ANNALES

DE

L'INSTITUT FOURIER

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Tome 57, nº 2 (2007), p. 361-377.

<http://aif.cedram.org/item?id=AIF_2007__57_2_361_0>

Ann. Inst. Fourier, Grenoble
57, 2 (2007) 361-377

A GENERALIZATION OF THE RECIPROCITY LAW OF MULTIPLE DEDEKIND SUMS

by Masahiro ASANO

Abstract. — Various multiple Dedekind sums were introduced by B.C.Berndt, L.Carlitz, S.Egami, D.Zagier and A.Bayad.
In this paper, noticing the Jacobi form in Bayad [4], the cotangent function in Zagier [23], Egami's result on cotangent functions [14] and their reciprocity laws, we study a special case of the Jacobi forms in Bayad [4] and deduce a generalization of Egami's result on cotangent functions and a generalization of Zagier's result. Further, we consider their reciprocity laws.

Résumé. — Plusieurs sommes multiples de Dedekind ont été introduites par B.C.Berndt, L.Carlitz, S.Egami, D.Zagier et A.Bayad. Dans cet article, après avoir remarqué la forme de Jacobi dans Bayad [4], la fonction cotangente dans Zagier [23], le résultat d'Egami sur les fonctions cotangentes [14] et leurs lois de réciprocité, nous étudions un cas spécial de la forme de Jacobi de Bayad [4] et déduisons une généralisation du résultat d'Egami sur les fonctions cotangentes et une généralisation du résultat de Zagier. De plus, nous considérons leurs lois de réciprocité.

1. Introduction

The Dedekind sum $s(d,c)$ $(c \in \mathbb{N}, d \in \mathbb{Z}, (c,d) = 1)$ is defined by

$$s(d,c) = \sum_{x \pmod{c}} \left(\left(\frac{x}{c}\right)\right)\left(\left(\frac{dx}{c}\right)\right).$$

Here $((x))$ is the sawtooth function defined by

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \notin \mathbb{Z}), \\ 0 & (x \in \mathbb{Z}). \end{cases}$$

It is known that Dedekind sums have many applications in various areas such as number theory (Carlitz[11],Rademacher[21]), problems of random

Keywords: Dedekind sums, reciprocity law, Jacobi forms.
Math. classification: 11A15, 11B68, 11F20, 11F23, 11F50.

numbers (Dieter[13]), the Atiyah-Singer index theorem (Atiyah and Hirzebruch [2], Atiyah and Singer[3]) and the Riemann-Roch theorem (Atiyah and Hirzebruch[1],Harder[16]).
Various generalized Dedekind sums were introduced by Carlitz[10, 12], Berndt[8], Zagier[23], Egami[14] and Bayad[4].
Applying the theory of Fourier analysis, we can write the Dedekind sum $s(p,q)$ $(p,q\in\mathbb{N},(p,q)=1)$ by means of cotangent functions as

$$s(p,q)=\frac{1}{4q}\sum_{k=1}^{q-1}\cot\left(\frac{\pi k}{q}\right)\cot\left(\frac{\pi pk}{q}\right). \tag{1.1}$$

Inspired by this expression of the Dedekind sum, Zagier defined

$$d(p;a_1,\cdots,a_n):=(-1)^{\frac{n}{2}}\frac{1}{p}\sum_{k=1}^{p-1}\cot\left(\frac{\pi ka_1}{p}\right)\cdots\cot\left(\frac{\pi ka_n}{p}\right) \tag{1.2}$$

for pairwise coprime positive integers $p,a_1,\cdots,a_n$ (n: even). Then he proved the following generalization of the reciprocity law:

Proposition 1.1 (Zagier [23], § 3, Theorem). — *Let $a_0,\cdots,a_n$ (n: even) be pairwise coprime positive integers. Then*

$$\sum_{j=0}^{n} d(a_j;a_0,\cdots,a_{j-1},a_{j+1},\cdots,a_n)=1-\frac{l_n(a_0,\cdots,a_n)}{a_0\cdots a_n},$$

where $l_n(a_0,\cdots,a_n)$ is a polynomial and can be written as $l_n(a_0,\cdots,a_n)=L_k(p_1,\cdots,p_k)$, where $k=\frac{n}{2}$, p_i $(i=1,\cdots,k)$ is the i-th elementary symmetric polynomial in $a_0^2,\cdots,a_n^2$ and L_k is the Hirzebruch polynomial (Hirzebruch[17]) which topologists know well.

Remark 1.2. — The polynomials $l_n(a_0,\cdots,a_n)$ play an important role in Section 5(see Theorem 5.1). Their very definition relates them to the sequence of Bernoulli numbers (Hirzebruch[17], §1.5; see also Beck[7] Corollary 10, §4 p.121, for an explicit expression).

On the other hand, an analogue of the Dedekind sum which was formed by replacing cotangent functions in the Dedekind sum (1.1) by some elliptic funtions was introduced by Sczech[22], Ito[19, 20] and its reciprocity law was proved. Egami[14] constructed the elliptic multiple Dedekind sum by replacing cotangent functions in the multiple Dedekind sum (1.2) by some elliptic functions and deduced its reciprocity law(cf. Proposition 3.4). And Egami also showed that some limit of his Dedekind sum agrees with (1.2)(cf. Proposition 3.7). Bayad and Robert introduced a Jacobi form (elliptic function) $D_L(z;\varphi)$ where $L=\mathbb{Z}\cdot\tau+\mathbb{Z}\cdot 1=[\tau,1]$ is a complex lattice,

$z, \varphi, \tau \in \mathbb{C}$ $(\operatorname{Im}\tau > 0)$(cf. Section 2,(2.1)), which is a generalization of the elliptic function used by Egami, and studied its properties[5, 6]. The case $\varphi = \frac{1}{2}$ coincides with Egami's case. In [4] Bayad defined a multiple Dedekind sum by replacing Egami's elliptic functions by $D_L(z;\varphi)$ and extended Egami's Theorem 1 in his paper[14] (cf. Proposition 3.4) to the case of d-division points on L(cf. Proposition 3.1).
In this paper, our aim is to obtain a generalization on Egami's Theorem 2 in [14](cf. Proposition 3.7) by taking some limit in the case of d-division point ($\varphi = \frac{m}{d}$, $d \nmid m$) of multiple Dedekind sum used by Bayad (cf. Theorem 4.1). Furthermore, we find that the case $\varphi = \frac{m\tau+j}{d}(j \in \mathbb{Z})$ can also be represented by Bernoulli polynomials(cf. Theorem 4.3).

