x, y \in A$
  (d) $(\{x \in A \mid \|x\| \leqslant r\} \mid r \in \mathbb{R}_{>0})$ is a fundamental system of neighbourhoods of 0 in A.

Assume that there is a topologically nilpotent unit s of A such that $\|s^{-1}\| = \|s\|^{-1}$ or, equivalently, $\|sa\| = \|s\| \cdot \|a\|$ for all $a \in A$. (Such a norm of A always exists).

Let I be an ideal of A with a finite system of generators $a_1, \dots, a_n$. Then the mapping $(*)$ in (i) is open if and only if

  $(**)$ there exists some $K \in \mathbb{R}_{>0}$ such that for every $x \in I$ there exist $x_1, \dots, x_n \in A$ with $x = x_1 a_1 + \ldots + x_n a_n$ and $\|x_i\| \leqslant K \cdot \|x\|$ for $i = 1, \dots, n$.

(iii) A complete Tate ring satisfies (N) if and only if it is noetherian ([8], 2.4.ii).

Lemma 2.4. — *Let A be a Tate ring that is hausdorff.*

(i) *If $A\langle T\rangle$ satisfies (N) then $A[T]$ satisfies (N).*
(ii) *If A and $\hat{A}\langle T\rangle$ satisfy (N) then $A\langle T\rangle$ and $A[T]$ satisfy (N).*
(iii) *Let $n \in \mathbb{N}$ such that A and $\hat{A}\langle T_1, \dots, T_n\rangle$ satisfy (N). For $i \in \{1, \dots, n\}$ let $\{^i\ ,\ ^i\}$ be $[\ ,\ ]$ or $\langle\ ,\ \rangle$. Then the Tate ring $A\{^1\,T_1\,^1\} \ldots \{^n\,T_n\,^n\}$ satisfies (N).*

Proof. We fix a norm $\| \quad \| : A \to \mathbb{R}_{\geqslant 0}$ of A such that $\|s^{-1}\| = \|s\|^{-1}$ for some topologically nilpotent unit s of A. We equip all rings occuring in (i) and (ii) with the Gauss norm with respect to $\| \quad \|$.

i) We will show that every ideal I of $A[T]$ has a finite system of generators $a_1, \dots, a_n$ for which $(**)$ in Remark 2.3(ii) holds.

First we reduce the situation to the case that I is T-saturated. So let I be an ideal of $A[T]$ and let I' be the T-saturation of I. Assume that the assertion holds for I', i.e., there exist a finite system of generators $e'_1, \dots, e'_n$ of I' and some $K \in \mathbb{R}_{>0}$ such that for every $x \in I'$ there exist $x_1, \dots, x_n \in A[T]$ with $x = x_1 e'_1 + \ldots + x_n e'_n$ and $\|x_i\| \leqslant K \cdot \|x\|$ for $i = 1, \dots, n$. Let $m \in \mathbb{N}$ such that $T^m e'_i \in I$ for $i = 1, \dots, n$. For every $r \in \mathbb{N}_0$ let π_r be the mapping

$$A[T] \to A,\ \sum a_p T^p \mapsto a_r.$$

By hypothesis $A\langle T\rangle$ satisfies (N) and hence A satisfies (N), too. Let $e_{r,1},\dots,e_{r,n(r)}$ be elements of $I\cap(T^r\cdot A[T])$ such that $\pi_r(e_{r,1}),\dots,\pi_r(e_{r,n(r)})$ generate the ideal $\pi_r(I\cap(T^r\cdot A[T]))$ of A. Applying (∗∗) in Remark 2.3(ii) to these generators, we obtain by induction on $r=0,1,\dots$ that for every $r\in\mathbb{N}_0$ there exists some $K_r\in\mathbb{R}_{>0}$ such that for every $x\in I$ there exists a family $x_{ij}, i=0,\dots,r, j=1,\dots,n(i)$ in A such that

$$x-\sum_{\substack{i=0,\dots,r\\ j=1,\dots,n(i)}} x_{ij}e_{ij}\ \in\ I\cap(T^{r+1}\cdot A[T])$$

and $\|x_{ij}\|\leqslant K_i\cdot\|x\|$ for $i=0,\dots,r, j=1,\dots,n(i)$. Therefore, in order to prove the assertion for I it is enough to consider the elements $x\in I\cap(T^m\cdot A[T])$. Then $T^{-m}\cdot x\in I'$ and so there exist $x_1,\dots,x_n\in A[T]$ with $T^{-m}\cdot x=x_1e_1'+\dots+x_ne_n'$ and $\|x_i\|\leqslant K\cdot\|T^{-m}\cdot x\|=K\cdot\|x\|$ for $i=1,\dots,n$. Then for $e_i:=T^me_i'\in I$ $(i=1,\dots,n)$ we get $x=x_1e_1+\dots+x_ne_n$. Thus we see that it is enough to prove the assertion for I', i.e., we may assume that I is T-saturated.
For $u,v\in\mathbb{N}_0$ with $u\leqslant v$ put

$$A[T]_{u,v}:=\{p\in T^u\cdot A[T]\mid \deg(p)\leqslant v\}.$$

For $x=\sum a_pT^p\in A\langle T\rangle$ and $u\in\mathbb{N}_0$ put

$${}_ux:=\sum_{p\leqslant u}a_pT^p\ \in\ A[T].$$

We consider the ideal of $A\langle T\rangle$ generated by I. Since $A\langle T\rangle$ satisfies (N), there exist $e_1,\dots,e_n\in I$ such that for every $x\in I$ there exist $x_1,\dots,x_n\in A\langle T\rangle$ with $x=x_1e_1+\dots+x_ne_n$ and $\|x_i\|\leqslant\|x\|$ for $i=1,\dots,n$. Then for $L:=\max(1,\|e_1\|,\dots,\|e_n\|)$, $m:=\max(\deg(e_1),\dots,\deg(e_n))$, $u:=\deg(x)$ and $y:=x-{}_u(x_1)e_1-\dots-{}_u(x_n)e_n$ hold

$$y\in I\cap A[T]_{u,u+m}\ \text{ and }\ \|y\|\leqslant L\cdot\|x\|.$$

Since I is T-saturated, we obtain

$$y=T^u\cdot z\ \text{ for some } z\in I\cap A[T]_{0,m}.$$

Therefore it suffices to consider the elements $z\in I\cap A[T]_{0,m}$. Similarly as in the reduction above we get that there exist $f_1,\dots,f_s\in I\cap A[T]_{0,m}$ and $K\in\mathbb{R}_{>0}$ such that for every $z\in I\cap A[T]_{0,m}$ there exist $z_1,\dots,z_s\in A$ with $z=z_1f_1+\dots+z_sf_s$ and $\|z_i\|\leqslant K\cdot\|z\|$ for $i=1,\dots,s$.
ii) By virtue of (i) it suffices to show that $A\langle T\rangle$ satisfies (N). Let I be an ideal of $A\langle T\rangle$. We will show that there exists a finite system of generators $a_1,\dots,a_n$ of I for which (∗∗) in Remark 2.3(ii) holds. As in the proof of (i) we may assume that I is T-saturated. We consider the ideal of $\hat{A}\langle T\rangle$

generated by I. Since $\hat{A}\langle T\rangle$ satisfies (N), there exist elements $g_1,\dots,g_m$ of I such that for every $x\in I$ and every $\varepsilon\in\mathbb{R}_{>0}$ there exist $x_1,\dots,x_m\in A\langle T\rangle$ with $\|x-\sum_{i=1}^m x_i g_i\|<\varepsilon$ and $\|x_i\|\leqslant\|x\|$ for $i=1,\dots,m$. Let π denote the ring homomorphism

$$A\langle T\rangle\to A,\ \sum a_iT^i\mapsto a_0.$$

Since A satisfies (N), there exist elements $g_{m+1},\dots,g_n$ of I such that for every $x\in\pi(I)$ there exist $x_{m+1},\dots,x_n\in A$ with $x=x_{m+1}\pi(g_{m+1})+\ldots+x_n\pi(g_n)$ and $\|x_i\|\leqslant\|x\|$ for $i=m+1,\dots,n$. We will show that for every $x\in I$ there exist $x_1,\dots,x_n\in A\langle T\rangle$ such that $x=x_1g_1+\ldots+x_ng_n$ and $\|x_i\|\leqslant\|x\|$ for $i=1,\dots,n$.
Let x be an element of I with $x\neq 0$. We choose a decreasing zero sequence $(\sigma_p)_{p\in\mathbb{N}_0}$ in $\mathbb{R}_{>0}$ with $\sigma_0=\|x\|$. We will construct, for every $p\in\mathbb{N}_0$, elements $y_{p,1},\dots,y_{p,n}$ of $A\langle T\rangle$ such that

a) $\|y_{p,i}\|\leqslant\sigma_p$ for $i=1,\dots,n$
b) If $x_{p,i}:=\sum_{q=0}^p T^q y_{q,i}\in A\langle T\rangle$ $(i=1,\dots,n)$ and $z_p:=x-\sum_{i=1}^n x_{p,i}\,g_i$ then $\|z_p\|\leqslant\sigma_{p+1}$ and $z_p\in T^{p+1}\cdot A\langle T\rangle$.

Then for

$$x_i:=\sum_{q\in\mathbb{N}_0}T^q y_{q,i}\in A\langle T\rangle\ \ (i=1,\dots,n)$$

we have $x=x_1g_1+\ldots+x_ng_n$ and $\|x_i\|\leqslant\|x\|$ for $i=1,\dots,n$.
Let $p\in\mathbb{N}_0$ such that $y_{q,1},\dots,y_{q,n}$ for $q=0,\dots,p-1$ are already constructed. Since $z_{p-1}\in I\cap(T^p\cdot A\langle T\rangle)$ (for $p=0$ put $z_{p-1}=x$) and I is T-saturated, we have $T^{-p}\cdot z_{p-1}\in I$. Choose $y_{p,1},\dots,y_{p,m}\in A\langle T\rangle$ such that if $z:=T^{-p}\cdot z_{p-1}-(y_{p,1}g_1+\ldots+y_{p,m}g_m)$ and $\lambda:=\max(1,\|g_{m+1}\|,\dots,\|g_n\|)$ then $\|z\|\cdot\lambda\leqslant\sigma_{p+1}$ and $\|y_{p,i}\|\leqslant\|T^{-p}\cdot z_{p-1}\|=\|z_{p-1}\|$ for $i=1,\dots,m$. Then choose $y_{p,m+1},\dots,y_{p,n}\in A$ such that $\pi(z)=y_{p,m+1}\pi(g_{m+1})+\ldots+y_{p,n}\pi(g_n)$ and $\|y_{p,i}\|\leqslant\|\pi(z)\|$ for $i=m+1,\dots,n$.
iii) For $i=1,\dots,n$ the Tate ring $(A\{{}^1\,T_1\,{}^1\}\ldots\{{}^{i-1}\,T_{i-1}\,{}^{i-1}\})^\wedge\langle T_i\rangle=\hat{A}\langle T_1,\dots,T_i\rangle$ satisfies (N), since $\hat{A}\langle T_1,\dots,T_n\rangle$ satisfies (N). Then the assertion follows from (ii) by induction on i. □

Proposition 2.5. — *Let A be a Tate ring that is complete and strictly noetherian. Then, for every $f\in A$, the Tate ring A^f is complete and strictly noetherian.*

Proof. We may assume that f is power bounded in A. Then we have the continuous ring homomorphism $\sigma:A\langle T\rangle\to A$ with $\sigma(a)=a$ for all $a\in A$ and $\sigma(T)=f$. Consider the induced continuous ring homomorphism $\tau:A\langle T\rangle^T\to A^f$. The mapping τ is surjective and open, as σ is surjective and open . Hence it suffices to show that $A\langle T\rangle^T$ is strictly noetherian.

Remark 2.2(i) immediately implies that, for every $n \in \mathbb{N}$, $A\langle T\rangle^T\langle X_1,\dots,X_n\rangle = A[X_1,\dots,X_n]\langle T\rangle^T$. By Lemma 2.4(iii) and Remark 2.3(iii) the ring $A[X_1,\dots,X_n]\langle T\rangle$ satisfies (N), and hence it is noetherian. Thus we obtain that $A\langle T\rangle^T\langle X_1,\dots,X_n\rangle$ is noetherian. □

Lemma 2.6. — *Let A be a strictly noetherian complete Tate ring, let f be an element of A, let $\rho : A \to A^f$ be the natural mapping and let B be a complete Tate ring of topologically finite type over A^f,*

$$A \xrightarrow{\rho} A^f \xrightarrow{\eta} B.$$

Let A_0 be a ring of definition of A, and so $\rho(A_0)$ is a ring of definition of A^f. Let B_0 be a ring of definition of B of topologically finite type over $\rho(A_0)$. Let C_0 be the ring B_0 equipped with the adic topology such that the ring homomorphism $\eta\circ\rho : A_0 \to C_0$ is adic. Let C be the subring of B generated by C_0 and $(\eta\circ\rho)(A)$, endowed with the group topology such that C_0 is an open topological subgroup of C. (Then C is a Tate ring, the mapping $\eta\circ\rho : A \to C$ is continuous and $B = C^{(\eta\circ\rho)(f)}$). Then

(i) *The topological ring C satisfies (N).*
(ii) *The ring homomorphism $\eta\circ\rho : A \to C^\wedge$ is of topologically finite type, more precisely, the ring homomorphism $\eta\circ\rho : A_0 \to C_0^\wedge$ is of topologically finite type.*

Proof. We may assume that $f \in A_0$. By virtue of the mapping $\tau : A\langle T\rangle^T \to A^f$ in the proof of Proposition 2.5 we may replace A, f, B, A_0, B_0 by $A\langle T\rangle, T$, $A\langle T\rangle^T\langle X_1,\dots,X_n\rangle, A_0\langle T\rangle, \rho_{A\langle T\rangle}(A_0\langle T\rangle)\langle X_1,\dots,X_n\rangle$. Then with Remark 2.2(ii) we obtain $C_0 = A_0[X_1,\dots,X_n]\langle T\rangle$, and hence $C = A[X_1,\dots,X_n]\langle T\rangle$ which satisfies (N) by Lemma 2.4(iii) and Remark 2.3(iii). Furthermore, $C_0^\wedge = A_0\langle X_1,\dots,X_n\rangle\langle T\rangle$ is of topologically finite type over $A_0\langle T\rangle$ □

For a ring A and a subset R of A, let us call an A-module M R-noetherian if for every sub-A-module P of M there exists some $n(P) \in \mathbb{N}$ such that for every $r \in R^{n(P)} := \{s^{n(P)} \mid s \in R\}$ there exists a finitely generated sub-A-module P' of M with $rP \subseteq P' \subseteq P$. The ring A is called R-noetherian if the A-module A is R-noetherian. (Remark. In our application (Proposition 2.8, the valuation of k not discrete) holds that for every $r \in R$ and every $n \in \mathbb{N}$ there exists some $t \in R$ with $r \in t^nA$. Then if M is R-noetherian then we can put $n(P) = 1$ for every sub-A-module P of M).

Lemma 2.7. — *Let A be a ring and R a subset of A.*

(i) *If A is R-noetherian then every finitely generated A-module is R-noetherian.*

(ii) *Assume that there exists a ring topology on A such that the following two conditions are satisfied*

 (a) *A is an open topological subring of some topological ring B which satisfies (N).*

 (b) *If $\lambda : A \to \hat{A}$ is the completion of A then the ring $\hat{A}$ is $\lambda(R)$-noetherian.*

 Then A is R-noetherian.

