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## ANDREEV'S THEOREM ON HYPERBOLIC POLYHEDRA

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*A la mémoire d'Adrien Douady, qui a tant inspiré les auteurs.*

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ABSTRACT. — In 1970, E.M. Andreev published a classification of all three-dimensional compact hyperbolic polyhedra (other than tetrahedra) having non-obtuse dihedral angles. Given a combinatorial description of a polyhedron,  $C$ , Andreev's Theorem provides five classes of linear inequalities, depending on  $C$ , for the dihedral angles, which are necessary and sufficient conditions for the existence of a hyperbolic polyhedron realizing  $C$  with the assigned dihedral angles. Andreev's Theorem also shows that the resulting polyhedron is unique, up to hyperbolic isometry.

Andreev's Theorem is both an interesting statement about the geometry of hyperbolic 3-dimensional space, as well as a fundamental tool used in the proof of Thurston's Hyperbolization Theorem for 3-dimensional Haken manifolds.

We correct a fundamental error in Andreev's proof of existence and also provide a readable new proof of the other parts of the proof of Andreev's Theorem, because Andreev's paper has the reputation of being "unreadable".

RÉSUMÉ. — E.M. Andreev a publié en 1970 une classification des polyèdres hyperboliques compacts de dimension 3 (autre que les tétraèdres) dont les angles dièdres sont non-obtus. Étant donné une description combinatoire d'un polyèdre  $C$ , le théorème d'Andreev dit que les angles dièdres possibles sont exactement décrits par cinq classes d'inégalités linéaires. Le théorème d'Andreev démontre également que le polyèdre résultant est alors unique à isométrie hyperbolique près.

D'une part, le théorème d'Andreev est évidemment un énoncé intéressant de la géométrie de l'espace hyperbolique en dimension 3; d'autre part c'est un outil essentiel dans la preuve du théorème d'hyperbolisation de Thurston pour les variétés Haken de dimension 3.

La démonstration d'Andreev contient une erreur importante. Nous corrigeons ici cette erreur et nous fournissons aussi une nouvelle preuve lisible des autres parties de la preuve, car l'article d'Andreev a la réputation d'être "illisible".

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*Keywords:* Hyperbolic polyhedra, dihedral angles, Andreev's Theorem, Whitehead move, hyperbolic orbifold.

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## 1. Statement of Andreev's Theorem

Andreev's Theorem provides a complete characterization of compact hyperbolic polyhedra having non-obtuse dihedral angles. This classification is essential for proving Thurston's Hyperbolization theorem for Haken 3-manifolds and is also a particularly beautiful and interesting result in its own right. Complete and detailed proofs of Thurston's Hyperbolization for Haken 3-manifolds are available written in English by Jean-Pierre Otal [20] and in French by Michel Boileau [6].

In this paper, we prove Andreev's Theorem based on the main ideas from his original proof [3]. However, there is an error in Andreev's proof of existence. We explain this error in Section 6 and provide a correction. Although the other parts of the proof are proven in much the same way as Andreev proved them, we have re-proven them and re-written them to verify them as well as to make the overall proof of Andreev's Theorem clearer. This paper is based on the doctoral thesis of the first author [26], although certain proofs have been streamlined, especially in Sections 4 and 5.

The reader may also wish to consider the similar results of Rivin & Hodgson [21], [16], Thurston [30, Chap. 13], Marden & Rodin [19], Bowers & Stephenson [8], Rivin [23], [22], [24], and Bao & Bonahon [5]. In [21], the authors prove a more general statement than Andreev's Theorem and in [16] Hodgson deduces Andreev's Theorem as a consequence of their previous work. The proof in [21] is similar to the one presented here, except that the conditions classifying the polyhedra are written in terms of measurements in the De Sitter space, the space dual to the hyperboloid model of hyperbolic space. Although a beautiful result, the main drawback of this proof is that the last sections of the paper, which are necessary for their proof that such polyhedra exist, are particularly hard to follow.

Marden & Rodin [19] and Thurston [30, Chapter 13] consider configurations of circles with assigned overlap angles on the Riemann Sphere and on surfaces of genus  $g$  with  $g > 0$ . Such a configuration on the Riemann Sphere corresponds to a configuration of hyperbolic planes in the Poincaré ball model of hyperbolic space. Thus, there is a direct connection between circle patterns and hyperbolic polyhedra. The proof of Thurston provides a classification of configurations of circles on surfaces of genus  $g > 0$ . The proof of Marden & Rodin [19] is an adaptation of Thurston's circle packing theorem to the Riemann Sphere, resulting in a theorem similar to Andreev's Theorem. Although Thurston's statement has analogous conditions to Andreev's classical conditions, Marden & Rodin require that the sum of angles

be less than  $\pi$  for every triple of circles for which each pair intersects. This prevents the patterns of overlapping circles considered in their theorem from corresponding to compact hyperbolic polyhedra.

Bowers & Stephenson [8] prove a “branched version” of Andreev’s Theorem, also in terms of circle patterns on the Riemann Sphere. Instead of the continuity method used by Thurston and Marden-Rodin, Bowers and Stephenson use ideas intrinsic to the famous Uniformization Theorem from complex analysis. The unbranched version of their theorem provides a complete proof of Andreev’s Theorem, which provides an alternative to the proof presented here.

Rivin has proven beautiful results on ideal hyperbolic polyhedra having arbitrary dihedral angles [23], [22] (see also Guéritaud [15] for an alternative viewpoint, with exceptionally clear exposition.) Similar nice results are proven for hyperideal polyhedra by Bao & Bonahon [5]. Finally, the papers of Vinberg on discrete groups of reflections in hyperbolic space [2], [32], [33], [34], [35] are also closely related, as well as the work of Bennett & Luo [9] and Schlenker [27], [28], [29].

Let  $E^{3,1}$  be  $\mathbb{R}^4$  with the indefinite metric

$$\|\mathbf{x}\|^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

The space of  $\mathbf{x}$  for which this indefinite metric vanishes is typically referred to as the lightcone, which we denote by  $C$ . In this paper, we work in the hyperbolic space  $\mathbb{H}^3$  given by the component of the subset of  $E^{3,1}$  given by

$$\|\mathbf{x}\|^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1$$

having  $x_0 > 0$ , with the Riemannian metric induced by the indefinite metric

$$-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

Hyperbolic space  $\mathbb{H}^3$  can be compactified by adding the set of rays in

$$\{\mathbf{x} \in C : x_0 \geq 0\},$$

which clearly form a topological space  $\partial\mathbb{H}^3$  homeomorphic to the sphere  $\mathbb{S}^2$ . We will refer to points in  $\partial\mathbb{H}^3$  as *points at infinity* and refer to the compactification as  $\overline{\mathbb{H}^3}$ . For more details, see [31, p. 66].

The hyperplane orthogonal to a vector  $\mathbf{v} \in E^{3,1}$  intersects  $\mathbb{H}^3$  if and only if  $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ . Let  $\mathbf{v} \in E^{3,1}$  be a vector with  $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ , and define

$$P_{\mathbf{v}} = \{\mathbf{w} \in \mathbb{H}^3 : \langle \mathbf{w}, \mathbf{v} \rangle = 0\}, \quad H_{\mathbf{v}} = \{\mathbf{w} \in \mathbb{H}^3 : \langle \mathbf{w}, \mathbf{v} \rangle \leq 0\}$$

to be the hyperbolic plane orthogonal to  $\mathbf{v}$  and the corresponding closed half space, oriented so that  $\mathbf{v}$  is the outward pointing normal.

If one normalizes  $\langle \mathbf{v}, \mathbf{v} \rangle = 1$  and  $\langle \mathbf{w}, \mathbf{w} \rangle = 1$  the planes  $P_{\mathbf{v}}$  and  $P_{\mathbf{w}}$  in  $\mathbb{H}^3$  intersect in a line if and only if  $\langle \mathbf{v}, \mathbf{w} \rangle^2 < 1$ , in which case their dihedral angle is  $\arccos(-\langle \mathbf{v}, \mathbf{w} \rangle)$ . They intersect in a single point at infinity if and only if  $\langle \mathbf{v}, \mathbf{w} \rangle^2 = 1$ ; in this case their dihedral angle is 0.

A *hyperbolic polyhedron* is an intersection

$$P = \bigcap_{i=0}^N H_{\mathbf{v}_i}$$

having non-empty interior. Throughout this paper we will make the assumption that  $\mathbf{v}_1, \dots, \mathbf{v}_N$  form a minimal set of vectors specifying  $P$ . That is, we assume that none of the half-spaces  $H_{\mathbf{v}_i}$  contains the intersection of all the others.

It is not hard to verify that if  $H_{\mathbf{v}_i}, H_{\mathbf{v}_j}, H_{\mathbf{v}_k}$  are three distinct halfspaces appearing in the definition of the polyhedron  $P$ , then the vectors  $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$  will always be linearly independent. For example, if  $\mathbf{v}_i = \mathbf{v}_j + \mathbf{v}_k$ , then  $H_{\mathbf{v}_i}$  would be a subset of  $H_{\mathbf{v}_j} \cap H_{\mathbf{v}_k}$ , contradicting minimality.

We will often use the Poincaré ball model of hyperbolic space, given by the open unit ball in  $\mathbb{R}^3$  with the metric

$$4 \frac{dx_1^2 + dx_2^2 + dx_3^2}{(1 - \|\mathbf{x}\|^2)^2}$$

and the upper half-space model of hyperbolic space, given by the subset of  $\mathbb{R}^3$  with  $x_3 > 0$  equipped with the metric

$$\frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}.$$

Both of these models are isomorphic to  $\mathbb{H}^3$ .

Hyperbolic planes in these models correspond to Euclidean hemispheres and Euclidean planes that intersect the boundary perpendicularly. Furthermore, these models are conformally correct, that is, the hyperbolic angle between a pair of such intersecting hyperbolic planes is exactly the Euclidean angle between the corresponding spheres or planes.

Figure 1.1 is an image of a hyperbolic polyhedron depicted in the Poincaré ball model with the sphere at infinity shown for reference. It was displayed in the excellent computer program Geomview [11].

#### *Abstract polyhedra and Andreev's Theorem*

Some elementary combinatorial facts about hyperbolic polyhedra are essential before we can state Andreev's Theorem. Notice that a compact hyperbolic polyhedron  $P$  is topologically a 3-dimensional ball, and its boundary

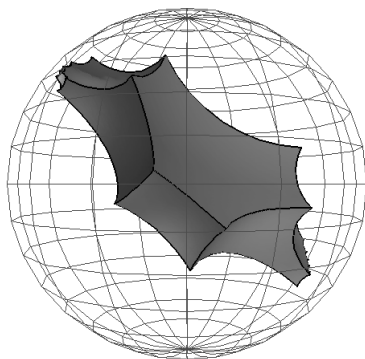


Figure 1.1. A hyperbolic polyhedron displayed by Geomview

a 2-sphere  $\mathbb{S}^2$ . The face structure of  $P$  gives  $\mathbb{S}^2$  the structure of a cell complex  $C$  whose faces correspond to the faces of  $P$ , and so forth.

Considering only hyperbolic polyhedra with non-obtuse dihedral angles simplifies the combinatorics of any such  $C$ :

PROPOSITION 1.1. — (a) A vertex of a non-obtuse hyperbolic polyhedron  $P$  is the intersection of exactly three faces.

(b) For such a  $P$ , we can compute the angles of the faces in terms of the dihedral angles; these angles are also  $\leq \frac{1}{2}\pi$ .

*Proof.* — Let  $v$  be a finite vertex where  $n$  faces of  $P$  meet. After an appropriate isometry, we can assume that  $v$  is the origin in the Poincaré ball model, so that the faces at  $v$  are subsets of Euclidean planes through the origin. A small sphere centered at the origin will intersect  $P$  in a spherical  $n$ -gon  $Q$  whose angles are the dihedral angles between faces. Call these angles  $\alpha_1, \dots, \alpha_n$ . Rescale  $Q$  so that it lies on the sphere of unit radius, then the Gauss-Bonnet formula gives

$$\alpha_1 + \dots + \alpha_n = \pi(n - 2) + \text{Area}(Q).$$

The restriction to  $\alpha_i \leq \frac{1}{2}\pi$  for all  $i$  gives  $\frac{1}{2}n\pi \geq \pi(n - 2) + \text{Area}(Q)$ . Hence,  $\frac{1}{2}n\pi < 2\pi$ . We conclude that  $n = 3$ .

The edge lengths of  $Q$  are precisely the angles in the faces at the origin. Supposing that  $Q$  has angles  $(\alpha_i, \alpha_j, \alpha_k)$  and edge lengths  $(\beta_i, \beta_j, \beta_k)$  with the edge  $\beta_\ell$  opposite of angle  $\alpha_\ell$  for each  $\ell$ , the law of cosines in spherical geometry states that

$$(1.1) \quad \cos(\beta_i) = \frac{\cos(\alpha_i) + \cos(\alpha_j) \cos(\alpha_k)}{\sin(\alpha_j) \sin(\alpha_k)}.$$

Hence, the face angles are calculable from the dihedral angles. They are non-obtuse, since the right-hand side of the equation is positive for  $\alpha_i, \alpha_j, \alpha_k$  non-obtuse. (Equation (1.1) will be used frequently throughout this paper.)  $\square$

The fundamental axioms of incidence place the following, obvious, further restrictions on the complex  $C$ :

- Every edge of  $C$  belongs to exactly two faces.
- A non-empty intersection of two faces is either an edge or a vertex.
- Every face contains not fewer than three edges.

We will call any trivalent cell complex  $C$  on  $\mathbb{S}^2$  that satisfies the three conditions above an *abstract polyhedron*. Notice that since  $C$  must be a trivalent cell complex on  $\mathbb{S}^2$ , its dual,  $C^*$ , has only triangular faces. The three other conditions above ensure that the dual complex  $C^*$  is a simplicial complex, which we embed in the same  $\mathbb{S}^2$  so that the vertex corresponding to any face of  $C$  is an element of the face, etc. (Andreev refers to this dual complex as the *scheme of the polyhedron*.) Figure 1.2 shows an abstract polyhedron  $C$  drawn in the plane (i.e. with one of the faces corresponding to the region outside of the figure.) The dual complex is also shown, in dashed lines.

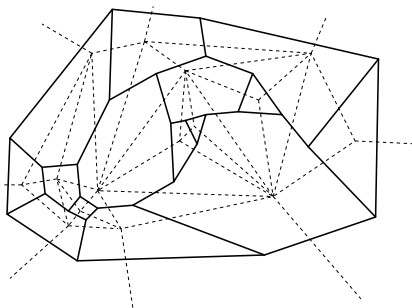


Figure 1.2. An abstract polyhedron drawn in the plane

We call a simple closed curve  $\Gamma$  formed of  $k$  edges of  $C^*$  a *k-circuit* and if all of the endpoints of the edges of  $C$  intersected by  $\Gamma$  are distinct, we call such a circuit a *prismatic k-circuit*. The figure below shows the same abstract polyhedron as above, except this time the prismatic 3-circuits are dashed, the prismatic 4-circuits are dotted, and the dual complex is not shown. Before stating Andreev's Theorem, we prove two basic lemmas about abstract polyhedra:

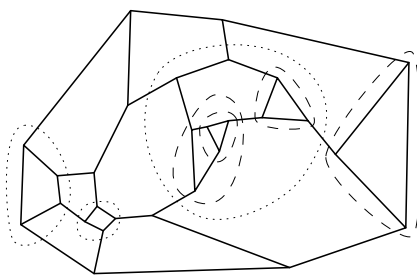


Figure 1.3. The same polyhedron where the prismatic 3-circuits are dashed and the prismatic 4-circuits are dotted, and without the dual complex

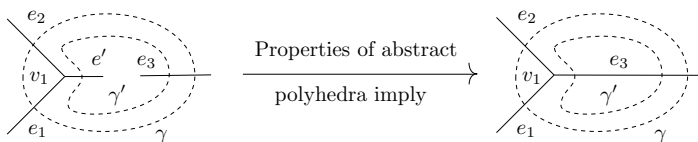


Figure 1.4

LEMMA 1.2. — *If  $\gamma$  is a 3-circuit that is not prismatic in an abstract polyhedron  $C$  intersecting edges  $e_1, e_2, e_3$ , then edges  $e_1, e_2$ , and  $e_3$  meet at a vertex.*

*Proof.* — Since  $\gamma$  is 3-circuit that is not prismatic, a pair of the edges meet at a vertex. We suppose that  $e_1$  and  $e_2$  meet at this vertex, which we label  $v_1$ . Since the vertices of  $C$  are trivalent, there is some edge  $e'$  meeting  $e_1$  and  $e_2$  at  $v_1$ . We suppose that  $e'$  is not the edge  $e_3$  to obtain a contradiction. Moving  $\gamma$  past the vertex  $v_1$ , we can obtain a new circuit  $\gamma'$  intersecting only the two edges  $e_3$  and  $e'$  (see Figure 1.4).

The curve  $\gamma'$  intersects only two edges, hence it only crosses two faces of  $C$ . However, this implies that these two faces of  $C$  intersect along the two distinct edges  $e'$  and  $e_3$ , contrary to fact that two faces of an abstract polyhedron which intersect do so along a single edge.  $\square$

LEMMA 1.3. — *Let  $C$  be an abstract polyhedron having no prismatic 3-circuits. If  $\gamma$  is a 4-circuit which is not prismatic, then  $\gamma$  separates exactly two vertices of  $C$  from the remaining vertices of  $C$ .*

*Proof.* — Suppose that  $\gamma$  crosses edges  $e_1, e_2, e_3$ , and  $e_4$  of  $C$ . Because  $\gamma$  is not a prismatic 4-circuit, a pair of these edges meet at a vertex. Without



loss of generality, we suppose that edges  $e_1$  and  $e_2$  meet at this vertex, which we denote  $v_1$ . Since  $C$  is trivalent, there is some edge  $e'$  meeting  $e_1$  and  $e_2$  at  $v_1$ . Let  $\gamma'$  be the 3-circuit intersecting edges  $e_3, e_4$  and  $e'$ , obtained by sliding  $\gamma$  past the vertex  $v_1$ . Since  $C$  has no prismatic 3-circuits,  $\gamma'$  is not prismatic, so by Lemma 1.2, edges  $e_3, e_4$ , and  $e'$  meet at another vertex  $v_2$ . The entire configuration is shown in Figure 1.5.

Therefore, the 4-circuit  $\gamma$  separates the two vertices  $v_1$  and  $v_2$  from the remaining vertices of  $C$ . □

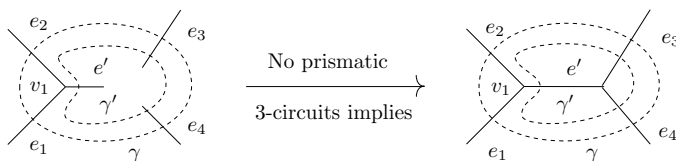


Figure 1.5

**THEOREM 1.4 (Andreev’s Theorem).** — *Let  $C$  be an abstract polyhedron with more than four faces and suppose that non-obtuse angles  $\alpha_i$  are given corresponding to each edge  $e_i$  of  $C$ . There is a compact hyperbolic polyhedron  $P$  whose faces realize  $C$  with dihedral angle  $\alpha_i$  at each edge  $e_i$  if and only if the following five conditions all hold:*

- 1) For each edge  $e_i$ ,  $\alpha_i > 0$ .
- 2) Whenever three distinct edges  $e_i, e_j, e_k$  meet at a vertex, then  $\alpha_i + \alpha_j + \alpha_k > \pi$ .
- 3) Whenever  $\Gamma$  is a prismatic 3-circuit intersecting edges  $e_i, e_j, e_k$ , then  $\alpha_i + \alpha_j + \alpha_k < \pi$ .
- 4) Whenever  $\Gamma$  is a prismatic 4-circuit intersecting edges  $e_i, e_j, e_k, e_l$ , then  $\alpha_i + \alpha_j + \alpha_k + \alpha_l < 2\pi$ .
- 5) Whenever there is a four sided face bounded by edges  $e_1, e_2, e_3, e_4$ , enumerated successively, with edges  $e_{12}, e_{23}, e_{34}, e_{41}$  entering the four vertices (edge  $e_{ij}$  connects to the ends of  $e_i$  and  $e_j$ ), then

$$\alpha_1 + \alpha_3 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi, \quad \text{and}$$

$$\alpha_2 + \alpha_4 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi.$$

Furthermore, this polyhedron is unique up to isometries of  $\mathbb{H}^3$ .