Section 2 contains some preparation on elliptic functions. Bayad's results [4] and Egami's results [14] are stated in Section 3. In Section 4, we state and prove the generalization of Egami's Theorem 2 in [14]. In Section 5, we consider the reciprocity law on it.
I thank Professor Kohji Matsumoto and Professor Yoshio Tanigawa for their advice on my study. I also thank Professor Shigeki Egami and Professor Kaori Ota for their useful suggestions about the Hardy sum(see [9] Corollary 5.5, §5). I also thank the referee for careful reading of the manuscript and many useful suggestions.

2. The Jacobi form of Bayad and Robert

Let $\tau, z, \varphi \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ in what follows. We put $q_\tau = e^{2\pi i\tau}, q_z = e^{2\pi iz}$, $L = [\tau, 1]$. We define the Jacobi theta function as follows:

$$\theta_\tau(z) := q_\tau^{\frac{1}{8}}(e^{\frac{z}{2}} - e^{-\frac{z}{2}}) \prod_{n=1}^{\infty}(1 - q_\tau^n)(1 - q_\tau^n e^z)(1 - q_\tau^n e^{-z}).$$

Let $L = [\tau, 1]$, $\operatorname{Im}(\tau) > 0$. Define $D_L(z;\varphi)$ by

$$D_L(z;\varphi) := 2\pi i\, q_z^{\frac{\operatorname{Im}\varphi}{\operatorname{Im}\tau}} \frac{\theta'_\tau(0)\theta_\tau(u+v)}{\theta_\tau(u)\theta_\tau(v)}, \qquad (z, \varphi \in \mathbb{C} - L) \tag{2.1}$$

where $u = 2\pi iz$, $v = 2\pi i\varphi$. This is the Jacobi form associated to L, which is introduced by Bayad and Robert[5, 6], Bayad[4]. From now on, when $L = [\tau, 1]$, we write $D_L(z;\varphi) = D_\tau(z;\varphi)$ for brevity. The function $D_\tau(z;\varphi)$ has the following properties:

Proposition 2.1 (Bayad [4], Proposition 1.13). —

i) *$D_\tau(z;\varphi)$ is periodic in φ, and meromorphic in z (with simple poles at L) and satisfies the following functional equation:*

$$D_\tau(z+\rho;\varphi) = \exp(2\pi i E_L(\rho,\varphi)) D_\tau(z;\varphi) \qquad \forall \rho \in L(=[\tau,1]),$$

where $E_L(\rho,\varphi) = \frac{\mathrm{Im}(\overline{\rho}\varphi)}{\mathrm{Im}\,\tau}$.

ii) *For any* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$D_{\frac{a\tau+b}{c\tau+d}}\left(\frac{z}{c\tau+d};\frac{\varphi}{c\tau+d}\right) = (c\tau+d) D_\tau(z;\varphi).$$

iii) (Product representation) *For* $z,\varphi \in \mathbb{C} - L$,

$$D_\tau(z;\varphi) = 2\pi i\, q_z^{\frac{\mathrm{Im}\,\varphi}{\mathrm{Im}\,\tau}} \frac{(q_{z+\varphi}^{\frac{1}{2}} - q_{z+\varphi}^{-\frac{1}{2}})}{(q_z^{\frac{1}{2}} - q_z^{-\frac{1}{2}})(q_\varphi^{\frac{1}{2}} - q_\varphi^{-\frac{1}{2}})}$$
$$\prod_{n\geqslant 1} \frac{(1-q_\tau^n)^2(1-q_\tau^n q_{z+\varphi})(1-q_\tau^n q_{z+\varphi}^{-1})}{(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1})(1-q_\tau^n q_\varphi)(1-q_\tau^n q_\varphi^{-1})}.$$

3. The elliptic analogue of the reciprocity law

Let $a_1,\cdots,a_n$ be pairwise coprime positive integers and p ($\geqslant 1$) be coprime with $a_i (1 \leqslant i \leqslant n)$. We put $E_p := \{x\tau + y\ ;\ (x,y) \neq (0,0),\ 0 \leqslant x,y \leqslant p-1\}$. Then Bayad[4] defined $d_\tau(p;a_1,\cdots,a_n;\varphi)$ by

$$d_\tau(p;a_1,\cdots,a_n;\varphi) :=$$
$$\frac{1}{p}\sum_{\omega\in E_p} \exp(2\pi i E_L(\omega,\varphi)) D_\tau\left(\frac{a_1\omega}{p};\varphi\right)\cdots D_\tau\left(\frac{a_n\omega}{p};\varphi\right).$$

We note that the multiple Dedekind sum d_τ does not depend on the representatives ω in L for the p-torsion points in L/pL, when the torsion point $(p+a_1+\cdots+a_n)\varphi$ is in L.
Bayad showed the following reciprocity law for $d_\tau(p;a_1,\cdots,a_n;\varphi)$:

Proposition 3.1 (Bayad [4] Theorem 2.2 reciprocity law). — *Let $a_0,a_1,\cdots,a_n$ be pairwise coprime positive integers and d be a divisor of $a_0+a_1+\cdots+a_n$. Then for all non-zero d-division point φ of $\mathbb{C}/L$,*

$$\sum_{k=0}^{n} d_\tau(a_k;a_0,\cdots,a_{k-1},a_{k+1},\cdots,a_n;\varphi) = -\frac{M_{n,\tau}(a_0;a_1,\cdots,a_n;\varphi)}{a_0\cdots a_n},$$

where $M_{n,\tau}(a_0;a_1,\cdots,a_n;\varphi)$ is the coefficient of z^n in the expansion of $\prod_{k=0}^n (a_k z) D_\tau(a_k z;\varphi)$, i.e. the residue of the function $\prod_{k=0}^n (a_k) D_\tau(a_k z;\varphi)$.

Remark 3.2. — $M_{n,\tau}(a_0; a_1, \cdots, a_n; \frac{1}{2})$ is written by means of elementary symmetric polynomials of $a_0^2, \cdots, a_n^2$ as a polynomial which occurs in elliptic genus theory (Hirzebruch et al.[18]). (See Cor. 5.2.)

By specializing φ to $\frac{1}{2}$ in the above result, one finds:

COROLLARY 3.3 (Bayad[4] Corollary 2.3). — *Let $a_0, a_1, \cdots, a_n$ be pairwise coprime positive integers such that $a_0 + \cdots + a_n$ is even. Then Proposition 3.1 for 2-division point $\varphi = \frac{1}{2}$ of $\mathbb{C}/L$ is equivalent to Theorem 1 of Egami*[14].