Proof. We proof (ii). Let I be an ideal of A. By hypothesis (b) there exists a $n(I \cdot \hat{A}) \in \mathbb{N}$ such that for every $r \in R^{n(I \cdot \hat{A})}$ there exists a finitely generated ideal J of $\hat{A}$ with $r \cdot I \cdot \hat{A} \subseteq J \subseteq I \cdot \hat{A}$. We will show that for every $r \in R^{n(I \cdot \hat{A})}$ there exists a finitely generated ideal K of A with $rI \subseteq K \subseteq I$.
By hypothesis (a) the ideal $I \cdot B$ of B is finitely generated. Hence there exists a finite subset S of I with $I \cdot B = S \cdot B$. Furthermore by (a) there exists an open subset U of B with $U \cap (I \cdot B) = S \cdot A$. Let $r \in R^{n(I \cdot \hat{A})}$ be given and let J be a finitely generated ideal of $\hat{A}$ with $r \cdot I \cdot \hat{A} \subseteq J \subseteq I \cdot \hat{A}$. We choose a finite subset T of I with $J \subseteq T \cdot \hat{A}$. Then $rI \subseteq (S \cup T) \cdot A \subseteq I$. Indeed, for every $x \in rI$ there exists a family $(a_t)_{t \in T}$ in A with $x - \sum_{t \in T} a_t \cdot t \in U \cap A$. Since $U \cap (I \cdot B) = S \cdot A$, we obtain $x - \sum_{t \in T} a_t \cdot t \in S \cdot A$. □

Proposition 2.8. — *We consider ring homomorphisms*

$$k \xrightarrow{\tau} A \xrightarrow{\rho} A^f \xrightarrow{\eta} B$$

where k is a complete non-archimedean field, A and B are complete Tate rings, f is an element of A, ρ is the natural ring homomorphism and τ and η are continuous ring homomorphisms of topologically finite type. Let A_0 be a ring of definition of A of topologically finite type over k° and let B_0 be a ring of definition of B of topologically finite type over the ring of definition $\rho(A_0)$ of A^f.
Then B_0 is $(\eta \circ \rho \circ \tau)(k^{\circ\circ})$-noetherian. (If the valuation of k is discrete then B_0 is noetherian).

Proof. Let C_0 and C be as in Lemma 2.6. Let $\lambda : C_0 \to C_0^\wedge$ be the completion of C_0. By Lemma 2.6(ii) the ring homomorphism $\sigma := \lambda \circ \eta \circ \rho \circ \tau : k^\circ \to C_0^\wedge$ is of topologically finite type. Then according to [14], Satz 5.1 the ring $C_0^\wedge$ is $\sigma(k^{\circ\circ})$-noetherian. By Lemma 2.6(i) the topological ring C satifies (N). Then we can conclude from Lemma 2.7(ii) that B_0 is $(\eta \circ \rho \circ \tau)(k^{\circ\circ})$-noetherian. □

Remark 2.9. — Let k be a non-archimedean field (not necessarily complete) and let $n \in \mathbb{N}$. Then

(i) The topological ring $k[X_1, \ldots, X_n]$ satisfies (N).
(ii) The ring $k^\circ[X_1, \ldots, X_n]$ is $k^{\circ\circ}$-noetherian

The first assertion follows from Lemma 2.4(iii). Since the completion $(k^\circ[X_1, \ldots, X_n])^\wedge = (k^\wedge)^\circ\langle X_1, \ldots, X_n\rangle$ is $(k^\wedge)^{\circ\circ}$- noetherian ([14], Satz 5.1), the second assertion follows from the first one and Lemma 2.7(ii).
I learned from S. Bosch that a better result than (i) holds. Namely, if V is a valuation ring then every $(V - \{0\})$-saturated ideal I of $V[X_1, \ldots, X_n]$ is finitely generated. Indeed, the ring $V[X_1, \ldots, X_n]/I$ is flat and of finite type over the integral domain V and hence of finite presentation over V by [19], 3.4.7.

3. Some analytic adic spaces

We fix an affinoid analytic adic space X and an element f of $\mathscr{O}_X(X)$. Put $A := \mathscr{O}_X(X)$ and $A^+ := \mathscr{O}_X^+(X)$.

According to Section 2 we have the Tate ring A^f. The integral closure $(A^+)^c$ of A^+ in A_f is a ring of integral elements of A^f. By Proposition 2.5 the Tate ring A^f is strictly noetherian. Hence we have an adic space associated with the affinoid ring $(A^f, (A^+)^c)$. We denote this space by X^f,

$$X^f := \operatorname{Spa}(A^f, (A^+)^c).$$

If Y is a further affinoid adic space and $h : Y \to X$ is a morphism of adic spaces then the continuous morphism of affinoid rings $h^* : (A, A^+) \to (B, B^+)$ with $B := \mathscr{O}_Y(Y)$ induces a continuous morphism of affinoid rings $(A^f, (A^+)^c) \to (B^{h^*(f)}, (B^+)^c)$ and so we get a morphism of adic spaces

$$h^f : Y^{h^*(f)} \to X^f.$$

(Remark. If h is of finite type, this does not imply in general that h^f is of finite type. This is the reason why in statement (II) of the introduction the morphism h is required to be locally algebraic.)

The aim of this section is to compare X and X^f. Similar considerations are contained in [21], 4.2.

Put $X_f := X - V(f)$. The A-algebra homomorphism $A^f \to \mathscr{O}_X(X_f)$ is continuous and maps $(A^+)^c$ to $\mathscr{O}_X^+(X_f)$. (Indeed, if $A \to B$ is a continuous ring homomorphism from A to a topological ring B which maps f to a unit then the A-algebra homomorphism $\sigma : A^f \to B$ is continuous, since the mapping $A \to A^f$ is open. Furthermore, if B^+ is a subring of B which is integrally closed in B and contains the image of A^+ in B then trivially $\sigma((A^+)^c) \subseteq B^+$). So we get by [8], 2.1(ii) a morphism of adic spaces

$$\pi_{X,f} : X_f \to X^f.$$

Proposition 3.1. — *$\pi_{X,f}$ is an open embedding of adic spaces.*

Proof. Let A_0 be a ring of definition of A and let s be a topologically nilpotent unit of A with $s \in A_0$. We may assume that $f \in sA_0$. Then A_0 is a ring of definition of A^f and f is a topologically nilpotent unit of A^f (more precisely, the images of A_0 and f in A_f). We consider rational subsets of X and X^f,

$$\begin{aligned} U_n &:= R_X(\frac{s^n}{f}) = \{x \in X \mid |s^n(x)| \leqslant |f(x)|\} \\ U'_n &:= R_{X^f}(\frac{s^n}{f}) = \{x \in X^f \mid |s^n(x)| \leqslant |f(x)|\}. \end{aligned}$$

Then $U_n \subseteq U_{n+1}$, $X_f = \bigcup_{n\in\mathbb{N}} U_n$, $U'_n \subseteq U'_{n+1}$ and $\pi^{-1}_{X,f}(U'_n) = U_n$. The morphism of affinoid rings $\pi^*_{X,f} : (\mathscr{O}_{X^f}(U'_n), \mathscr{O}^+_{X^f}(U'_n)) \to (\mathscr{O}_X(U_n), \mathscr{O}^+_X(U_n))$ is an isomorphism, since both affinoid rings are completions of the affinoid ring $(A_f, A^+[\frac{s^n}{f}]^c)$ where $A^+[\frac{s^n}{f}]^c$ is the integral closure of $A^+[\frac{s^n}{f}]$ in A_f and A_f is equipped with the topology such that $A_0[\frac{s^n}{f}]$ is a ring of definition with $s \cdot A_0[\frac{s^n}{f}]$ (or, equivalently, $f \cdot A_0[\frac{s^n}{f}]$) an ideal of definition ([8], Proposition 1.3). ☐

We put, for s a topologically nilpotent unit of A,

$$\begin{aligned} (X^f)_\varepsilon &:= \{x \in X^f \mid |s(x)| < 1\} \\ (X^f)_\iota &:= \{x \in X^f \mid |s(x)| \geqslant 1\} = \{x \in X^f \mid |s(x)| = 1\}. \end{aligned}$$

These subsets of X^f are independent of the choice of s. The set X^f_ι is rational in X^f and the set X^f_ε is closed and constructible in X^f. Let $(X^f_\varepsilon)^\circ$ denote the interior of X^f_ε in X^f.

Proposition 3.2. — (i) $(X^f_\varepsilon)^\circ = \mathrm{im}(\pi_{X,f} : X_f \to X^f)$

(ii) *Let B be the completion of A in the fA-adic topology and let B' be the image of B in B_f. Then*

$$\begin{aligned} \mathscr{O}_{X^f}(X^f_\iota) &= B_f, \textit{ equipped with the topology such that } B' \textit{ is a ring} \\ &\qquad \textit{of definition and } fB' \textit{ is an ideal of definition} \\ \mathscr{O}^+_{X^f}(X^f_\iota) &= \textit{integral closure of } B \textit{ in } B_f. \end{aligned}$$

Hence X^f_ι is the analytic adic space associated with the formal completion of the scheme Spec A *along its closed subset* $V(f)$.

Proof. i) We use the notations of the proof of Proposition 3.1 (in particular, f is a topologically nilpotent unit of A^f). By the proof of Proposition 3.1 we have

$$\operatorname{im}(\pi_{X,f}) = \bigcup_{n\in\mathbb{N}} \{x \in X^f \mid |s^n(x)| \leqslant |f(x)|\}$$

which is the interior of $\{x \in X^f \mid |s(x)| < 1\}$ in X^f ([11], Lemma 1.3.ii).
ii) is obvious. □

For an analytic adic space Y and an element t of $\mathscr{O}_Y(Y)$ and finite subsets D, E of $\mathscr{O}_Y(Y)$ with $(D \cup E \cup \{t\}) \cdot \mathscr{O}_Y(Y) = \mathscr{O}_Y(Y)$ we put

$$\begin{aligned} S_Y(\frac{D|E}{t}) \quad := \quad & \{y \in Y \mid |d(y)| \leqslant |t(y)| \text{ for every } d \in D \\ & \text{and } |e(y)| < |t(y)| \text{ for every } e \in E\}. \end{aligned}$$

Then $S_Y(\frac{D|E}{t})$ is a locally closed constructible subset of Y. If $E = \emptyset$ (resp. $D = \emptyset$) then $S_Y(\frac{D|E}{t})$ is open (resp. closed) ([9], 3.1).

Proposition 3.3. — *There is a mapping*

$$\sigma_{X,f} : X^f_\varepsilon \to X$$

such that

(i) *For any element t of A and finite subsets D, E of A with $(D \cup E \cup \{t\}) \cdot A = A$,*

$$\sigma^{-1}_{X,f}(S_X(\frac{D|E}{t})) = S_{X^f}(\frac{D|E}{t}) \cap X^f_\varepsilon.$$

(ii) *The mapping $\sigma_{X,f}$ is spectral, i.e., $\sigma_{X,f}$ is continuous and for any quasi-compact open subset U of X the preimage $\sigma^{-1}_{X,f}(U)$ is quasi-compact.*

(iii) *The composite map $\sigma_{X,f} \circ \pi_{X,f} : X_f \to X$ is the inclusion of X_f into X.*

(iv) *$\sigma_{X,f}$ is functorial in X, i.e., for any morphism $r : S \to T$ of affinoid analytic adic spaces and for any $g \in \mathscr{O}_T(T)$ the diagram*

$$\begin{array}{ccc} (S^{r^*(g)})_\varepsilon & \xrightarrow{\sigma_{S,r^*(g)}} & S \\ \downarrow{\scriptstyle r^g} & & \downarrow{\scriptstyle r} \\ (T^g)_\varepsilon & \xrightarrow[\sigma_{T,g}]{} & T \end{array}$$

commutes.

The mapping $\sigma_{X,f}$ is uniquely determined by (i). If A has a noetherian ring of definition that is contained in A^+ then $\sigma_{X,f}$ is uniquely determined by (ii) and (iii). The family $(\sigma_{X,f})_{X,f}$ of all $\sigma_{X,f}$ is uniquely determined by (ii),(iii),(iv).

Proof. Let s be a topologically nilpotent unit of A. We consider the valuation spectrum $\operatorname{Spv} A$ of A and the subset

$$W := \{v \in \operatorname{Spv} A \mid v(a) \leqslant 1 \text{ for every } a \in A^+ \text{ and } v(s) < 1\}.$$

According to [7] we have

$$X = \operatorname{Spa}(A, A^+) = \{v \in W \mid \Gamma_v = c\Gamma_v\}$$

and we have the retraction

$$r : W \to X,\ v \mapsto v|c\Gamma_v.$$

The ring homomorphism $\rho : A \to A_f$ induces the mapping

$$p := \operatorname{Spv}(\rho) : \operatorname{Spv} A_f \to \operatorname{Spv} A.$$

Since $X^f_\varepsilon \subseteq p^{-1}(W)$, we get the mapping

$$\sigma_{X,f} := r \circ p : X^f_\varepsilon \to X.$$

For any element t of A and finite subsets D, E of A with $(D \cup E \cup \{t\}) \cdot A = A$, the subset of $\operatorname{Spv} A$

$$\begin{aligned}&\{v \in \operatorname{Spv} A \mid v(d) \leqslant v(t) \text{ for every } d \in D\\ &\text{and } v(e) < v(t) \text{ for every } e \in E\}\end{aligned}$$

is closed under primary specializations and primary generalizations in $\operatorname{Spv} A$. Hence

$$\sigma^{-1}_{X,f}(S_X(\frac{D|E}{t})) = S_{X^f}(\frac{D|E}{t}) \cap X^f_\varepsilon\ ,$$

i.e., (i) holds. (ii) follows from (i). (iii) and (iv) are easily checked.
For any $x \in X$, the set $\{x\}$ is the intersection of all $S_X(\frac{D|E}{t})$ which contain x. Hence $\sigma_{X,f}$ is uniquely determined by (i).
The constructible topology of a spectral space is hausdorff. So the constructible topology of X is hausdorff. If A has a noetherian ring of definition that is contained in A^+ (and so $A^+ = A^\circ$, [13], 2.4.16) then the set of all maximal points of X^f is dense in the constructible topology of X^f ([7], Lemma 3.4), in particular $(X^f_\varepsilon)^\circ$ is dense in the constructible topology of X^f_ε. By Proposition 3.2(i) we have $(X^f_\varepsilon)^\circ = \operatorname{im}(\pi_{X,f})$. Therefore $\sigma_{X,f}$ is uniquely determined by (ii) and (iii) if A has a noetherian ring of definition that is contained in A^+.
Let S be as in (iv) and let h be an element of $\mathscr{O}_S(S)$. Let $(r_i : S \to T_i)_{i \in I}$

be the family of all morphisms from S to an affinoid analytic adic space T_i such that $\mathscr{O}_{T_i}(T_i)$ has a noetherian ring of definition contained in $\mathscr{O}^+_{T_i}(T_i)$ and there is some $t_i \in \mathscr{O}_{T_i}(T_i)$ with $h = r_i^*(t_i)$. Any $x \in S$ is uniquely determined by the family $(r_i(x))_{i\in I}$ (i.e., if x, y are elements of S with $r_i(x) = r_i(y)$ for all $i \in I$ then $x = y$). Hence the family of all $\sigma_{X,f}$ is uniquely determined by (ii),(iii),(iv). □

Lemma 3.4. — *Put* $U := \{x \in \operatorname{Spec} A \mid f(x) \neq 0\}$ *and* $V := \{x \in X \mid f(x) = 0\}$.

(i) *The mapping* $\sigma = \sigma_{X,f} : X^f_\varepsilon \to X$ *is generalizing, specializing and closed.*

(ii) *If* $i : Y \hookrightarrow X$ *is the closed adic subspace of* X *corresponding to the scheme-theoretic closure of* U *in* $\operatorname{Spec} A$ *then* $i^f : Y^{i^*(f)} \to X^f$ *is an isomorphism.*

(iii) $\sigma^{-1}(V) = X^f_\varepsilon - (X^f_\varepsilon)^\circ$. *If* U *is dense in* $\operatorname{Spec} A$ *then* $\sigma(X^f_\varepsilon - (X^f_\varepsilon)^\circ) = V$ *(and hence* $\sigma : X^f_\varepsilon \to X$ *is surjective).*

Proof. We use the notations of the proofs of Proposition 3.1 and Proposition 3.3.

i) Let x be an element of X^f_ε and let v (resp. w) be a specialization (resp. generalization) of $\sigma(x)$. We consider $\sigma(x), v, w$ as elements of $\operatorname{Spv} A$. Then v (resp. w) is a secondary specialization (resp. secondary generalization) of $\sigma(x)$ in $\operatorname{Spv} A$, and $p(x) \in \operatorname{Spv} A$ is a primary generalization of $\sigma(x)$. Hence there exist $v', w' \in \operatorname{Spv} A$ such that v' (resp. w') is a primary generalization of v (resp. w) and a secondary specialization (resp. secondary generalization) of $p(x)$ ([12], Lemma 1.2.5.ii,iv). Let v'' and w'' be the elements of $\operatorname{Spv} A_f$ with $p(v'') = v'$ and $p(w'') = w'$. Then $v'', w'' \in X^f_\varepsilon$ and v'' (resp. w'') is a specialization (resp. generalization) of x in X^f_ε and $\sigma(v'') = v$ and $\sigma(w'') = w$. This shows that σ is specializing and generalizing. As σ is spectral (Proposition 3.3(ii)), we obtain that σ is closed.

ii) is obvious

iii) If $x \in X^f_\varepsilon - (X^f_\varepsilon)^\circ$ then $|f(x)| < |s(x)|^n$ for every $n \in \mathbb{N}$ (by the proof of Proposition 3.2) and hence $|f(\sigma(x))| < |s(\sigma(x))|^n$ for every $n \in \mathbb{N}$ which implies $f(\sigma(x)) = 0$, i.e. $\sigma(x) \in V$.