In addition to the role that Andreev’s Theorem plays, as a bootstrap in the proof of Thurston’s hyperbolization Theorem, it is worth noting that,

in the context of orbifolds, the former can be thought of as a very special case of the latter (extended to Haken orbifolds as in [7, Chap. 8] or [10]). Consider closed 3-orbifolds with underlying topological space a 3-ball, and with singular set equal to the boundary sphere. That singular set will consist of a trivalent graph, together with “mirrors” on the complementary regions. Each edge of the graph is labeled with an integer  $k > 1$ , corresponding to a dihedral angle of  $\pi/k$ . The definition of a 3-orbifold implies that the angle sum at each vertex will satisfy condition 2) in Andreev's Theorem [17, Sections 6.1 and 6.3].

Restrict the combinatorics of the singular set, slightly more than in the statement of Andreev's Theorem:  $C$  must be an abstract polyhedron with more than five faces. Such an orbifold is Haken if and only if it is irreducible [30, Prop. 13.5.2]. Condition 3) in Andreev's Theorem guarantees irreducibility, and also, together with condition 4), guarantees that every Euclidean 2-suborbifold is compressible. Therefore, for Haken orbifolds of this topological type, Andreev's Theorem says precisely that having no incompressible Euclidean 2-suborbifolds is equivalent to the existence of a hyperbolic structure [30, Section 13.6]; see also [17, Section 6.4].

For a given  $C$  let  $E$  be the number of edges of  $C$ . The subset of  $(0, \frac{1}{2}\pi]^E$  satisfying these linear inequalities will be called the *Andreev Polytope*,  $A_C$ . Since  $A_C$  is determined by linear inequalities, it is convex.

Andreev's restriction to non-obtuse dihedral angles is emphatically necessary to ensure that  $A_C$  be convex. Without this restriction, the corresponding space of dihedral angles,  $\Delta_C$ , of compact (or finite volume) hyperbolic polyhedra realizing a given  $C$  is not convex [12]. In fact, the recent work by Díaz [13] provides a detailed analysis of this space of dihedral angles  $\Delta_C$  for the class of abstract polyhedra  $C$  obtained from the tetrahedron by successively truncating vertices. Her work nicely illustrate the types of non-linear conditions that are necessary in a complete analysis of the larger space of dihedral angles  $\Delta_C$ .

The work of Rivin [23], [22] shows that the space of dihedral angles for ideal polyhedra forms a convex polytope, *without the restriction to non-obtuse angles* (see also [15]).

Notice also that the hypothesis that the number of faces is greater than four is also necessary because the space of non-obtuse dihedral angles for compact tetrahedra is not convex [25]. Conditions 1)–5) remain necessary conditions for compact tetrahedra, but they are no longer sufficient.

**PROPOSITION 1.5.** — *If  $C$  is not the triangular prism, condition 5) of Andreev's Theorem is a consequence of conditions 3) and 4).*

*Proof.* — Given a quadrilateral face, if the four edges leading from it form a prismatic 4-circuit,  $\Gamma_1$ , as depicted on the left hand side of Figure 1.6, clearly condition 5) is a result of condition 4). Otherwise, at least one pair of the edges leading from it meet at a vertex. If only one pair meets at a point, we have the diagram of Figure 1.6 in the middle. In this case, the curve  $\Gamma_2$  can easily be shown to be a prismatic 3-circuit, so that  $\alpha_{34} + \alpha_{41} + \beta < \pi$ , so that condition 5) is satisfied because  $\alpha_{34}$  and  $\alpha_{41}$  cannot both be  $\frac{1}{2}\pi$ .

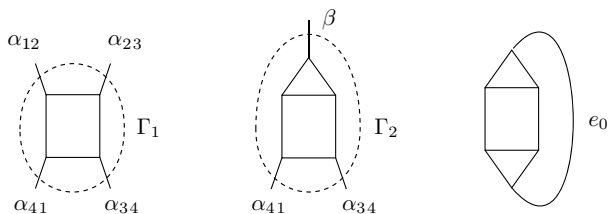


Figure 1.6

Otherwise, if two pairs of the edges leaving the quadrilateral face meet at vertices, we have the diagram on the right-hand side of Figure 1.6. The only way to complete this diagram is with the edge labeled  $e_0$ , resulting in the triangular prism.  $\square$

Hence, we need only check condition 5) for the triangular prism, which corresponds to the only five-faced  $C$ .

Given some  $C$ , it may be a difficult problem to determine whether  $A_C = \emptyset$  and correspondingly, whether there are any hyperbolic polyhedra realizing  $C$  with non-obtuse dihedral angles. In fact, for the abstract polyhedron in Figure 1.7, conditions 2) and 3) imply respectively that

$$\alpha_1 + \dots + \alpha_{12} > 4\pi \quad \text{and} \quad \alpha_1 + \dots + \alpha_{12} < 4\pi.$$

So, for this  $C$ , we have  $A_C = \emptyset$ . However, for more complicated  $C$ , it can be significantly harder to determine whether  $A_C = \emptyset$ .

Luckily, there are special cases, including:

**COROLLARY 1.6.** — *If there are no prismatic 3-circuits in  $C$ , there exists a unique hyperbolic polyhedron realizing  $C$  with dihedral angles  $\frac{2}{5}\pi$ .*

*Proof.* — Since there are no prismatic 3-circuits in  $C$ , condition 3) of the theorem is vacuous and clearly  $\alpha_i = \frac{2}{5}\pi$  satisfy conditions 1), 2), 4), and 5).  $\square$

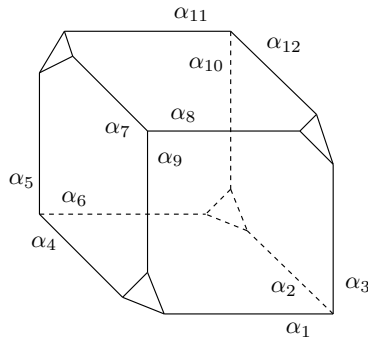


Figure 1.7

### 2. Setup of the proof

Let  $C$  be a trivalent abstract polyhedron with  $N$  faces. We say that a hyperbolic polyhedron  $P \subset \mathbb{H}^3$  realizes  $C$  if there is a cellular homeomorphism from  $C$  to  $\partial P$  (i.e., a homeomorphism mapping faces of  $C$  to faces of  $P$ , edges of  $C$  to edges of  $P$ , and vertices of  $C$  to vertices of  $P$ ). We will call each isotopy class of cellular homeomorphisms  $\phi : C \rightarrow \partial P$  a marking on  $P$ .

We will define  $\mathcal{P}_C$  to be the set of pairs  $(P, \phi)$  so that  $\phi$  is a marking with the equivalence relation that  $(P, \phi) \sim (P', \phi')$  if there exists an isomorphism  $\alpha : \mathbb{H}^3 \rightarrow \mathbb{H}^3$  such that  $\alpha(P) = P'$  and both  $\phi'$  and  $\alpha \circ \phi$  represent the same marking on  $P'$ .

PROPOSITION 2.1. — *The space  $\mathcal{P}_C$  is a manifold of dimension  $3N - 6$  (perhaps empty).*

*Proof.* — Let  $\mathcal{H}$  be the space of closed half-spaces of  $\mathbb{H}^3$ ; clearly  $\mathcal{H}$  is a 3-dimensional manifold. Let  $\mathcal{O}_C$  be the set of marked hyperbolic polyhedra realizing  $C$ . Using the marking to number the faces from 1 to  $N$ , an element of  $\mathcal{O}_C$  is an  $N$ -tuple of half-spaces that intersect in a polyhedron realizing  $C$ . This induces a mapping from  $\mathcal{O}_C$  to  $\mathcal{H}^N$  whose image is an open set. We give  $\mathcal{O}_C$  the topology that makes this mapping from  $\mathcal{O}_C$  into  $\mathcal{H}^N$  a local homeomorphism. Since  $\mathcal{H}^N$  is a  $3N$ -dimensional manifold,  $\mathcal{O}_C$  must be a  $3N$ -dimensional manifold as well.

If  $\alpha(P, \phi) = (P, \phi)$ , we have that  $\alpha \circ \phi$  is isotopic to  $\phi$  through cellular homeomorphisms. Hence, the isomorphism  $\alpha$  must fix all vertices of  $P$ , and consequently restricts to the identity on all edges and faces. However,

an isomorphism of  $\mathbb{H}^3$  which fixes four non-coplanar points must be the identity. Therefore  $\text{Isom}(\mathbb{H}^3)$  acts freely on  $\mathcal{O}_C$ . This quotient space of this action is  $\mathcal{P}_C$ , hence  $\mathcal{P}_C$  is a manifold with dimension equal to  $\dim(\mathcal{O}_C) - \dim(\text{Isom}(\mathbb{H}^3)) = 3N - 6$ .  $\square$

An  $m$ -sided polygon  $Q \subset \mathbb{H}^2$  with sides  $s_i$  supported by lines  $\ell_i$  for  $i = 1, \dots, m$  is called a *parallelogram* if, after adjoining any ideal endpoints in  $\partial\mathbb{H}^2$  to these sides and lines,  $s_i \cap s_j = \emptyset$  implies  $\ell_i \cap \ell_j = \emptyset$ . In other words, if two sides of  $Q$  don't meet in  $\overline{\mathbb{H}^2}$ , then their supporting lines have a common perpendicular. We then define  $\mathcal{P}_C^1$  to be the subset of  $\mathcal{P}_C$  consisting of those polyhedra all of whose faces are parallelograms.

Let's check that  $\mathcal{P}_C^1$  is an open subset of  $\mathcal{P}_C$ . If  $P_{v_i}$ ,  $i = 1, 2, 3$  are three planes carrying faces of  $P \in \mathcal{P}_C$ , then  $\{v_i, i = 1, 2, 3\}$  will be a linearly independent set spanning a subspace  $V$ . Such a triple of planes has no common intersection point in  $\overline{\mathbb{H}^3}$  if and only if the metric is indefinite when restricted to  $V$ , or equivalently, if and only if every vector orthogonal to  $V$  has positive inner product with itself. This is an open condition on triples of half-spaces; hence, it is an open condition on  $\mathcal{P}_C$  to require that the planes supporting three fixed faces of  $P$  have no intersection in  $\overline{\mathbb{H}^3}$ .

Requiring that a single face of  $P$  be a parallelogram is a finite intersection of such open conditions, for triples formed of that face and two faces whose intersections with that face form non-adjacent edges. For  $P$  to lie in  $\mathcal{P}_C^1$  is a further finite intersection over its faces, so  $\mathcal{P}_C^1$  is an open subset of  $\mathcal{P}_C$ , and hence  $\mathcal{P}_C^1$  is a manifold of dimension  $3N - 6$ , as well.

In fact, we are most interested in the subset  $\mathcal{P}_C^0$  of polyhedra with non-obtuse dihedral angles. Notice that  $\mathcal{P}_C^0$  is not, *a priori*, a manifold or even a manifold with boundary. However, as a consequence of Proposition 1.1 (b) and the fact that polygons with non-obtuse interior angles are parallelograms, we have the inclusion  $\mathcal{P}_C^0 \subset \mathcal{P}_C^1$ .

Using the fact that the edge graph of  $C$  is trivalent, one can check that  $E$ , the number of edges of  $C$ , is the same as the dimension of  $\mathcal{P}_C^1$ . Since exactly three edges enter each vertex and each edge enters exactly two vertices,  $3V = 2E$ . The Euler characteristic gives  $2 = N - E + V = N - E + \frac{2}{3}E$  implying  $E = 3(N - 2)$ , the dimension of  $\mathcal{P}_C$  and  $\mathcal{P}_C^1$ .

Given any  $P \in \mathcal{P}_C$ , let  $\alpha(P) = (\alpha_1, \alpha_2, \alpha_3, \dots)$  be the  $E$ -tuple consisting of the dihedral angles of  $P$  at each edge (according to some fixed numbering of the edges of  $C$ ). This map  $\alpha$  is obviously continuous with respect to the topology on  $\mathcal{P}_C$ , which it inherits from its manifold structure.

Our goal is to prove the following theorem, of which Andreev's Theorem is a consequence:

**THEOREM 2.2.** — *For every abstract polyhedron  $C$  having more than four faces, the mapping  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is a homeomorphism.*

We will say that *Andreev's Theorem holds for  $C$*  if  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is a homeomorphism for a specific abstract polyhedron  $C$ .

We begin the proof of Theorem 2.2 by checking that  $\alpha(\mathcal{P}_C^0) \subset A_C$  in Section 3. In Section 4, we prove that  $\alpha$  restricted to  $\mathcal{P}_C^1$  is injective, and in Section 5, we prove that  $\alpha$  restricted to  $\mathcal{P}_C^0$  is proper. In the beginning of Section 6 we combine these results to show that  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is a homeomorphism onto its image and that this image is a component of  $A_C$ . The remaining — and most substantial — part of Section 6 is to show that  $A_C \neq \emptyset$  implies  $\mathcal{P}_C^0 \neq \emptyset$ .

### 3. The inequalities are satisfied.

**PROPOSITION 3.1.** — *Given  $P \in \mathcal{P}_C^0$ , the dihedral angles  $\alpha(P)$  satisfy conditions 1)–5).*

We will need the following two lemmas about the basic properties of hyperbolic geometry.

**LEMMA 3.2.** — *Suppose that three planes  $P_{\mathbf{v}_1}, P_{\mathbf{v}_2}, P_{\mathbf{v}_3}$  intersect pairwise in  $\mathbb{H}^3$  with non-obtuse dihedral angles  $\alpha, \beta,$  and  $\gamma$ . Then,  $P_{\mathbf{v}_1}, P_{\mathbf{v}_2}, P_{\mathbf{v}_3}$  intersect at a vertex in  $\overline{\mathbb{H}^3}$  if and only if  $\alpha + \beta + \gamma \geq \pi$ . The planes intersect in  $\mathbb{H}^3$  if and only if the inequality is strict.*

*Proof.* — The planes intersect in a point of  $\overline{\mathbb{H}^3}$  if and only if the inner product is either positive definite or semi-definite on the subspace  $V$  spanned by  $\{\mathbf{v}_i, i = 1, 2, 3\}$ . In the former case the intersection point is in  $\mathbb{H}^3$ , and in the latter case it is in  $\partial\mathbb{H}^3$ ; in both cases the point is determined by the orthogonal complement of  $V$ . The matrix describing the inner product on  $V$  is

$$\begin{bmatrix} 1 & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \\ \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & 1 & \langle \mathbf{v}_2, \mathbf{v}_3 \rangle \\ \langle \mathbf{v}_1, \mathbf{v}_3 \rangle & \langle \mathbf{v}_2, \mathbf{v}_3 \rangle & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\cos \alpha & -\cos \beta \\ -\cos \alpha & 1 & -\cos \gamma \\ -\cos \beta & -\cos \gamma & 1 \end{bmatrix}$$

where  $\alpha, \beta,$  and  $\gamma$  are the dihedral angles between the pairs of faces  $(P_{\mathbf{v}_1}, P_{\mathbf{v}_2}), (P_{\mathbf{v}_1}, P_{\mathbf{v}_3}),$  and  $(P_{\mathbf{v}_2}, P_{\mathbf{v}_3}),$  respectively.

Since the principal minor is positive definite for  $0 < \alpha \leq \frac{1}{2}\pi$ , it is enough to find out when the following determinant is non-negative:

$$1 - 2 \cos \alpha \cos \beta \cos \gamma - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma.$$

A bit of trigonometric trickery (we used complex exponentials) shows that the expression above is equal to

$$(3.1) \quad -4 \cos\left(\frac{\alpha + \beta + \gamma}{2}\right) \cos\left(\frac{\alpha - \beta + \gamma}{2}\right) \times \cos\left(\frac{\alpha + \beta - \gamma}{2}\right) \cos\left(\frac{-\alpha + \beta + \gamma}{2}\right).$$

Let  $\delta = \alpha + \beta + \gamma$ . When  $\delta < \pi$ , (3.1) is strictly negative; when  $\delta = \pi$ , (3.1) is clearly zero; and when  $\delta > \pi$ , (3.1) is strictly positive. Hence the inner product on the space spanned by  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is positive semidefinite if and only if  $\delta \geq \pi$ . It is positive definite if and only if  $\delta > \pi$ .

Then it is easy to see that the three planes  $P_{\mathbf{v}_1}, P_{\mathbf{v}_2}, P_{\mathbf{v}_3} \subset \mathbb{H}^3$  intersect at a point in  $\mathbb{H}^3$  if and only if they intersect pairwise in  $\mathbb{H}^3$  and the sum of the dihedral angles  $\delta \geq \pi$ . It is also clear that they intersect at a finite point if and only if the inequality is strict.  $\square$

LEMMA 3.3. — *Let  $P_1, P_2, P_3 \subset \mathbb{H}^3$  be planes carrying faces of a polyhedron  $P$  that has all dihedral angles  $\leq \frac{1}{2}\pi$ .*

(a) *If  $P_1, P_2, P_3$  intersect at a point in  $\mathbb{H}^3$ , then the point  $p = P_1 \cap P_2 \cap P_3$  is a vertex of  $P$ .*

(b) *If  $P_1, P_2, P_3$  intersect at a point in  $\partial\mathbb{H}^3$ , then  $P$  is not compact, and the point of intersection is in the closure of  $P$ .*

*Proof.* — (a) Consider what we see in the plane  $P_1$ . Let  $H_i$  be the half-space bounded by  $P_i$  which contains the interior of  $P$ , and let  $Q = P_1 \cap H_2 \cap H_3$ . If  $p \notin P$ , then let  $U$  be the component of  $Q - P$  that contains  $p$  in its closure. This is a non-convex polygon; let  $p, p_1, \dots, p_k$  be its vertices. The exterior angles of  $U$  at  $p_1, \dots, p_k$  are the angles of the face of  $P$  carried by  $P_1$ , hence  $\leq \frac{1}{2}\pi$  by part (b) of Proposition 1.1 (see Figure 3.1).

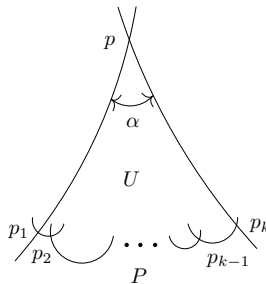


Figure 3.1

Suppose that  $\alpha_1, \dots, \alpha_k$  are the angles of  $P$  at  $p_1, \dots, p_k$ , and let  $\alpha$  be the angle at  $p$ . Then the Gauss-Bonnet formula tells us that

$$(\pi - \alpha) + \alpha_1 - ((\pi - \alpha_2) + \dots + (\pi - \alpha_{k-1})) + \alpha_k - \text{Area}(U) = 2\pi,$$

which can be rearranged to read

$$(\alpha_1 + \alpha_k - \pi) - \alpha - \sum_{j=2}^{k-1} (\pi - \alpha_j) = \text{Area}(U).$$

This is clearly a contradiction. All of the terms on the left are non-positive, and  $\text{Area}(U) > 0$ . If  $p$  is at infinity (i.e.,  $\alpha = 0$ ), this expression is still a contradiction, proving part (b). □

*Proof of Proposition 3.1.* — For condition 1), notice that if two adjacent faces intersect at dihedral angle 0, they intersect at a point at infinity. If this were the case,  $P$  would be non-compact.

For condition 2), let  $x$  be a vertex of  $P$ . Since  $P$  is compact,  $x \in \mathbb{H}^3$  and by Lemma 3.2, the sum of the dihedral angles between the three planes intersecting at  $x$  must be  $> \pi$ .