Here, Theorem 1 of Egami[14] is stated as follows:

PROPOSITION 3.4 (Egami[14] Theorem 1). — *Let $a_0, a_1, \cdots, a_n$ be pairwise coprime positive integers such that $a_0 + \cdots + a_n$ is even. Then*

$$\sum_{k=0}^{n} \frac{1}{a_k} D'_\tau(a_k; a_0, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n) = -\frac{M'_{n,\tau}(a_0, \cdots, a_n)}{a_0 \cdots a_n},$$

where

$$D'_\tau(p; a_1, \cdots, a_n) = \sum_{\substack{\omega \in E_p \\ \omega = x\tau + y}} (-1)^x \prod_{k=1}^{n} \varphi\left(\tau, \frac{2\pi i a_k \omega}{p}\right),$$

$$\varphi(\tau, z) = \frac{1}{2} \frac{e^{\frac{z}{2}} + e^{-\frac{z}{2}}}{e^{\frac{z}{2}} - e^{-\frac{z}{2}}} \prod_{n=1}^{\infty} \frac{(1 + q_\tau^n e^z)(1 + q_\tau^n e^{-z})(1 - q_\tau^n)^2}{(1 - q_\tau^n e^z)(1 - q_\tau^n e^{-z})(1 + q_\tau^n)^2},$$

and $M'_{n,\tau}(a_0, \cdots, a_n)$ is the coefficient of z^n in $\prod_{k=0}^{n}(a_k z)\varphi(\tau, a_k z)$.

Remark 3.5. — The definitions of $M'_{n,\tau}(a_0, \cdots, a_n)$ and $M_{n,\tau}(a_0; a_1, \cdots, a_n; \varphi)$ are slightly different from their original ones. The necessity of this change has been pointed out by Fukuhara and Yui [15].

Remark 3.6. — From Proposition 2.1 iii), since $\varphi(\tau, z) = \frac{1}{2\pi i} D_\tau\left(\frac{z}{2\pi i}; \frac{1}{2}\right) = D_{2\pi i L}(z; \pi i)$, we find $\left(\frac{1}{2\pi i}\right) n^p\, d_\tau(p; a_1, \cdots, a_n; \frac{1}{2}) = D'_\tau(p; a_1, \cdots, a_n)$ and $\left(\frac{1}{2\pi i}\right)^n M_{n,\tau}\left(a_0; a_1, \cdots, a_n; \frac{1}{2}\right) = M'_{n,\tau}(a_0, \cdots, a_n)$. (i.e. the case $\varphi = \frac{1}{2}$ is essentially Egami's result.)

Egami showed that some limit of D'_τ is written as a sum of product of cotangent functions similarly to the right hand side of (1.2):

Proposition 3.7 (Egami[14] Theorem 2, Bayad[4] Corollary 2.5). — *Let* $p, a_1, \cdots, a_n$ *(n: even) be pairwise coprime positive integers. Then*

$$\left(\frac{1}{2i}\right)^n \sum_{k=1}^{p-1} \cot\left(\frac{\pi k a_1}{p}\right) \cdots \cot\left(\frac{\pi k a_n}{p}\right) \tag{3.1}$$

$$= \lim_{\mathrm{Im}(\tau)\to\infty} D'_\tau(p; a_1, \cdots, a_n) + \frac{(-1)^{n+1}}{2^n} p \sum_{t=1}^{p-1} (-1)^{t+\left[\frac{a_1 t}{p}\right]+\cdots+\left[\frac{a_n t}{p}\right]}.$$

Remark 3.8. — When n is odd, the left hand side of (3.1) vanishes identically.

4. A generalization to the case of d-division points

Generalizing Egami's argument[14] in his proof of Proposition 3.7, we can prove a generalization of Proposition 3.7 as follows:

Theorem 4.1. — *Let* $p, a_1, \cdots, a_n$ *be pairwise coprime positive integers and* d *be a positive integer such that* $d \geqslant 2$. *Then for any* d*-division point* $\frac{m}{d}$ $(m \in \mathbb{N}, d \nmid m)$,

(4.1)

$$\left(\frac{1}{2\pi i}\right)^n \lim_{\mathrm{Im}(\tau)\to\infty} p\, d_\tau\left(p; a_1, \cdots, a_n; \frac{m}{d}\right)$$

$$= \sum_{n'=1}^{p-1} \prod_{q=1}^{n} \frac{\xi e^{2\pi i \frac{a_q n'}{p}} - 1}{(e^{2\pi i \frac{a_q n'}{p}} - 1)(\xi - 1)} + \left(\frac{1}{\xi - 1}\right)^n p \sum_{m'=1}^{p-1} (\xi^{-m'}) \prod_{k=1}^{n} (\xi^{-1})^{\left[\frac{a_k m'}{p}\right]}$$

$$= \left(\frac{1}{2i}\right)^n \sum_{n'=1}^{p-1} \prod_{q=1}^{n} \left(\cot \pi \left(\frac{m}{d}\right) + \cot \pi \left(\frac{a_q n'}{p}\right)\right)$$

$$+ \left(\frac{1}{\xi - 1}\right)^n p \sum_{m'=1}^{p-1} (\xi^{-m'}) \prod_{k=1}^{n} (\xi^{-1})^{\left[\frac{a_k m'}{p}\right]},$$

where $\xi = e^{2\pi i \frac{m}{d}}$.

Remark 4.2. — If $\cot \pi \left(\frac{m}{d}\right) = 0$ $\left(\frac{m}{d} = \frac{1}{2}\right)$, then the above result coincides with Proposition 3.7 (Egami[14](Theorem 2)).

Proof. — From Proposition 2.1 iii), we deduce

$$(4.2)\quad D_\tau\left(z;\frac{m}{d}\right)=2\pi i\,\frac{\xi^{\frac{1}{2}}q_z^{\frac{1}{2}}-\xi^{-\frac{1}{2}}q_z^{-\frac{1}{2}}}{(q_z^{\frac{1}{2}}-q_z^{-\frac{1}{2}})(\xi^{\frac{1}{2}}-\xi^{-\frac{1}{2}})}\times\prod_{n\geqslant 1}\frac{(1-q_\tau^n)^2(1-q_\tau^n q_z\cdot\xi)(1-q_\tau^n q_z^{-1}\xi^{-1})}{(1-q_\tau^n q_z)(1-q_\tau^n\cdot q_z^{-1})(1-q_\tau^n\cdot\xi)(1-q_\tau^n\xi^{-1})}.$$

Let $z=x\tau+y\notin L$ $(x,y\in\mathbb{R})$, and write $\tau=k+il$. Then for $0\leqslant x<1$,

$$q_\tau^n\to 0,\quad q_z\to 0\qquad(x\neq 0)$$
$$\text{and}\ \ q_\tau^n q_z^{-1}=e^{2\pi i(nk-kx-y)}\cdot e^{-2\pi(n-x)l}\to 0\qquad\text{as}\ \ \operatorname{Im}\tau=l\to\infty.$$