Assume that U is dense in $\operatorname{Spec} A$. Let x be an element of V. We show that there exists some $y \in X^f_\varepsilon$ with $x = \sigma(y)$. Since U is dense in $\operatorname{Spec} A$, there exists a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \subseteq \operatorname{supp}(x)$ and $f \notin \mathfrak{p}$. By [12], Lemma 1.2.6 there exists a primary generalization z of x in $\operatorname{Spv} A$ with $\mathfrak{p} = \operatorname{supp}(z)$. Then for $z(f) \in \Gamma_z \cup \{0\}$ we have $z(f) \neq 0$ and $z(f) < c\Gamma_z$. Let H be the smallest convex subgroup of Γ_z which contains $z(f)$. Then $c\Gamma_z \subseteq H$ and so we have the primary specialization $y := z|H \in \operatorname{Spv} A$.

Since $y(f) \neq 0$, we can consider y as an element of $\operatorname{Spv} A_f$. It is easily seen that $y \in X^f_\varepsilon$ and $x = \sigma(y)$. $\square$

Example 3.5. — Assume that A is a Dedekind ring and $f \neq 0$. We want to describe the topological space X^f.
For this we use the following natural inclusions of topological spaces

$$X^f \subseteq \operatorname{Spv} A_f \subseteq \operatorname{Spv} A \supseteq X.$$

So we consider X^f and X as topological subspaces of $\operatorname{Spv} A$. The elements of $\operatorname{Spv} A$ are written as pairs $(\mathfrak{p}, B)$ where $\mathfrak{p}$ is a prime ideal of A and B is a valuation ring of the quotient field $qf(A/\mathfrak{p})$. Put

$$\begin{aligned} W &:= \{x \in \operatorname{Spec} A \mid f(x) = 0\} = \{\mathfrak{m} \in \operatorname{Max} A \mid f \in \mathfrak{m}\} \\ V &:= \{x \in X \mid f(x) = 0\} = \{(\mathfrak{m}, B) \in X \mid \mathfrak{m} \in W\}. \end{aligned}$$

If $\mathfrak{m}$ is a maximal ideal of A and B is a valuation ring of $A/\mathfrak{m}$ then $(\mathfrak{m}, B) \in \operatorname{Spv} A$ is an element of X if and only if the image of A^+ in $A/\mathfrak{m}$ is contained in B and, for some (and then for any) topologically nilpotent unit s of A, the image of s in $A/\mathfrak{m}$ is contained in the maximal ideal of B.
We divide X^f into the three subsets $(X^f_\varepsilon)^\circ$, X^f_ι, $R := X^f_\varepsilon - (X^f_\varepsilon)^\circ = (X^f_\iota)^- - X^f_\iota$ where $(X^f_\iota)^-$ denotes the closure of X^f_ι in X^f.
a) By Proposition 3.1 and 3.2(i) the adic spaces $(X^f_\varepsilon)^\circ$ and X_f are isomorphic. Identifying X^f and X as topological subspaces of $\operatorname{Spv} A$, we get the equality

$$(X^f_\varepsilon)^\circ = X - V.$$

b) By Proposition 3.2(ii) the mapping

$$\psi : W \to X^f_\iota,\ \mathfrak{m} \mapsto (\{0\}, A_\mathfrak{m})$$

is a homeomorphism. So X^f_ι is a finite discrete topological space. For every $\mathfrak{m} \in W$, the subset $\{\psi(\mathfrak{m})\}$ of X^f is rational and $\mathscr{O}_{X^f}(\{\psi(\mathfrak{m})\}) = qf(\hat{A}^\mathfrak{m})$ and $\mathscr{O}^+_{X^f}(\{\psi(\mathfrak{m})\}) = \hat{A}^\mathfrak{m}$ where $\hat{A}^\mathfrak{m}$ denotes the completion of A in the $\mathfrak{m}$-adic topology.
c) For a maximal ideal $\mathfrak{m}$ of A and a valuation ring B of $A/\mathfrak{m}$, let $P(\mathfrak{m}, B)$ denote the preimage of B under the mapping $A_\mathfrak{m} \to A/\mathfrak{m}$. Then $P(\mathfrak{m}, B)$ is a valuation ring of $qf(A)$. The mapping

$$V \to R,\ (\mathfrak{m}, B) \mapsto (\{0\}, P(\mathfrak{m}, B))$$

is a homeomorphism.
d) The restriction of the mapping $\sigma_{X,f} : X^f_\varepsilon \to X$ to R is a homeomorphism from R onto V, namely it is the inverse mapping of the bijection in (c). Then $\sigma_{X,f} : X^f_\varepsilon \to X$ is bijective. As $\sigma_{X,f}$ is closed, we obtain that $\sigma_{X,f} : X^f_\varepsilon \to X$ is a homeomorphism.

Remark 3.6. — Let B be a Tate ring (not necessarily complete), let f be an element of B and let B^+ be a ring of integral elements of B. According to Section 2 we have the Tate ring B^f. The integral closure $(B^+)^c$ of B^+ in B_f is a ring of integral elements of B^f. Assume that the completions $B^\wedge$ and $(B^f)^\wedge$ of B and B^f are strictly noetherian. Then we have the adic spaces

$$\begin{aligned} Y &:= \operatorname{Spa}(B, B^+) \\ Z &:= \operatorname{Spa}(B^f, (B^+)^c). \end{aligned}$$

(The natural morphism $(B^f)^\wedge \to (B^\wedge)^f$ is not an isomorphism in general (see Remark 2.2(i)), and hence the natural morphism of adic spaces $Y^f \to Z$ is not an isomorphism in general.)
Let s be a topologically nilpotent unit of B and put

$$\begin{aligned} Z_\varepsilon &:= \{z \in Z \mid |s(z)| < 1\} \\ Z_\iota &:= \{z \in Z \mid |s(z)| \geqslant 1\} = \{z \in Z \mid |s(z)| = 1\}. \end{aligned}$$

The sets Z_ε and Z_ι are independent of s. As above (i.e., as in the case that B is complete) one can define a morphism of adic spaces

$$\pi : Y - V(f) \to Z$$

and a mapping

$$\sigma : Z_\varepsilon \to Y$$

for which (3.1)-(3.5) hold analogously.

4. Some constructible sheaves

Let A be a noetherian ring.

Definition 4.1. — For an analytic pseudo-adic space (X, L) with X affinoid, $\mathscr{X}(X, L)$ denotes the class of all A-modules F on $(X, L)_{\text{ét}}$ which satisfy the following equivalent conditions ($Y := \operatorname{Spec} \mathscr{O}_X(X)$)

(i) For every $y \in Y$ there exist a morphism of schemes $f : S \to Y$ and a constructible A-module G on $S_{\text{ét}}$ such that f is of finite type, $y \in f(S)$ and the restrictions of F and G to the étale site of $(X, L) \times_Y S$ are isomorphic.

(ii) For every $y \in Y$ there exists a morphism of schemes $f : S \to Y$ such that f is of finite type, $y \in f(S)$ and the restriction of F to the étale site of $(X, L) \times_Y S$ is a constant A-module of finite type.

(iii) For every $y \in Y$ there exist a locally closed subscheme R of Y and a surjective finite étale morphism of schemes $S \to R$ such that $y \in R$ and the restriction of F to the étale site of $(X, L) \times_Y S$ is a constant A-module of finite type.

(iv) There exist a decreasing sequence of closed adic subspaces of X

$$X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n = \emptyset$$

and, for every $i \in \{0, \ldots, n-1\}$, a finite morphism of adic spaces $f_i : T_i \to X_i$ such that $f_i^{-1}(X_i - X_{i+1}) \to X_i - X_{i+1}$ is surjective and étale and the restriction of F to the étale site of $f_i^{-1}(L \cap (X_i - X_{i+1}))$ is a constant A-module of finite type.

(The equivalence of (ii) and (iii) follows from [5], 17.16.4).

Definition 4.2. — For an analytic pseudo-adic space (X, L), let $\mathscr{C}(X, L)$ denote the class of all A-modules F on $(X, L)_{\text{ét}}$ such that, for every $x \in L$, there exist a locally closed locally constructible subset P of L with $x \in P$ and a surjective étale morphism of pseudo-adic spaces $(Y, M) \to (X, P)$ with Y affinoid such that the restriction of F to the étale site of (Y, M) is an element of $\mathscr{Z}(Y, M)$.

If (X, L) is a pseudo-adic space then an A-module F on $(X, L)_{\text{ét}}$ is called constructible if, for every $x \in L$, there exists a locally closed locally constructible subset P of L such that $x \in P$ and the restriction of F to the étale site of (X, P) is a locally constant A-module finite type ([9], 2.7) Every constructible A-module on $(X, L)_{\text{ét}}$ is an element of $\mathscr{C}(X, L)$.

Example 4.3. — Let k be a non-archimedean field, let T be a 1-dimensional normal affinoid adic space of finite type over $\mathrm{Spa}(k, k^\circ)$ and let f be an element of $\mathscr{O}_T(T)$. According to Section 3 we have the analytic adic space $X := T^f$ (cf. Example 3.5). Let L be a convex locally pro-constructible subset of X. An A-module F on $(X, L)_{\text{ét}}$ is an element of $\mathscr{C}(X, L)$ if and only if the following two conditions are satisfied

(i) For every $x \in L$ such that the support $\mathrm{supp}(x) := \{s \in \mathscr{O}_X(X) \mid s(x) = 0\} \in \mathrm{Spec}\, \mathscr{O}_X(X)$ is a generic point of $\mathrm{Spec}\, \mathscr{O}_X(X)$ there exists a locally closed constructible subset P of L such that $x \in P$ and $F|P$ is a locally constant A-module of finite type on $P_{\text{ét}}$.

(ii) For every $x \in L$ such that the support $\mathrm{supp}(x) \in \mathrm{Spec}\, \mathscr{O}_X(X)$ is not a generic point of $\mathrm{Spec}\, \mathscr{O}_X(X)$, the restriction of F to $\{x\}_{\text{ét}}$ is a locally constant A-module of finite type and there exist an open neighbourhood Y of x in X and a finite surjective morphism of adic spaces $g : Z \to Y$ such that g is étale over $Y - \{x\}$ and the restriction of F to $g^{-1}(Y \cap L - \{x\})_{\text{ét}}$ is a constant A-module of finite type.

In order to check property (ii) of Example 4.3 we will use in the proof of Theorem 5.2 the following criterion. This criterion is the reason why we introduced in Section 3 the adic spaces X^f.

LEMMA 4.4. — *Let X be an affinoid analytic adic space such that $B := \mathscr{O}_X(X)$ is normal, let L be a convex pro-constructible subset of X and let F be an A-module on $(X, L)_{\text{ét}}$. Let f be a non zero divisor of B and let x be an element of L such that the support $\operatorname{supp}(x) \in \operatorname{Spec} B$ is a generic point of $\{\mathfrak{p} \in \operatorname{Spec} B \mid f \in \mathfrak{p}\}$.*
According to Section 3 we have the mapping $\sigma : X^f_\varepsilon \to X$ and we consider $X - V(f)$ as an open subspace both of X and X^f. Let u be the element of X^f_ε with $\sigma(u) = x$. Then $\operatorname{supp}(u) \in \operatorname{Spec} B_f$ is a generic point of $\operatorname{Spec} B_f$.
Assume that there exist a locally closed constructible subset Q of $\sigma^{-1}(L)$ and a locally constant A-module G of finite type on $Q_{\text{ét}} = (X^f, Q)_{\text{ét}}$ such that $u \in Q$ and the restrictions of F and G to the étale site of $Q \cap (X - V(f))$ are isomophic.
Then there exist a locally closed constructible subset P of L with $x \in P$ and an étale morphism $h : Y \to X$ with $P \subseteq \operatorname{im}(h)$ and a surjective finite morphism $g : Z \to Y$ which is étale over $Y - V(f)$ such that the restriction of F to the étale site of $(g \circ h)^{-1}(P - V(f))$ is a constant A-module of finite type. If x is a maximal point of X (i.e., the valuation corresponding to x has rank 1) then one can choose $h : Y \to X$ as an open embedding and $P = h(Y) \cap L$.

Proof. We may assume that there exist an affinoid adic space V and an étale morphism $r : V \to X^f$ such that $Q \subseteq r(V)$ and the restriction of G to $r^{-1}(Q)$ is a constant A-module of finite type. Since there exists an affine scheme S étale over $T := \operatorname{Spec} B_f$ such that V is an open subspace of $S \times_T X^f$ ([9], 1.7.3), there exists a finite morphism $s : W \to X$ such that V is an open subspace of $W^{s^*(f)}$. We may assume that $\mathscr{O}_W(W)$ is normal. Let v be an element of V with $r(v) = u$. Since $u \in X^f_\varepsilon$, we have $v \in W^{s^*(f)}_\varepsilon$. The mapping $\sigma_W : W^{s^*(f)}_\varepsilon \to W$ is closed (Lemma 3.4(i)). Furthermore, for $z := \sigma_W(v) \in W$ we have $\sigma_W^{-1}(z) = \{v\}$. Hence there exists an open subset Z of W such that $z \in Z$ and $\sigma_W^{-1}(Z) \subseteq V$. The mapping $s : W \to X$ is open (even universally open, [13], 3.4.7 and [12], 2.1.6). Hence $Y := s(Z)$ is an open neighbourhood of $s(z) = x$ in X. Let $g : Z \to Y$ be the restriction of s

and let P be a locally closed constructible subset of L such that $x \in P \subseteq Y$ and $\sigma^{-1}(P) \subseteq Q$. Since $G|r^{-1}(Q)$ is a constant A-module of finite type and since $G|Q\cap(X-V(f)) \cong F|Q\cap(X-V(f))$, we obtain that $F|g^{-1}(P-V(f))$ is a constant A-module of finite type. If x is a maximal point of X then there exists an open neighbourhood U of x in Y such that $g^{-1}(U) \to U$ is finite. For an arbitrary $x \in X$, replacing (Y, x) by an étale neighbourhood of (Y, x) we may assume that $g : Z \to Y$ is finite ([11], 3.2). □

Proposition 4.5. — *Let $f : X \to Y$ be a finite morphism between affinoid analytic adic spaces. Let M be a convex locally pro-constructible subset of Y and put $L := f^{-1}(M)$. Let $g : (X, L) \to (Y, M)$ be the morphism of pseudo-adic spaces induced by f.*

(i) *If $F \in \mathscr{Z}(X, L)$ then $g_*F \in \mathscr{Z}(Y, M)$.*
(ii) *Assume that $M \subseteq f(X)$. If F is an A-module on $(Y, M)_{\text{ét}}$ with $g^*F \in \mathscr{Z}(X, L)$ then $F \in \mathscr{Z}(Y, M)$.*

Proof. (i) can be proved as the corresponding statement for constructible sheaves on schemes, cf. [4], Lemma 4.11.
(ii) follows immediately from Definition 4.1. □

Proposition 4.6. — *Let $f : X \to Y$ be a separated quasi-finite morphism of finite type between analytic adic spaces. Let M be a convex locally pro-constructible subset of Y and let L be a locally closed constructible subset of $f^{-1}(M)$. Let $g : (X, L) \to (Y, M)$ be the morphism of pseudo-adic spaces induced by f.*

(i) *If $F \in \mathscr{C}(X, L)$ then $g_!F \in \mathscr{C}(Y, M)$ and $R^n g_! F = 0$ for $n > 0$.*
(ii) *Assume that $f(L) = M$. If F is an A-module on $(Y, M)_{\text{ét}}$ with $g^*F \in \mathscr{C}(X, L)$ then $F \in \mathscr{C}(Y, M)$.*

Proof. i) By [9], 5.5.6, $R^n g_! F = 0$ for $n > 0$. We fix some $y \in M$ and show that there exist a locally closed locally constructible subset P of M with $y \in P$ and a surjective étale morphism of pseude-adic spaces $(Y', P') \to (Y, P)$ with Y' affinoid such that the restriction of $g_!F$ to (Y', P') is an element of $\mathscr{Z}(Y', P')$.
We may assume that Y is affinoid and M is convex and pro-constructible in Y. First we show

(∗) We may assume that $L = f^{-1}(M)$ and that there exists an étale morphism $h : Z \to X$ such that Z is affinoid and $L \subseteq h(Z)$ and $F \,|\, (Z, h^{-1}(L)) \in \mathscr{Z}(Z, h^{-1}(L))$.