For condition 3), note first that by Lemma 3.2 if three faces forming a 3-circuit have dihedral angles summing to a number  $\geq \pi$ , then they meet in  $\overline{\mathbb{H}^3}$ . If they meet at a point in  $\mathbb{H}^3$ , by Lemma 3.3 (a) this point is a vertex of  $P$ , so these three faces do not form a prismatic 3-circuit. Alternatively, if the three planes meet in  $\partial\mathbb{H}^3$  by Lemma 3.3 (b), then  $P$  is non-compact, contrary to assumption. Hence, any three faces forming a prismatic 3-circuit in  $P$  must have dihedral angles summing to  $< \pi$ .

For condition 4), let  $H_{\mathbf{v}_1}, H_{\mathbf{v}_2}, H_{\mathbf{v}_3}, H_{\mathbf{v}_4}$  be half spaces corresponding to the faces which form a prismatic 4-circuit; obviously condition 4) is satisfied unless all of the dihedral angles are  $\frac{1}{2}\pi$ , so we suppose that they are. We will assume the normalization  $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$  for each  $i$ . The Gram matrix

$$Q = [\langle \mathbf{v}_i, \mathbf{v}_j \rangle]_{i,j} = \begin{bmatrix} 1 & 0 & \langle \mathbf{v}_1, \mathbf{v}_3 \rangle & 0 \\ 0 & 1 & 0 & \langle \mathbf{v}_2, \mathbf{v}_4 \rangle \\ \langle \mathbf{v}_3, \mathbf{v}_1 \rangle & 0 & 1 & 0 \\ 0 & \langle \mathbf{v}_4, \mathbf{v}_2 \rangle & 0 & 1 \end{bmatrix}$$

has determinant 0 if the  $\mathbf{v}$ 's are linearly dependent, and otherwise represents the inner product of  $E^{3,1}$  and hence has negative determinant. In both cases we have

$$\det Q = (1 - \langle \mathbf{v}_1, \mathbf{v}_3 \rangle^2)(1 - \langle \mathbf{v}_2, \mathbf{v}_4 \rangle^2) \leq 0.$$



So  $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle^2 \leq 1$  and  $\langle \mathbf{v}_2, \mathbf{v}_4 \rangle^2 \geq 1$  or vice versa (perhaps one or both are equalities). This means that one of the opposite pairs of faces of the 4-circuit intersect, perhaps at a point at infinity. We can suppose that this pair is  $H_{\mathbf{v}_1}$  and  $H_{\mathbf{v}_3}$ .

If  $H_{\mathbf{v}_1}$  and  $H_{\mathbf{v}_3}$  intersect in  $\mathbb{H}^3$ , they do so with positive dihedral angle. Since  $H_{\mathbf{v}_2}$  intersects each  $H_{\mathbf{v}_1}$  and  $H_{\mathbf{v}_3}$  orthogonally, the three faces pairwise intersect and have dihedral angle sum  $> \pi$ . By Lemmas 3.2 and 3.3 these three faces intersect at a point in  $\mathbb{H}^3$  which is a vertex of  $P$ . In this case, the 4-circuit  $H_{\mathbf{v}_1}, H_{\mathbf{v}_2}, H_{\mathbf{v}_3}, H_{\mathbf{v}_4}$  is not prismatic.

Otherwise,  $H_{\mathbf{v}_1}$  and  $H_{\mathbf{v}_3}$  intersect at a point at infinity. In this case, since  $H_{\mathbf{v}_2}$  intersects each  $H_{\mathbf{v}_1}$  and  $H_{\mathbf{v}_3}$  with dihedral angle  $\frac{1}{2}\pi$  the three faces intersect at this point at infinity by Lemma 3.2 and then by Lemma 3.3  $P$  is not compact, contrary to assumption.

Hence, if  $H_{\mathbf{v}_1}, H_{\mathbf{v}_2}, H_{\mathbf{v}_3}, H_{\mathbf{v}_4}$  forms a prismatic 4-circuit, the sum of the dihedral angles cannot be  $2\pi$ .

For condition 5), suppose that the quadrilateral is formed by edges  $e_1, e_2, e_3, e_4$ . Violation of one of the inequalities would give that the dihedral angles at each of the edges  $e_{ij}$  leading to the quadrilateral is  $\frac{1}{2}\pi$  and that the dihedral angles at two of the opposite edges of the quadrilateral are  $\frac{1}{2}\pi$  (see Figure 3.2).

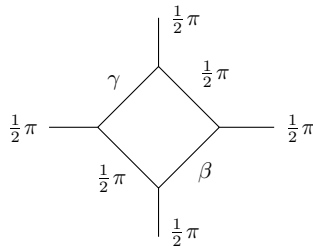


Figure 3.2

Each vertex of this quadrilateral had three incident edges labeled  $e_i, e_j$ , and  $e_{ij}$ . Violation of the inequality gives that  $\alpha_{ij} = \frac{1}{2}\pi$  and either  $\alpha_i = \frac{1}{2}\pi$  or  $\alpha_j = \frac{1}{2}\pi$ . Using Equation (1.1) from Section 1, we see that each face angle in the quadrilateral must be  $\frac{1}{2}\pi$ . So, we have that each of the face angles of the quadrilateral is  $\frac{1}{2}\pi$ , which is a contradiction to the Gauss-Bonnet Theorem. Hence both of the inequalities in condition 5) must be satisfied.

This was the last step in proving Proposition 3.1. □

#### 4. The mapping $\alpha : \mathcal{P}_C^1 \rightarrow \mathbb{R}^E$ is injective.

Recall from Section 2 an  $m$ -sided polygon  $Q \subset \mathbb{H}^2$  with sides  $s_i$  supported by lines  $\ell_i$  for  $i = 1, \dots, m$  is called a *parallelogram* if, after adjoining any ideal endpoints in  $\partial\mathbb{H}^2$  to these sides and lines,  $s_i \cap s_j = \emptyset$  implies  $\ell_i \cap \ell_j = \emptyset$ .

In Section 2 we defined  $\mathcal{P}_C^1$  to be the space of polyhedra realizing  $C$  whose faces are parallelograms. We then checked that  $\mathcal{P}_C^1$  is an open subset of  $\mathcal{P}_C$  and that  $\mathcal{P}_C^0 \subset \mathcal{P}_C^1$ . The goal of this section is to prove

PROPOSITION 4.1. — *The mapping  $\alpha : \mathcal{P}_C^1 \rightarrow \mathbb{R}^E$  is injective.*

*Proof.* — Suppose that  $P, P' \in \mathcal{P}_C^1$  are two polyhedra such that  $\alpha(P) = \alpha(P')$ . We can label each edge  $e$  of  $C$  by  $-$ ,  $0$ , or  $+$  depending on whether the length of  $e$  in  $P'$  is less than, equal to, or greater than the length of  $e$  in  $P$ .

We will prove that if  $\alpha(P) = \alpha(P')$  then each pair of corresponding edges has the same length. This gives that the faces of  $P$  and  $P'$  are congruent since the face angles are determined by the dihedral angles (Proposition 1.1). Then, since  $P$  and  $P'$  have congruent faces and the same dihedral angles, they are themselves congruent.

Each edge of  $C$  corresponds to a unique edge of the dual complex,  $C^*$ , which we label with  $-$ ,  $0$ , or  $+$ , accordingly. Consider the graph  $\mathcal{G}$  consisting of the edges of  $C^*$  labeled either  $+$  or  $-$ , but not  $0$ , together with the vertices incident to these edges. Since  $C^*$  is a simplicial complex on  $\mathbb{S}^2$ ,  $\mathcal{G}$  is a simple planar graph. (Here, simple means that there is at most one edge between any distinct pair of vertices and no edges from a vertex to itself.) We assume that  $\mathcal{G}$  is non-empty, in order to find a contradiction.

PROPOSITION 4.2. — *Let  $\mathcal{G}$  be a simple planar graph whose edges are labeled with  $+$  and  $-$ . There is a vertex of  $\mathcal{G}$  with at most two sign changes when following the cyclic order of the edges meeting at that vertex.*

Proposition 4.2 provides the global statement necessary for Cauchy's rigidity theorem on Euclidean polyhedra, see [1, Chap. 12], and also the global statement necessary here. The proof, see [1, p. 68], is a clever, yet elementary counting argument, combined with Euler's Formula.

Therefore, at some vertex of  $\mathcal{G}$ , there are either zero or two changes of sign when following the cyclic order of edges meeting at that vertex. In the case that there are zero changes in sign, we may assume, without loss in generality that all of the signs at this vertex are  $+$ 's, by switching the roles of  $P$  and  $P'$ , if necessary. Thus, in either case, there is a face  $F$  of  $C$

not marked entirely with 0's so that, once the edges labeled  $-$  are removed from  $\partial F$ , the edges labeled  $+$  all lie in the same component of what remains.

Let  $Q$  and  $Q'$  be the faces in  $P$  and  $P'$  corresponding to  $F$ . By the assumption that  $P, P' \in \mathcal{P}_C^1$ ,  $Q$  and  $Q'$  are parallelograms and because  $P$  and  $P'$  have the same dihedral angles,  $Q$  and  $Q'$  must have the same face angles. We will now show that  $Q$  and  $Q'$  cannot have side lengths differing according to the distribution of  $+$ 's and  $-$ 's on  $\partial F$  that was deduced above.

The following lemma, from [3, p. 422] but with a new proof, shows that (roughly speaking) stretching edges in a piece of the boundary will pull apart the two edges at the ends of that piece. It is important to keep in mind that the parallelograms  $R$  and  $R'$  need not be compact and need not have finite volume, since there are no restrictions on whether the first and  $m$ -th sides intersect.

LEMMA 4.3 (Andreev's Auxiliary Lemma). — *Let  $R$  and  $R'$  be  $m$ -sided parallelograms further assume that  $R$  and  $R'$  have finite vertices  $A_i = s_i \cap s_{i+1}$  and  $A'_i = s'_i \cap s'_{i+1}$  for  $i = 1, \dots, m - 1$ . If*

- *the interior angle at vertex  $A_i$  and vertex  $A'_i$  are equal for  $i = 2, \dots, m - 1$ , and*
- *$|s_j| \leq |s'_j|$  for  $j = 2, \dots, m - 1$ ,*

*with at least one of the inequalities strict, then*

$$\langle \mathbf{v}_1, \mathbf{v}_m \rangle > \langle \mathbf{v}'_1, \mathbf{v}'_m \rangle,$$

*where  $\mathbf{v}_i$  and  $\mathbf{v}'_m$  are the outward pointing normal to the edge  $s_i$  of  $R$  and  $R'$ , respectively.*

*Proof.* — We will prove the lemma first in the case where the side lengths differ only at one side  $|s_j| < |s'_j|$  and then observe that the resulting polygon again satisfies the hypotheses of the lemma so that one can repeat as necessary for each pair of sides that differ in length.

We can situate side  $s_j$  on the line  $x_2 = 0$  centered at  $(1, 0, 0)$  within the upper sheet of the hyperboloid  $-x_0^2 + x_1^2 + x_2^2 = -1$  and assume that  $R$  is entirely "above" this line, that is at points with  $x_2 \geq 0$ . We also assume that the sides of  $R$  are labeled counterclockwise, i.e.,  $s_{i+1}$  is counterclockwise from  $s_i$  for each  $i$ . Applying the isometry

$$I(t) = \begin{bmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to the sides  $s_i$  with index  $i > j$  for  $t > 0$  performs the desired deformation of  $R$ .

If we write  $\mathbf{v}_1 = (v_{10}, v_{11}, v_{12})$  and  $\mathbf{v}_m = (v_{m0}, v_{m1}, v_{m2})$  one checks that  $\frac{d}{dt} \langle \mathbf{v}_1, I(t)\mathbf{v}_m \rangle = \sinh(t)(v_{m1}v_{11} - v_{m0}v_{10}) - \cosh(t)(v_{m1}v_{10} - v_{m0}v_{11})$ .

Since  $(1, 0, 0)$  is in the interior of  $s_j$  we must have that  $\langle (1, 0, 0), \mathbf{v}_1 \rangle < 0$ , which is equivalent to  $v_{10} > 0$ . For the same reason we also have  $v_{m0} > 0$ .

We will first check that this derivative  $d\langle \mathbf{v}_1, I(t)\mathbf{v}_m \rangle/dt$  is negative at  $t = 0$ , or equivalently that

$$0 < \det \begin{bmatrix} v_{10} & 0 & v_{m0} \\ v_{11} & 0 & v_{m1} \\ v_{12} & -1 & v_{m2} \end{bmatrix} = v_{m1}v_{10} - v_{m0}v_{11}.$$

Imagine the three column vectors, in order, in a right-handed coordinate system with the  $x_0$  axis pointing up, the  $x_1$  axis pointing forward and the  $x_2$  axis pointing to the right. Because of the choice of orientation made above, the duals to the geodesics carrying  $s_1, s_i, s_m$ , when viewed from “above” and in that order, will also turn counter-clockwise. Since these duals all have non-negative  $x_0$  coordinates, and two of these are positive, they form a right-handed frame, and the determinant is therefore positive.

We now check that  $d\langle \mathbf{v}_1, I(t)\mathbf{v}_m \rangle/dt < 0$  for an arbitrary  $t > 0$ . Because  $v_{m1}v_{10} - v_{m0}v_{11} > 0$  this is equivalent to

$$(4.1) \quad \frac{v_{m1}v_{11} - v_{m0}v_{10}}{v_{m1}v_{10} - v_{m0}v_{11}} < \frac{\cosh(t)}{\sinh(t)}.$$

Furthermore, since  $\cosh(t)/\sinh(t) > 1$  it is sufficient to show that

$$\frac{v_{m1}v_{11} - v_{m0}v_{10}}{v_{m1}v_{10} - v_{m0}v_{11}} \leq 1.$$

Since  $R$  is a parallelogram oriented counter-clockwise,  $\ell_1$  and  $\ell_j$  cannot intersect in  $\mathbb{H}^2$  to the right of  $O$ , since only  $\ell_{j+1}$  can intersect  $\ell_j$  there; nor can  $\ell_1$  and  $\ell_j$  be asymptotic at the right ideal endpoint of  $\ell_j$  (represented by the vector  $(1, 1, 0) \in E^{2,1}$ ). This means that we never intersect the boundary of the half-space in  $E^{2,1}$  corresponding to  $\mathbf{v}_1$  when we move in a straight line from  $(1, 0, 0)$  to  $(1, 1, 0)$ , forcing the latter point to lie in the interior of that half-space. In other words,

$$0 > \langle (1, 1, 0), (v_{10}, v_{11}, v_{12}) \rangle = -v_{10} + v_{11}.$$

The diagrams of Figure 4.1 (radially projecting  $E^{2,1}$  to  $\{x_0 = 1\}$ ) illustrate some of the ways  $\ell_1, \ell_j$ , and  $\ell_m$  can be arranged, but are not intended to be a comprehensive list. Configuration 1 is allowed (but will only occur if  $2 < j < m - 1$ ). Configuration 2 is only allowed when  $m = j + 1$ . Configuration 3 is ruled out by the intersection of  $\ell_1$  with  $\ell_j$  to the right of  $O$ ,

violating the parallelogram condition, and Configuration 4 is forbidden by the orientation condition.

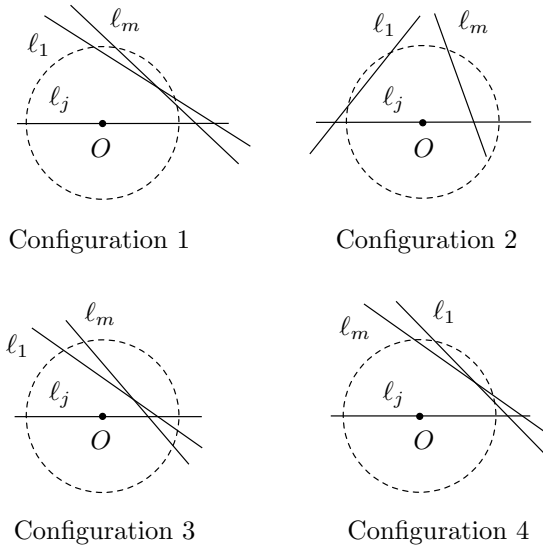


Figure 4.1

An analogous argument shows  $\ell_j$  and  $\ell_m$  cannot intersect in  $\mathbb{H}^2$  to the left of  $O$  and that they cannot be asymptotic at  $(1, -1, 0)$ , so  $(1, -1, 0)$  is contained in the interior of the half-space dual to  $(v_{m0}, v_{m1}, v_{m2})$ , or in other words,  $-v_{m0} - v_{m1} < 0$ .

Combining these two observations yields

$$0 > (v_{m1} + v_{m0})(v_{11} - v_{10}) = v_{m1}v_{11} + v_{m0}v_{11} - v_{m1}v_{10} - v_{m0}v_{10}$$

which is equivalent to  $v_{m1}v_{11} - v_{m0}v_{10} < v_{m1}v_{10} - v_{m0}v_{11}$ . In combination with the fact that the right hand side of this inequality is positive, this shows that Equation (4.1) holds.

For  $i = 1, \dots, m - 1$  the adjacent sides  $s_i$  and  $s_{i+1}$  of  $R$  continue to intersect at finite vertices with the same interior angles as before this deformation. Applying what we have just proved to an appropriate sub-polygon of  $R$  we can see that  $\langle \mathbf{v}_k, \mathbf{v}_\ell \rangle$  is non-increasing for pairs of sides of  $s_k$  and  $s_\ell$  that did not intersect before this deformation. Because these sides satisfied  $\langle \mathbf{v}_k, \mathbf{v}_\ell \rangle < 0$  before the deformation, they continue to do so, and the resulting polygon satisfies the hypotheses of Lemma 4.3. Hence, one can

increase the lengths of sides  $s_j$  sequentially, in order to prove Lemma 4.3 in full generality.  $\square$

We continue the proof of Proposition 4.1.

Suppose that  $F$  has  $n$  sides. We can renumber the sides of  $F$  so that the second through  $(m - 1)$ -st sides of  $F$  are labeled with  $+$ 's and  $0$ 's and at least one of them is labeled with a  $+$ , so that the first and  $m$ -th sides are arbitrarily labeled, and, if  $m < n$ , so that the remaining sides are all labeled with  $-$ 's and  $0$ 's. There is usually more than one way to do this (often with differing values of  $m$ ), any of which will suffice (see Figure 4.2 for examples). In the former, there are many alternate choices; in the latter, there is essentially one choice, up to combinatorial symmetry.

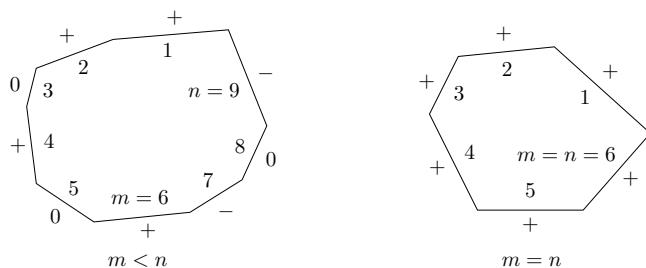


Figure 4.2

Because  $Q$  and  $Q'$  are parallelograms, the (possibly non-compact) polygons bounded by the union of sides  $s_1, \dots, s_m$  from  $Q$  and the union of the sides  $s'_1, \dots, s'_m$  from  $Q'$  satisfy the hypotheses of Lemma 4.3. Hence, if we denote the outward pointing normals to  $s_1$  and  $s_m$  in  $Q$  by  $\mathbf{v}_1$  and  $\mathbf{v}_m$  and in  $Q'$  by  $\mathbf{v}'_1$  and  $\mathbf{v}'_m$ , the lemma guarantees that  $\langle \mathbf{v}'_1, \mathbf{v}'_m \rangle > \langle \mathbf{v}_1, \mathbf{v}_m \rangle$ .

However, either  $n = m$  so that the first and  $m$ -th sides of  $F$  meet at a vertex giving  $\langle \mathbf{v}'_1, \mathbf{v}'_m \rangle = \langle \mathbf{v}_1, \mathbf{v}_m \rangle$ , or, if  $n > m$  all of the sides in  $F$  with index greater than  $m$  are labeled  $0$  or  $-$ . Applying Lemma 4.3 to the polygons bounded by the sides  $s_{m+1}, \dots, s_n$  from  $Q$  and  $s'_{m+1}, \dots, s'_n$  from  $Q'$  we find that  $\langle \mathbf{v}'_1, \mathbf{v}'_m \rangle \leq \langle \mathbf{v}_1, \mathbf{v}_m \rangle$ . In both cases we obtain a contradiction.