Therefore, when $x\neq 0$, the infinite product on the right hand side of (4.2) tends to 1 as $\operatorname{Im}\tau=l\to\infty$. We note that if $x=0$, then q_z does not tend to zero, but since q_τ tends to zero, the same conclusion holds in this case. Further, as $\operatorname{Im}\tau=l\to\infty$, we find

$$\frac{\xi q_z^{\frac{1}{2}}-q_z^{-\frac{1}{2}}}{(q_z^{\frac{1}{2}}-q_z^{-\frac{1}{2}})(\xi-1)}\to\begin{cases}\frac{\xi e^{\pi iy}-e^{-\pi iy}}{(e^{\pi iy}-e^{-\pi iy})(\xi-1)} & (x=0),\\ \frac{1}{\xi-1} & (x>0).\end{cases}$$

Therefore,

$$\left(\frac{1}{2\pi i}\right)\lim_{\operatorname{Im}\tau\to\infty}D_\tau\left(x\tau+y;\frac{m}{d}\right)=\begin{cases}\frac{\xi e^{\pi iy}-e^{-\pi iy}}{(e^{\pi iy}-e^{-\pi iy})(\xi-1)} & (x=0),\\ \frac{1}{\xi-1} & (0<x<1).\end{cases}$$

We have

$$\left(\frac{1}{2\pi i}\right)\lim_{\operatorname{Im}\tau\to\infty}D_\tau\left(x\tau+y;\frac{m}{d}\right)=\frac{1}{\xi-1}\cdot(\xi^{-1})^{[x]}\qquad(x\notin\mathbb{Z},\ x>0).$$

Thus from the definition of $d_\tau\left(p;a_1,\cdots,a_n;\frac{m}{d}\right)$, we have

$$\begin{aligned}&\left(\frac{1}{2\pi i}\right)^n\lim_{\operatorname{Im}(\tau)\to\infty}p\,d_\tau\left(p;a_1,\cdots,a_n;\frac{m}{d}\right)\\&=\lim_{\operatorname{Im}(\tau)\to\infty}\sum_{\omega\in E_p}\exp\left(2\pi iE_L\left(\omega,\frac{m}{d}\right)\right)\\&\times\left(\frac{1}{2\pi i}\right)^n D_\tau\left(\frac{a_1\omega}{p};\frac{m}{d}\right)\cdots D_\tau\left(\frac{a_n\omega}{p};\frac{m}{d}\right),\end{aligned}$$

where $E_p=\{x\tau+y\ ;\ (x,y)\neq(0,0)\ ,\ 0\leqslant x,y\leqslant p-1\}$, $E_L\left(\omega,\frac{m}{d}\right)=\frac{\operatorname{Im}\left(\overline{\omega}\cdot\frac{m}{d}\right)}{\operatorname{Im}\tau}$.

When we write $\omega=m'\tau+n'$, we have $\exp\left(2\pi iE_L\left(\omega,\frac{m}{d}\right)\right)=\xi^{-m'}$.

Therefore,

$$\left(\frac{1}{2\pi i}\right)^n \lim_{\operatorname{Im}\tau\to\infty} p\, d_\tau\left(p; a_1, \cdots, a_n; \frac{m}{d}\right)$$
$$= \sum_{n'=1}^{p-1}\prod_{q=1}^{n} \frac{\xi e^{\pi i \frac{a_q n'}{p}} - e^{-\pi i \frac{a_q n'}{p}}}{(e^{\pi i \frac{a_q n'}{p}} - e^{-\pi i \frac{a_q n'}{p}})(\xi - 1)}$$
$$+ \left(\frac{1}{\xi-1}\right)^n \sum_{m'=1}^{p-1}(\xi^{-m'}) \sum_{n'=0}^{p-1}\prod_{k=1}^{n}(\xi^{-1})^{\left[\frac{a_k m'}{p}\right]},$$

which shows the theorem immediately. □

Further, with respect to τ, we can show the following theorem.

Theorem 4.3. — *Assuming the same condition as in Theorem 4.1, for any d-division point $\frac{m\tau+j}{d}$ ($m \in \mathbb{N}, d \nmid m,\ j \in \mathbb{Z}$) we have*

(4.3)
$$\left(\frac{1}{2\pi i}\right)^n \lim_{\operatorname{Im}(\tau)\to\infty} p d_\tau\left(p; a_1, \cdots, a_n; \frac{m\tau + j}{d}\right)$$
$$= \sum_{n'=1}^{p-1}(\xi^{n'})\prod_{k=1}^{n}\frac{e^{2\pi i \frac{m}{d}\cdot\frac{a_k n'}{p}}}{e^{2\pi i \frac{a_k n'}{p}} - 1}$$
$$= \sum_{n'=1}^{p-1}(\xi^{n'})\prod_{k=1}^{n}\sum_{t=0}^{\infty} B_t\left(\frac{m}{d}\right)\frac{\left(2\pi i \frac{a_k n'}{p}\right)^{t-1}}{t!},$$

where $B_n(x)$ is the n-th Bernoulli polynomial defined by $\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$ and $\xi = e^{2\pi i \frac{m}{d}}$.

Proof. — We first consider the case $1 \leqslant t \leqslant d-1$ (t:fix). From Proposition 2.1 iii),

(4.4)
$$D_\tau\left(z; \frac{t\tau+j}{d}\right) = 2\pi i\, q_z^{\frac{t}{d}} \frac{q_z^{\frac12}\cdot q_\tau^{\frac{t}{d}}\zeta - q_z^{-\frac12}}{(q_z^{\frac12} - q_z^{-\frac12})(q_\tau^{\frac{t}{d}}\zeta - 1)}$$
$$\times \prod_{n\geqslant 1}\frac{(1-q_\tau^n)^2(1-q_\tau^{n+\frac{t}{d}}q_z\zeta)(1-q_\tau^{n-\frac{t}{d}}q_z^{-1}\zeta^{-1})}{(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1})(1-q_\tau^{n+\frac{t}{d}}\zeta)(1-q_\tau^{n-\frac{t}{d}}\zeta^{-1})},$$

where $\zeta = e^{2\pi i \frac{j}{d}}$. We write $z = x\tau + y \notin L \quad (x, y \in \mathbb{R})$ and $\tau = k + il$. Then for $0 \leqslant x < 1 - \frac{t}{d}$,

$$q_\tau^n \to 0 \quad (q_\tau^{n-\frac{t}{d}} \to 0), \;\; q_z \to 0 \quad (x \neq 0), \;\; q_\tau^n q_z^{-1} \to 0$$
$$\text{and} \quad q_\tau^{n-\frac{t}{d}} q_z^{-1} = e^{2\pi i\alpha} \cdot e^{-2\pi l(n-\frac{t}{d}-x)} \to 0 \quad \text{as } \operatorname{Im}\tau = l \to \infty,$$