Proof of (∗). The topological space $f^{-1}(y)$ is finite and discrete. For every $x \in f^{-1}(y) \cap L$ we fix a quasi-compact open subset U_x of X and a locally closed constructible subset L_x of $U_x \cap L$ and a surjective étale morphism $h_x : Z_x \to U_x$ with Z_x affinoid such that $x \in L_x$ and $f^{-1}(y) \cap U_x = \{x\}$ and $F \mid (Z_x, h_x^{-1}(L_x)) \in \mathscr{Z}(Z_x, h_x^{-1}(L_x))$. Let Q be a locally closed constructible subset of M such that $y \in Q$ and $f^{-1}(Q) \cap L \subseteq \bigcup_{x \in f^{-1}(y) \cap L} U_x$ and $f^{-1}(Q) \cap U_x \subseteq L_x$ (for all $x \in f^{-1}(y) \cap L$) and $f^{-1}(Q) \cap U_x \cap U_{x'} = \emptyset$ (for all $x, x' \in f^{-1}(y) \cap L$ with $x \neq x'$). Then we may replace

- M by Q
- X by $X' := \coprod_{x \in f^{-1}(y) \cap L} U_x$
- f by the morphism $f' : X' \to Y$ with $f'|U_x = f|U_x$
- L by $L' := f'^{-1}(Q)$
- g by the morphism $g' : (X', L') \to (Y, Q)$ induced by f'.

Put $Z := \coprod_{x \in f^{-1}(y) \cap L} Z_x$ and let $h : Z \to X'$ be the morphism with $h|Z_x = h_x$. Then h is étale and $L' \subseteq h(Z)$ and Z is affinoid and $F \mid (Z, h^{-1}(L')) \in \mathscr{Z}(Z, h^{-1}(L'))$. Thus $(*)$ is shown.
Replacing (Y, y) by an étale neighbourhood of (Y, y) and replacing M by a locally closed constructible subset of M containing y, we may assume that the quasi-finite morphisms $f : X \to Y$ and $f \circ h : Z \to Y$ are finite ([11], 3.2). Then h is finite. Then Proposition 4.5 gives that $F \in \mathscr{Z}(X, L)$ and $g_! F \in \mathscr{Z}(Y, M)$.
ii) Let y be an element of M. There exist a separated quasi-finite morphism of finite type $p : Z \to Y$ and a locally closed locally constructible subset Q of M with $y \in Q$ such that $Q \subseteq p(Z)$ and $F \mid (Z, p^{-1}(Q)) \in \mathscr{Z}(Z, p^{-1}(Q))$. (Indeed, for the A-module $g^* F \in \mathscr{C}(X, L)$ construct X', Z, f', h as in the proof of $(*)$ above and put $p := f' \circ h : Z \to Y$). Replacing (Y, y) by an étale neighbourhood of (Y, y) and replacing Q by a locally closed constructible subset of Q containing y, we may assume that $p : Z \to Y$ is finite ([11], 3.2). Then $F|(Y, Q) \in \mathscr{Z}(Y, Q)$ by Proposition 4.5. □

Proposition 4.7. — *Let k be a non-archimedean field, let T be an affinoid adic space of finite type over* $\mathrm{Spa}(k, k^\circ)$ *with* $\dim T \leqslant 1$, *let $f \in \mathscr{O}_T(T)$ and let L be a locally closed constructible subset of $X := T^f$.*

(i) *If $F, G \in \mathscr{C}(X, L)$ and $h : F \to G$ is a morphism of A-modules then* $\ker(h)$,$\mathrm{im}(h)$,$\mathrm{coker}(h) \in \mathscr{C}(X, L)$.
(ii) *If $0 \to F \to G \to H \to 0$ is an exact sequence of A-modules on $(X, L)_{\text{ét}}$ and $F, H \in \mathscr{C}(X, L)$ then $G \in \mathscr{C}(X, L)$.*

Proof. By Proposition 4.6(ii) we may assume that T is normal. Then we can use Example 3.5 and 4.3, and the assertion is easily seen. □

In [11] we defined, for an adic space X of finite type over $\mathrm{Spa}(k,k^\circ)$ with k a non-archimedean field of characteristic zero and for a finite ring A, quasi-constructible A-modules on $X_{\text{ét}}$. Obviously, every element of $\mathscr{C}(X)$ is quasi-constructible. Riemann's existence theorem ([16], [18]) implies that if $\dim(X)=1$ then every quasi-constructible A-module on $X_{\text{ét}}$ is an element of $\mathscr{C}(X)$.

5. Results

We fix a non-archimedean field k and a noetherian torsion ring A with torsion prime to $\mathrm{char}(k^\circ/k^{\circ\circ})$.

The following two theorems are the main results of this paper.

THEOREM 5.1. — *Let (Y,M) be a quasi-compact pseudo-adic space such that $Y=T^f$ for some affinoid adic space T of finite type over $\mathrm{Spa}(k,k^\circ)$ with $\dim T \leqslant 1$ and for some $f\in \mathscr{O}_T(T)$. Let $g:(X,L)\to(Y,M)$ be a morphism of pseudo-adic spaces such that the morphism of adic spaces $\underline{g}:X\to Y$ is of finite type and separated and L is a locally closed constructible subset of $\underline{g}^{-1}(M)$. Let $F\in\mathscr{C}(X,L)$.*
Then, for every $m\in\mathbb{N}_0$, the A-module $R^m g_! F$ is generically constructible on $(Y,M)_{\text{ét}}$, i.e., there exists an open subset U of M such that the restriction $R^m g_! F|U$ is constructible on U and every $x\in M$ whose support $\mathrm{supp}(x)=\{c\in\mathscr{O}_Y(Y)\mid c(x)=0\}\in\mathrm{Spec}\,\mathscr{O}_Y(Y)$ is a generic point of $\mathrm{Spec}\,\mathscr{O}_Y(Y)$ is contained in U.

Theorem 5.1 will be proved in Section 7.

THEOREM 5.2. — *Let (Y,M) be as in Theorem 5.1. Let $S\to\mathrm{Spec}\,\mathscr{O}_Y(Y)$ be a separated scheme of finite type over $\mathrm{Spec}\,\mathscr{O}_Y(Y)$, let $X:=S\times_{\mathrm{Spec}\,\mathscr{O}_Y(Y)} Y$ be the adic space over Y associated with the scheme S over $\mathrm{Spec}\,\mathscr{O}_Y(Y)$, and let $\underline{g}:X\to Y$ be the projection. Let L be a quasi-compact locally closed constructible subset of $\underline{g}^{-1}(M)$ and let $g:(X,L)\to(Y,M)$ be the morphism of pseudo-adic spaces induced by $\underline{g}$. Let G be a constructible A-module on $S_{\text{ét}}$ and let F be the pullback of G on $(X,L)_{\text{ét}}$ under the natural morphism of sites $(X,L)_{\text{ét}}\to S_{\text{ét}}$.*
Then, for every $m\in\mathbb{N}_0$, $R^m g_! F\in\mathscr{C}(Y,M)$.

For the proof of Theorem 5.2 we may assume that Y is normal (Proposition 4.6(ii)). Furthermore we may assume that M is constructible in Y. (Even more, we may assume that $M = Y$. Indeed, let L' be a quasi-compact locally closed constructible subset of X with $L' \cap \underline{g}^{-1}(M) = L$ and consider $(X, L') \to (Y, Y)$ instead of $(X, L) \to (Y, M)$). Let $y \in M$. If $\mathrm{supp}(y)$ is a generic point of $\mathrm{Spec}\,\mathscr{O}_Y(Y)$ then by Theorem 5.1 there exists a locally closed constructible subset P of M such that $y \in P$ and $(R^m g_! F)|P$ is a locally constant A-module of finite type. If $\mathrm{supp}(y)$ is not a generic point of $\mathrm{Spec}\,\mathscr{O}_Y(Y)$ then $(R^m g_! F)|\{y\}$ is a locally constant A-module of finite type (apply Theorem 5.1 to the morphism $X \times_Y \mathrm{Spa}\,\kappa(y) \to \mathrm{Spa}\,\kappa(y)$ with $\kappa(y) := (k(y), k(y)^+)$). Then Theorem 5.2 follows from the following lemma.

Lemma 5.3. — *In the situation of Theorem 5.2 assume that Y is normal and M is constructible in Y. Let $y \in M$ such that* $\mathrm{supp}(y)$ *is not a generic point of* $\mathrm{Spec}\,\mathscr{O}_Y(Y)$. *Then there exist an open neighbourhood W of y in Y and a surjective finite morphism $q : Z \to W$ which is étale over $W - \{y\}$ such that, for every $m \in \mathbb{N}_0$, the restriction of $R^m g_! F$ to $q^{-1}(W \cap M - \{y\})$ is a constant A-module of finite type.*

Proof. Put $(B, B^+) := (\mathscr{O}_Y(Y), \mathscr{O}_Y^+(Y))$. By Proposition 4.7 we my assume that $S = \mathrm{Spec}\,E$ is affine. Let $\varphi : B \to E$ be the ring homomorphism corresponding to the morphism of schemes $S \to \mathrm{Spec}\,\mathscr{O}_Y(Y)$. We can equip E with the structure of an affinoid ring, (E, E^+), such that $\varphi : (B, B^+) \to (E, E^+)$ is a continuous morphism of affinoid rings which is of algebraically finite type (i.e., there exists a finite subset L of E such that E^+ is the integral closure of $B^+[L]$ in E and, for every ring of definition B_0 of B, $B_0[L]$ is a ring of definition of E) and such that $\mathrm{Spa}\,(E, E^+)$ is an open subspace of X containing L.
Let r be a non zero divisor of B contained in $\mathrm{supp}(y)$. According to Section 2 and 3 we have the affinoid rings $\tilde{B} := (B^r, (B^+)^c)$ and $\tilde{E} := (E^{\varphi(r)}, (E^+)^c)$ where $(B^+)^c$ and $(E^+)^c$ are the integral closures of B^+ and E^+ in B_r and $E_{\varphi(r)}$. The morphism of affinoid rings $\tilde{\varphi} : \tilde{B} \to \tilde{E}$ induced by φ is of algebraically finite type and hence the completion $\tilde{\varphi}^\wedge : \tilde{B} \to \tilde{E}^\wedge$ is of topologically finite type. Since by Proposition 2.5 the Tate ring B^r is stricty noetherian, we get that the Tate ring $(E^{\varphi(r)})^\wedge$ is strictly noetherian, and so we have the adic spaces $\tilde{Y} := \mathrm{Spa}\,\tilde{B} = Y^r$ and $\tilde{X} := \mathrm{Spa}\,\tilde{E}$. Let $\sigma_2 : \tilde{Y}_\varepsilon \to Y$ and $\sigma_1 : \tilde{X}_\varepsilon \to \mathrm{Spa}\,(E, E^+)$ be the mappings from Proposition 3.3 and Remark 3.6. Put

$$\begin{aligned} \underline{\tilde{g}} &:= \mathrm{Spa}\,(\tilde{\varphi}) : \tilde{X} \to \tilde{Y} \\ \tilde{L} &:= \sigma_1^{-1}(L) \subseteq \tilde{X}_\varepsilon \subseteq \tilde{X} \\ \tilde{M} &:= \sigma_2^{-1}(M) \subseteq \tilde{Y}_\varepsilon \subseteq \tilde{Y}. \end{aligned}$$

Then $\tilde{M}$ is a locally closed constructible subset of $\tilde{Y}$ and $\tilde{L}$ is a locally closed constructible subset of $\underline{\tilde{g}}^{-1}(\tilde{M})$. Let

$$\tilde{g} : (\tilde{X}, \tilde{L}) \to (\tilde{Y}, \tilde{M})$$

be the morphism of pseudo-adic spaces induced by $\underline{\tilde{g}}$. Let $\tilde{F}$ be the pullback of G under the natural morphism of sites $(\tilde{X}, \tilde{L})_{\text{ét}} \to (\operatorname{Spec} E)_{\text{ét}} = S_{\text{ét}}$ (where G is as in Theorem 5.2). According to Proposition 3.1 and Remark 3.6 we consider $Y - V(r)$ and $\operatorname{Spa}(E, E^+) - V(\varphi(r))$ as open subspaces of $\tilde{Y}$ and $\tilde{X}$. Then $\underline{\tilde{g}}^{-1}(Y - V(r)) = \operatorname{Spa}(E, E^+) - V(\varphi(r))$, $M \cap (Y - V(r)) = \tilde{M} \cap (Y - V(r))$, $L \cap (\operatorname{Spa}(E, E^+) - V(\varphi(r))) = \tilde{L} \cap (\operatorname{Spa}(E, E^+) - V(\varphi(r)))$ and $F \,|\, L \cap (\operatorname{Spa}(E, E^+) - V(\varphi(r))) = \tilde{F} \,|\, \tilde{L} \cap (\operatorname{Spa}(E, E^+) - V(\varphi(r)))$. Hence $(R^m g_! F) \,|\, M \cap (Y - V(r)) = (R^m \tilde{g}_! \tilde{F}) \,|\, \tilde{M} \cap (Y - V(r))$. By Theorem 5.1 the A-module $R^m \tilde{g}_! \tilde{F}$ on $(\tilde{Y}, \tilde{M})_{\text{ét}}$ is generically constructible. Now the assertion follows from Lemma 4.4. □

In the following we deduce some consequences from Theorem 5.1 and Theorem 5.2. These consequences concern results of [11] and [10] which were proved there for $\operatorname{char}(k) = 0$ and $|A| < \infty$ and which can be proved now without any restriction on $\operatorname{char}(k)$ and $|A|$.
For the rest of this section, k is assumed to be algebraically closed.

Theorem 5.1 with $Y = \operatorname{Spa}(k, k^\circ)$ says

Corollary 5.4. — *Let X be a separated adic space of finite type over $\operatorname{Spa}(k, k^\circ)$, let L be a locally closed constructible subset of X and let $F \in \mathscr{C}(X, L)$. Then, for every $m \in \mathbb{N}_0$, the A-module $H^m_c((X, L), F)$ is of finite type.*

The next corollary is a consequence of Corollary 5.4 (cf. proof of [11], Prop. 2.12).

Corollary 5.5. — *Let $\mathscr{X}$ be a formal scheme locally of finite type over $\operatorname{Spf} k^\circ$, let $d(\mathscr{X})$ be the analytic adic space associated with $\mathscr{X}$, let $\lambda : d(\mathscr{X})_{\text{ét}} \to \mathscr{X}_{\text{ét}}$ be the natural morphism of sites and let $F \in \mathscr{C}(d(\mathscr{X}))$. Then, for every $m \in \mathbb{N}_0$, the A-module $R^m \lambda_* F$ on $\mathscr{X}_{\text{ét}}$ is constructible.*

Corollary 5.6. — *Let X, L, F be as in Corollary 5.4 (in fact, it suffices that X is quasi-separated instead of separated). Then, for every $m \in \mathbb{N}_0$, the A-module $H^m((X, L), F)$ is of finite type.*

Proof. For a constructible A-module G on the étale site of a scheme Y of finite type over a separably algebraically closed field, the A-module $H^n(Y, G)$ is of finite type ([3], Th.finitude 1.10). Hence if $L = X$ then the assertion follows immediately from Corollary 5.5. The general case can obviously be reduced to this case. Namely, let U be a quasi-compact open subset of X such that L is closed in U and let $i : L \to U$ be the inclusion. Then $i_*F = i_!F \in \mathscr{C}(X,U) = \mathscr{C}(U)$ and $H^m((X,L),F) = H^m((X,U),i_*F) = H^m(U,i_*F)$. $\square$

5.7. — Let X be a separated adic space of finite type over $\mathrm{Spa}\,(k,k^\circ)$ and let L be a closed constructible subset of X. Then there exist finitely many quasi-compact open subsets $U_1,\dots,U_p$ of X and $f_{ij} \in \mathscr{O}_X(U_i)$ ($i = 1,\dots,p$, $j = 1,\dots,q(i)$) and $s^{ij}_{k\ell} \in \mathscr{O}_X(U_i \cap U_k)$ ($i,k = 1,\dots,p$, $j = 1,\dots,q(i)$, $\ell = 1,\dots,q(k)$) such that

- $X = U_1 \cup \ldots \cup U_p$
- $L \cap U_i = \{x \in U_i \mid |f_{ij}(x)| < 1 \text{ for } j = 1,\dots,q(i)\}$ $(i = 1,\dots,p)$
- $f_{ij}|U_i \cap U_k = \sum\limits_{\ell=1}^{q(k)} s^{ij}_{k\ell}\cdot(f_{k\ell}|U_i\cap U_k)$ $(i,k = 1,\dots,p,\ j = 1,\dots,q(i))$
- $|s^{ij}_{k\ell}(x)| \leqslant 1$ for all $x \in U_i \cap U_k$ $(i,k = 1,\dots,p,\ j = 1,\dots,q(i),$ $\ell = 1,\dots,q(k))$.