We have not used any restriction on the dihedral angles, only the restriction that  $P_1$  and  $P_2$  are have parallelogram faces, so we shown that  $\alpha : \mathcal{P}_C^1 \rightarrow \mathbb{R}^E$  is injective.  $\square$

Because  $\mathcal{P}_C^0 \subset \mathcal{P}_C^1$ , it follows immediately that

COROLLARY 4.4. —  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is injective.

This gives the uniqueness part of Andreev's Theorem.

## 5. The mapping $\alpha : \mathcal{P}_C^0 \rightarrow A_C$ is proper.

In this section, we prove that the mapping  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is a proper map. In fact, we will prove a more general statement (Proposition 5.3) which will be useful later in the paper.

LEMMA 5.1. — *Let  $\mathcal{F}$  be a face of a hyperbolic polyhedron  $P$  with non-obtuse dihedral angles. If a face angle of  $\mathcal{F}$  equals  $\frac{1}{2}\pi$  at the vertex  $v$ , then the dihedral angle of the edge opposite the face angle (the edge that enters  $v$  and is not in  $\mathcal{F}$ ) is  $\frac{1}{2}\pi$  and the dihedral angle of one of the two edges in  $\mathcal{F}$  that enters  $v$  is  $\frac{1}{2}\pi$ .*

*Proof.* — This will follow from Equation (1.1) in Proposition 1.1, which one can use to calculate face angles from the dihedral angles at a vertex. In Equation (1.1), if  $\beta_i = \frac{1}{2}\pi$  we have

$$0 = \frac{\cos(\alpha_i) + \cos(\alpha_j) \cos(\alpha_k)}{\sin(\alpha_j) \sin(\alpha_k)},$$

where  $\alpha_i$  is the dihedral angle opposite the face angle  $\beta_i$  and  $\alpha_j, \alpha_k$  are the dihedral angles of the other two edges entering  $v$ . Both  $\cos(\alpha_i) \geq 0$  and  $\cos(\alpha_j) \cos(\alpha_k) \geq 0$  for non-obtuse  $\alpha_i, \alpha_j$ , and  $\alpha_k$ , so that  $\cos(\alpha_i) = 0$  and  $\cos(\alpha_j) \cos(\alpha_k) = 0$ . Hence  $\alpha_i = \frac{1}{2}\pi$  and either  $\alpha_j = \frac{1}{2}\pi$  or  $\alpha_k = \frac{1}{2}\pi$ .  $\square$

LEMMA 5.2. — *Given three points  $v_1, v_2, v_3$  that form a non-obtuse, non-degenerate triangle in the Poincaré model of  $\mathbb{H}^3$ , there is a unique isometry taking  $v_1$  to a positive point on the  $x$ -axis,  $v_2$  to a positive point on the  $y$ -axis, and  $v_3$  to a positive point on the  $z$ -axis.*

*Proof.* — The points  $v_1, v_2$ , and  $v_3$  form a triangle  $T$  in a plane  $P_T$ . It is sufficient to show that there is a plane  $Q_T$  in the Poincaré ball model that intersects the positive octant in a triangle isomorphic to  $T$ . The isomorphism taking  $v_1, v_2$ , and  $v_3$  to the  $x, y$ , and  $z$ -axes will then be the one that takes the plane  $P_T$  to the plane  $Q_T$  and the triangle  $T$  to the intersection of  $Q_T$  with the positive octant.

Let  $s_1, s_2$ , and  $s_3$  be the side lengths of  $T$ . The plane  $Q_T$  must intersect the  $x, y$ , and  $z$ -axes distances  $a_1, a_2$ , and  $a_3$  satisfying the hyperbolic Pythagorean theorem

$$\begin{aligned} \cosh(s_1) &= \cosh(a_2) \cosh(a_3), \\ \cosh(s_2) &= \cosh(a_3) \cosh(a_1), \\ \cosh(s_3) &= \cosh(a_1) \cosh(a_2). \end{aligned}$$

These equations can be solved for  $(\cosh^2(a_1), \cosh^2(a_2), \cosh^2(a_3))$ , obtaining

$$\left( \frac{\cosh(s_2) \cosh(s_3)}{\cosh(s_1)}, \frac{\cosh(s_3) \cosh(s_1)}{\cosh(s_2)}, \frac{\cosh(s_1) \cosh(s_2)}{\cosh(s_3)} \right),$$

The only concern in solving for  $a_i$  is that each of these expressions is  $\geq 1$ . However, this follows from the triangle  $T$  being non-obtuse, using the hyperbolic law of cosines. □

All of the results in this chapter are corollaries to the following

**PROPOSITION 5.3.** — *Given a sequence of compact polyhedra  $P_i$  realizing  $C$  with  $\alpha(P_i) = \mathbf{a}_i \in A_C$ . If  $\mathbf{a}_i$  converges to  $\mathbf{a} \in \bar{A}_C$ , satisfying conditions 1), 3)–5), then there exists a polyhedron  $P_0$  realizing  $C$  with dihedral angles  $\mathbf{a}$ .*

*Proof.* — Throughout this proof, we will denote the vertices of  $P_i$  by  $v_1^i, \dots, v_n^i$ . Let  $v_a^i, v_b^i, v_c^i$  be three vertices on the same face of  $P_i$ , which will form a triangle with non-obtuse angles for all  $i$ . According to Lemma 5.2, for each  $i$  we can normalize  $P_i$  in the Poincaré ball model so that  $v_a^i$  is on the  $x$ -axis,  $v_b^i$  is on the  $y$ -axis, and  $v_c^i$  is on the  $z$ -axis.

For each  $i$ , the vertices  $v_1^i, \dots, v_n^i$  are in  $\bar{\mathbb{H}}^3$ , which is a compact space under the Euclidean metric. Therefore, by taking a subsequence, if necessary, we can assume that each vertex of  $P_i$  converges to some point in  $\bar{\mathbb{H}}^3$ . We denote the collection of all of these limit points of  $v_1^i, \dots, v_n^i$  by  $\mathcal{A}_1, \dots, \mathcal{A}_q \in \bar{\mathbb{H}}^3$ . Let  $P_0$  be the convex hull of  $\mathcal{A}_1, \dots, \mathcal{A}_q$ . Since each  $P_i$  realizes  $C$ , if we can show that each  $\mathcal{A}_m$  is the limit of a single vertex of  $P_i$ , then  $P_0$  will realize  $C$  and have dihedral angles  $\mathbf{a}$ .

In summary, we must show that no more than one vertex converges to each  $\mathcal{A}_m$ , using that  $\mathbf{a}$  satisfies conditions 1), 3), 4), and 5).

- *We first check that if  $\mathcal{A}_m$  belongs to  $\partial\mathbb{H}^3$ , then there is a single vertex of  $P_i$  converging to  $\mathcal{A}_m$ .*

Therefore, suppose that there are  $k > 1$  vertices of  $P_i$  converging to  $\mathcal{A}_m$  to find a contradiction to the fact that  $\mathbf{a}$  satisfies conditions 1) and 3)–5). Without loss of generality, we assume that  $v_1^i, \dots, v_k^i$  converge to  $\mathcal{A}_m$  and  $v_{k+1}^i, \dots, v_n^i$  converge to other points  $\mathcal{A}_j$  for  $j \neq m$ .

Since  $\mathcal{A}_m$  is at infinity, our normalization (restricting  $v_a^i, v_b^i$ , and  $v_c^i$  to the  $x, y$ , and  $z$ -axes respectively) ensures that at least two of these vertices ( $v_a^i, v_b^i$ , and  $v_c^i$ ) do not converge to  $\mathcal{A}_m$ . This fact will be essential throughout this part of the proof.



Doing a Euclidean rotation of the entire Poincaré ball, we can assume that  $\mathcal{A}_m$  is at the north pole of the sphere, without changing the fact that there are at least two vertices of  $P_i$  that do not converge to  $\mathcal{A}_m$ .

We will do a sequence of normalizations of the position of  $P_i$  in the Poincaré ball model to study the geometry of  $P_i$  near  $\mathcal{A}_m$  as  $i$  increases.

For all sufficiently large  $i$ , there is some hyperbolic plane  $Q$ , which is both perpendicular to the  $z$ -axis and has  $\mathcal{A}_m$  and  $v_1^i, \dots, v_k^i$  on one side of  $Q$  and the remaining  $\mathcal{A}_j$  ( $j \neq m$ ) and all of the vertices  $v_{k+1}^i, \dots, v_n^i$  that do not converge to  $\mathcal{A}_m$  on the other side of  $Q$ . This is possible because  $\mathcal{A}_1, \dots, \mathcal{A}_q$  are distinct points in  $\overline{\mathbb{H}^3}$ , and because  $\mathcal{A}_m \in \partial\mathbb{H}^3$ .

For each  $i$ , let  $R_i$  be the hyperbolic plane which intersects the  $z$ -axis perpendicularly, and at the point farthest from the origin such that the closed half-space toward  $\mathcal{A}_m$  contains all vertices which will converge to  $\mathcal{A}_m$  (see Figure 5.1). Let  $D_i$  denote the distance from  $R_i$  to  $Q$  along the  $z$ -axis; as  $i \rightarrow \infty$  and the vertices  $v_1^i, \dots, v_k^i$  tend to  $\mathcal{A}_m \in \partial\mathbb{H}^3$ ,  $D_i \rightarrow \infty$ . Let  $S_i$  be the hyperbolic plane intersecting the  $z$ -axis perpendicularly, halfway between  $R_i$  and  $Q$ .

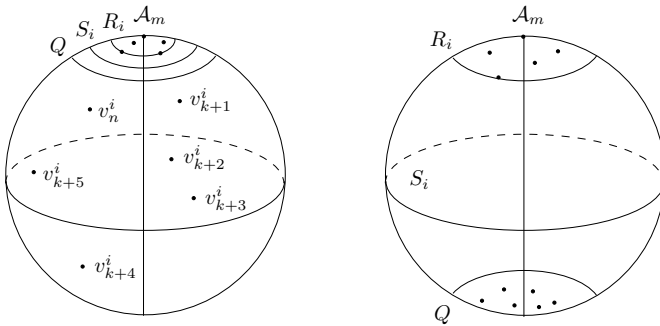


Figure 5.1

For each  $i$ , we normalize the polyhedra  $P_i$  by translating the plane  $S_i$  along the  $z$ -axis to the equatorial plane,  $H = \{z = 0\}$ . We consider this a change of viewpoint, so the translated points and planes retain their former names. Hence, under this normalization, we have planes  $R_i$  and  $Q$  both perpendicular to the  $z$ -axis, and hyperbolic distance  $\frac{1}{2}D_i$  from the origin. Each vertex  $v_1^i, \dots, v_k^i$  is bounded above the plane  $R_i$ , and all of the vertices  $v_{k+1}^i, \dots, v_n^i$  of  $P_i$  that do not converge to  $\mathcal{A}_m$  are bounded below the plane  $Q$ .

As  $i$  tends to infinity,  $R_i$  and  $Q$  intersect the  $z$ -axis at arbitrarily large hyperbolic distances from the origin. Denote by  $\mathcal{N}$  and by  $\mathcal{S}$  the half-spaces that these planes bound away from the origin. Given arbitrarily small (Euclidean) neighborhoods of the pole, for all sufficiently large  $i$  the half-spaces will be contained in these neighborhoods. Hence, any edge running from  $\mathcal{N}$  to  $\mathcal{S}$  will intersect  $H$  almost orthogonally, and close to the origin, as illustrated in Figure 5.2. Let  $e_1^i, \dots, e_\ell^i$  denote the collection of such edges.

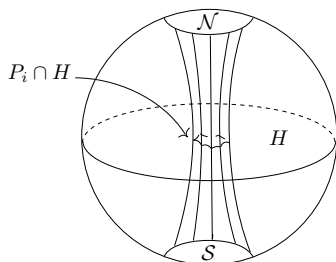


Figure 5.2

By assumption, there are  $k \geq 2$  vertices in  $\mathcal{N}$  and due to our normalization, there are two or more vertices in  $\mathcal{S}$ .

Thus the intersection  $P_i \cap H$  will be almost a Euclidean polygon and its angles will be almost the dihedral angles  $\alpha(e_1^i), \dots, \alpha(e_\ell^i)$ ; in particular, for  $i$  sufficiently large they will be at most only slightly larger than  $\frac{1}{2}\pi$ . This implies that  $\ell_i$ , the number of such edges, is 3 or 4 for  $i$  sufficiently large, as a Euclidean polygon with at least 5 faces has at least one angle  $\geq \frac{3}{5}\pi$ .

If  $\ell = 3$ , the edges  $e_1^i, e_2^i, e_3^i$  intersecting  $H$  are a prismatic circuit because there are two or more vertices in both  $\mathcal{N}$  and in  $\mathcal{S}$ . The sum  $\alpha(e_1^i) + \alpha(e_2^i) + \alpha(e_3^i)$  tends to  $\pi$  as  $i$  tends to infinity, hence  $\mathbf{a}$  cannot satisfy condition 3), contrary to assumption.

Similarly, if  $\ell_i = 4$ , and if  $e_1^i, e_2^i, e_3^i, e_4^i$  form a prismatic 4-circuit, then the corresponding sum of dihedral angles tends to  $2\pi$  violating condition 4).

So we are left with the possibility that  $\ell = 4$ , and that  $e_1^i, e_2^i, e_3^i, e_4^i$  do not form a prismatic 4-circuit. In this case, a pair of these edges meet at a vertex which may be in either  $\mathcal{N}$ , as shown in the diagram below, or in  $\mathcal{S}$ . Without loss of generality, we assume that  $e_1^i$  and  $e_2^i$  meet at this vertex, which we call  $x^i$  and we assume that  $x^i \in \mathcal{N}$ . Because we assume that there are two or more vertices converging to  $\mathcal{A}_m$ , there must be some edge  $e_j^i$  ( $j \neq 1, 2, 3, 4$ ) meeting  $e_1^i$  and  $e_2^i$  at  $x^i$ . We denote by  $f^i$  the face

of  $P_i$  containing  $e_1^i, e_2^i$  and  $x^i$ . An example of this situation is drawn in the diagram below, although the general situation can be more complicated.

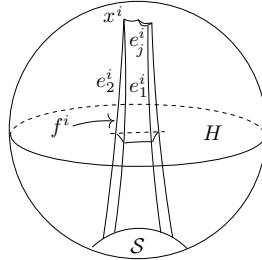


Figure 5.3

Since the sum of the dihedral angles along this 4-circuit limits to  $2\pi$ , and each dihedral angle is non-obtuse, each of the dihedral angles  $\alpha(e_1^i), \dots, \alpha(e_4^i)$  limits to  $\frac{1}{2}\pi$ . One can use Equation (1.1) to check that the dihedral angle  $\alpha(e_j^i)$  will converge to the face angle  $\beta^i$  in the face  $f^i$  at vertex  $v^i$ . This is because the right-hand side of this equation limits to  $\cos(\alpha(e_j^i))$ , while the right hand side of the equation equals  $\cos(\beta_i)$ . Then, as  $i$  goes to infinity, the neighborhood  $\mathcal{N}$  (containing  $x_1^i$ ) converges to the north pole while the neighborhood  $\mathcal{S}$  (containing the other two vertices in  $f^i$  which form the face angle  $\beta_i$  at  $x_1^i$ ) converges to the south pole. This forces the face angle  $\beta^i$  to tend to zero, and hence the dihedral angle  $\alpha(e_j^i)$  must tend to zero as well, contradicting the fact that all coordinates of the limit point  $\mathbf{a}$  are positive, by condition 1).

Therefore, we can conclude that any  $\mathcal{A}_m \in \partial\mathbb{H}^3$  is the limit of a single vertex of the  $P_i$ . Hence, the vertices of  $P_i$  that converge to points at infinity (in the original normalization) converge to distinct points at infinity.

- *It remains to show that if  $\mathcal{A}_m$  belongs to  $\mathbb{H}^3$ , then there is a single vertex of  $P_i$  converging to  $\mathcal{A}_m$ .*

First, we check that none of the faces of the  $P_i$  can degenerate to either a point, a line segment, or a ray.

Because we have already proven that each of the vertices of  $P_i$  that converge to points in  $\partial\mathbb{H}^3$  converge to distinct points, any face  $\mathcal{F}_i$  of  $P_i$  that degenerates to a point, a line segment, or a ray, must have area that limits to zero. Hence, by the Gauss-Bonnet formula, the sum of the face angles would have to converge to  $\pi(n - 2)$ , if the face has  $n$  sides. The fact that the face angles are non-obtuse, allows us to see that  $\pi(n - 2) \leq \frac{1}{2}n\pi$

implying that  $n \leq 4$ . This restricts such a degenerating face  $\mathcal{F}_i$  to either a triangle or a quadrilateral, the only Euclidean polygons having non-obtuse angles.

If  $\mathcal{F}_i$  is a triangle, then the three edges leading to  $\mathcal{F}_i$  form a prismatic 3-circuit, because our hypothesis  $N > 4$  implies that  $C$  is not the simplex. If  $\mathcal{F}_i$  degenerates to a point, the three faces adjacent to  $\mathcal{F}_i$  would meet at a vertex, in the limit. Therefore, by Lemma 3.2, the sum of the dihedral angles at the edges leading to  $\mathcal{F}_i$  would limit to a value  $\geq \pi$ , contrary to condition 3). Otherwise, if  $\mathcal{F}_i$  is a triangle which degenerates to a line segment or ray in the limit, then two of its face angles will tend to  $\frac{1}{2}\pi$ . Then, by Lemma 5.1, the dihedral angles at the edges opposite of these face angles become  $\frac{1}{2}\pi$ . However, these edges are part of the prismatic 3-circuit of edges leading to  $\mathcal{F}_i$ , resulting in an angle sum  $\geq \pi$ , contrary to condition 3).

If, on the other hand,  $\mathcal{F}_i$  is a quadrilateral, each of the face angles would have to limit to  $\frac{1}{2}\pi$ . By Lemma 5.1, the dihedral angles at each of the edges leading from  $\mathcal{F}_i$  to the rest of  $P_i$  would limit to  $\frac{1}{2}\pi$ , as well as at least one edge of  $\mathcal{F}_i$  leading to each vertex of  $\mathcal{F}_i$ . Therefore, the dihedral angles at each of the edges leading from  $\mathcal{F}_i$  to the rest of  $P_i$  and at least one opposite pair of edges of  $\mathcal{F}_i$  limit to  $\frac{1}{2}\pi$ , in violation of condition 5).

Since none of the faces of the  $P_i$  can degenerate to a point a line segment, or a ray, neither can the  $P_i$ . Suppose that the  $P_i$  degenerate to a polygon,  $\mathcal{G}$ . Because the dihedral angles are non-obtuse, only two of the faces of the  $P_i$  can limit to the polygon  $\mathcal{G}$ . Therefore the rest of the faces of the  $P_i$  must limit to points, line segments, or rays, contrary to our reasoning above.

We are now ready to show that any  $\mathcal{A}_m$  that is a point in  $\mathbb{H}^3$  is the limit of a single vertex of the  $P_i$ . Let  $v_1, \dots, v_k$  be the distinct vertices that converge to the same point  $\mathcal{A}_m$ . Then, since the  $P_i$  do not shrink to a point, a line segment, a ray, or a polygon, there are at least three vertices  $v_p^i, v_q^i, v_r^i$  that don't converge to  $\mathcal{A}_m$  and that don't converge to each other. Perform the appropriate isometry taking  $\mathcal{A}_m$  to the origin in the ball model. Place sphere  $S$  centered at the origin, so small that  $v_p^i, v_q^i, v_r^i$  never enter  $S$ .