where α is a quantity independent of l. Therefore, when $x \neq 0$, the above infinite product on the right hand side of (4.4) tends to 1 as $\operatorname{Im}\tau = l \to \infty$. We note that if $x = 0$, then q_z does not tend to zero, but since q_τ tends to zero, the same conclusion holds in this case. Furthermore, since

$$\text{(4.5)} \qquad \frac{q_\tau^{\frac{t}{d}} q_z^{\frac{t}{d}+\frac{1}{2}}\zeta - q_z^{-\left(\frac{1}{2}-\frac{t}{d}\right)}}{(q_z^{\frac{1}{2}} - q_z^{-\frac{1}{2}})(q_\tau^{\frac{t}{d}}\zeta - 1)} \to \begin{cases} 0 & (x > 0), \\ \frac{e^{\frac{2\pi i y t}{d}}}{e^{2\pi i y}-1} & (x = 0), \end{cases}$$

as $l \to \infty$, we have

$$\left(\frac{1}{2\pi i}\right) \lim_{\operatorname{Im}\tau\to\infty} D_\tau\left(x\tau + y; \frac{t\tau + j}{d}\right) = \begin{cases} \frac{e^{\frac{2\pi i y t}{d}}}{e^{2\pi i y}-1} & (x = 0), \\ 0 & (0 < x < 1 - \frac{t}{d}). \end{cases}$$

Next we prove that if $x \notin \mathbb{N}$ $(x > 0)$, then

$$\text{(4.6)} \qquad \left(\frac{1}{2\pi i}\right) \lim_{\operatorname{Im}\tau\to\infty} D_\tau\left(x\tau + y; \frac{t\tau + j}{d}\right) = 0.$$

Since other parts in the infinite product on the right hand side of (4.4) tend to 1 as $l \to \infty$, it is enough to consider the following part of the infinite product:

$$\text{(4.7)} \qquad \prod_{n\geqslant 1} \frac{1 - q_\tau^{n-\frac{t}{d}} q_z^{-1}\zeta^{-1}}{1 - q_\tau^n q_z^{-1}}.$$

The above product (4.7) is equal to

$$\text{(4.8)} \qquad \prod_{n\geqslant 1} \frac{1 - e^{2\pi i\alpha'} \cdot e^{-2\pi l\left(n-\frac{t}{d}-x\right)}}{1 - e^{2\pi i(nk-y-kx)}e^{-2\pi(n-x)l}},$$

where α' is a quantity independent of l. Let

$$\text{(4.8)} = \prod_{\substack{n\geqslant 1\\ n-x<0}} \times \prod_{\substack{n\geqslant 1\\ 0<n-x<\frac{t}{d}}} \times \prod_{\substack{n\geqslant 1\\ n-x>\frac{t}{d}}} \times \prod_{\substack{n\geqslant 1\\ n-x=\frac{t}{d}}} =: P_1 \times P_2 \times P_3 \times P_4,$$

say. Here, the empty product is to be regarded as 1.

First, we consider the case $n - x < 0$ (x is fixed). Then, since $n - \frac{t}{d} - x < 0$, $P_1 = O\left(e^{2\pi l[x]\frac{t}{d}}\right) \qquad (l \to \infty)$.

Secondly, we consider the case $0 < n - x < \frac{t}{d}$. Then since $n - \frac{t}{d} - x < 0$,

$P_2 = O\left(e^{-2\pi l\left(n-\frac{t}{d}-x\right)}\right) = O\left(e^{-2\pi l\left([x]+1-\frac{t}{d}-x\right)}\right)\ (l \to \infty)$.
Thirdly, we consider the case $n - x > \frac{t}{d}$. Since $n - x - \frac{t}{d} > 0$, P_3 tends to 1 as $l \to \infty$.
Fourthly, we consider the case $n - x = \frac{t}{d}$. Since $n - x - \frac{t}{d} = 0$, $P_4 = O(1)\ (l \to \infty)$. And when $x > 0$, the order of (4.5) is at most $O\left(e^{-\frac{2\pi l x t}{d}}\right)$ $(l \to \infty)$. Hence the order of $\left(\frac{1}{2\pi i}\right) D_\tau\left(x\tau + y; \frac{t\tau+j}{d}\right)\quad (l \to \infty)$ is

$$\begin{aligned} &O\left(\exp\left\{-2\pi l\left(\frac{xt}{d} - \frac{[x]t}{d} + n - \frac{t}{d} - x\right)\right\}\right) \\ &= O\left(\exp\left\{-2\pi l\left(\frac{(d-t)([x]+1-x)}{d}\right)\right\}\right). \end{aligned}$$

Therefore (4.6) holds.

Here, writing $\omega = m'\tau + n'$, we have $\exp\left(2\pi i E_L\left(\omega, \frac{t\tau+j}{d}\right)\right) = \xi^{n'} \cdot \zeta^{-m'}$, where $\xi = e^{2\pi i \frac{t}{d}}$. Therefore, by the definition of d_τ,

$$\begin{aligned} &\left(\frac{1}{2\pi i}\right)^n \lim_{\operatorname{Im}\tau\to\infty} p\, d_\tau\left(p; a_1, \cdots, a_n; \frac{t\tau+j}{d}\right) \\ &= \sum_{\substack{m',n'=0 \\ (m',n')\neq(0,0)}}^{p-1} (\xi^{n'}\zeta^{-m'}) \\ &\times \left(\frac{1}{2\pi i}\right)^n \prod_{k=1}^{n} \lim_{\operatorname{Im}\tau\to\infty} D_\tau\left(\frac{a_k m'\tau + a_k n'}{p}; \frac{t\tau+j}{d}\right) \\ &= \sum_{n'=1}^{p-1} (\xi^{n'}) \prod_{k=1}^{n} \frac{e^{\frac{2\pi i t}{d}\cdot\frac{a_k n'}{p}}}{e^{2\pi i \frac{a_k n'}{p}} - 1}. \end{aligned}$$

For the case $m > d$ ($m \in \mathbb{N}$), though (4.6) holds only $m < d$, we can reduce to $D_\tau\left(z; \frac{t\tau+j}{d}\right)\quad (1 \leqslant t \leqslant d-1)$ because of the periodity of $D_\tau(z;\varphi)$ with φ. Therefore we can replace $pd_\tau\left(p; a_1, \cdots, a_n; \frac{m\tau+j}{d}\right)$ by $pd_\tau\left(p; a_1, \cdots, a_n; \frac{t\tau+j}{d}\right)$. Hence the theorem follows.