For every $\varepsilon \in |k^*|$ and $a \in k^*$ with $\varepsilon = |a|$, let L_ε be the subset of X such that for every $i \in \{1,\dots,p\}$

$$L_\varepsilon \cap U_i = \{x \in U_i \mid |f_{ij}(x)| \leqslant |a(x)| \text{ for } j = 1,\dots,q(i)\}.$$

Then L_ε is a quasi-compact open subset of X, and $L_\varepsilon \subseteq L_{\varepsilon'}$ for $\varepsilon \leqslant \varepsilon'$, and if L° denotes the interior of L in X then

$$L^\circ = \bigcup_{\substack{\varepsilon\in|k^*|,\\ \varepsilon<1}} L_\varepsilon.$$

From Theorem 5.1 one can deduce (cf. [11], Theorem 2.9)

Corollary 5.8. — *In the situation of* (5.7), *for every $F \in \mathscr{C}(X)$ there is an $\varepsilon_0 \in |k^*|, \varepsilon_0 < 1$ such that, for every $\varepsilon \in |k^*|$ with $\varepsilon_0 < \varepsilon < 1$ and every $n \in \mathbb{N}_0$, the natural mapping*

$$H^n_c(L_\varepsilon,F) \longrightarrow H^n_c(L^\circ,F)$$

is bijective.

5.9. — Let X be a separated scheme of finite type over $\mathrm{Spec}\,k$ and let $X^{\mathrm{ad}} := X \times_{\mathrm{Spec}\,k} \mathrm{Spa}\,(k,k^\circ)$ be the associated adic space over $\mathrm{Spa}\,(k,k^\circ)$. Let T be a closed subscheme of X. Then there exist finitely many open subsets $U_1,\dots,U_p$ of X and $f_{ij} \in \mathscr{O}_X(U_i)$ ($i = 1,\dots,p$, $j = 1,\dots,q(i)$) and $s^{ij}_{k\ell} \in \mathscr{O}_X(U_i \cap U_k)$ ($i,k = 1,\dots,p$, $j = 1,\dots,q(i)$, $\ell = 1,\dots,q(k)$) and constructible open subsets $W_1,\dots,W_p$ of X^{ad} such that

- $W_i \subseteq U_i^{\mathrm{ad}} \quad (i = 1, \dots, p)$
- $X^{\mathrm{ad}} = W_1 \cup \ldots \cup W_p, \;\; X = U_1 \cup \ldots \cup U_p$
- $T \cap U_i = V(f_{i1}, \dots, f_{iq(i)}) \;\; (i = 1, \dots, p)$
- $f_{ij}|U_i \cap U_k = \sum_{\ell=1}^{q(k)} s_{k\ell}^{ij} \cdot (f_{k\ell}|U_i \cap U_k) \;\; (i, k = 1, \dots, p, \; j = 1, \dots, q(i))$
- $|s_{k\ell}^{ij}(x)| \leqslant 1$ for all $x \in W_i \cap W_k \;\; (i, k = 1, \dots, p, \; j = 1, \dots, q(i),$ $\ell = 1, \dots, q(k))$.

For every $\varepsilon \in |k^*|$ and $a \in k^*$ with $\varepsilon = |a|$, let E_ε be the subset of X^{ad} such that for every $i \in \{1, \dots, p\}$

$$E_\varepsilon \cap W_i = \{x \in W_i \mid |f_{ij}(x)| < |a(x)| \text{ for } j = 1, \dots, q(i)\}.$$

Then E_ε is a closed constructible subset of X^{ad}, and $E_\varepsilon \subseteq E_{\varepsilon'}$ for $\varepsilon \leqslant \varepsilon'$, and

$$T^{\mathrm{ad}} = \bigcap_{\varepsilon \in |k^*|} E_\varepsilon.$$

Let L be a quasi-compact locally closed constructible subset of X^{ad}. Put

$$Z := T^{\mathrm{ad}} \cap L \quad , \quad U := L - Z$$
$$S_\varepsilon := E_\varepsilon \cap L \quad , \quad U_\varepsilon := L - S_\varepsilon.$$

From Theorem 5.2 (more precisely, from Lemma 5.3) one can deduce (cf. [11], 2.4-2.7)

Corollary 5.10. — *In the situation of* (5.9) *let F be a constructible A-module on $X_{\text{ét}}$ and let F^{ad} be the associated A-module on $(X^{\mathrm{ad}})_{\text{ét}}$. Then there is an $\varepsilon_0 \in |k^*|$ such that, for every $\varepsilon \in |k^*|$ with $\varepsilon \leqslant \varepsilon_0$, the following equivalent properties are satisfied*

(i) *For every $n \in \mathbb{N}_0$, $H_c^n(S_\varepsilon - Z, F^{\mathrm{ad}}) = 0$.*
(ii) *For every $n \in \mathbb{N}_0$, $H_c^n(S_\varepsilon, F^{\mathrm{ad}}) \xrightarrow{\sim} H_c^n(Z, F^{\mathrm{ad}})$.*
(iii) *For every $n \in \mathbb{N}_0$, $H_c^n(U_\varepsilon, F^{\mathrm{ad}}) \xrightarrow{\sim} H_c^n(U, F^{\mathrm{ad}})$.*

Corollary 5.4 and Corollary 5.10 easily imply (cf. [1], [10])

Corollary 5.11. — *Let R be a complete discrete valuation ring with* $\mathrm{char}(R/\mathfrak{m}_R) \neq \mathrm{char}(k^\circ/k^{\circ\circ})$ *and* $\mathrm{char}(R/\mathfrak{m}_R) > 0$. *In the situation of* (5.9), *for every constructible R.-module $(F_n)_{n \in \mathbb{N}}$ on $X_{\text{ét}}$ and every $q \in \mathbb{N}_0$ the natural mapping*

$$H_c^q(U, (F_n^{\mathrm{ad}})_{n \in \mathbb{N}}) \longrightarrow \varprojlim_n H_c^q(U, F_n^{\mathrm{ad}})$$

is bijective.

Applying Corollary 5.11 to a compactification of a k-scheme, we get the following comparison theorem (cf. [1], [10])

Corollary 5.12. — *Let R be as in* (5.11). *For every separated scheme X of finite type over k and every constructible $R.$-module $(F_n)_{n\in\mathbb{N}}$ on $X_{\text{ét}}$ and every $q \in \mathbb{N}_0$ there is a natural isomorphism*

$$H_c^q(X, (F_n)_{n\in\mathbb{N}}) \xrightarrow{\sim} H_c^q(X^{\mathrm{ad}}, (F_n^{\mathrm{ad}})_{n\in\mathbb{N}}).$$

6. GAGA

For the proof of Theorem 5.1 we will need the classical GAGA theorem for some projective curves over affinoid analytic adic spaces Y. The aim of this section is to explain a proof of this theorem. If $\mathscr{O}(Y)$ has a noetherian ring of definition then the GAGA theorem for proper schemes over Y is an immediate consequence of the GAGA theorem for formal completions of noetherian schemes in [5], III.5.1. But we can not restrict ourselves to the case that $\mathscr{O}(Y)$ has a noetherian ring of definition, since the non-archimedean field k in Theorem 5.1 is not assumed to be discretely valued. Our proof of the GAGA theorem follows ideas of [14], [15],[20].

For the whole section we fix an affinoid analytic adic space Y. For a scheme S locally of finite type over $\operatorname{Spec} \mathscr{O}(Y)$, let $S^{\mathrm{ad}} := S \times_{\operatorname{Spec} \mathscr{O}(Y)} Y$ be the adic space over Y associated with the scheme S over $\operatorname{Spec} \mathscr{O}(Y)$ and let $p : S^{\mathrm{ad}} \to S$ be the projection. For an $\mathscr{O}_S$-module F, the $\mathscr{O}_{S^{\mathrm{ad}}}$-module p^*F is denoted by F^{ad}.

Lemma 6.1. — *The morphism of locally ringed spaces $p : S^{\mathrm{ad}} \to S$ is flat.*

Proof. We may assume that $S = \mathbb{A}^n_{\operatorname{Spec} \mathscr{O}(Y)}$. In the proof of [9], 1.7.6 is shown that the ring homomorphism $\mathscr{O}(Y)[T_1, \dots, T_n] \subseteq \mathscr{O}(Y)\langle T_1, \dots, T_n\rangle$ is flat. Hence there is a covering $(U_i)_{i\in I}$ of S^{ad} by affinoid open subspaces U_i such that $p^* : \mathscr{O}(S) \to \mathscr{O}(U_i)$ is flat for all $i \in I$. For affinoid analytic adic spaces U, V where V is an open subspace of U, the ring homomorphism $\mathscr{O}(U) \to \mathscr{O}(V)$ is flat ([13], 3.3.8 or (II.1.iv) in the proof of [8], 2.5). Hence $p : S^{\mathrm{ad}} \to S$ is flat. □

Theorem 6.2. — *Let S be a projective* $\operatorname{Spec} \mathscr{O}(Y)$*-scheme.*
Then hold

(I_S) *For every coherent $\mathscr{O}_S$-module F and every $q \in \mathbb{N}_0$ the mapping $H^q(S, F) \to H^q(S^{\mathrm{ad}}, F^{\mathrm{ad}})$ is bijective.*

(II$_S$) *For all coherent $\mathscr{O}_S$-modules F, G the mapping* $\mathrm{Hom}_{\mathscr{O}_S}(F,G) \to \mathrm{Hom}_{\mathscr{O}_{S^{\mathrm{ad}}}}(F^{\mathrm{ad}}, G^{\mathrm{ad}})$ *is bijective.*

If for every coherent $\mathscr{O}_{S^{\mathrm{ad}}}$-module F the $\mathscr{O}(Y)$-module $H^1(S^{\mathrm{ad}}, F)$ is finitely generated then holds

(III$_S$) *For every coherent $\mathscr{O}_{S^{\mathrm{ad}}}$-module F there exists a coherent $\mathscr{O}_S$-module G such that F and G^{ad} are isomorphic.*

Proof. Köpf proves in [15] the GAGA theorem for proper schemes over affinoid rigid analytic spaces. The arguments of the proofs of 4.1, 4.7 and 4.11 in loc. cit. easily imply (I$_S$) and (II$_S$).
We show by noetherian induction on the set of all closed subschemes of S that (III$_{S'}$) holds for every closed subscheme S' of S. So we may assume that (III$_{S'}$) holds for every closed subscheme S' of S with $S' \neq S$.
Let F be a coherent $\mathscr{O}_{S^{\mathrm{ad}}}$-module. In order to show that there exists a coherent $\mathscr{O}_S$-module G with $F \cong G^{\mathrm{ad}}$ it suffices to show that there exists a surjective morphism of $\mathscr{O}_{S^{\mathrm{ad}}}$-modules $H^{\mathrm{ad}} \to F$ with H a coherent $\mathscr{O}_S$-module. There exists a largest sub-$\mathscr{O}_{S^{\mathrm{ad}}}$-module F' of F such that there exists a surjective morphism of $\mathscr{O}_{S^{\mathrm{ad}}}$-modules $H^{\mathrm{ad}} \to F'$ with H a coherent $\mathscr{O}_S$-module. Let T be the support of F/F'. We have to show that $T = \emptyset$.
For an invertible $\mathscr{O}_S$-module L and a section $f \in \Gamma(S, L)$, let $Z(f) \subseteq S$ denote the zero scheme of f. We have

(1) Let L be a very ample invertible sheaf on S relative to $\mathrm{Spec}\,\mathscr{O}(Y)$ and let $f \in \Gamma(S, L)$ such that $f^k \in \Gamma(S, L^k)$ is not the zero section for every $k \in \mathbb{N}$. Then there exists a morphism of $\mathscr{O}_{S^{\mathrm{ad}}}$-modules $H^{\mathrm{ad}} \to F$ such that $(H^{\mathrm{ad}})_x \to F_x$ is surjective for every $x \in p^{-1}(Z(f))$ and H is of the shape $\bigoplus_{i=1}^m L^{-n}$ with $m, n \in \mathbb{N}$.

Indeed, by induction hypothesis (III$_{Z(f^k)}$) holds for every $k \in \mathbb{N}$. Then one can prove (1) with the arguments of [15], 5.9 and 5.10.
Fix a L and a f as in (1). The scheme $S - Z(f)$ is affine and hence $S^{\mathrm{ad}} - p^{-1}(Z(f))$ is the union of an increasing sequence of affinoid open subspaces of S^{ad}. By (1) we have $T \subseteq S^{\mathrm{ad}} - p^{-1}(Z(f))$. Hence we can equip T with the structure of an adic space such that T is affinoid and a closed adic subspace of S^{ad}. Then T is finite over Y ([9], 1.4.7), and therefore there exists a closed subscheme D of $S - Z(f)$ such that D is finite over $\mathrm{Spec}\,\mathscr{O}(Y)$ and $T = p^{-1}(D)$.
Every closed point of S is contained in the image of $p : S^{\mathrm{ad}} \to S$. Hence from (1) we can deduce

(2) For every very ample invertible sheaf L on S relative to $\operatorname{Spec} \mathscr{O}(Y)$ and every $f \in \Gamma(S, L)$ such that $f^k \in \Gamma(S, L^k)$ is not the zero section for all $k \in \mathbb{N}$, we have $D \cap Z(f) = \emptyset$.

If D is a proper subset of S then $D = \emptyset$ by (2). If D is not a proper subset of S and hence S is finite over $\operatorname{Spec} \mathscr{O}(Y)$ then obviously (III_S) holds. □

For an affinoid adic space X and a morphism $X \to Y$ of finite type, we call $a_1, \ldots, a_n \in \mathscr{O}^+(X)$ a system of generators of X over Y if the continuous $(\mathscr{O}(Y), \mathscr{O}^+(Y))$-morphism of affinoid rings $(\mathscr{O}(Y), \mathscr{O}^+(Y))\langle T_1, \ldots, T_n\rangle \to (\mathscr{O}(X), \mathscr{O}^+(X))$ with $T_i \mapsto a_i$ is a quotient mapping. Let $\mathring{X}^Y$ be the set of all $x \in X$ for which the following equivalent conditions are satisfied ([13], 3.13)

- for every γ of the value group Γ_x of the valuation $|\ \ |_x$ associated with x there exists a system of generators $a_1, \ldots, a_n$ of X over Y such that $|a_i(x)|_x \leqslant \gamma$ for $i = 1, \ldots, n$
- there exist a system of generators $a_1, \ldots, a_n$ of X over Y and monic polynomials $p_1, \ldots, p_n \in \mathscr{O}^+(Y)[T]$ such that $\alpha_i := |p_i(a_i)(x)|_x$ is cofinal in Γ_x (,i. e., for every $\gamma \in \Gamma_x$ there exists some $m \in \mathbb{N}$ with $\alpha_i^m \leqslant \gamma$) for $i = 1, \ldots, n$
- for every $a \in \mathscr{O}^+(X)$ there exists a monic polynomial $p \in \mathscr{O}^+(Y)[T]$ such that $|p(a)(x)|_x$ is cofinal in Γ_x.

A morphism of adic spaces $g : X \to Y$ is called proper if g is of finite type, separated and universally closed, and let us call g Kiehl-proper if the following equivalent conditions are satisfied ([13], 3.13)

- g is proper and for every $x \in X$ there exists an affinoid open subset of X that contains the closure of $\{x\}$ in X
- g is of finite type and separated and for every $x \in X$ there exists an affinoid open subset U of X such that $x \in \mathring{U}^Y$.