For all sufficiently large values of  $i$ , the intersection  $P_i \cap S$  approximates a spherical polygon whose angles approximate the dihedral angles between the faces of  $P_i$  that enter  $S$ . These spherical polygons cannot degenerate to a point or a line segment because the polyhedra  $P_i$  do not degenerate to a line segment, a ray, or a polygon. By reasoning similar to that of Proposition 1.1, one can check that this polygon must have only three sides

and angle sum  $> \pi$ . If  $k > 1$ , the edges of this triangle form a prismatic 3-circuit in  $C^*$ , since for each  $i$ ,  $P_i$  has more than one vertex inside the sphere and at least the three vertices corresponding to the points represented by  $v_p^i, v_q^i, v_r^i$ . So, the  $P_i$  would have a prismatic 3-circuit whose angle sum limits to a value  $> \pi$ . However, this contradicts our assumption that  $\mathbf{a}$  satisfies condition 3). Therefore, we must have  $k = 1$ .

Therefore, we conclude that each  $\mathcal{A}_m$  is the limit of a single vertex of  $P_i$ .  $\square$

**COROLLARY 5.4.** — *The mapping  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is proper.*

*Proof.* — Suppose that  $P_i$  is a sequence of polyhedra in  $\mathcal{P}_C^0$  with  $\alpha(P_i) = \mathbf{a}_i \in A_C$ . If the sequence  $\mathbf{a}_i$  converges to a point in  $\mathbf{a} \in A_C$ , we must show that there is a subsequence of the  $P_i$  that converges in  $\mathcal{P}_C^0$ .

Since  $\mathbf{a}$  satisfies conditions 1) and 3)–5), by Proposition 5.3, there is a subsequence of the  $P_i$  converging to a polyhedron  $P_0$  that realizes  $C$  with dihedral angles  $\mathbf{a}$ . The sum of the dihedral angles at each vertex of  $P_0$  is  $> \pi$  since  $\mathbf{a}$  satisfies condition 2) as well. Therefore, by Lemma 3.2, each vertex of  $P_0$  is at a finite point in  $\mathbb{H}^3$ , giving that  $P_0$  is compact. Hence  $P_0$  realizes  $C$ , is compact, and has non-obtuse dihedral angles. Therefore  $P_0 \in \mathcal{P}_C^0$ .  $\square$

**COROLLARY 5.5.** — *Suppose that Andreev's Theorem holds for  $C$  and suppose that there is a sequence  $\mathbf{a}_i \in A_C$  converging to  $\mathbf{a} \in \partial A_C$ . If  $\mathbf{a}$  satisfies conditions 1), 3)–5) and if condition 2) is satisfied for vertices  $v_1, \dots, v_k$  of  $C$ , but not for  $v_{k+1}, \dots, v_n$  for which the dihedral angle sum is exactly  $\pi$ , then there exists a non-compact polyhedron  $P_0$  realizing  $C$  with dihedral angles  $\mathbf{a}$ .  $P_0$  has vertices  $v_1, \dots, v_k$  at distinct finite points and the  $v_{k+1}, \dots, v_n$  at distinct points at infinity.*

*Proof.* — Because we assume that Andreev's Theorem holds for  $C$ , there exists a sequence of polyhedra  $P_i$  with  $\alpha(P_i) = \mathbf{a}_i$ . The proof continues in the same way as that of Corollary 5.4, except that in this case, it then follows from Lemma 3.2 that  $v_1, \dots, v_k$  lie in  $\mathbb{H}^3$ , while  $v_{k+1}, \dots, v_n$  lie in  $\partial\mathbb{H}^3$ .  $\square$

Notice that if an abstract polyhedron has no prismatic 3-circuits, it has no triangular faces, so collapsing a single edge to a point results in a cell complex satisfying all of the conditions of an abstract polyhedron, except that one of the vertices is now 4-valent.

**PROPOSITION 5.6.** — *Let  $C$  be an abstract polyhedron having no prismatic 3-circuits for which Andreev's Theorem is satisfied. For any edge  $e_0$*

of  $C$ , let  $C_0$  be the complex obtained by contracting  $e_0$  to a point. Then, there exists a non-compact polyhedron  $P_0$  realizing  $C_0$  with the edge  $e_0$  contracted to a single vertex at infinity and the rest of the vertices at finite points in  $\mathbb{H}^3$ .

*Proof.* — Let  $v_1$  and  $v_2$  be the vertices at the ends of  $e_0$ , let  $e_1, e_2, e_3, e_4$  the edges emanating from the ends of  $e_0$ , and  $f_1, f_2, f_3, f_4$  be the four faces containing either  $v_1$  or  $v_2$ , as illustrated below.

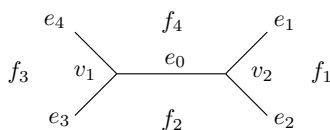


Figure 5.4

For  $\epsilon \in (0, \frac{1}{2}\pi]$  the angles  $\alpha(e_0) = \epsilon$ ,  $\alpha(e_1) = \alpha(e_2) = \alpha(e_3) = \alpha(e_4) = \frac{1}{2}\pi$ , and  $\alpha(e) = \frac{2}{5}\pi$ , for all other edges  $e$ , are in  $A_C$  since  $C$  has no prismatic 3-circuits. Therefore, because we assume that Andreev's Theorem holds for  $C$ , there is a polyhedron  $P_\epsilon \in \mathcal{P}_C^0$  realizing these angles. Choose a sequence  $\epsilon_i > 0$  converging to 0.

As in the proof of Proposition 5.3, we choose three vertices of  $P_i$  that are on the same face (but not on  $f_2$  or  $f_4$ ) and normalize the polyhedra so that each  $P_i$  has these three vertices on the  $x$ -axis, the  $y$ -axis, and the  $z$ -axis, respectively. The vertices  $v_1^i, \dots, v_n^i$  of each  $P_i$  are in the compact space  $\overline{\mathbb{H}^3}$  so, as before, we can take a subsequence so that each vertex converges to some point in  $\mathbb{H}^3$ .

The strategy of the proof is to consider the limit points  $\mathcal{A}_1, \dots, \mathcal{A}_q$  of these vertices and to show that  $v_1^i$  and  $v_2^i$  converge to the same limit point, say  $\mathcal{A}_1$  and that exactly one of each of the other vertices  $v_j^i$  (for  $j > 2$ ) converges to each of the other  $\mathcal{A}_m$ , with  $m > 1$ .

We first check that that  $v_1^i$  and  $v_2^i$  converge to the same point at infinity. Since the dihedral angle at edge  $e_0$  converges to 0, the two planes carrying faces  $f_2$  and  $f_4$  will intersect at dihedral angles converging to 0. In the limit, these planes will intersect with dihedral angle 0, therefore, at a single point in  $\partial\mathbb{H}^3$ . The edge  $e_0$  is contained within the intersection of these two planes, hence the entire edge  $e_0$  converges to a single point in  $\partial\mathbb{H}^3$ , and hence  $v_1^i$  and  $v_2^i$  converge to this point. We label this point by  $\mathcal{A}_1$ .

We must now show  $v_1^i$  and  $v_2^i$  are the only vertices that converge to  $\mathcal{A}_1$ . As in the proof of Proposition 5.3, one can do further normalizations so

that all of the vertices that converge to  $\mathcal{A}_1$  are in an arbitrarily small neighborhood  $\mathcal{N}$  of the north pole and all of the other vertices are in an arbitrarily small neighborhood  $\mathcal{S}$  of the south pole. Since  $\mathbf{a}$  satisfies conditions 3) and 4) we deduce that there are exactly four edges  $e_1^i, e_2^i, e_3^i, e_4^i$  that connect  $\mathcal{N}$  to  $\mathcal{S}$  and that do not form a prismatic 4-circuit. Then, by Lemma 1.3, this 4-circuit separates exactly two vertices from the remaining vertices of  $P_i$ , hence only  $v_1^i$  and  $v_2^i$  converge to  $\mathcal{A}_1$ .

The proof that a single vertex of  $P_i$  converges to each  $\mathcal{A}_m$ , for  $m > 1$  is almost the same as the proof of Proposition 5.3, because the dihedral angles  $\mathbf{a}_i$  are non-zero for all edges other than  $e_0$  and because conditions 3)–5) are satisfied.

The only difference is that one must directly check that faces  $f_2$  and  $f_4$  do not collapse to line segments or rays, even though on each of these faces, two vertices converge to the same point at infinity. A vertex of  $f_2$  or  $f_4$  that is not  $v_1$  or  $v_2$  must have face angle that is strictly acute. Otherwise, Lemma 5.1 would give the dihedral angles at two of the edges meeting at this vertex is  $\frac{1}{2}\pi$ , contrary to the fact that at least two of these dihedral angles are assigned to be  $\frac{2}{5}\pi$ . From here, the Gauss-Bonnet Theorem can be used to check that neither  $f_2$  nor  $f_4$  collapses to a line segment or a ray.

Therefore, we conclude that the vertices  $v_j^i$  ( $i > 3$ ) converge to distinct points in  $\overline{\mathbb{H}^3}$  away from the limit of  $v_1^i$  and  $v_2^i$ , which converge to the same point in  $\partial\mathbb{H}^3$ . Since condition 2) is satisfied at each vertex  $v_j^i$  ( $i > 3$ ), Lemma 3.2 guarantees that each of these  $\mathcal{A}_m$  is at a finite point. Therefore,  $v_1^i$  and  $v_2^i$  converge to the same point at infinity, and each of the other vertices converges to a distinct finite point.  $\square$

## 6. $A_C \neq \emptyset$ implies $\mathcal{P}_C^0 \neq \emptyset$

At this point, we know the following result:

PROPOSITION 6.1. — *If  $\mathcal{P}_C^0 \neq \emptyset$ , then  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is a homeomorphism.*

*Proof.* — We have shown in preceding sections that  $\alpha : \mathcal{P}_C^1 \rightarrow \mathbb{R}^E$  is a continuous and injective map whose domain and range are manifolds (without boundary) of the same dimension, so it follows from invariance of domain that  $\alpha$  is a local homeomorphism. Local homeomorphisms restrict nicely to subspaces, giving that  $\alpha : \mathcal{P}_C^0 \rightarrow \mathbb{R}^E$  is a local homeomorphism, as well. In fact, because  $\alpha : \mathcal{P}_C^0 \rightarrow \mathbb{R}^E$  is also injective it is a homeomorphism onto its image, which we showed in Section 3 is a subset of  $A_C$ .

Since  $A_C$  is convex and therefore connected, it suffices to show that  $\alpha(\mathcal{P}_C^0)$  is both open and closed in  $A_C$ , for it will then follow (since the image is non-empty by hypothesis) that the image is the entire set  $A_C$ . But local homeomorphisms are open maps, so  $\alpha(\mathcal{P}_C^0)$  is open in  $A_C$ ; and since Corollary 5.4 shows that  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is proper, it immediately follows that the limit of any sequence in the image of  $\alpha$  which converges in  $A_C$  must lie in the image of  $\alpha$ , so  $\alpha(\mathcal{P}_C^0)$  is closed in  $A_C$ .  $\square$

Indeed, any local homeomorphism between metric spaces which is also proper will be a finite-sheeted covering map [14, p. 23] and [18, p. 127]. This gives an alternative route to proving the above result.

But what is left is absolutely not obvious, and is the hardest part of the whole proof: proving that if  $A_C \neq \emptyset$ , then  $\mathcal{P}_C^0 \neq \emptyset$ . We have no tools with which to approach it and must use bare hands. We follow the proof of Andreev, although the proof of his key lemma contains a significant error. We provide our own correction.

First recall that in Corollary 1.6, we saw that  $A_C \neq \emptyset$  if  $C$  has no prismatic 3-circuits. We will call polyhedra that have no prismatic 3-circuits *simple polyhedra*. We will also say that the dual graph  $C^*$  is simple if it satisfies the dual condition, that every 3-cycle is the boundary of a single face. (This usage of “simple” follows Andreev, but differs from that used by others, including Vinberg [33, p. 47], for polyhedra in all dimensions greater than 2.)

We first prove that  $\mathcal{P}_C^0 \neq \emptyset$  for simple polyhedra, and hence by Proposition 6.1 that Andreev's Theorem holds for simple polyhedra. We then show that for any  $C$  having prismatic 3-circuits, if  $A_C \neq \emptyset$ , then  $\mathcal{P}_C^0 \neq \emptyset$  by making a polyhedron realizing  $C$  from (possibly many) simple polyhedra. By Proposition 6.1, this final step will complete the proof of Andreev's Theorem.

### Proof of Andreev's Theorem for Simple Polyhedra

PROPOSITION 6.2. — *If  $C$  is simple and has  $N > 5$  faces,  $\mathcal{P}_C^0 \neq \emptyset$ . In words: every simple polyhedron is realizable.*

We need three lemmas. We will first state the lemmas and prove this proposition using them. Then we will prove the lemmas.

LEMMA 6.3. — *Let  $Pr_N$  and  $D_N$  be the abstract polyhedra corresponding to the  $N$ -faced prism and the  $N$ -faced “split prism”, as illustrated below. If  $N > 4$ ,  $\mathcal{P}_{Pr_N}^0$  is nonempty and if  $N > 7$ ,  $\mathcal{P}_{D_N}^0$  is nonempty.*



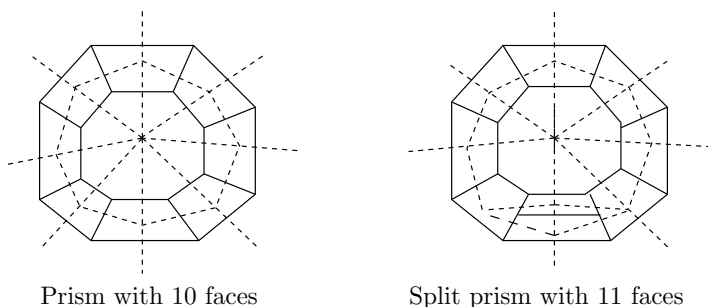


Figure 6.1

A Whitehead move on an edge  $e$  of an abstract polyhedron is given by the local change  $\text{Wh}(e)$  described by Figure 6.2. The Whitehead move in the dual complex is dashed. (Sometimes we will find it convenient to describe the Whitehead move entirely in terms of the dual complex, in which case we write  $\text{Wh}(f)$ ).

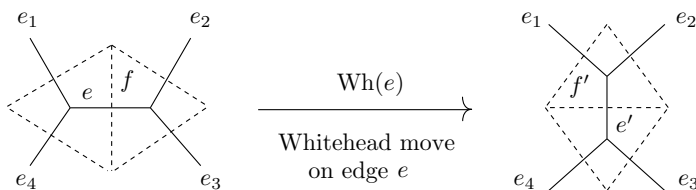


Figure 6.2

LEMMA 6.4. — *Let the abstract polyhedron  $C'$  be obtained from the simple abstract polyhedron  $C$  by a Whitehead move  $\text{Wh}(e)$ . Then if  $\mathcal{P}_C^0$  is non-empty, so is  $\mathcal{P}_{C'}^0$ .*

LEMMA 6.5 (Whitehead Sequence). — *Let  $C$  be a simple abstract polyhedron on  $\mathbb{S}^2$  which is not a prism. If  $C$  has  $N > 7$  faces, one can simplify  $C$  by a finite sequence of Whitehead moves to  $D_N$  such that all of the intermediate abstract polyhedra are simple.*

*Proof of Proposition 6.2, assuming these three lemmas.* — Given simple  $C$  with  $N > 5$  faces; if  $C$  is a prism, the statement is proven by Lemma 6.3. One can check that if  $C$  has 7 or fewer faces (and is not the tetrahedron) it is a prism. So, if  $C$  is not a prism, we have  $N > 7$ .

Then, according to Lemma 6.5, one finds a reduction by (say  $m$ ) Whitehead moves to  $D_N$ , with each intermediate abstract polyhedron simple. Applying Lemma 6.4  $m$  times, one sees that  $\mathcal{P}_C^0$  is non-empty if and only if  $\mathcal{P}_{D_N}^0$  is non-empty. However,  $\mathcal{P}_{D_N}^0$  is non-empty by Lemma 6.3.  $\square$

Theorem 6 from Andreev's original paper corresponds to our Proposition 6.2. The hard technical part of this is the proof of Lemma 6.5. Andreev's original proof of Theorem 6 in [3], [4] provides an algorithm giving the Whitehead moves needed for this lemma but the algorithm *just doesn't work*. It was implemented as a computer program by the first author and failed on the first test case,  $C$  being the dodecahedron. On one of the final steps, it produced an abstract polyhedron which had a prismatic 3-circuit. This error was then traced back to a false statement in Andreev's proof of the lemma. We will explain the details of this error in the proof of Lemma 6.5.

*Proof of Lemma 6.3.* — We construct the  $N$ -faced prism explicitly. First, construct a regular polygon with  $N - 2$  sides centered at the origin in the disc model for  $\mathbb{H}^2$ . ( $N - 2 \geq 3$ , since  $N \geq 5$ .) We can do this with the angles arbitrarily small. Now view  $\mathbb{H}^2$  as the equatorial plane the ball model of  $\mathbb{H}^3$ ; and consider the hyperbolic planes which are perpendicular to the equatorial plane and contain one side of the polygon. In Euclidean geometry these are hemispheres with centers on the boundary of the equatorial disc. The dihedral angles between intersecting pairs of these planes are the angles of the polygon.

Now consider two hyperbolic planes close to the equatorial plane, one slightly above and one slightly beneath, both perpendicular to the  $z$ -axis. These will intersect the previous planes at angles slightly smaller than  $\frac{1}{2}\pi$ . The region defined by these  $N$  planes makes a hyperbolic polyhedron realizing the cell structure of the prism, so  $\mathcal{P}_{Pr_N} \neq \emptyset$ . In particular, using Proposition 6.1, Andreev's Theorem holds for  $C = Pr_N$ ,  $N \geq 5$ .

When  $N > 7$ , the split prism  $D_N$  can be constructed by gluing together a prism and its mirror image, each having  $N - 1$  faces. The dihedral angles are given in Figure 6.3.

These angles satisfy Andreev's conditions 1)–5), and Andreev's Theorem holds for  $Pr_{N-1}$  since  $N - 1 > 6 > 5$ , so there exists such a hyperbolic prism. When this prism is glued to its mirror image, along the  $(N - 3)$ -gon given by the outermost edges in the figure, the corresponding dihedral angles all double. So the edges on the outside which were labeled  $\frac{1}{2}\pi$  “disappear” into the interior of a “merged” face, and the edge which was labeled  $\frac{1}{4}\pi$  now corresponds to a dihedral angle of  $\frac{1}{2}\pi$ . Hence,  $\mathcal{P}_{D_N}^0 \neq \emptyset$ ,

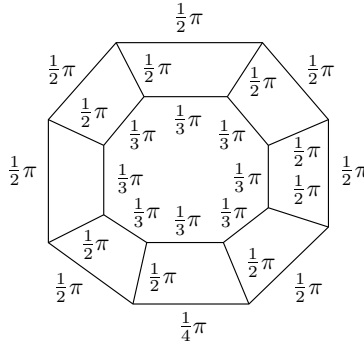


Figure 6.3

when  $N > 7$ . Notice that when  $N = 7$ , the construction yields  $Pr_7$  (which is combinatorially equivalent to  $D_7$ ). □

*Proof of Lemma 6.4.* — We are given  $C$  and  $C'$  simple with  $C'$  obtained by a Whitehead move on the edge  $e_0$  and we are given that  $P_C \neq \emptyset$ . By Proposition 6.1, since  $P_C \neq \emptyset$ , we conclude that Andreev's Theorem holds for  $C$ . Let  $C_0$  be the complex obtained from  $C$  by shrinking the edge  $e_0$  down to a point. By Proposition 5.6, there exists a non-compact polyhedron  $P_0$  realizing  $C_0$  since Andreev's Theorem holds for  $C$ .

We use the upper half-space model of  $\mathbb{H}^3$ , and normalize so that  $e_0$  has collapsed to the origin of  $\mathbb{C} \subset \partial\mathbb{H}^3$ . The faces incident to  $e_0$  are carried by four planes  $H_1, \dots, H_4$  each intersecting the adjacent ones at right angles, and all meeting at the origin. Their configuration will look like the center of the following Figure 6.4. (Recall that planes in the upper half-space model of  $\mathbb{H}^3$  are hemispheres which intersect  $\partial\mathbb{H}^3$  in their boundary circles. The dihedral angle between a pair of planes is the angle between the corresponding pair of circles in  $\partial\mathbb{H}^3$ .)