□

5. Consideration of the generalized reciprocity law

Denote the first term on the right hand side of (4.1) by

$$K\left(p; a_1, \cdots, a_n; \frac{m}{d}\right) := \left(\frac{1}{2i}\right)^n \sum_{n'=1}^{p-1} \prod_{q=1}^{n} \left(\cot\pi\left(\frac{m}{d}\right) + \cot\pi\left(\frac{a_q n'}{p}\right)\right),$$

and the second term of (4.1) by

$$g\left(p; a_1, \cdots a_n; \frac{m}{d}\right) := \left(\frac{1}{\xi - 1}\right)^n p \sum_{m'=1}^{p-1} (\xi^{-m'}) \prod_{k=1}^{n} (\xi^{-1})^{\left[\frac{a_k m'}{p}\right]}.$$

Then $K\left(p; a_1, \cdots, a_n; \frac{m}{d}\right)$ is written as a sum of product of cotangent functions, expanding the above product. Hence the following theorem holds. From now on we put $c = \cot \pi \left(\frac{m}{d}\right)$ $(d, m$: fix) for brevity.

THEOREM 5.1. — *Let n be an even natural number. Consider a_0, $a_1, \cdots, a_n$ be pairwise coprime positive integers such that $d | (a_0 + a_1 + \cdots + a_n)$ and $d \geqslant 2$. Then for any d-division point $\frac{m}{d}$ $(m \in \mathbb{N}$, $d \nmid m)$,*

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^n \sum_{k=0}^{n} \lim_{\operatorname{Im}\tau \to \infty} d_\tau \left(a_k; a_0, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n; \frac{m}{d}\right) \\
&= -\left(\frac{1}{2\pi i}\right)^n \lim_{\operatorname{Im}\tau\to\infty} \frac{M_{n,\tau}\left(a_0; a_1, \cdots, a_n; \frac{m}{d}\right)}{a_0 \cdots a_n} \\
&= -\left(\tilde{l_n} - \left(\frac{c}{2}\right)^2 \sum \tilde{l}_{n-2} + \left(\frac{c}{2}\right)^4 \sum \tilde{l}_{n-4} + \cdots \right.\\
&\qquad\qquad \left. + i^{n-2} \left(\frac{c}{2}\right)^{n-2} \sum \tilde{l_2} + i^n \left(\frac{c}{2}\right)^n \sum_{k=0}^{n} \frac{1}{a_k}\right),
\end{aligned}$$

where $c = \cot \pi \left(\frac{m}{d}\right)$, $\tilde{l_n} := \frac{1}{2^n} \frac{l_n(a_0, \cdots, a_n)}{a_0 \cdots a_n}$ and for even s satisfying $2 \leqslant s \leqslant n-2$, we put

$$\sum \tilde{l_s} := \frac{1}{2^s} \sum_{\substack{(m_1, \cdots, m_{s+1}) \in \mathbb{Z}^{s+1} \\ 0 \leqslant m_1 < \cdots < m_{s+1} \leqslant n}} \frac{l_s(a_{m_1}, \cdots, a_{m_{s+1}})}{a_{m_1} \cdots a_{m_{s+1}}}.$$

Proof. — We first illustrate the essence of the argument by considering the case $n = 4$.
Let k be an integer $(0 \leqslant k \leqslant 4)$ and put

$$f(a_k; a_0, \cdots, a_{k-1}, a_{k+1}, \cdots, a_4) := \left(\frac{1}{2i}\right)^4 \frac{1}{a_k} \sum_{n'=1}^{a_k - 1} \cot \pi \left(\frac{a_0 n'}{a_k}\right) \cdots \overset{\vee}{k} \cdots \cot \pi \left(\frac{a_4 n'}{a_k}\right).$$

Here $\overset{\vee}{k}$ means that we omit the k-th cotangent function. We notice that if the number of cotangent functions is odd, then $f = 0$ by Remark 3.8.

Hence for any fixed k, we have

$$\frac{1}{a_k}K\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right)$$

$$=\left(\frac{1}{2i}\right)^4\frac{1}{a_k}\sum_{n'=1}^{a_k-1}\prod_{\substack{q=0\\q\neq k}}^{4}\left(c+\cot\pi\left(\frac{a_q n'}{a_k}\right)\right)$$

$$=f(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4)+c^2\sum_{l_1=0}^{4}{}'\sum_{l_2=0}^{4}{}' f(a_k;a_0,\overset{\vee}{a_{l_1}},\cdots,\overset{\vee}{a_{l_2}},\overset{\vee}{a_k},a_4)$$

$$+\left(\frac{1}{2i}\right)^4c^4\,\frac{a_k-1}{a_k},$$

where $\overset{\vee}{a_k}$ means that we omit the term a_k, $\sum{}'$ means the sum with the condition $l_1\neq k, l_2\neq k$ and $l_1\neq l_2$ and we let c:=$\cot\pi\left(\frac{m}{d}\right)$ $(m,d$: fix).

Therefore we have

$$\left(\frac{1}{2\pi i}\right)^4\sum_{k=0}^{4}\lim_{\operatorname{Im}\tau\to\infty}d_\tau\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right)$$

$$=\sum_{k=0}^{4}\left(\frac{1}{a_k}K\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right)+\frac{1}{a_k}g\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right)\right)$$

$$=\sum_{k=0}^{4}f(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4)$$

$$+c^2\sum_{k=0}^{4}\sum_{l_1=0}^{4}{}'\sum_{l_2=0}^{4}{}' f(a_k;a_0,\cdots,\overset{\vee}{a_{l_1}},\cdots,\overset{\vee}{a_{l_2}},\cdots,\overset{\vee}{a_k},\cdots,a_4)$$

$$+\left(\frac{1}{2i}\right)^4\cdot c^4\cdot\sum_{k=0}^{4}\left(1-\frac{1}{a_k}\right)+\sum_{k=0}^{4}\frac{1}{a_k}g\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right).$$

By Proposition 1.1, denoting $1-\frac{l_i(a_{m_0},\cdots,a_{m_i})}{a_{m_0}\cdots a_{m_i}}=:l'_i(a_{m_0},\cdots,a_{m_i})\quad(0\leqslant m_0<\cdots<m_i\leqslant 4,\ \ i=2,4)$, we have

(5.1)

$$\left(\frac{1}{2\pi i}\right)^4\sum_{k=0}^{4}\lim_{\operatorname{Im}\tau\to\infty}d_\tau\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right)$$

$$=\frac{1}{2^4}\left\{l'_4(a_0,\cdots,a_4)-c^2\sum_{0\leqslant m_1<m_2<m_3\leqslant 4}l'_2(a_{m_1},a_{m_2},a_{m_3})\right\}$$

$$+\left(\frac{1}{2i}\right)^4\cdot c^4\cdot\sum_{k=0}^{4}\left(1-\frac{1}{a_k}\right)+\sum_{k=0}^{4}\frac{1}{a_k}g\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right).$$