Theorem 6.3. — *Let X be an adic space, let $g : X \to Y$ be a morphism of adic spaces and let F be a coherent $\mathscr{O}_X$-module. If one of the following conditions is satisfied*

(a) *g is proper and the Tate ring $\mathscr{O}(Y)$ has a noetherian ring of definition*

(b) *g is Kiehl-proper and there exist a complete Tate ring D of topologically finite type over a complete non-archimedean field and a $f \in D$ such that $\mathscr{O}(Y)$ is of topologically finite type over D^f*

then for every $q \in \mathbb{N}_0$ hold

(i) *The $\mathscr{O}(Y)$-module $H^q(X,F)$ is finitely generated.*
(ii) *Let $H^q(X,F)\otimes\mathscr{O}_Y$ denote the $\mathscr{O}_Y$-module associated with the $\mathscr{O}_Y(Y)$-module $H^q(X,F)$. Then the natural morphism of $\mathscr{O}_Y$-modules*

$$H^q(X,F)\otimes\mathscr{O}_Y\longrightarrow R^qg_*F$$

is an isomorphism.

Proof. i) Let s be a topologically nilpotent unit of $\mathscr{O}(Y)$, let $U_1,\dots,U_n$ be affinoid open subsets of X and, for $i\in\{1,\dots,n\}$, let $a_{i1},\dots,a_{im(i)}\in\mathscr{O}^+(U_i)$ be a system of generators of U_i over Y. Put $V_i:=\{x\in U_i\mid |a_{ij}(x)|\leqslant|s(x)|\text{ for }j=1,\dots,m(i)\}$. Assume that $X=V_1\cup\ldots\cup V_n$. Let $C=C(\mathfrak{V},F)$ be the alternating Čech complex to F and the covering $\mathfrak{V}:=(V_i)_{i\in\{1,\dots,n\}}$. The arguments of [14], §1, §2 give

(α) Let $q\in\mathbb{N}_0$ such that $C^q=Z^q(C)$ (e.g., $q=n-1$). Then the $\mathscr{O}(Y)$-module $H^q(X,F)$ is finitely generated and $B^q(C)$ is closed in C^q.
(β) Assume that there exist a ring of definition B of $\mathscr{O}(Y)$ and an $r\in B\cap\mathscr{O}(Y)^*$ such that, for every finitely generated B-module M and every sub-B-module N of M, there exists a finitely generated sub-B-module N' of M with $rN\subseteq N'\subseteq N$.
Then, for every $q\in\mathbb{N}_0$, the $\mathscr{O}(Y)$-module $H^q(X,F)$ is finitely generated and $B^q(C)$ is closed in C^q (the latter is equivalent to the fact that the differential $C^{q-1}\to C^q$ is strict).

By virtue of (β), Lemma 2.7(i) and Proposition 2.8 we obtain that (i) holds if condition (b) is satisfied. If condition (a) is satisfied then (i) holds by [13], 3.12.13.
ii) Again by [13], 3.12.13 we know that (ii) holds if (a) is satisfied. So let us assume that condition (b) is fulfilled. Let $U_i,V_i,C(\mathfrak{V},F)$ be as in the proof of (i). For an open subset U of Y put $H(U):=H^q(g^{-1}(U),F)$, and for open subsets U and V of Y with $U\subseteq V$ let

$$c_{V,U}:H(V)\otimes_{\mathscr{O}(V)}\mathscr{O}(U)\longrightarrow H(U)$$

be the natural $\mathscr{O}(U)$-linear mapping. We have to show that, for every rational subset U of Y, the mapping $c_{Y,U}$ is bijective. First we show

(1) $c_{Y,U}$ is bijective if there exists a power bounded element t of $\mathscr{O}(Y)$ such that $U=\{y\in Y\mid 1\leqslant|t(y)|\}$.

Proof. As t is power bounded, there exists a ring of definition of $\mathscr{O}(Y)$ containing t. Let $C(\mathfrak{V},F)_t$ be the localization of the complex $C(\mathfrak{V},F)$ with respect to the multiplicative system $\{1, t, t^2, \ldots\}$. There are natural topologies on the components $C^q(\mathfrak{V}, F)_t$ such that

$$C(\mathfrak{V} \cap g^{-1}(U), F) = (C(\mathfrak{V}, F)_t)^\wedge$$

(see [8], §1,§2). From the proof of (i) we know that the differentials of $C(\mathfrak{V}, F)$ are strict. Then the differentials of $C(\mathfrak{V}, F)_t$ are strict, too. Hence if we endow $Z^q(C(\mathfrak{V}, F)_t)$ with the subspace topology of $C^q(\mathfrak{V}, F)_t$ and $H^q(C(\mathfrak{V}, F)_t)$ with the quotient topology $\mathscr{T}$ with respect to $Z^q(C(\mathfrak{V}, F)_t) \to H^q(C(\mathfrak{V}, F)_t)$ then we have

$$H^q((C(\mathfrak{V}, F)_t)^\wedge) = (H^q(C(\mathfrak{V}, F)_t))^\wedge$$

([2], III.2.12, Lemma 2 or [13], 3.14(B.3)). For every finitely generated $\mathscr{O}(Y)$-module M, the natural $\mathscr{O}(Y)$-module topology of M is the unique complete $\mathscr{O}(Y)$-module topology of M which has a countable fundamental system of neighbourhoods of 0. Hence the quotient topology of $H^q(C(\mathfrak{V}, F))$ with respect to $Z^q(C(\mathfrak{V}, F)) \to H^q(C(\mathfrak{V}, F))$ (where $Z^q(C(\mathfrak{V}, F))$ carries the subspace topology of $C^q(\mathfrak{V}, F))$ equals the natural $\mathscr{O}(Y)$-module topology of $H^q(C(\mathfrak{V}, F))$. This implies that $\mathscr{T}$ equals the natural module topology of $H^q(C(\mathfrak{V}, F)_t) = H^q(C(\mathfrak{V}, F))_t$ over the topological ring $\mathscr{O}(Y)(\frac{1}{t})$ (with $\mathscr{O}(Y)(\frac{1}{t})$ defined as in [8], §1). Then

$$\begin{aligned}(H^q(C(\mathfrak{V}, F)_t))^\wedge &= H^q(C(\mathfrak{V}, F))_t \otimes_{\mathscr{O}(Y)(\frac{1}{t})} (\mathscr{O}(Y)(\frac{1}{t}))^\wedge \\ &= H^q(C(\mathfrak{V}, F)) \otimes_{\mathscr{O}(Y)} \mathscr{O}(U).\end{aligned}$$

This shows (1).

We may assume that $\mathscr{O}^+(Y) = \mathscr{O}(Y)^\circ$. Then according to hypothesis (b) there exist a complete non-archimedean field k, an affinoid adic space T of finite type over $\mathrm{Spa}(k, k^\circ)$, a $f \in \mathscr{O}_T(T)$ and a morphism of adic spaces $\ell : Y \to T^f$ such that ℓ has a factorization $Y \xrightarrow{u} Y' \xrightarrow{v} T^f$ where Y' is an affinoid adic space, v is of finite type and $u^* : \mathscr{O}(Y') \to \mathscr{O}(Y)$ is an isomorphism of topological rings. Put

$$\begin{aligned}Y_\varepsilon &:= \ell^{-1}(T^f_\varepsilon) \\ Y_\iota &:= \ell^{-1}(T^f_\iota).\end{aligned}$$

We have

(2) $c_{Y,U}$ is bijective if U is a rational subset of Y such that $U \subseteq Y_\varepsilon$ or $U \subseteq Y_\iota$.

Proof. First let us assume that $U \subseteq Y_\varepsilon$. Then $U \subseteq (Y_\varepsilon)^\circ = \ell^{-1}((T^f_\varepsilon)^\circ)$. According to Proposition 3.1 and 3.2(i), $\mathscr{O}(U)$ is of topologically finite type over k. Then the bijectivity of $c_{Y,U}$ can be be proved as the corresponding statement in [14], 3.5. Now let $U \subseteq Y_\iota$. By virtue of Proposition 3.2(ii), Y_ι is an affinoid adic space such that $\mathscr{O}(Y_\iota)$ has a noetherian ring of definition. Then $c_{Y_\iota,U}$ is bijective. By (1), c_{Y,Y_ι} is bijective. Thus we obtain that $c_{Y,U}$ is bijective and (2) is proved.

Let U be a rational subset of Y. We show that $c_{Y,U}$ is bijective. It is enough to show that, for every maximal ideal $\mathfrak{m}$ of $\mathscr{O}(U)$, the mapping $c_{Y,U} \otimes_{\mathscr{O}(U)} \mathscr{O}(U)_\mathfrak{m}$ is bijective. For this we choose a maximal point y of U with $\mathrm{supp}(y) = \mathfrak{m}$ and a rational subset V of Y such that $y \in V \subseteq U$ and $V \subseteq Y_\varepsilon$ or $V \subseteq Y_\iota$. By (2) the mapping $c_{Y,V}$ is bijective. Replacing Y by U, we get correspondingly that $c_{U,V}$ is bijective. As $c_{Y,V} = c_{U,V} \circ (c_{Y,U} \otimes_{\mathscr{O}(U)} \mathscr{O}(V))$, we obtain that $c_{Y,U} \otimes_{\mathscr{O}(U)} \mathscr{O}(V)$ is bijective. The ring homomorphism $\mathscr{O}(U) \to \mathscr{O}(V)$ is flat, and $\mathfrak{m} = \mathrm{supp}(y)$ lies in the image of $\mathrm{Spec}\,\mathscr{O}(V) \to \mathrm{Spec}\,\mathscr{O}(U)$ since y lies in the image of the inclusion $V \subseteq U$. This shows that $c_{Y,U} \otimes_{\mathscr{O}(U)} \mathscr{O}(U)_\mathfrak{m}$ is bijective. □

Corollary 6.4. — *If the Tate ring $\mathscr{O}(Y)$ satisfies one of the following conditions*

(a) *$\mathscr{O}(Y)$ has a noetherian ring of definition*

(b) *there exist a complete Tate ring D of topologically finite type over a complete non-archimedean field and a $f \in D$ such that $\mathscr{O}(Y)$ is of topologically finite type over D^f*

then $(\mathrm{I}_S), (\mathrm{II}_S), (\mathrm{III}_S)$ *hold for every proper* $\mathrm{Spec}\,\mathscr{O}(Y)$*-scheme S.*

Proof. If S is a projective $\mathrm{Spec}\,\mathscr{O}(Y)$-scheme then $(\mathrm{I}_S), (\mathrm{II}_S), (\mathrm{III}_S)$ hold by Theorem 6.2 and Theorem 6.3(i). From this and Chow's lemma (and Theorem 6.3(ii)) one obtains the general case of a proper $\mathrm{Spec}\,\mathscr{O}(Y)$-scheme, cf. [6], XII.4. □

Corollary 6.5. — *Let S be a* $\mathrm{Spec}\,\mathscr{O}(Y)$*-scheme such that there exists a finite* $\mathrm{Spec}\,\mathscr{O}(Y)$*-morphism $S \to \mathbb{P}^1_{\mathrm{Spec}\,\mathscr{O}(Y)}$. Then* $(\mathrm{I}_S), (\mathrm{II}_S), (\mathrm{III}_S)$ *hold.*

Proof. By virtue of (i.α) in the proof of Theorem 6.3, for every coherent $\mathscr{O}_{S^{\mathrm{ad}}}$-module F the $\mathscr{O}(Y)$-module $H^1(S^{\mathrm{ad}}, F)$ is finitely generated. Then the assertion follows from Theorem 6.2. □

7. Proof of Theorem 5.1

We fix a noetherian torsion ring A and assume that, for every pseudo-adic space W, the torsion of A is prime to $\mathrm{char}^+(W)$.

The starting-point of the proof of Theorem 5.1 is the following finiteness

Proposition 7.1. — *Let $f : X \to Y$ be a smooth separated quasi-compact morphism of analytic pseudo-adic spaces and let F be a constructible A-module on $X_{\text{ét}}$. Then, for every $n \in \mathbb{N}_0$, the A-module $R^n f_! F$ on $Y_{\text{ét}}$ is constructible.*

Proof. In [9], 6.2.2 this statement is proved under the hypothesis that there exists a morphism $Y \to \operatorname{Spa}(K, K^\circ)$ with K a non-archimedean field, and it is remarked that this hypothesis is not necessary if Lütkebohmert's result [17], 5.3 on compactifications of smooth morphisms of affinoid rigid analytic spaces holds, more generally, for smooth morphisms of affinoid analytic adic spaces. An inspection of the proof of loc. cit. shows that this is fulfilled. $\square$

Next we want to deduce, for some morphisms of analytic pseudo-adic spaces $f : X \to Y$ such that $\dim(f) \leqslant 1$ and $\{x \in X \mid f \text{ is not smooth at } x\}$ is quasi-finite over Y and for constant A-modules F of finite type on $X_{\text{ét}}$, a finiteness of $R^n f_! F$ (Lemma 7.5).

Lemma 7.2. — *Let S be an analytic adic space which is local, i.e., there exists a point $s_0 \in S$ such that every element of S is a generalization of s_0. We assume that the valuation ring $k(s_0)^+$ is henselian. Let $f : X \to S$ be a partially proper morphism of adic spaces and let U be a constructible open subset of $f^{-1}(s_0)$. Put $B := \{x \in U \mid x$ has a specialization in $f^{-1}(s_0) - U\}$.*
Then, for every constructible open subset W of X with $B \subseteq W$, there exists a smallest subset P_W of X such that $W \cup P_W$ is an open subset of X that contains U. The set P_W is constructible and closed in X. If G denotes the set of all generalizations in X of all elements of U then $P_W = G - W$.

Remark. If the ring $k(s_0)^+$ is finite dimensional but not necessarily henselian then $P_W := G - W$ is the smallest subset of X such that $W \cup P_W$ is an open subset of X that contains U. Furthermore, P_W is constructible in X. But P_W is not closed in X in general.

Proof. We have

(1) For every $x \in X$, the set $\overline{\{x\}} \cap f^{-1}(s_0)$ is connected.

Proof of (1). Let K be the residue class field of $k(x)^+$ and let k be the residue class field of $k(f(x))^+$. So we consider k as a subfield of K. Let D be the valuation ring of k corresponding to $s_0 \in S$. Since f is partially proper, $\overline{\{x\}} \cap f^{-1}(s_0)$ is homeomorhpic to $\{C \in \operatorname{Spv} K \mid C \cap k = D\}$. By assumption $k(s_0)^+$ is henselian. Then D is henselian, too. So by [12], 2.4.4 the topological space $\{C \in \operatorname{Spv} K \mid C \cap k = D\}$ is connected. This proves (1).

Put

$$\begin{aligned} H &:= \{x \in X \mid x \text{ specializes to an element of } B\} \\ I &:= \{x \in X \mid x \text{ specializes to an element of } U \text{ and} \\ &\qquad \text{to an element of } f^{-1}(s_0) - U\}. \end{aligned}$$

Then

(2) (i) $H = I$
(ii) $G - H = \{x \in X \mid \overline{\{x\}} \cap f^{-1}(s_0) \subseteq U\}$
(iii) $G - H$ is ind-constructible in X
(iv) G is pro-constructible in X.

Proof of (2). i) Obviously, $H \subseteq I$. Let x be an element of I and put $T := \overline{\{x\}} \cap f^{-1}(s_0)$. Then $U \cap T$ is a constructible open subset of T with $\emptyset \neq U \cap T \neq T$, and by (1) T is connected. Hence there exists some $y \in U \cap T$ that has a specialization in $T - U$, i.e., $y \in B$. This shows that $x \in H$.
ii) Since $\overline{\{x\}} \cap f^{-1}(s_0) \neq \emptyset$ for every $x \in X$, we have $G - I = \{x \in X \mid \overline{\{x\}} \cap f^{-1}(s_0) \subseteq U\}$. Then the assertion follows from (i).
iii) Let x be an element of $G - H$. X is taut ([9], 5.1.4), i.e., the closure of every quasi-compact subset of X is quasi-compact. Therefore there exist quasi-compact open subsets V_1, V_2 of X such that $\overline{\{x\}} \subseteq V_1$ and $\overline{V_1} \subseteq V_2$. By (ii), $\overline{\{x\}} \cap f^{-1}(s_0) \subseteq U$. Hence there exits a closed constructible subset D of V_2 with $x \in D \subseteq V_1$ and $D \cap f^{-1}(s_0) \subseteq U$. Again by (ii), $D \subseteq G - H$.
iv) Let V_1, V_2 be quasi-compact open subsets of X with $\overline{V_1} \subseteq V_2$. Let $(U_i)_{i \in I}$ be the family of all constructible open subsets of V_2 containing $U \cap V_2$. Then $V_1 \cap G = V_1 \cap \bigcap_{i \in I} U_i$.
This concludes the proof of (2).