The pattern of circles in the center of the figure can be modified forming the figures on the left and the right with each of the four circles intersecting the adjacent two circles orthogonally. If we leave the other faces of  $P_0$  fixed we can make a small enough modification that the edges  $e_1, e_2, e_3, e_4$  still have finite non-zero lengths. Since each of the dihedral angles corresponding to edges other than  $e_0, e_1, e_2, e_3,$  and  $e_4$  were chosen to be  $\frac{2}{5}\pi$ , this small modification will not increase any of these angles past  $\frac{1}{2}\pi$ . One of these modified patterns of intersecting circles will correspond to a polyhedron in  $\mathcal{P}_C^0$  and the other to a polyhedron in  $\mathcal{P}_{C'}^0$ . □

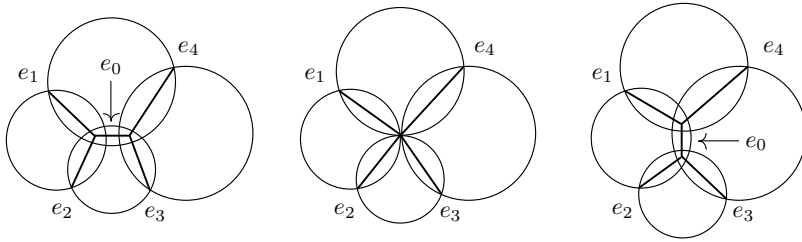


Figure 6.4

*Proof of Lemma 6.5.* — We assume that  $C \neq Pr_N$  is a simple abstract polyhedron with  $N > 7$  faces. We will construct a sequence of Whitehead moves that change  $C$  to  $D_N$ , so that no intermediate complex has a prismatic 3-circuit.

Find a vertex  $v_\infty$  of  $C^*$  which is connected to the greatest number of other vertices. We will call the link of  $v_\infty$ , a cycle of  $k$  vertices and  $k$  edges, the *outer polygon*. Notice that  $k \leq N - 2$ , with equality precisely when  $C = Pr_N$ . Therefore, since  $C \neq Pr_N$  by hypothesis, our first goal is to find Whitehead moves which increase  $k$  to  $N - 3$  without introducing any prismatic 3-circuits along the way. Once this is completed, an easy sequence of Whitehead moves changes the resulting complex to  $D_N^*$ .

Let us set up some notation. Draw the dual complex  $C^*$  in the plane with the vertex  $v_\infty$  at infinity, so that the outer polygon  $P$  surrounding the remaining vertices and triangles. We call the vertices inside of  $P$  *interior vertices*. All of the edges inside of  $P$  which do not have an endpoint on  $P$  are called *interior edges*.

Note that the graph of interior vertices and edges is connected, since  $C^*$  is simple. An interior vertex which is connected to only one other interior vertex will be called an *endpoint*.

Throughout this proof we will draw  $P$  in black and we draw interior edges and vertices of  $C^*$  in black, as well. The connections between  $P$  and the interior vertices will be in grey. Connections between  $P$  and  $v_\infty$  will be black, if shown at all.

The link of an interior vertex  $v$  will intersect  $P$  in a number of components  $F_v^1, \dots, F_v^n$  (possibly  $n = 0$ ). See the above Figure. We will say that  $v$  is *connected to  $P$  in these components*. Notice that since  $C^*$  is simple, an endpoint is always connected to  $P$  in exactly one such component.

**SUB-LEMMA 6.6.** — *If a Whitehead move on the dual  $C^*$  of an abstract polyhedron yields  $C'^*$  (replacing  $f$  by  $f'$ ), and if  $\delta$  is a simple closed path*

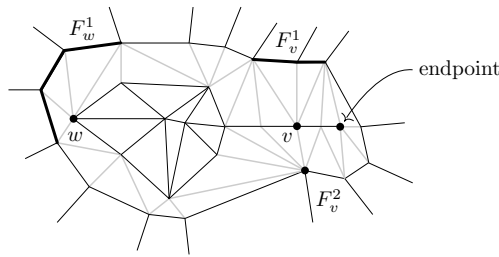


Figure 6.5

in  $C^*$ , which separates one endpoint of  $f'$  from the other, then any newly-created 3-circuit will contain some vertex of  $\delta$  which shares edges with both endpoints of  $f'$ .

*Proof.* — A newly created 3-circuit  $\gamma$  must contain the new edge  $f'$  as well as two additional edges  $e_1$  and  $e_2$  connecting from a single vertex  $V$  to the two endpoints of  $f'$ . By the Jordan Curve Theorem, since  $\delta$  separates the endpoints of  $f'$ , the path  $e_1e_2$  intersects  $\delta$ . The path  $e_1e_2$  is entirely in  $C^*$  since  $f'$  is the only new edge in  $C'^*$ , so the vertex  $V$  is in  $\delta$ .  $\square$

We now prove three additional sub-lemmas that specify certain Whitehead moves that, when performed on a simple abstract polyhedron  $C$  (which is not a prism and has more than seven faces), do not introduce any prismatic 3-circuits. Hence the resulting abstract polyhedron  $C'^*$  is simple. More specifically, we will use sub-Lemma 6.6 to see that each Whitehead move introduces exactly two newly-created 3-circuits in  $C'^*$ , the two triangles containing the new edge  $f'$ .

**SUB-LEMMA 6.7.** — *Suppose that there is an interior vertex  $A$  of  $C^*$  which is connected to  $P$  in exactly one component consisting of exactly two consecutive vertices  $Q$  and  $R$ . The Whitehead move  $\text{Wh}(QR)$  on  $C^*$  increases the length of the outer polygon by one, and introduces no prismatic 3-circuit.*

*Proof.* — Clearly this Whitehead move increases the length of  $P$  by one. We apply sub-Lemma 6.6 to see that this move introduces no prismatic 3-circuits. We let  $\delta = P$ , the outer polygon, which clearly separates the interior vertex  $A$  from  $v_\infty$  in  $C^*$ . Any new 3-circuit would consist of a point on  $P$  connected to both  $A$  and  $v_\infty$ . By hypothesis, there were only the two points  $Q$  and  $R$  on  $P$  connected to  $A$ . These two points result in

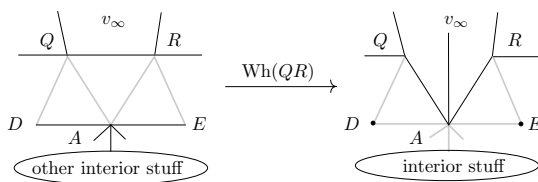


Figure 6.6

the new triangles  $QAv_\infty$  and  $RAv_\infty$  in  $C'^*$ . Therefore  $\text{Wh}(QR)$  results in no prismatic 3-circuits.  $\square$

In the above sub-lemma, the condition that  $A$  is connected to exactly two consecutive vertices of  $P$  prevents  $A$  from being an endpoint. For if  $A$  is an endpoint, let  $B$  denote the unique interior vertex connected to  $A$ . Then  $BQR$  would be a 3-circuit in  $C'^*$  separating  $A$  from the other vertices and hence would contradict the hypothesis that  $C$  is simple. Therefore any endpoint must be connected to  $P$  in a single component having three or more vertices.

**SUB-LEMMA 6.8.** — Suppose that there is an interior vertex  $A$  that is connected to  $P$  in a component consisting of  $M$  consecutive vertices  $Q_1, \dots, Q_M$  of  $P$  (and possibly other components).

(a) If  $A$  is not an endpoint and  $M > 2$ , the sequence of Whitehead moves  $\text{Wh}(AQ_M), \dots, \text{Wh}(AQ_3)$  results in a complex in which  $A$  is connected to the same component of  $P$  in only  $Q_1$  and  $Q_2$ . These moves leave  $P$  unchanged, and introduce no prismatic 3-circuit.

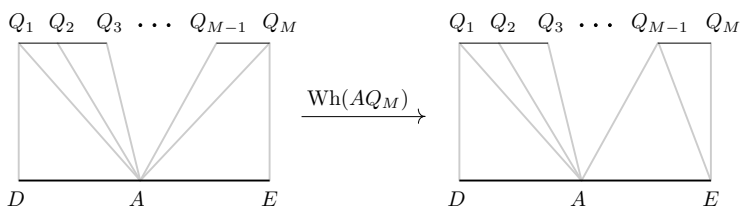


Figure 6.7

(b) If  $A$  is an endpoint and  $M > 3$ , the sequence of Whitehead moves  $\text{Wh}(AQ_M), \dots, \text{Wh}(AQ_4)$  results in a complex in which  $A$  is connected to the same component of  $P$  in only  $Q_1, Q_2$ , and  $Q_3$ . These moves leave  $P$  unchanged and introduce no prismatic 3-circuits.

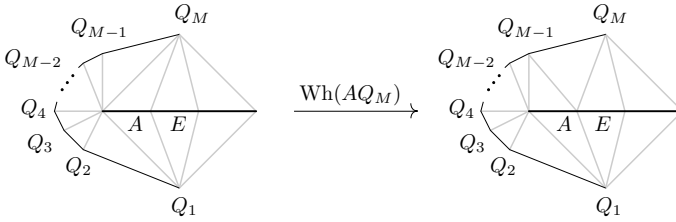


Figure 6.8

*Proof.* — Part (a)  $A$  is not an endpoint. Clearly the move  $\text{Wh}(AQ_M)$  decreases  $M$  by one. We check that if  $M > 2$ , this move introduces no prismatic 3-circuits. We let  $\delta$  be the path  $v_\infty Q_{M-2} A Q_M$  which separates  $Q_{M-1}$  from  $E$  in  $C^*$ . By sub-Lemma 6.6, any new 3-circuit contains a vertex on  $\delta$  connected to both  $E$  and  $Q_{M-1}$ . Clearly  $v_\infty$  is not connected to the interior vertex  $E$ . If  $M > 3$ ,  $Q_{M-2}$  is connected only to  $Q_{M-1}$ ,  $A$ ,  $Q_{M-3}$ , and  $v_\infty$ . Otherwise, when  $M = 3$ , a connection of  $Q_1$  to  $E$  would mean that  $C^*$  had a 3-cycle  $E Q_1 A$  which would have to separate  $D$  (in Figure 6.7) from  $Q_2$ , contrary to the hypothesis that  $C^*$  is simple.

Hence, the only two vertices on  $\delta$  that are connected to both  $E$  and  $Q_{M-1}$  are  $A$  and  $Q_M$ , forming the two triangles  $A Q_{M-1} E$  and  $Q_M Q_{M-1} E$  in  $C'^*$ . Hence there are no new prismatic 3-circuits, so we can reduce  $M$  by one, when  $M > 2$ .

Part (b)  $A$  is an endpoint. We again use  $\delta = v_\infty Q_{M-2} A Q_M$  to check that the move  $\text{Wh}(AQ_M)$  introduces no prismatic 3-circuits. The proof is identical to part (a), except that  $M > 3$  is needed to conclude that  $Q_{M-2}$  is not connected to  $E$  since  $A$  is now an endpoint.

Therefore, as long as  $M > 3$  we can reduce  $M$  by one without introducing prismatic 3-circuits. Recall that an endpoint of a simple complex cannot be connected to fewer than three points of  $P$ , so this is optimal.  $\square$

*Note.* — In both parts (a) and (b), each of the Whitehead moves  $\text{Wh}(AQ_M)$  transfers the connection between  $A$  and  $Q_M$  to a connection between the neighboring interior vertex  $E$  and  $Q_{M-1}$ . In fact  $Q_{M-1}$  gets added to the component containing  $Q_M$  in which  $E$  is connected to  $P$ . This is helpful later on, in Case 2) of Lemma 6.5.

**SUB-LEMMA 6.9.** — *Suppose that there is an interior vertex  $A$  whose link contains two distinct vertices  $X$  and  $Y$  of  $P$ . Then there are Whitehead moves which eliminate any component in which  $A$  is connected to  $P$ , if that*

component does not contain  $X$  or  $Y$ .  $P$  is unchanged, and no prismatic 3-circuits will be introduced.

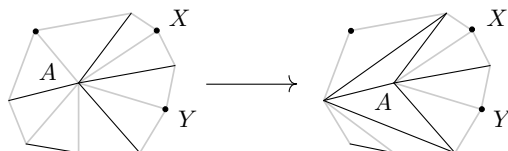


Figure 6.9

Here  $A$  is connected to  $P$  in four components containing six vertices. We can eliminate connections of  $A$  to all of the components except for the single-point components  $X$  and  $Y$ .

*Proof.* — Let  $O$  be a component not containing  $X$  or  $Y$ . If  $O$  contains more than two vertices, we can reduce it to two vertices by sub-Lemma 6.8(a).

Suppose that  $O$  contains exactly two vertices,  $V$  and  $W$ . We check that the move  $\text{Wh}(AW)$  eliminates the connection from  $A$  to  $W$  without introducing prismatic 3-circuits. Let  $D$  be the unique interior vertex forming triangle  $ADW$ , as in Figure 6.10. The move  $\text{Wh}(AW)$  creates the new edge  $DV$ . Let  $\delta$  be the loop  $v_\infty YAW$  which separates  $D$  from  $V$  in  $C^*$ . See the dashed curve in Figure 6.10.

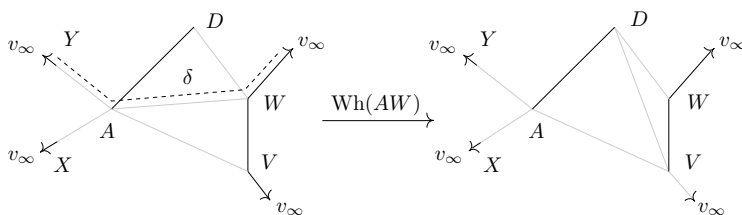


Figure 6.10

By sub-Lemma 6.6, any new 3-circuit contains a point on  $\delta$  that is connected to both  $D$  and  $V$ . Clearly  $v_\infty$  is not connected to the interior vertex  $D$ . Since  $Y$  and  $V$  are in different components of connection between  $A$  and  $P$ ,  $Y$  is not connected to  $V$ . Therefore, only  $A$  and  $W$  are connected to both  $D$  and  $V$ , forming the triangles  $ADV$  and  $WVD$  in  $C'^*$ . Therefore,  $\text{Wh}(AW)$  results in no prismatic 3-circuits.



Thus, we can suppose that  $O$  contains a single vertex  $V$ . We check that the move  $\text{Wh}(AV)$ , which eliminates this connection, does not introduce any prismatic 3-circuits. Let  $D$  and  $E$  be the unique interior vertices forming the triangles  $ADV$  and  $AEV$ . Let  $\delta_1$  be the curve  $v_\infty YAV$  and  $\delta_2$  be the curve  $v_\infty XAV$  in  $C^*$ . See the two dashed curves in Figure 6.11.

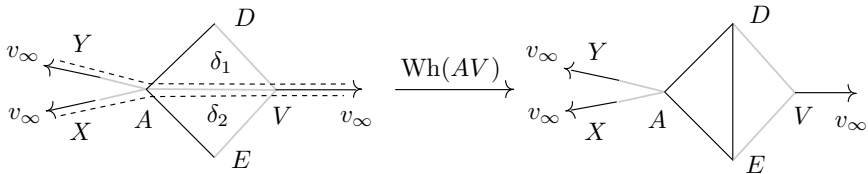


Figure 6.11

Both of these curves separate  $D$  and  $E$  in  $C^*$ . Applying sub-Lemma 6.6 twice, we conclude that any newly created 3-circuit contains a point that is both on  $\delta_1$  and on  $\delta_2$  and that connects to both  $D$  and  $E$ . The only points on both  $\delta_1$  and  $\delta_2$  are  $v_\infty, A$ , and  $V$ . Since  $D$  and  $E$  are interior,  $v_\infty$  cannot connect to either of them. The connections from  $A$  and from  $V$  to  $D$  and  $E$  result in the triangles  $ADE$  and  $VDE$  in  $C'^*$ . Therefore,  $\text{Wh}(AV)$  results in no prismatic 3-circuits.  $\square$

The proof that this move does not introduce any prismatic 3-circuit depends essentially on the fact that  $A$  is connected to  $P$  in at least two other vertices  $X$  and  $Y$ . Andreev describes a nearly identical process to sub-Lemma 6.9 in his paper [3] on pages 333-334. However, he merely assumes that  $A$  is connected to  $P$  in at least one component in addition to the components being eliminated. He does not require that  $A$  is connected to  $P$  in at least *two vertices* outside of the components being eliminated. Andreev then asserts: “It is readily seen that all of the polyhedra obtained in this way are simple ...” In fact, the Whitehead move demonstrated below clearly creates a prismatic 3-circuit. (Here,  $M$  and  $N$  lie in  $P$ .)

Having assumed this stronger (and incorrect) version of sub-Lemma 6.9, the remainder of Andreev’s proof is relatively easy. Unfortunately, the situation pictured above is not uncommon (as we will see in Case 3 below!) Restricted to the weaker hypotheses of sub-Lemma 6.9 we will have to work a little bit harder.

We will now use these three sub-lemmas to show that if the length of  $P$  is less than  $N - 3$  (so that there are at least three interior vertices), then

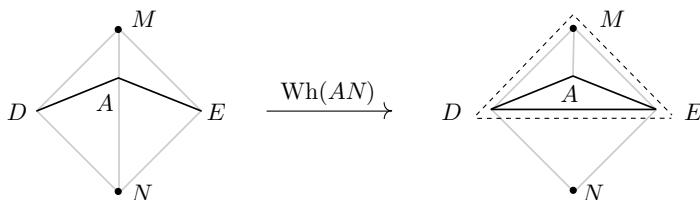


Figure 6.12

we can do Whitehead moves to increase the length of  $P$  by one, without introducing any prismatic 3-circuits.

*Case 1.* — An interior point which isn't an endpoint connects to  $P$  in a component with two or more vertices, and possibly in other components.

Apply sub-Lemma 6.8(a) decreasing this component to two vertices. We can then apply sub-Lemma 6.9, eliminating any other components since this component contains two vertices. Finally, apply sub-Lemma 6.7 to increase the length of the outer polygon by 1.

*Case 2.* — An interior vertex that is an endpoint is connected to more than three vertices of  $P$ .

We assume that each of the interior points that are not endpoints are connected to  $P$  in components consisting of single points, otherwise we are in Case 1. Let  $A$  be the endpoint which is connected to more than three vertices of  $P$ . By sub-Lemma 6.8(b), there is a Whitehead move that transfers one of these connections to the interior vertex  $E$  that is next to  $A$ . The point  $E$  is not an endpoint since the interior graph is connected and the assumption  $N - k > 3$  implies that there are at least three interior vertices. Now, one of the components in which  $E$  is connected to  $P$  has exactly two vertices, so we can then apply Case 1 for vertex  $E$ .

*Case 3.* — Each interior vertex which is an endpoint is connected to exactly three points of  $P$  and every other interior vertex is connected to  $P$  in components each consisting of a single vertex.

First, notice that if the interior vertices and edges form a line segment, this restriction on how interior points are connected to  $P$  results in the prism, contrary to hypothesis of this lemma. However, there are many complexes satisfying the hypotheses of this case which have interior vertices and edges forming a graph more complicated than a line segment:

For such complexes we need a very special sequence of Whitehead moves to increase the length of  $P$ .