Then by using the analytic continuation of the formula for the Hardy sum (Corollary 5.5 in [9]) and the condition that $d|(a_0+a_1+\cdots+a_4)$, we have

$$\sum_{k=0}^{4}\frac{1}{a_k}g\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right)$$
$$=\left(\frac{1}{\xi-1}\right)^4\sum_{k=0}^{4}\sum_{m'=1}^{a_k-1}(\xi^{-m'})\prod_{\substack{j=0\\j\neq k}}^{4}(\xi^{-1})^{\left[\frac{a_jm'}{a_k}\right]}$$
$$=-\left(\frac{1}{\xi-1}\right)^5\cdot\left(\xi^5-1\right),$$

where $\xi=e^{2\pi i\frac{m}{d}}$. By the definitions of l_i' $(i=2,4)$ and $c=i(\xi+1)/(\xi-1)$, we see that the terms on the right hand side of (5.1), which do not depend on $a_0,\cdots,a_4$, are

$$\frac{1}{2^4}\left\{1-\binom{5}{2}c^2+5\cdot c^4\right\}-\frac{\xi^5-1}{(\xi-1)^5}$$
$$=\frac{1}{2^5}\left((1+ic)^5+(1-ic)^5\right)-\frac{\xi^5-1}{(\xi-1)^5}$$
$$=0.$$

Therefore we have

$$\left(\frac{1}{2\pi i}\right)^4\sum_{k=0}^{4}\lim_{\operatorname{Im}\tau\to\infty}d_\tau\left(a_k;a_0,\cdots,\overset{\vee}{a_k},\cdots,a_4;\frac{m}{d}\right)$$
$$=-\left(\tilde{l_4}-\left(\frac{c}{2}\right)^2\sum\tilde{l_2}\ +\left(\frac{c}{2}\right)^4\sum_{k=0}^{4}\frac{1}{a_k}\right),$$

where $\tilde{l_4}:=\frac{1}{2^4}l_4(a_0,\cdots,a_4)/a_0a_1\cdots a_4$ and

$$\sum\tilde{l_2}:=\frac{1}{2^2}\sum_{0\leqslant m_1<m_2<m_3\leqslant 4}\frac{l_2(a_{m_1},a_{m_2},a_{m_3})}{a_{m_1}a_{m_2}a_{m_3}}.$$

We showed the theorem in the case $n=4$.

The structure of the proof of the general case is the same. In fact, let k be an integer $(0\leqslant k\leqslant n)$, and put

$$f(a_k;a_0,\cdots,a_{k-1},a_{k+1},\cdots,a_n):=$$
$$\left(\frac{1}{2i}\right)^n\frac{1}{a_k}\sum_{n'=1}^{a_k-1}\cot\pi\left(\frac{a_0n'}{a_k}\right)\cdots\overset{\vee}{k}\cdots\cot\pi\left(\frac{a_nn'}{a_k}\right).$$

Then, for any fixed k,

$$\begin{aligned}
&\frac{1}{a_k} K\left(a_k; a_0, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n; \frac{m}{d}\right) \\
&= \left(\frac{1}{2i}\right)^n \frac{1}{a_k} \sum_{n'=1}^{a_k-1} \prod_{\substack{q=0 \\ q\neq k}}^{n} \left(c + \cot \pi \left(\frac{a_q n'}{a_k}\right)\right) \\
&= f(a_k; a_0, \cdots, \overset{\vee}{a_k}, \cdots, a_n) \\
&+ c^2 \sum_{l_1=0}^{n}{}' \sum_{l_2=0}^{n}{}' f(a_k; a_0, \cdots, \overset{\vee}{a}_{l_1}, \cdots, \overset{\vee}{a}_{l_2}, \cdots, \overset{\vee}{a_k}, \cdots, a_n) \\
&+ c^4 \sum_{l_1=0}^{n}{}' \sum_{l_2=0}^{n}{}' \sum_{l_3=0}^{n}{}' \sum_{l_4=0}^{n}{}' f(a_k; a_0, \dots, \overset{\vee}{a}_{l_1}, \dots, \overset{\vee}{a}_{l_2}, \dots, \overset{\vee}{a}_{l_3}, \dots, \overset{\vee}{a}_{l_4}, \dots, \overset{\vee}{a_k}, \dots, a_n) \\
&+ \cdots + c^{n-2} \sum_{l_1=0}^{n}{}' \cdots \sum_{l_{n-2}=0}^{n}{}' f(a_k; a_0, \cdots, \overset{\vee}{a}_{l_1}, \cdots, \overset{\vee}{a}_{l_{n-2}}, \overset{\vee}{a_k}, \cdots, a_n) \\
&+ \left(\frac{1}{2i}\right)^n \cdot c^n \cdot \frac{a_k - 1}{a_k}.
\end{aligned}$$

We note that if the number of cotangent functions is odd, f vanishes by Remark 3.8.

Then we have

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^n \sum_{k=0}^{n} \lim_{\operatorname{Im}\tau\to\infty} d_\tau \left(a_k; a_0, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n; \frac{m}{d}\right) \\
&= \sum_{k=0}^{n} \left(\frac{1}{a_k} K\left(a_k; a_0, \cdots, \overset{\vee}{a_k}, \cdots, a_n; \frac{m}{d}\right) + \frac{1}{a_k} g\left(a_k; a_0, \cdots, \overset{\vee}{a_k}, \cdots, a_n; \frac{m}{d}\right)\right) \\
&= \sum_{k=0}^{n} f(a_k; a_0, \cdots, \overset{\vee}{a_k}, \cdots, a_n) \\
&+ c^2 \sum_{k=0}^{n} \sum_{l_1=0}^{n}{}' \sum_{l_2=0}^{n}{}' f(a_k; a_0, \cdots, \overset{\vee}{a}_{l_1}, \cdots, \overset{\vee}{a}_{l_2}, \cdots, \overset{\vee}{a_k}, \cdots, a_n) \\
&+ c^4 \sum_{k=0}^{n} \sum_{l_1=0}^{n}{}' \cdots \sum_{l_4=0}^{n}{}' f(a_k; a_0, \cdots, \overset{\vee}{a}_{l_1}, \cdots, \overset{\vee}{a}_{l_2}, \cdots, \overset{\vee}{a}_{l_3}, \cdots, \overset{\vee}{a}_{l_4}, \cdots, \overset{\vee}{a_k}, \cdots, a_n) \\
&+ \cdots + c^{n-2} \sum_{k=0}^{n} \sum_{l_1=0}^{n}{}' \cdots \sum_{l_{n-2}=0}^{n}{}' f(a_k; a_0, \cdots, \overset{\vee}{a}_{l_1}, \cdots, \overset{\vee}{a}_{l_{n-2}}, \cdots, \overset{\vee}{a_k}, \cdots, a_n) \\
&+ \left(\frac{1}{2i}\right)^n \cdot c^n \cdot \sum_{k=0}^{n} \left(1 - \frac{1}{a_k}\right) + \sum_{k=0}^{n} \frac{1}{a_k} g\left(a_k; a_0, \cdots, \overset{\vee}{a_k}, \cdots, a_n; \frac{m}{d}\right).
\end{aligned}$$