Let W be a constructible open subset of X containing B. Put

$$P_W := G - W.$$

Since $H \subseteq W$, we have $P_W = (G - H) \cap (X - W)$. Then by (2.iii,iv) P_W is ind-constructible and pro-constructible and so constructible, and by (2.ii) P_W is closed under specializations in X and hence closed in X. The set $W \cup P_W = W \cup G$ is closed under generalizations in X and hence open in X. If C is a subset of X such that $W \cup C$ is open in X and contains U then $G \subseteq W \cup C$ and therefore $P_W = G - W \subseteq C$. □

Lemma 7.3. — *Let $X = \operatorname{Spa}(B, B^+)$ be an affinoid analytic adic space, let S be a separated $\operatorname{Spec} B$-scheme such that there exists a quasi-finite $\operatorname{Spec} B$-morphism $S \to \mathbb{P}^1_{\operatorname{Spec} B}$, let $S^{\mathrm{ad}} := S \times_{\operatorname{Spec} B} X$ be the adic space over X associated with the scheme S over $\operatorname{Spec} B$, let U be a quasi-compact*

open subset of S^{ad}*, and let* $g : Z \to U$ *be a finite morphism of adic spaces such that* $\Delta := \{z \in Z \mid g \text{ is not étale at } z\}$ *is quasi-finite over* X *and* g *is of constant degree over* $U - g(\Delta)$*.*
Then, for every $x_0 \in X$*, there exist an étale neighbourhood* $(X' = \mathrm{Spa}\,(B', B'^+), x_0')$ *of* (X, x_0) *and a finite morphism of schemes*

$$t : T \longrightarrow S' := S \times_{\mathrm{Spec}\,B} \mathrm{Spec}\,B'$$

such that the following holds: If

$$Z' \xrightarrow{g'} U' \subseteq (S^{\mathrm{ad}})'$$

denotes the base extension of $Z \xrightarrow{g} U \subseteq S^{\mathrm{ad}}$ *from* X *to* X'*, and*

$$t^{\mathrm{ad}} : T \times_{\mathrm{Spec}\,B'} X' \longrightarrow S' \times_{\mathrm{Spec}\,B'} X' = (S^{\mathrm{ad}})'$$

denotes the morphism induced by t*, and*

$$f' : (S^{\mathrm{ad}})' \longrightarrow X'$$

denotes the structure morphism of $(S^{\mathrm{ad}})'$ *then there exists an open subset* V *of* U' *such that* $g' : g'^{-1}(V) \to V$ *and* $t^{\mathrm{ad}} : (t^{\mathrm{ad}})^{-1}(V) \to V$ *are isomorphic over* V *and* $f'^{-1}(x_0') \cap U' \subseteq V$*.*

Remark. i) The proof of Lemma 7.3 relies on Lemma 7.2. If the valuation ring $k(x_0)^+$ is henselian (or, equivalently, the completion $(k(x_0)^+)^\wedge$ is henselian) (for example, this is fulfilled if x_0 is a maximal point of X) then X' can be choosen as an open neighbourhood of x_0 in X. If x_0 is a maximal point of X then there exists an affinoid open neighbourhood W of x_0' in X' such that $f'^{-1}(W) \cap U' \subseteq V$, and therefore, replacing X' by W we may assume that $V = U'$.
ii) For the proof of Lemma 7.3 we will use Corollary 6.5. If we can apply Corollary 6.4 (,i.e., if B satisfies one of the conditions (a) and (b) of Corollary 6.4) then S can be an arbitrary separated scheme of finite type over $\mathrm{Spec}\,B$ with $\dim(S/\mathrm{Spec}\,B) \leqslant 1$.

Proof. Let $P \to \mathrm{Spec}\,B$ be a scheme over $\mathrm{Spec}\,B$ such that there exists a finite $\mathrm{Spec}\,B$-morphism $P \to \mathbb{P}^1_{\mathrm{Spec}\,B}$ and S is $\mathrm{Spec}\,B$-isomorphic to an open subscheme of P. We consider the open subspaces

$$U \subseteq S^{\mathrm{ad}} \subseteq P^{\mathrm{ad}} := P \times_{\mathrm{Spec}\,B} X.$$

Let $e : P^{\mathrm{ad}} \to X$ be the structure morphism. Put

$$\begin{aligned} R &:= \{x \in U \cap e^{-1}(x_0) \mid x \text{ has a specialization in} \\ &\quad\ e^{-1}(x_0) - U\} \\ &= \{x \in U \cap e^{-1}(x_0) \mid x \text{ has a specialization in} \\ &\quad\ (e^{-1}(x_0) \cap S^{\mathrm{ad}}) - U\}. \end{aligned}$$

The set R is finite since $\dim(S/\operatorname{Spec} B) \leqslant 1$. Put

$$\begin{aligned} R_g &:= \{x \in P^{\mathrm{ad}} \mid x \text{ is a maximal point of } P^{\mathrm{ad}} \text{ and} \\ &\qquad \text{a generalization of some element of } R\}. \end{aligned}$$

Then $R_g \subseteq U$. So we have a natural morphism of adic spaces

$$q :\ L := \coprod_{x \in R_g} \operatorname{Spa}(k(x), k(x)^+) \longrightarrow U.$$

Composing this morphism with the natural morphism of locally ringed spaces $U \to P$, we obtain a morphism of locally ringed spaces

$$p : L \longrightarrow P.$$

From the approximation theorem for independent valuations ([2], VI.7.2) we can conclude that there exists a finite morphism of schemes

$$r : W \to P$$

which is étale over every point of $\operatorname{im}(p)$ and such that there exists a L-isomorphism

$$L \times_U Z \ \cong\ L \times_P W.$$

For every $x \in R_g$ and every $y \in R$ with $x \succ y$, let A_y be the valuation ring of $k(x)$ such that $A_y \subseteq k(x)^+$ and $A_y \cap k(y) = k(y)^+$. Put

$$A(x) := \bigcap_{\substack{y \in R, \\ x \succ y}} A_y.$$

Then $A(x)$ is a ring of definition of the Tate ring $k(x)$ and so we have the adic space

$$L' := \coprod_{x \in R_g} \operatorname{Spa}(k(x), A(x)).$$

By [2], VI.7.1, Cor.3 every valuation ring of $k(x)$ containing $A(x)$ contains one of the A_y. Hence we get a morphism of adic spaces

$$q' : L' \longrightarrow U$$

whose underlying continuous mapping is a homeomorphism from L' to the set of all generalizations of the elements of R. Let

$$p' : L' \longrightarrow P$$

be the composition of q' and the morphism $U \to P$. The isomorphism $L \times_U Z \cong L \times_P W$ from above extends to an isomorphism

$$L' \times_U Z \;\cong\; L' \times_P W,$$

and this isomorphism extends to a F-isomorphism

$$g^{-1}(F) = F \times_U Z \;\cong\; F \times_P W = (r^{\mathrm{ad}})^{-1}(F)$$

where F is an open neighbourhood of R in U and $r^{\mathrm{ad}} : W^{\mathrm{ad}} \to P^{\mathrm{ad}}$ is the morphism of adic spaces associated with $r : W \to P$.
Let $H := \mathrm{Spa}(K, K^+) \to X$ be the strict henselization of X at x_0 (so K is a separable algebraic closure of $k(x_0)$ and K^+ is a valuation ring of K extending $k(x_0)^+$ ([9], 2.5.13.i)). For an adic space Y over X put $Y_{(H)} := Y \times_X H$. The open inclusions $F \subseteq U \subseteq P^{\mathrm{ad}}$ give the open inclusions

$$F_{(H)} \subseteq U_{(H)} \subseteq (P^{\mathrm{ad}})_{(H)}.$$

Let $\ell : (P^{\mathrm{ad}})_{(H)} \to H$ be the structure morphism and let h_0 be the closed point of H. We apply Lemma 7.2 to $\ell : (P^{\mathrm{ad}})_{(H)} \to H, \ell^{-1}(h_0) \cap U_{(H)}, F_{(H)}$ and obtain that there exists a closed constructible subset C of $(P^{\mathrm{ad}})_{(H)}$ such that $C \subseteq U_{(H)}, F_{(H)} \cap C = \emptyset$, $F_{(H)} \cup C$ is open in $(P^{\mathrm{ad}})_{(H)}$ and $\ell^{-1}(h_0) \cap U_{(H)} \subseteq F_{(H)} \cup C$.
H is the projective limit of all pointed étale neighbourhoods of (X, x_0). Hence there exists an affinoid étale neighbourhood (M, m_0) of (X, x_0) and a closed constructible subset D of $(P^{\mathrm{ad}})_{(M)} := P^{\mathrm{ad}} \times_X M$ such that $D \subseteq U_{(M)}, F_{(M)} \cap D = \emptyset$, $F_{(M)} \cup D$ is open in $(P^{\mathrm{ad}})_{(M)}$ and $w^{-1}(m_0) \cap U_{(M)} \subseteq F_{(M)} \cup D$ (with $w : (P^{\mathrm{ad}})_{(M)} \to M$ the structure morphism).
We cover $(P^{\mathrm{ad}})_{(M)}$ by the open subsets

$$\begin{aligned} E_1 &:= F_{(M)} \cup D \\ E_2 &:= (P^{\mathrm{ad}})_{(M)} - D. \end{aligned}$$

The finite morphisms of adic spaces $g : Z \to U$ and $r^{\mathrm{ad}} : W^{\mathrm{ad}} \to P^{\mathrm{ad}}$ give by base extensions the finite morphisms

$$\begin{aligned} \pi_1 :\;& Z \times_U E_1 \longrightarrow E_1 \\ \pi_2 :\;& W^{\mathrm{ad}} \times_{P^{\mathrm{ad}}} E_2 \longrightarrow E_2. \end{aligned}$$

π_1 and π_2 are isomorphic over $E_1 \cap E_2 = F_{(M)}$ because of the F-isomorphism $g^{-1}(F) \cong (r^{\mathrm{ad}})^{-1}(F)$ from above. Hence π_1 and π_2 glue together to a finite morphism of adic spaces $\tilde{\pi} : \tilde{E} \to (P^{\mathrm{ad}})_{(M)}$. Put $P' :=$

$P \times_{\operatorname{Spec} B} \operatorname{Spec} \mathscr{O}_M(M)$. Then $(P^{\mathrm{ad}})_{(M)} = P' \times_{\operatorname{Spec} \mathscr{O}_M(M)} M$, and by Corollary 6.5 there exists a finite morphism of schemes $\pi : E \to P'$ such that $\tilde{\pi} = \pi^{\mathrm{ad}} := \pi \times_{\operatorname{Spec} \mathscr{O}_M(M)} M$. □

For an analytic pseudo-adic space (X, L), let $\mathscr{D}^b(X, L)$ be the bounded derived category of the category of A-modules on $(X, L)_{\text{ét}}$. We define subclasses $\mathscr{S}_1(X, L) \subseteq \mathscr{S}_2(X, L) \subseteq \mathscr{S}_3(X, L) \subseteq \mathscr{S}_4(X, L)$ of the class of objects of $\mathscr{D}^b(X, L)$ as follows.

$\mathscr{S}_1(X, L)$ is the smallest subclass of the class of objects of $\mathscr{D}^b(X, L)$ which satisfies the following three properties

(i) Every object F of $\mathscr{D}^b(X, L)$ such that, for every $n \in \mathbb{Z}$, the A-module $H^n(F)$ on $(X, L)_{\text{ét}}$ is constructible is an element of $\mathscr{S}_1(X, L)$.

(ii) If F is an object of $\mathscr{D}^b(\operatorname{Spec} \mathscr{O}_X(X))$ such that, for every $n \in \mathbb{Z}$, the A-module $H^n(F)$ on $(\operatorname{Spec} \mathscr{O}_X(X))_{\text{ét}}$ is constructible then $i^*F \in \mathscr{S}_1(X, L)$ where $i : (X, L)_{\text{ét}} \to (\operatorname{Spec} \mathscr{O}_X(X))_{\text{ét}}$ is the natural morphism of sites.

(iii) If (B, C, D) is a distinguished triangle in $\mathscr{D}^b(X, L)$ and two of the three objects B, C, D belong to $\mathscr{S}_1(X, L)$ then also the third.

$\mathscr{S}_2(X, L)$ is the class of all objects of $\mathscr{D}^b(X, L)$ which are direct summands of elements of $\mathscr{S}_1(X, L)$.

$\mathscr{S}_3(X, L)$ is the class of all objects F of $\mathscr{D}^b(X, L)$ such that, for every $x \in L$, there exists a locally closed locally constructible subset M of L such that $x \in M$ and $F|(X, M) \in \mathscr{S}_2(X, M)$.

$\mathscr{S}_4(X, L)$ is the class of all objects F of $\mathscr{D}^b(X, L)$ such that, for every $x \in L$, there exists a locally closed locally constructible subset M of L and a surjective étale morphism of pseudo-adic spaces $(Y, N) \to (X, M)$ such that $x \in M$ and $F|(Y, N) \in \mathscr{S}_2(Y, N)$.

Lemma 7.4. — *Let*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & P \arrow[d, "h"] \\ Y \arrow[r, "b"] & Q \end{tikzcd}$$

be a commutative diagram of locally ringed spaces where $h : P \to Q$ is a separated morphism of finite type between schemes, $f : X \to Y$ is a morphism of analytic adic spaces, Q is affin and Y is affinoid, and $X = P \times_Q Y$. Let M be a convex pro-constructible subset of Y and let L be *a quasi-compact locally closed constructible subset of* $f^{-1}(M)$. *Let* $g : (X, L) \to (Y, M)$ *be the morphism of pseudo-adic spaces induced by* f, *and let* F *be the constant* A*-module on* $(X, L)_{\text{ét}}$ *associated with a finitely generated* A*-module* C. *Put* $X' := \{x \in X \mid f \text{ is not smooth at } x\}$ *and* $X'' := \{x \in X' \mid f|X' : X' \to Y \text{ is quasi-finite at } x\}$. *Then*

(i) *If* $X' \cap f^{-1}(M) \subseteq L$ *and* h *is proper then* $Rg_!F \in \mathscr{S}_1(Y, M)$.
(ii) *If* $X' \cap L \subseteq X''$ *then* $Rg_!F \in \mathscr{S}_3(Y, M)$.