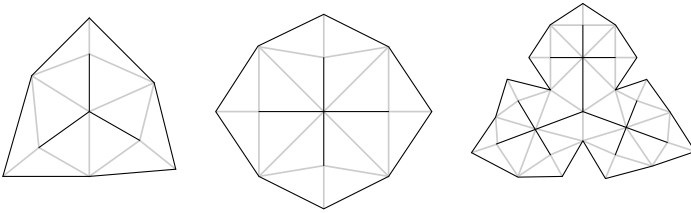


Figure 6.13

Pick an interior vertex which is an endpoint and label it  $I_1$ . Denote by  $P_1, P_2,$  and  $P_3$  the three vertices of  $P$  to which  $I_1$  connects.  $I_1$  will be connected to a sequence of interior vertices  $I_2, I_3, \dots, I_m, m \geq 2,$  with  $I_m$  the first interior vertex in the sequence that is connected to more than two other interior vertices. Vertex  $I_m$  must exist by the assumption that the interior vertices don't form a line segment, the configuration that we ruled out above. By hypothesis,  $I_2, \dots, I_m$  can only connect to  $P$  in components which each consist of a vertex, hence each must be connected to  $P_1$  and to  $P_3$ . Similarly, there is an interior vertex (call it  $X$ ) which connects both to  $I_m$  and to  $P_1$  and another vertex  $Y$  which connects to  $I_m$  and  $P_3$ . Vertex  $I_m$  may connect to other vertices of  $P$  and other interior vertices, as shown on the left side of the Figure 6.14.

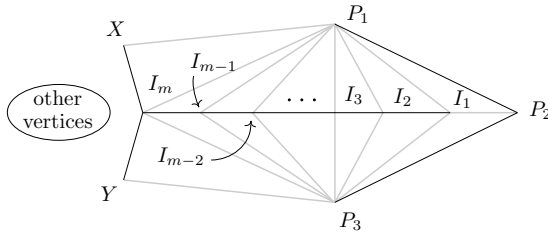


Figure 6.14

Now we describe a sequence of Whitehead moves that can be used to connect  $I_m$  to  $P$  in only  $P_1$  and  $P_2$ . This will allow us to use sub-Lemma 6.7 to increase the length of  $P$  by 1.

First, using sub-Lemma 6.9, one can eliminate all possible connections of  $I_m$  to  $P$  in places other than  $P_1$  and  $P_3$ . Next, we do the move  $Wh(I_m P_3)$  so that  $I_m$  connects to  $P$  only in  $P_1$ . We check that this Whitehead move does not create any prismatic 3-circuits. Let  $\delta$  be the curve  $v_\infty P_1 I_m P_3$  separating

$I_{m-1}$  from  $Y$ . By sub-Lemma 6.6, any newly created prismatic 3-circuit would contain a point on  $\delta$  connected to both  $I_{m-1}$  and  $Y$ . Since  $Y$  and  $I_{m-1}$  are interior,  $v_\infty$  does not connect to them. Also,  $P_1$  is not connected to  $Y$  as this would correspond to a pre-existing prismatic 3-circuit  $P_1I_mY$ , contrary to assumption. So, the only vertices of  $\delta$  that are connected to both  $I_{m-1}$  and  $Y$  are  $I_m$  and  $P_3$ , which result in the triangles  $I_mI_{m-1}Y$  and  $P_3I_{m-1}Y$ , hence do not correspond to newly created prismatic 3-circuits. Therefore  $\text{Wh}(I_mP_3)$  introduces no prismatic 3-circuits.

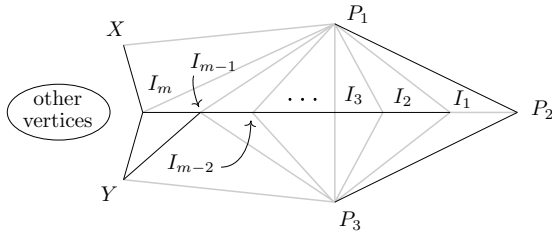


Figure 6.15

Next, one must do the moves  $\text{Wh}(I_{m-1}P_1), \dots, \text{Wh}(I_1P_1)$ , in that order (see Figure 6.16). We check that each of these moves creates no prismatic 3-circuits. Fix  $1 \leq k \leq m - 1$ , and let  $\delta$  be the loop  $v_\infty P_1 I_k P_3$ .  $\text{Wh}(I_k P_1)$  creates a new edge  $I_{k-1}I_m$  if  $k > 1$ , or  $P_2I_m$  if  $k = 1$ , the vertices of which are separated by  $\delta$ . Since  $I_m$  is interior,  $v_\infty$  does not connect to  $I_m$ . Also,  $I_m$  is no longer connected to  $P_3$ . Therefore the only points of  $\delta$  that are both connected to  $I_m$  and  $I_{k-1}$  are  $I_k$  and  $P_1$ . Those connections form the new triangles  $P_1I_mI_{k-1}$  and  $I_kI_{k-1}I_m$  (when  $k = 1$ , replace  $I_{k-1}$  with  $P_2$ ). Hence no prismatic 3-circuits were created, as claimed.

After this sequence of Whitehead moves, we obtain the upper diagram in Figure 6.17, with  $I_m$  connected to  $P$  exactly at  $P_1$  and  $P_2$ . We can then apply sub-Lemma 6.7 to increase the length of  $P$  by the move  $\text{Wh}(P_1P_2)$ , as shown in Figure 6.17. This concludes Case 3.

Since  $C^*$  must belong to one of these cases, we have seen that if the length of  $P$  is less than  $N - 3$ , we can do Whitehead moves to increase it to  $N - 3$  without creating prismatic 3-circuits. Hence we can reduce to the case of two interior vertices, both of which must be endpoints. Then we can apply sub-Lemma 6.8(b) to decrease the number of connections between one of these two interior vertices and  $P$  to exactly 3. The result is the complex  $D_N$ , as shown to the right of Figure 6.18.

Lemma 6.5 is proved. □

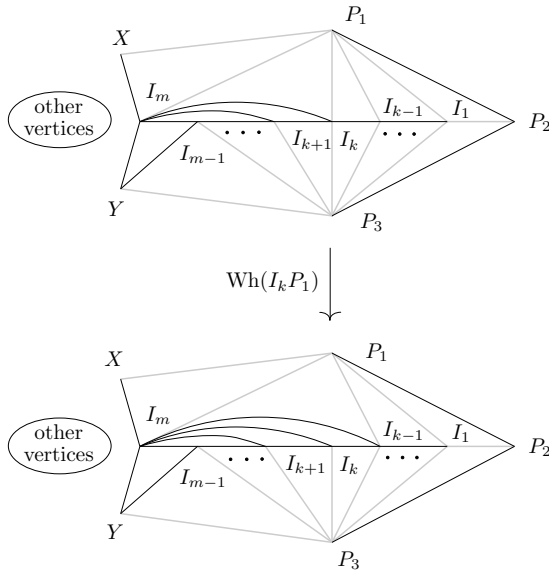


Figure 6.16

**Proof of Andreev’s Theorem for General Polyhedra**

We have seen that Andreev’s Theorem holds for every simple abstract polyhedron  $C$ . Now we consider the case of  $C$  having prismatic 3-circuits. So far, the only example we have seen has been the triangular prism. Recall that there are some such  $C$  for which  $A_C = \emptyset$ , so we can only hope to prove that  $\mathcal{P}_C^0 \neq \emptyset$  when  $A_C \neq \emptyset$ .

LEMMA 6.10. — *If  $A_C \neq \emptyset$ , then there are points in  $A_C$  arbitrarily close to  $(\frac{1}{3}\pi, \frac{1}{3}\pi, \dots, \frac{1}{3}\pi)$ .*

*Proof.* — Let  $\mathbf{a} \in A_C$  and let  $\mathbf{a}_t = \mathbf{a}(1 - t) + (\frac{1}{3}\pi, \frac{1}{3}\pi, \dots, \frac{1}{3}\pi)t$ . To see that for each  $t \in [0, 1)$ ,  $\mathbf{a}_t \in A_C$ , we check conditions 1)–5): each coordinate is clearly  $> 0$  so condition 1) is satisfied. Given edges  $e_i, e_j, e_k$  meeting at a vertex we have  $(\mathbf{a}_i + \mathbf{a}_j + \mathbf{a}_k)(1 - t) + \pi t > \pi(1 - t) + \pi t = \pi$  for  $t < 1$ , since  $(\mathbf{a}_i + \mathbf{a}_j + \mathbf{a}_k) > \pi$ . So, condition 2) is satisfied. Similarly, given a prismatic 3-circuit intersecting edges  $e_i, e_j, e_k$  we have  $(\mathbf{a}_i + \mathbf{a}_j + \mathbf{a}_k)(1 - t) + \pi t < \pi(1 - t) + \pi t = \pi$  for  $t < 1$ , so condition 3) is satisfied. Conditions 4) and 5) are satisfied since each component of  $\mathbf{a}_t$  is  $< \frac{1}{2}\pi$  for  $t > 0$  and since  $\mathbf{a}$  satisfies these conditions for  $t = 0$ . □

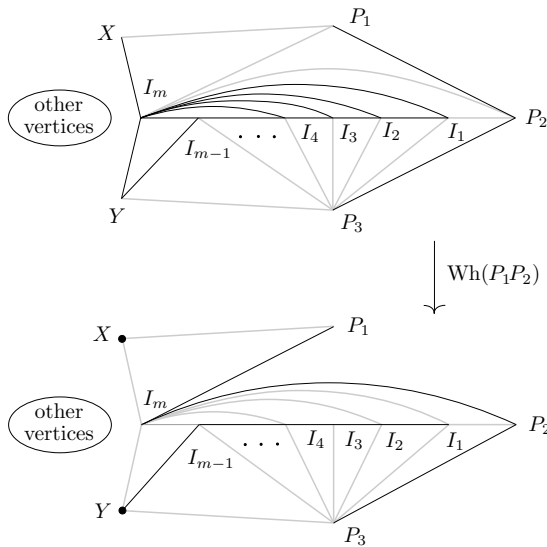


Figure 6.17

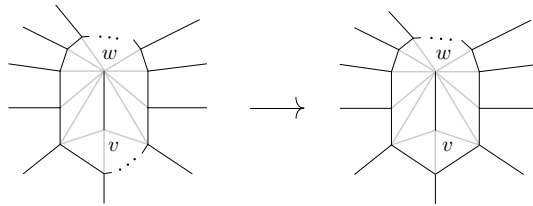


Figure 6.18

The polyhedra corresponding to these points in  $A_C$ , if they exist, will have “spiky” vertices and thin “necks”, wherever there is a prismatic 3-circuit.

We will distinguish two types of prismatic 3-circuits. If a prismatic 3-circuit in  $C^*$  separates one point from the rest of  $C^*$ , we will call it a *truncated triangle*, otherwise we will call it an *essential 3-circuit*. The name truncated triangle comes from the fact that such a 3-circuit in  $C^*$  corresponds geometrically to the truncation of a vertex, forming a triangular face. We will first prove the following sub-case:

PROPOSITION 6.11. — *Let  $C$  be an abstract polyhedron (with  $N > 4$  faces) in which every prismatic 3-circuit in  $C^*$  is a truncated triangle. If  $A_C$  is non-empty, then  $\mathcal{P}_C^0$  is non-empty.*

We will need the following three elementary lemmas in the proof:

LEMMA 6.12. — *Given three planes in  $\mathbb{H}^3$  that intersect pairwise, but which do not intersect at a point in  $\overline{\mathbb{H}^3}$ , there is a fourth plane that intersects each of these planes at right angles.*

*Proof.* — Suppose that the three planes are given by  $P_{\mathbf{v}_1}$ ,  $P_{\mathbf{v}_2}$ , and  $P_{\mathbf{v}_3}$ . Since there is no common point of intersection in  $\overline{\mathbb{H}^3}$ , the line  $P_{\mathbf{v}_1} \cap P_{\mathbf{v}_2} \cap P_{\mathbf{v}_3}$  in  $E^{3,1}$  is outside of the light-cone, so the hyperplane  $(P_{\mathbf{v}_1} \cap P_{\mathbf{v}_2} \cap P_{\mathbf{v}_3})^\perp$  intersects  $\mathbb{H}^3$  and hence defines a plane orthogonal to  $P_{\mathbf{v}_1}$ ,  $P_{\mathbf{v}_2}$ , and  $P_{\mathbf{v}_3}$ .  $\square$

LEMMA 6.13. — *Given two halfspaces  $H_1$  and  $H_2$  intersecting with dihedral angle  $a \in (0, \frac{1}{2}\pi]$  and a point  $p$  in the interior of  $H_1 \cap H_2$ . Let  $\ell_1$  be the ray from  $p$  perpendicular to  $\partial H_1$  and let  $\widetilde{H}_1$  be the half-space obtained by translating  $\partial H_1$  a distance  $\delta$  further from  $p$  perpendicularly along  $\ell_1$ . For  $\delta > 0$  and sufficiently small,  $\widetilde{H}_1$  and  $H_2$  intersect with dihedral angle  $\alpha(\delta)$  where  $\alpha$  is a decreasing continuous function of  $\delta$ .*

*Proof.* — In this and the following lemma, we use the fact that in a polyhedron  $P$  with non-obtuse dihedral angles (here  $P = H_1 \cap H_2$ ), the foot of the perpendicular from an arbitrary interior point of  $P$  to a plane containing a face of  $P$  will lie in that face (and indeed will be an interior point of that face). See, for example, [33, p. 48].

We assume that  $\delta$  is sufficiently small so that  $\partial \widetilde{H}_1$  and  $\partial H_2$  intersect. Let  $\ell_2$  be the ray from  $p$  perpendicular to  $\partial H_2$  and let  $Q$  be the plane containing  $\ell_1$  and  $\ell_2$ . By construction,  $Q$  intersects  $\partial H_1$ ,  $\partial H_2$  and  $\partial \widetilde{H}_1$  each perpendicularly so that  $Q \cap H_1$  and  $Q \cap H_2$  are half-planes in  $Q$  intersecting with angle  $a$  and  $Q \cap \widetilde{H}_1$  and  $Q \cap H_2$  are half-planes in  $Q$  intersecting with angle  $\alpha(\delta)$  (see Figure 6.19).

Let  $R$  be the compact polygon in  $Q$  bounded by  $\partial \widetilde{H}_1$ ,  $\partial H_2$ ,  $\ell_1$  and  $\ell_2$  ( $R$  is the shaded region in Figure 6.19) Because  $R$  has non-obtuse interior angles it is a parallelogram. If we denote the outward pointing normal vectors to  $\widetilde{H}_1$  and  $H_2$ , by  $\mathbf{v}_1(\delta)$  and  $\mathbf{v}_2$ , then Lemma 4.3 gives that  $\langle \mathbf{v}_1(\delta), \mathbf{v}_2 \rangle$  is a decreasing continuous function of  $\delta$ . Since  $\alpha(\delta) = \arccos(-\langle \mathbf{v}_1(\delta), \mathbf{v}_2 \rangle)$ , we see that  $\alpha$  is a decreasing continuous function of  $\delta$ , as well.  $\square$

LEMMA 6.14. — *Given a finite volume hyperbolic polyhedron  $P$  with dihedral angles in  $(0, \frac{1}{2}\pi]$  and with trivalent ideal vertices. Suppose that the vertices  $v_1, \dots, v_n$  are at distinct points at infinity and the remaining*

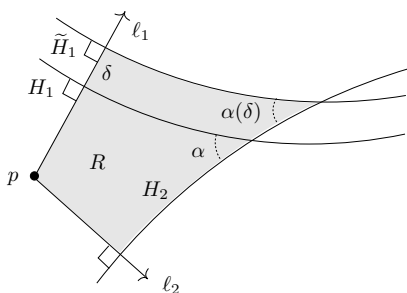


Figure 6.19

are at finite points in  $\mathbb{H}^3$ . Then there exists a polyhedron  $P'$  which is combinatorially equivalent to the result of truncating  $P$  at its ideal vertices, such that the new triangular faces of  $P'$  are orthogonal to each adjacent face, and each of the remaining dihedral angles of  $P'$  lies in  $(0, \frac{1}{2}\pi]$ .

*Proof.* — Let  $p$  be an arbitrary point in the interior of  $P = \bigcap_{i=0}^N H_i$  and let  $\ell_i$  be the ray from  $p$  to  $\partial H_i$  that is perpendicular to  $\partial H_i$ .

By Lemma 6.13, we can decrease the dihedral angles between the face carried by  $\partial H_i$  and all adjacent faces an arbitrarily small non-zero amount by translating  $\partial H_i$  a sufficiently small distance further from  $p$ . Appropriately repeating for each  $i = 1, \dots, N$ , we can shift each of the half-spaces an appropriate small distance further away from  $p$  in order to decrease all of the dihedral angles by some non-zero amount bounded above by any given  $\epsilon > 0$ .

We choose  $\epsilon$  sufficiently small that the sum of the dihedral angles at finite vertices of  $P$  remains  $> \pi$  and the dihedral angle between each pair of faces remains  $> 0$ . At each infinite vertex  $v_i$  of  $P$  the sum of dihedral angles becomes  $< \pi$  because each dihedral angle is decreased by a non-zero amount. Consequently, Lemma 6.12 gives a fourth plane perpendicular to each of the three planes previously meeting at  $v_i$ . This resulting polyhedron  $P'$  has the same combinatorics as  $P$ , except that each of the infinite vertices of  $P$  is replaced by a triangular face perpendicular to its three adjacent faces. By construction the dihedral angles of  $P'$  are in  $(0, \frac{1}{2}\pi]$ .  $\square$

*Proof of Proposition 6.11.* — By the hypothesis that  $N > 4$  from Andreev's Theorem,  $C$  is not the tetrahedron and since we have already seen that Andreev's Theorem holds for the triangular prism, we assume that  $C$  has more than five faces. In this case, one can replace all (or all but one) of the truncated triangles by single vertices to reduce  $C$  to a simple abstract



polyhedron (or to a triangular prism). The latter case is necessary when replacing all of the truncated triangles would lead to a tetrahedron, ( $Pr_5$  is a truncated tetrahedron). In either case, we call the resulting abstract polyhedron  $C^0$ .

The idea of the proof quite simple. Since Andreev's Theorem holds for  $C^0$ , we construct a polyhedron  $P^0$  realizing  $C^0$  with appropriately chosen dihedral angles. We then decrease the dihedral angles of  $P^0$ , using Lemmas 6.14 and 6.12 to truncate vertices of  $P^0$  as they "go past  $\infty$ ," eventually obtaining a compact polyhedron realizing  $C$  with non-obtuse dihedral angles.

Using that  $A_C \neq \emptyset$  and Lemma 6.10, choose a point  $\beta \in A_C$  so that each coordinate of  $\beta$  is within  $\delta$  of  $\frac{1}{3}\pi$ , with  $\delta < \frac{1}{18}\pi$ . It will be convenient to number the edges of  $C$  and  $C^0$  in the following way: If there is a prismatic 3-circuit in  $C^0$  (i.e.,  $C^0 = Pr_5$ ), we number these edges 1, 2, and 3 in  $C$  and  $C^0$ .

Otherwise, we can use the facts that  $C^0$  is trivalent and is not a tetrahedron, to find three edges whose endpoints are six distinct vertices. Next, we number the remaining edges common to  $C$  and  $C^0$  by 4, 5,  $\dots$ ,  $k$ . Finally, we number the edges of  $C$  that were removed to form  $C^0$  by  $k+1, \dots, n$  so that the edges surrounded by prismatic 3-circuits of  $C$  with smaller angle sum (given by  $\beta$ ) come before those surrounded by prismatic 3-circuits with larger angle sum.

To see that the point  $\gamma = (\beta_1, \beta_2, \beta_3, \beta_4 + 2\delta, \beta_5 + 2\delta, \dots, \beta_k + 2\delta)$  is an element of  $A_{C^0}$ , we check conditions 1)–5). Each of the dihedral angles specified by  $\gamma$  is in  $(0, \frac{1}{2}\pi)$  because  $0 < \beta_i + 2\delta < \frac{1}{3}\pi + 3\delta < \frac{1}{3}\pi + \frac{1}{6}\pi = \frac{1}{2}\pi$ . Therefore, condition 1) is satisfied, as well as conditions 4) and 5) because the angles are acute. Two of the edges labeled 4 and higher will enter any vertex of  $C^0$  so the sum of the three dihedral angles at each vertex is at least  $4\delta$  greater than the sum of the three dihedral angles given by  $\beta$ , which is  $> \pi - 3\delta$ . Therefore condition 2) is satisfied. If there is a prismatic 3-circuit in  $C^0$ , it crosses the first three edges of  $C^0$  and is also a prismatic 3-circuit in  $C$ . Since  $\beta \in A_C$ ,  $\beta_1 + \beta_2 + \beta_3 < \pi$ , and it follows that condition 3) is satisfied by  $\gamma$ .