By Proposition 1.1, denoting $1-\frac{l_i(a_{m_0},\cdots,a_{m_i})}{a_{m_0}\cdots a_{m_i}} = l'_i(a_{m_0},\cdots,a_{m_i}) \quad (0 \leqslant m_0 < \cdots < m_i \leqslant n,\ \ i=2,4,\cdots,n)$, we have

(5.2)
$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^n \sum_{k=0}^{n} \lim_{\operatorname{Im}\tau\to\infty} d_\tau\left(a_k; a_0,\cdots,\overset{\vee}{a_k},\cdots,a_n;\frac{m}{d}\right)\\
&= \frac{1}{2^n}\Bigg\{ l'_n(a_0,\cdots,a_n) - c^2 \sum_{0\leqslant m_1<\cdots<m_{n-2}<m_{n-1}\leqslant n} l'_{n-2}(a_{m_1},\cdots,a_{m_{n-2}},a_{m_{n-1}})\\
&+ c^4 \sum_{0\leqslant m_1<\cdots<m_{n-4}<m_{n-3}\leqslant n} l'_{n-4}(a_{m_1},\cdots,a_{m_{n-4}},a_{m_{n-3}})\\
&+\cdots + i^{n-2}c^{n-2} \sum_{0\leqslant m_1<m_2<m_3\leqslant n} l'_2(a_{m_1},a_{m_2},a_{m_3})\Bigg\}\\
&+\left(\frac{1}{2i}\right)^n \cdot c^n \cdot \sum_{k=0}^{n}\left(1-\frac{1}{a_k}\right) + \sum_{k=0}^{n}\frac{1}{a_k} g\left(a_k; a_0,\cdots,\overset{\vee}{a_k},\cdots,a_n;\frac{m}{d}\right).
\end{aligned}$$

Then by using the analytic continuation of the formula for the Hardy sum (Corollary 5.5 in [9]) and the condition that $d|(a_0+\cdots+a_n)$, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{1}{a_k} g\left(a_k; a_0,\cdots,\overset{\vee}{a_k},\cdots,a_n;\frac{m}{d}\right)\\
&\quad = \left(\frac{1}{\xi-1}\right)^n \sum_{k=0}^{n}\sum_{m'=1}^{a_k-1}\left(\xi^{-m'}\right) \prod_{\substack{j=0\\ j\neq k}}^{n} (\xi^{-1})^{\left[\frac{a_j m'}{a_k}\right]}\\
&\quad = -\left(\frac{1}{\xi-1}\right)^{n+1}\cdot(\xi^{n+1}-1),
\end{aligned}$$

where $\xi = e^{2\pi i\frac{m}{d}}$. By the definitions of l'_i $(i=2,4,\cdots,n)$ and c, we see that the terms on the right hand side of (5.2), which do not depend on $a_0,\cdots,a_n$ are

$$\begin{aligned}
&\frac{1}{2^n}\Bigg\{1-\binom{n+1}{2}c^2+\binom{n+1}{4}c^4-\cdots\\
&\qquad + i^{n-2}\binom{n+1}{n-2}c^{n-2} + i^n\binom{n+1}{n}c^n\Bigg\} - \frac{\xi^{n+1}-1}{(\xi-1)^{n+1}}\\
&= \frac{1}{2^{n+1}}\left((1+ic)^{n+1}+(1-ic)^{n+1}\right) - \frac{\xi^{n+1}-1}{(\xi-1)^{n+1}} = 0.
\end{aligned}$$

Therefore (5.2) becomes

$$-\left(\tilde{l_n} - \left(\frac{c}{2}\right)^2 \sum \tilde{l}_{n-2} + \left(\frac{c}{2}\right)^4 \sum \tilde{l}_{n-4} + \cdots + i^{n-2}\left(\frac{c}{2}\right)^{n-2} \sum \tilde{l_2} + i^n \left(\frac{c}{2}\right)^n \sum_{k=0}^{n} \frac{1}{a_k}\right). \tag{5.3}$$

Hence the theorem is proved. □

And for $\frac{m}{d} = \frac{1}{2}$, the following corollary holds:

Corollary 5.2. — *Let $a_0, a_1, \cdots, a_n$ be pairwise coprime positive integers. Both n and $a_0 + \cdots + a_n$ are supposed to be even. Then for $d = 2, m = 1$,*

$$\begin{aligned} &\left(\frac{1}{2\pi i}\right)^n \sum_{k=0}^{n} \lim_{\operatorname{Im}\tau\to\infty} d_\tau\left(a_k; a_0, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n; \frac{1}{2}\right) \\ &= -\lim_{\operatorname{Im}\tau\to\infty} \frac{M'_{n,\tau}(a_0, a_1, \cdots, a_n)}{a_0 \cdots a_n} \\ &= -\frac{1}{2^n} \cdot \frac{l_n(a_0, \cdots, a_n)}{a_0 \cdots a_n} \end{aligned}$$

where $M'_{n,\tau}$ has been defined in Proposition 3.4.

Remark 5.3. — As noted in [14] and [4] this statement comes directly from Proposition 3.7 (one has to use Corollary 5.5 in [9]).

Note added in proof

After the present paper had been accepted the author noticed the existence of Bayad's two papers (Applications aux sommes elliptiques d'Apostol-Dedekind-Zagier, C. R. Acad. Sci. Paris, Ser.I 339 (2004) 529-532; Sommes elliptiques multiples d'Apostol-Dedkind-Zagier, C. R. Acad. Sci. Paris, Ser.I 339 (2004) 547-462). In the former paper, Bayad also deduced the limit of $D_\tau(z;\varphi)$(as $Im_\tau \to \infty$) and in the latter paper, Bayad obtained a different expression of $M_{n,\tau}(a_0; a_1, \ldots, a_n; \varphi)$.

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Manuscrit reçu le 31 août 2004,
révisé le 24 mars 2006,
accepté le 11 mai 2006.

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