Proof. i) Since h is proper, f is quasi-compact and separated. Let U be a quasi-compact open subset of $f^{-1}(M)$ such that L is a closed subset of U, and put $V := f^{-1}(M) - U$ and $W := U - L$. Let $s : (X, f^{-1}(M)) \to (Y, M)$ be the morphism of pseudo-adic spaces induced by f, and let G and H be the constant A-modules on $(X, f^{-1}(M))_{\text{ét}}$ and $P_{\text{ét}}$ associated with the A-module C. Since we have the distinguished triangle $(R(s|U)_!(G|U), Rs_!G, R(s|V)_!(G|V))$ in $\mathscr{D}^b(Y, M)$ and since $R(s|V)_!G \in \mathscr{S}_1(Y, M)$ (by Proposition 7.1) and $Rs_!G = j^*Rh_!H \in \mathscr{S}_1(Y, M)$ (with $j : (Y, M)_{\text{ét}} \to Q_{\text{ét}}$ the morphism induced by $b : Y \to Q$), we obtain that $R(s|U)_!(G|U) \in \mathscr{S}_1(Y, M)$. Similarly, since we have the distinguished triangle $(R(s|W)_!(G|W), R(s|U)_!(G|U), Rg_!F)$ in $\mathscr{D}^b(Y, M)$ and since $R(s|W)_!(G|W) \in \mathscr{S}_1(Y, M)$ (by Proposition 7.1), we obtain that $Rg_!F \in \mathscr{S}_1(Y, M)$.
ii) Replacing h by a compactification of h we may assume that h is proper. We fix an element m of M. As X' is Zariski-closed in X, X' is pro-constructible in X and closed under specializations and generalizations in X. Furthermore, for every $x \in X'' \cap f^{-1}(m)$, the set $\{x\}$ is closed under specializations and generalizations in $X' \cap f^{-1}(m)$. By assumption, $L \cap X' \subseteq X''$. Hence $(L \cup X') \cap f^{-1}(M)$ is a convex pro-constructible subset of $f^{-1}(M)$ and $L \cap f^{-1}(m)$ is closed under specializations and generalizations in $(L \cup X') \cap f^{-1}(m)$, i.e., $L \cap f^{-1}(m)$ is closed and open in $(L \cup X') \cap f^{-1}(m)$. Then there exist a locally closed constructible subset V of M and a locally closed constructible subset T of $f^{-1}(V)$ such that $m \in V$, $(L \cup X') \cap f^{-1}(V) \subseteq T$ and $L \cap f^{-1}(V)$ is open and closed in T. Let $r : (X, T) \to (Y, V)$ be the morphism of pseudo-adic spaces induced by f and let G be the constant A-module on $(X, T)_{\text{ét}}$ with $G|L \cap f^{-1}(V) = F|L \cap f^{-1}(V)$. Since $X' \cap f^{-1}(V) \subseteq T$, we obtain from (i) that $Rr_!G \in \mathscr{S}_1(Y, V)$. As $L \cap f^{-1}(V)$ is a direct summand of T, we get that $(Rg_!F)|V = R(r|L \cap f^{-1}(V))_!(G|L \cap f^{-1}(V))$ is a direct summand of $Rr_!G$ and therefore an element of $\mathscr{S}_2(Y, V)$. This shows that $Rg_!F \in \mathscr{S}_3(Y, M)$. □

Lemma 7.5. — *Let* $X = \operatorname{Spa}(B, B^+)$ *be an affinoid analytic adic space. Let* S *be a separated* $\operatorname{Spec} B$*-scheme such that there exists a quasi-finite*

$\operatorname{Spec} B$*-morphism* $S \to \mathbb{P}^1_{\operatorname{Spec} B}$. (Remark. If B satisfies one of the conditions (a) and (b) of Corollary 6.4 then S can be an arbitrary separated scheme of finite type over $\operatorname{Spec} B$ with $\dim(S/\operatorname{Spec} B) \leqslant 1$). *Assume that* $\{s \in S \mid S$ is not smooth over $\operatorname{Spec} B$ at $s\}$ *is quasi-finite over* $\operatorname{Spec} B$. *Let* $S^{\mathrm{ad}} := S \times_{\operatorname{Spec} B} X$ *be the adic space over* X *associated with the scheme* S *over* $\operatorname{Spec} B$ *and let* $f : S^{\mathrm{ad}} \to X$ *be the structure morphism. Let* U *be a quasi-compact open subset of* S^{ad} *and let* $g : Z \to U$ *be a finite morphism of adic spaces such that* $\Delta := \{z \in Z \mid g$ is not étale at $z\}$ *is quasi-finite over* X *and* g *is of constant degree over* $U - g(\Delta)$. *Let* M *be a convex locally pro-constructible subset of* X *and let* L *be a locally closed constructible subset of* $(f \circ g)^{-1}(M)$. *Let* $h : (U, L) \to (X, M)$ *be the morphism of pseudo-adic spaces induced by* $f \circ g$. *Let* F *be a constant* A*-module of finite type on* $(U, L)_{\text{ét}}$.
Then $Rh_! F \in \mathscr{S}_4(X, M)$.

Proof. The assertion follows from Lemma 7.3 and Lemma 7.4(ii). □

In order to arrange the quasi-finiteness of Δ over X in Lemma 7.5 we will need the following two lemmata.

Lemma 7.6. — (i) *Let* $f : X \to Y$ *be a morphism locally of finite type between analytic adic spaces. Then every* $x \in X$ *has an open neighbourhood* U *in* X *such that the restriction* $f|U$ *has a factorization*

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathbb{B}^n_Y \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

where g *is locally quasi-finite and* q *is the natural morphism and* $n \leqslant \dim(f)$.

(ii) *Let* Y *be an analytic adic space, let* n *be a natural number, let* W *be an analytic adic space locally of finite type over* Y *with* $\dim(W/Y) < n$, *and let* $g : W \to \mathbb{B}^n_Y$ *be a locally quasi-finite* Y*-morphism. Then, for every* $w \in W$, *there exist an open neighbourhood* U *of* w *in* W *and a* Y*-automorphism* $h : \mathbb{B}^n_Y \to \mathbb{B}^n_Y$ *such that if* $q : \mathbb{B}^n_Y \to \mathbb{B}^{n-1}_Y$ *denotes the projection then*

$$q \circ h \circ (g|U) : U \longrightarrow \mathbb{B}^{n-1}_Y$$

is locally quasi-finite.

Proof. This lemma is stated in [11], 3.3 under the assumption that Y is locally of finite type over some non-archimedean field. But this assumption is not needed in the proof of loc. cit.. □

LEMMA 7.7. — *Let Y be an adic space of the shape $Y = T^f$ where T is a 1-dimensional normal affinoid adic space of finite type over $\mathrm{Spa}(k, k^\circ)$ with k a non-archimedean field and $f \in \mathscr{O}_T(T)$. Let X be an affinoid adic space which is connected (or, equivalently, $\tilde{X} := \mathrm{Spec}\,\mathscr{O}_X(X)$ is connected), and let $g : X \to Y$ be a smooth morphism of adic spaces. Let $\tilde{Z}$ be an irreducible closed subset of $\tilde{X}$ with $\tilde{Z} \neq \tilde{X}$ and let Z be the corresponding closed subset of X. Then $\dim(Z/Y) < \dim(X/Y)$ or there exists a maximal ideal $\mathfrak{m}$ of $\mathscr{O}_Y(Y)$ with $Z \subseteq g^{-1}(V(\mathfrak{m}))$.*

Proof. We consider the subsets $Y_\iota = (T^f)_\iota$ and $Y_\varepsilon = (T^f)_\varepsilon$ of Y. By Proposition 3.1 and Proposition 3.2(i) we have $Y_\varepsilon^\circ = T - V(f)$. According to Example 3.5 the rational subset Y_ι of Y is finite and discrete and, for every $y \in Y_\iota$, the rational subspace $\{y\}$ of Y is of the shape $\{y\} = \mathrm{Spa}(L, L^\circ)$ with L a discretely valued non-archimedean field.

For every $x \in X$, $\mathscr{O}_{X,x}$ is a normal integral domain. Since the ring homomorphism $\mathscr{O}_X(X)_{\mathrm{supp}(x)} \to \mathscr{O}_{X,x}$ is flat, we can conclude that $\mathscr{O}_X(X)$ is a normal integral domain.

We have

$$(*)\ \ \dim(g^{-1}(y) \cap Z) < \dim(g^{-1}(y)) \text{ for every } y \in Y_\iota \text{ with } g^{-1}(y) \neq \emptyset.$$

Indeed, if $\dim(g^{-1}(y) \cap Z) = \dim(g^{-1}(y))$ then $g^{-1}(y) \cap Z$ contains a connected component U of $g^{-1}(y)$. Since U is a rational subset of X, the ring homomorphism $\mathscr{O}_X(X) \to \mathscr{O}_X(U)$ is flat. Then we obtain that $\tilde{Z}$ contains the generic point of $\mathrm{Spec}\,\mathscr{O}_X(X)$, in contradiction to our assumption that $\tilde{Z} \neq \mathrm{Spec}\,\mathscr{O}_X(X)$.

Assume that there exists a maximal point y of Y such that $g^{-1}(y) \neq \emptyset$ and $\dim(g^{-1}(y) \cap Z) = \dim(g^{-1}(y))$. By $(*)$ we have $y \in Y_\varepsilon^\circ = T - V(f)$. Then according to [9], 1.8.10 there exists a maximal ideal $\mathfrak{m}$ of $\mathscr{O}_Y(Y)$ such that, for the point s of Y with $\{s\} = V(\mathfrak{m})$, we have $g^{-1}(s) \neq \emptyset$ and $\dim(g^{-1}(s) \cap Z) = \dim(g^{-1}(s))$. So there exists an irreducible component W of $V(\mathfrak{m} \cdot \mathscr{O}_X X)) \subseteq \mathrm{Spec}\,\mathscr{O}_X(X) = \tilde{X}$ that is contained in $\tilde{Z}$. Let a be a non-zero element of the image of $\mathfrak{m}$ in $\mathscr{O}_X(X)$. Then W is an irreducible component of $V(a) \subseteq \tilde{X}$ and hence W is 1-codimensional in $\tilde{X}$. Since $W \subseteq \tilde{Z} \subsetneqq \tilde{X}$ and $\tilde{Z}$ is irreducible, we obtain $\tilde{Z} = W$ and therefore $Z \subseteq g^{-1}(V(\mathfrak{m}))$. □

Proof of Theorem 5.1. Replacing T by its normalization we may assume that T is normal. We use induction on $\dim(X/Y)$. If $\dim(X/Y) = 0$ then the assertion follows from Proposition 4.6 and Example 4.3. Let $n := \dim(X/Y) > 0$. By Lemma 7.6(i) we may assume that $\underline{g} : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[rr, "\underline{\ell}"] \arrow[dr, "\underline{g}"'] & & \mathbb{B}^n_Y \arrow[dl, "\underline{r}"] \\ & Y & \end{tikzcd}$$

with $\underline{\ell}$ quasi-finite and $\underline{r}$ the natural morphism. Then g has a factorization

$$\begin{tikzcd} (X,L) \arrow[rr, "\ell"] \arrow[dr, "g"'] & & (\mathbb{B}^n_Y, \underline{r}^{-1}(M)) \arrow[dl, "r"] \\ & (Y,M) & \end{tikzcd}$$

where ℓ and r are the morphisms of pseudo-adic spaces induced by $\underline{\ell}$ and $\underline{r}$. Then $Rg_!F = Rr_!(\ell_!F)$ and $\ell_!F \in \mathscr{C}(\mathbb{B}^n_Y, \underline{r}^{-1}(M))$ (by Proposition 4.6). Hence it suffices to prove the assertion for r and $\ell_!F$, i.e., we may assume that $X = \mathbb{B}^n_Y$ and $L = \underline{g}^{-1}(M)$.

Étale locally and constructible locally F is an element of $\mathscr{Z}(-)$. Therefore we may assume that X is affinoid, $F \in \mathscr{Z}(X,L)$ and $g : (X,L) \to (Y,M)$ has a factorization

$$\begin{tikzcd} (X,L) \arrow[rr, "\ell"] \arrow[dr, "g"'] & & (\mathbb{B}^n_Y, \underline{r}^{-1}(M)) \arrow[dl, "r"] \\ & (Y,M) & \end{tikzcd}$$

where $\underline{\ell} : X \to \mathbb{B}^n_Y$ is an étale morphism of adic spaces and $\underline{r} : \mathbb{B}^n_Y \to Y$ is the natural morphism. Furthermore we may assume that Y and X are connected.

Since $F \in \mathscr{Z}(X,L)$, there exist a finite morphism $\underline{h} : X' \to X$ and a Zariski-closed subset X'' of X such that $\underline{h}$ is étale of constant degree over $X - X''$ and $X'' \neq X$ and $F|\underline{h}^{-1}(L \cap (X - X''))$ is the constant A-module on $\underline{h}^{-1}(L \cap (X - X''))_{\text{ét}}$ associated to a finitely generated A-module P. By Lemma 7.7 there exists a Zariski-closed subset Y' of Y such that $Y' \neq Y$ and $\underline{g}^{-1}(Y - Y') \cap X''$ is over $Y - Y'$ of relative dimension $\leqslant n - 1$.

Replacing X by an affinoid open subspace of $\underline{g}^{-1}(Y - Y')$, we may assume that $\dim(X''/Y) \leqslant n-1$. By Lemma 7.6(ii) we may assume that the composition

$$X'' \xrightarrow{\ell} \mathbb{B}^n_Y \xrightarrow{q} \mathbb{B}^{n-1}_Y$$

is quasi-finite.

Let $h : (X', \underline{h}^{-1}(L)) \to (X, L)$ be the morphism of pseudo-adic spaces induced by $\underline{h} : X' \to X$ and let F' be the constant A-module on $(X', \underline{h}^{-1}(L))_{\text{ét}}$ associated with the A-module P. Since $\dim(X''/Y) \leqslant n-1$, the induction hypothesis implies that it suffices to show that, for every $m \in \mathbb{N}_0$, the A-module $R^m(g \circ h)_! F'$ on $(Y, M)_{\text{ét}}$ is generically constructible (more precisely, we have to show this for $X' \times_X X' \times \ldots \times_X X'$ instead of X'). Let $\underline{t} : \mathbb{B}^{n-1}_Y \to Y$ be the projection, and let $t : (\mathbb{B}^{n-1}_Y, \underline{t}^{-1}(M)) \to (Y, M)$ and $q : (\mathbb{B}^n_Y, \underline{r}^{-1}(M)) \to (\mathbb{B}^{n-1}_Y, \underline{t}^{-1}(M))$ be the morphisms of pseudo-adic spaces induced by $\underline{t}$ and $\underline{q}$. Then

$$g = r \circ \ell = t \circ q \circ \ell$$

and so

$$g \circ h = t \circ (q \circ \ell \circ h).$$

Put $B := \mathscr{O}_{\mathbb{B}^{n-1}_Y}(\mathbb{B}^{n-1}_Y)$. Since $\underline{\ell} : X \to \mathbb{B}^n_Y$ is étale, there exists by [9], 1.7.3 an affine $\operatorname{Spec} B$-scheme,

$$S \to \operatorname{Spec} B,$$

such that there exists an étale $\operatorname{Spec} B$-morphism $S \to \mathbb{A}^1_{\operatorname{Spec} B}$ and such that X is an open subspace of $S \times_{\operatorname{Spec} B} \mathbb{B}^{n-1}_Y$ and $\underline{q} \circ \underline{\ell}$ is the restriction of the projection $S \times_{\operatorname{Spec} B} \mathbb{B}^{n-1}_Y \to \mathbb{B}^{n-1}_Y$ to X. Then by Lemma 7.5, $R(q \circ \ell \circ h)_!(F') \in \mathscr{S}_4(\mathbb{B}^{n-1}_Y, \underline{t}^{-1}(M))$. This and the induction hypothesis imply that $R^m t_!(R(q \circ \ell \circ h)_!(F'))$ is generically constructible on $(Y, M)_{\text{ét}}$ for every $m \in \mathbb{N}_0$. $\square$

Remark 7.8. — The proof of Theorem 5.1 shows that the open subset U of M in Theorem 5.1 is of the following shape.

Let Y_0 be the set of all points $y \in Y$ whose support $\operatorname{supp}(y) \in \operatorname{Spec} \mathscr{O}_Y(Y)$ is not a generic point of $\operatorname{Spec} \mathscr{O}_Y(Y)$. Put $M_0 = Y_0 \cap M$. We define inductively, for every $n \in \mathbb{N}_0$, a set $\mathscr{M}_n$ of subsets of M: Let $\mathscr{M}_0$ be the set of all finite subsets of M_0 and, for every $n > 0$, let $\mathscr{M}_n$ be the set of all subsets P of M such that there exists a finite subset L of M_0 such that $L \subseteq P$ and, for every quasi-compact open subset W of $M - L$, the intersection $P \cap W$ is an element of $\mathscr{M}_{n-1}$. Then for every $n \in \mathbb{N}_0$

(i) Every element of $\mathscr{M}_n$ is contained in M_0 and is closed in M (even closed in Y). Every $P \in \mathscr{M}_n$ is a 0-dimensional spectral space, and therefore a subset Q of P is pro-constructible if and only Q is closed in P. For every $P \in \mathscr{M}_n$, a subset Q of P is an element of $\mathscr{M}_n$ if and only if Q is closed in P. (This follows from the fact that every element of Y_0 is a closed point of Y).

(ii) If $P, Q \in \mathscr{M}_n$ then $P \cup Q \in \mathscr{M}_n$.

(iii) $\mathscr{M}_n \subseteq \mathscr{M}_{n+1}$.

The proof of Theorem 5.1 shows that if U is the greatest open subset of M such that, for every $m \in \mathbb{N}_0$, $R^m g_! F|U$ is constructible then $M - U \in \mathscr{M}_{\dim(X/Y)}$.

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Manuscrit reçu le 9 juin 2006,
accepté le 28 septembre 2006.

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