Now define  $\alpha(t) = (1-t)\gamma + t(\beta_1, \dots, \beta_k)$ . Let  $T_0, \dots, T_{\ell-1} \in (0, 1)$  be the values of  $t$  at which there is at least one vertex of  $C^0$  for which  $\alpha(t)$  gives an angle sum of  $\pi$ . We also define  $T_{-1} = 0$  and  $T_\ell = 1$ . We will label the vertices that have angle sum  $\pi$  at  $T_i$  by  $v_1^i, \dots, v_{n(i)}^i$ . Let  $C^{i+1}$  be  $C^i$  with  $v_1^i, \dots, v_{n(i)}^i$  truncated for  $i = 0, \dots, \ell-1$ . Hence  $C^\ell$  is combinatorially equivalent to  $C$ .

Since the number of edges increases as we move from  $C^0$  toward  $C$ , we will redefine  $\alpha(t)$ , appending  $\sum_{j=1}^i n(j)$  coordinates, all constant and equal to  $\frac{1}{2}\pi$ , for values of  $t$  between  $T_{i-1}$  and  $T_i$ .

We know that Andreev's Theorem holds for  $C^0$  because  $C^0$  is either simple, or the triangular prism. So, it is sufficient to show that if Andreev's Theorem is satisfied for  $C^i$  then, for each  $i = 0, \dots, \ell - 1$ , it is satisfied for  $C^{i+1}$ . To do this, we must generate a polyhedron realizing  $C^i$  with the vertices  $v_1^i, \dots, v_{n(i)}^i$  at infinity and the other vertices at finite points in  $\mathbb{H}^3$ . This will be easy with our definition of  $\alpha(t)$  and Corollary 5.5. We will then use Lemma 6.14 to truncate these vertices. Details follow.

To use Corollary 5.5, we must check that  $\alpha(t) \in A_{C^i}$  when  $t \in (T_{i-1}, T_i)$ . This follows directly from the definition of  $\alpha(t)$ . To check condition 1), notice that both  $\beta_j$  and  $\gamma_j$  are non-zero and non-obtuse, for each  $j$  ( $1 \leq j \leq k$ ), so  $\alpha_j(t)$  must be as well. To check condition 2), note first that if a vertex of  $C^i$  belongs also to both  $C$  and  $C^0$ , then it will have dihedral angle sum greater than  $\pi$  because both the corresponding sums in both  $\beta$  and  $\gamma$  have that property. If the vertex corresponds to a truncated triangle of  $C$ , but not of  $C^i$ , then by the definition of  $T_i$ , the angle sum is greater than  $\pi$ . If the vertex lies on one of the truncated triangles of  $C^i$ , then at least two of the incident edges will have angles equal to  $\frac{1}{2}\pi$ . In each case, condition 2) is satisfied.

Any prismatic 3-circuit in  $C^i$  is either a prismatic 3-circuit in both  $C^0$  and  $C$  (the special case where  $C^0$  is the triangular prism) or is a prismatic circuit of  $C^i$  which wasn't a prismatic circuit of  $C^0$ . In the first case, the dihedral angle sum is  $< \pi$  because condition 3) is satisfied by both  $\beta$  and  $\gamma$  and in the second case, the angle sum is  $< \pi$  by definition of the  $T_i$ .

For each  $j = 1, \dots, k$  we have  $\beta_j, \gamma_j \in (0, \frac{1}{2}\pi)$ , so  $\alpha_j(t) \in (0, \frac{1}{2}\pi)$ . However,  $\alpha_j(t) = \frac{1}{2}\pi$  for  $j > k$ , corresponding to the edges of the added triangular faces. Fortunately, a prismatic 4-circuit cannot cross edges of these triangular faces, since it would have to cross two edges from the same triangle, which meet at a vertex, which is contrary to the definition of a prismatic circuit. Thus, a prismatic 4-circuit can only cross edges numbered less than or equal to  $k$ , each of which is assigned an acute dihedral angle, giving that condition 4) is satisfied.

Lemma 1.5 gives that condition 5) is a consequence of conditions 3) and 4) for  $C^1, \dots, C^\ell$ , because they cannot be triangular prisms. If  $C^0$  happens to be the triangular prism, condition 5) holds, since each of its edges  $e_1, \dots, e_k$  is assigned an acute dihedral angle.

Consider the sequence  $\alpha_{m,i} = \alpha(T_{i-1} + (1 - \frac{1}{m})(T_i - T_{i-1}))$  of dihedral angles. By our above analysis,  $\alpha_{m,i} \in A_{C^i}$  for each  $m, i$ . In fact, by definition  $\alpha_{m,i}$  limits to the point  $\alpha(T_i) \in \partial A_{C^i}$  (as  $m \rightarrow \infty$ ), which satisfies conditions (1-5) to be in  $A_{C^i}$ , except that the sum of the dihedral angles at each vertex  $v_1^i, \dots, v_{n(i)}^i$  is exactly  $\pi$ . We assume that Andreev's Theorem holds for  $C^i$ , so by Corollary 5.5, there exists a non-compact polyhedron  $P^i$  realizing  $C^i$  with each of the vertices  $v_1^i, \dots, v_{n(i)}^i$  at infinity and the rest of the vertices at finite points.

By Lemma 6.14, the existence of  $P^i$  implies that there is a polyhedron realizing  $C^{i+1}$  and therefore, by Proposition 6.1, that Andreev's Theorem holds for the abstract polyhedron  $C^{i+1}$ . Repeating this process until  $i + 1 = \ell$ , we see that Andreev's Theorem holds for  $C^\ell$ , which is our original abstract polyhedron  $C$ .  $\square$

PROPOSITION 6.15. — *If  $A_C \neq \emptyset$ , then  $\mathcal{P}_C^0 \neq \emptyset$ .*

Combined with Proposition 6.1, this proposition concludes the proof of Andreev's Theorem.

*Proof.* — By Proposition 6.2 and Proposition 6.11 we are left with the case that there are  $k > 0$  essential 3-circuits. We will show that a polyhedron realizing  $C$  with dihedral angles  $\mathbf{a} \in A_C$  can be formed by gluing together  $k + 1$  appropriate sub-polyhedra, each of which has only truncated triangles.

We will work entirely within the dual complex  $C^*$ . Label the essential 3-circuits  $\gamma_1, \dots, \gamma_k$ . The idea will be to replace  $C^*$  with  $k + 1$  separate abstract polyhedra  $C_1^*, \dots, C_{k+1}^*$  each of which has no essential 3-circuits. The  $\gamma_i$  separate the sphere into exactly  $k + 1$  components. Let  $C_i^*$  be the  $i$ -th of these components. To make  $C_i^*$  a simplicial complex on the sphere we must fill in the holes, each of which is bounded by 3 edges (some  $\gamma_\ell$ ). We glue in Figure 6.20 (the dark outer edge is  $\gamma_\ell$ ) so that where there was an essential prismatic 3-circuit in  $C^*$  there are now two truncated triangles in distinct  $C_i^*$ . Notice that none of the  $C_i$  is a triangular prism, since we have divided up  $C$  along essential prismatic 3-circuits.

In  $C_i$ , we will call each vertex, edge, or face obtained by such filling in a *new vertex*, *new edge*, or *new face* respectively. We will call all of the other edges *old edges*. We label each such new vertex with the number  $\ell$  corresponding to the 3-circuit  $\gamma_\ell$  that was filled in. For each  $\ell$ , there will be exactly two new vertices labeled  $\ell$  which are in two different  $C_i^*, C_j^*$  (see Figure 6.21 for an example).

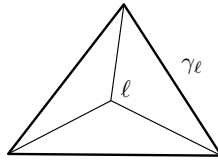


Figure 6.20

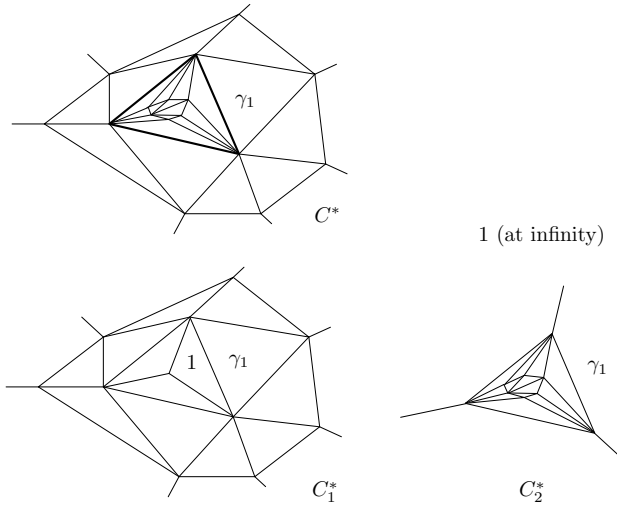


Figure 6.21

The choice of angles  $\mathbf{a} \in A_C$  gives dihedral angles assigned to each old edge in each  $C_i^*$ . Assign to each of the new edges  $\frac{1}{2}\pi$ . This gives a choice of angles  $\mathbf{a}_i$  for for which it is easy to check that  $\mathbf{a}_i \in A_{C_i}$  for each  $i$ :

Clearly condition 1) is satisfied since these angles are non-zero and none of them obtuse. The angles along each triangle of old edges in  $C_i^*$  already satisfy condition 2) since  $\mathbf{a} \in A_C$ . For each of the new triangles added, two of the edges are assigned  $\frac{1}{2}\pi$  and the third was already assigned a non-zero angle, according to  $\mathbf{a}$ , so condition 2) is satisfied for these triangles, too. None of the new edges in  $C_i^*$  can be in a prismatic 3-circuit or a prismatic 4-circuit since such a circuit would have to involve two such new edges, which form two sides of a triangle. Therefore, each prismatic 3- or 4-circuit in  $C_i^*$  has come from such a circuit in  $C^*$ , so the choice of angles  $\mathbf{a}_i$  will satisfy 3) and 4). Since none of the  $C_i^*$  is a triangular prism, condition 5) is a consequence of condition 4), and hence is satisfied.

Therefore by Proposition 6.11 there exist polyhedra  $P_i$  realizing the data  $(C_i, \mathbf{a}_i)$ , for  $1 \leq i \leq k + 1$ . For each pair of new vertices labeled  $\ell$ , the two faces dual to them are isomorphic, since by Proposition 1.1, the face angles are the same (giving congruent triangular faces). So one can glue all of the  $P_i$  together according to the labeling by  $\ell$ . Since the edges of these triangles were assigned dihedral angles of  $\frac{1}{2}\pi$ , the faces coming together from opposite sides of such a glued pair fit together smoothly. The result is a polyhedron  $P$  realizing  $C$  and angles  $\mathbf{a}$ . See the Figure 6.22.

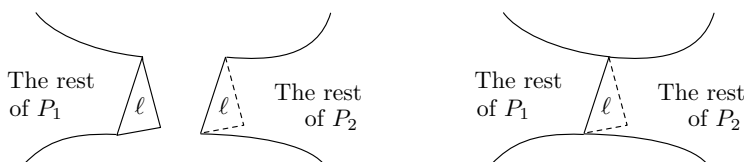


Figure 6.22

That concludes the proof that  $\alpha : \mathcal{P}_C^0 \rightarrow A_C$  is a homeomorphism for every abstract polyhedron  $C$  having more than four faces and hence concludes the proof of Andreev's Theorem.  $\square$

## 7. Example of the combinatorial algorithm from Lemma 6.5

We include an example of the combinatorial algorithm described in Lemma 6.5, which gives a sequence of Whitehead moves to reduce the dual complex of a simple abstract polyhedron,  $C^*$ , to  $D_N^*$ . We then explain how the sequence of Whitehead moves described in Andreev's paper [3] would result in prismatic 3-circuits for this  $C$  and for many others.

It is interesting to note that Andreev's version of our Lemma 6.5 (his Theorem 6 in [3]) would fail for this abstract polyhedron  $C^*$ . The major difficulty is to achieve the first increase in the length of the outer polygon  $P$ . We carefully chose the vertex  $v$  where the graph of interior vertices and edges branches and then did Whitehead moves to reduce the number of components where this vertex is connected to  $P$ . This is done in sub-Figures (1)–(4). If we had started with any other interior vertex and tried to decrease the number of components where it is connected to  $P$ , prismatic 3-circuits would develop as shown below (the dashed curve) in sub-Figures (1) and (2).

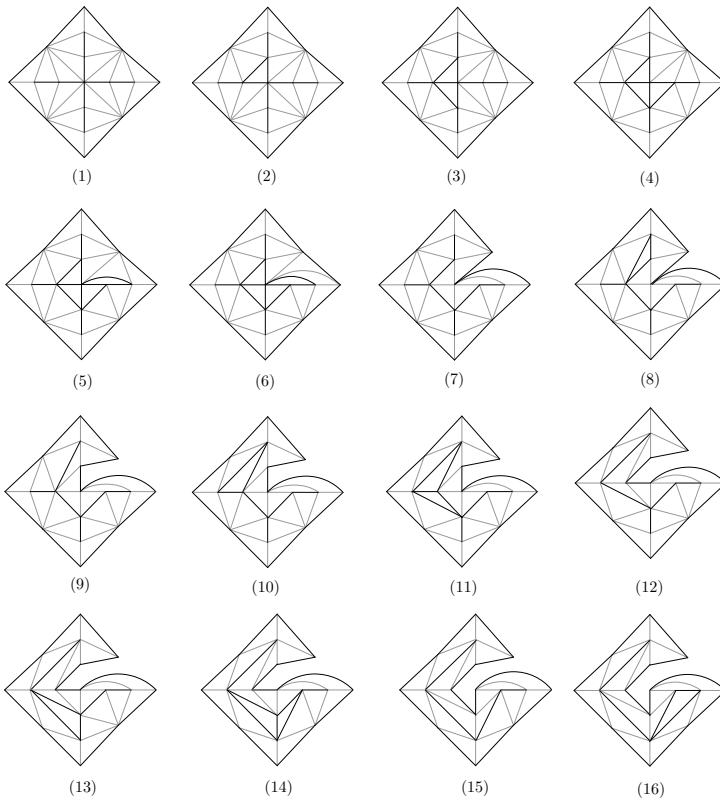


Figure 7.1. Case 3 from the proof of Lemma 6.5 is done in sub-Figures (1)–(7). Case 1 follows from sub-Figures (7)–(9) and again in sub-Figures (9)–(12). Case 1 is repeated two times, first in sub-Figures (12)–(15), then in (15)–(17).

Andreev states that one must do Whitehead moves until each interior vertex is connected to  $P$  in a single component (creating what he calls the inner polygon), but he *does not* indicate that one must do this for certain interior vertices before others. Using a very specific order of Whitehead moves to reduce the number of components of connection from each interior vertex to  $P$  to be either one or zero would work, but Andreev does not prove this. Instead of doing this, we find that it is simpler just to do Whitehead moves so that  $v$  is connected to  $P$  in a single component consisting of two vertices and an edge, instead of creating the whole “inner polygon” as

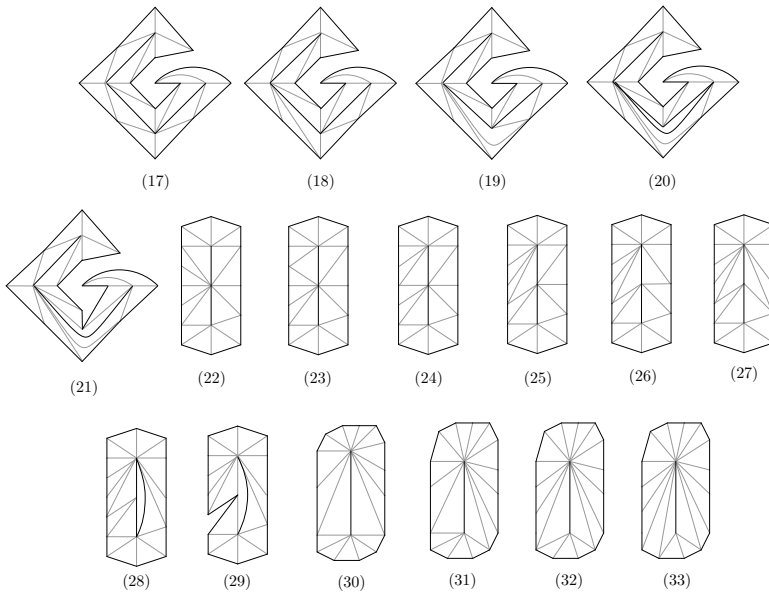


Figure 7.2. Case 1 is done in sub-Figures (17)–(21). The diagram is straightened out between sub-Figures (21) and (22), then Case 1 is repeated again in (22)–(29). The diagram is straightened out between sub-Figures (29) and (30). Apply sub-Lemmas 6.8(b) in (30)–(33) so that one of the two interior vertices is only connected to three points on the outer polygon. This reduces the complex to  $D_{18}^*$ .

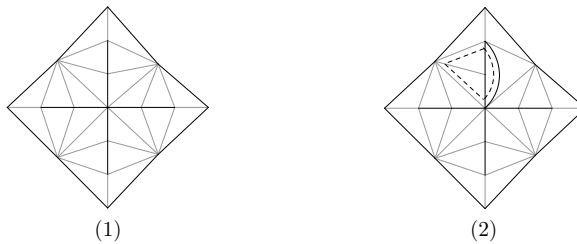


Figure 7.3

Andreev would. Once this is done, doing the Whitehead move on this edge of  $P$  increases the length of  $P$  by 1.

There are cases where the graph of interior vertices and edges has no branching points and Andreev's proof could not work, even having chosen to do Whitehead moves to decrease the components of connection between

each interior vertex and  $P$  to one in some specific order. This is true for the following abstract polyhedron  $C^*$ , shown in sub-Figure (1) below:



Figure 7.4

Each interior vertex of  $C^*$  is either an endpoint, or connected to  $P$  in exactly two components, each of which is a single point. Doing a Whitehead move to eliminate any of these connections would result in a prismatic 3-circuit like the dashed one in sub-Figure (2) above.

Instead, Case 2 of our Lemma 6.5 “borrows” a point of connection from one of the endpoints (sub-Figures (1) and (2) below), making one of the interior vertices connected to  $P$  in two components, one consisting of two vertices and an edge, and the other consisting of one vertex. One then can eliminate the single point of connection as in the Whitehead move 2) to 3). Then, it is simple to increase the length of  $P$  by doing the Whitehead move on the single edge in the single component of connection between this interior vertex and  $P$ , as shown in the change from sub-Figure (3) to (4).

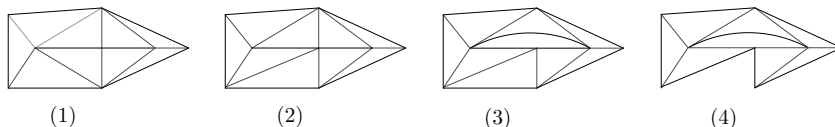


Figure 7.5

**Acknowledgments.** — This paper is a abridged version of the thesis written by the first author for the Université de Provence [26]. He would like to thank first of all John Hamal Hubbard, his thesis director, for getting him interested and involved in the beautiful subject of hyperbolic geometry.

In 2002, J.H. Hubbard suggested that the first author write a computer program that implements Andreev’s Theorem using the ideas from Andreev’s proof. He was given a version of this proof written by J.H. Hubbard and William D. Dunbar as a starting point. From the ideas in that manuscript, he wrote a program which, while faithfully implementing Andreev’s



algorithm, was unsuccessful in computing the polyhedra with given dihedral angles and combinatorial structure. From there, he found the error in Andreev's proof, which led him to first to a way to correct the program so that the polyhedra for which it was tested were actually computed correctly, and later to the arguments needed close the gap in Andreev's proof.

All of the work that the first author has done has developed out of ideas that he learned from the manuscript written by J.H. Hubbard and W.D. Dunbar. Therefore, he thanks both J.H. Hubbard and Bill Dunbar for their extensive work on that preliminary manuscript, and for their significant subsequent help and interest in his corrections of the proof by including them as co-authors.

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