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PALINDROMIC COMPLEXITY OF INFINITE WORDS ASSOCIATED WITH SIMPLE PARRY NUMBERS

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Abstract. — A simple Parry number is a real number $\beta > 1$ such that the Rényi expansion of 1 is finite, of the form $d_\beta(1) = t_1 \cdots t_m$. We study the palindromic structure of infinite aperiodic words u_β that are the fixed point of a substitution associated with a simple Parry number β. It is shown that the word u_β contains infinitely many palindromes if and only if $t_1 = t_2 = \cdots = t_{m-1} \geqslant t_m$. Numbers β satisfying this condition are the so-called *confluent* Pisot numbers. If $t_m = 1$ then u_β is an Arnoux-Rauzy word. We show that if β is a confluent Pisot number then $\mathcal{P}(n+1) + \mathcal{P}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) + 2$, where $\mathcal{P}(n)$ is the number of palindromes and $\mathcal{C}(n)$ is the number of factors of length n in u_β. We then give a complete description of the set of palindromes, its structure and properties.

Résumé. — Un nombre de Parry simple est un nombre réel $\beta > 1$ tel que le développement de Rényi de 1 est fini, de la forme $d_\beta(1) = t_1 \cdots t_m$. Nous étudions la structure palindromique des mots infinis apériodiques u_β qui sont point fixe d'une substitution associée à un nombre de Parry simple β. Nous montrons que le mot u_β contient un nombre infini de palindromes si et seulement si $t_1 = t_2 = \cdots = t_{m-1} \geqslant t_m$. Les nombres β satisfaisant cette condition sont connus sous le nom de nombres de Pisot confluents. Si de plus $t_m = 1$ alors u_β est un mot d'Arnoux-Rauzy. Nous montrons que si β est un nombre de Pisot confluent alors $\mathcal{P}(n+1) + \mathcal{P}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) + 2$, où $\mathcal{P}(n)$ est le nombre de facteurs de longueur n de u_β. Nous donnons aussi une description complète de l'ensemble des palindromes, de sa structure et de ses propriétés.

1. Introduction

Infinite aperiodic words over a finite alphabet are suitable models for one-dimensional quasicrystals, i.e. non-crystallographic materials displaying long-range order, since they define one-dimensional Delaunay sets with

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finite local complexity. The first quasicrystal was discovered in 1984: it is a solid structure presenting a local symmetry of order 5, i.e. a local invariance under rotation of $\pi/5$, and it is linked to the golden ratio and to the Fibonacci substitution. The Fibonacci substitution, given by

$$0 \mapsto 01, \ 1 \mapsto 0,$$

defines a quasiperiodic selfsimilar tiling of the positive real line, and is a historical model of a one-dimensional quasicrystal. The fixed point of the substitution is the infinite word

$$010010101 \cdots$$

The description and the properties of this tiling use a number system in base the golden ratio.

A more general theory has been elaborated with Pisot numbers(1) for base, see [8, 16]. Note that so far, all the quasicrystals discovered by physicists present local symmetry of order 5 or 10, 8, and 12, and are modelized using quadratic Pisot units, namely the golden ratio for order 5 or 10, $1+\sqrt{2}$ for order 8, and $2+\sqrt{3}$ for order 12.

For the description of physical properties of these materials it is important to know the combinatorial properties of the infinite aperiodic words, such as the factor complexity, which corresponds to the number of local configurations of atoms in the material, or the palindromic structure of the aperiodic words, describing local symmetry of the material. The palindromic structure of the infinite words has been proven important for the description of the spectra of Schrödinger operators with potentials adapted to aperiodic structures [25].

The most studied infinite aperiodic word is the Fibonacci word, which is the paradigm of the notion of Sturmian words. Sturmian words are binary aperiodic words with minimal factor complexity, i.e. $\mathcal{C}(n) = n+1$ for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. There exist several equivalent definitions of Sturmian words see [10], or [28, Chapter 2]. From our point of view the characterization of Sturmian words using palindromes [20] is particularly interesting.

Sturmian words can be generalized in several different ways to words over m-letter alphabet, namely to Arnoux-Rauzy words of order m, see [3, 10], or to infinite words coding m-interval exchange [27, 33]. The Sturmian case is included for $m = 2$.

(1) A *Pisot number* is an algebraic integer > 1 such that the other roots of its minimal polynomial have a modulus less than 1. The golden ratio and the natural integers are Pisot numbers.

Arnoux-Rauzy words and words coding generic m-interval exchange have factor complexity $\mathcal{C}(n) = (m-1)n + 1$ for $n \in \mathbb{N}$, [27]. For Arnoux-Rauzy words the palindromic structure is also known [26, 18]: for every n the number $\mathcal{P}(n)$ of palindromes of length n is equal to $\mathcal{P}(n) = 1$ if n is even and to $\mathcal{P}(n) = m$ if n is odd. The palindromic structure of infinite words coding m-interval exchange is more complicated. The existence of palindromes of arbitrary length depends on the permutation which exchanges the intervals. For $m = 3$ and the permutation $\pi = (321)$ the result is given in [18], for general m in [6].

As we have seen for the Fibonacci word, infinite aperiodic words can also be obtained as the fixed point of a substitution canonically associated with a number system where the base is an irrational number β, the so-called β-expansions introduced by Rényi [34]. The words u_β are defined in the case that β is a Parry number, that is to say when the Rényi expansion of 1 is eventually periodic or finite, see Section 2 for definitions. These words provide a good model of one-dimensional quasicrystals [8]. The factor complexity of these words is at most linear, because they are fixed points of primitive substitutions [32]. The exact values of the complexity function $\mathcal{C}(n)$ for a large class of Parry numbers β can be found in [23] and some partial results about other Parry numbers β are to be found in [24].

This paper is devoted to the description of the palindromic structure of the infinite words u_β, when β is a simple Parry number, with the Rényi expansion of 1 being of the form $d_\beta(1) = t_1 \cdots t_m$. We first show that the word u_β contains infinitely many palindromes if and only if $t_1 = t_2 = \cdots = t_{m-1} \geqslant t_m$. Numbers β satisfying this condition have been introduced and studied in [22] from the point of view of linear numeration systems. Confluent linear numeration systems are exactly those for which there is no propagation of the carry to the right in the process of normalization, which consists of transforming a non-admissible representation on the canonical alphabet of a number into the admissible β-expansion of that number. Such a number β is known to be a Pisot number, and will be called a *confluent* Pisot number. We also know from [23] that the infinite word u_β is an Arnoux-Rauzy sequence if and only if it is a confluent Pisot number with the last coefficient t_m being equal to 1; then β is an algebraic unit.

In the sequel β is a confluent Pisot number. We then determine the palindromic complexity, that is $\mathcal{P}(n)$, the number of palindromes in u_β of length n. In the description of $\mathcal{P}(n)$ we use the notions introduced in [23] for the factor complexity. The connection of the factor and palindromic complexity is not surprising. For example, in [2] the authors give an upper estimate of the palindromic complexity $\mathcal{P}(n)$ in terms of $\mathcal{C}(n)$.

In this paper we show that if the length of palindromes is not bounded, which is equivalent to $\limsup_{n\to\infty} \mathcal{P}(n) > 0$, then

$$\mathcal{P}(n+1) + \mathcal{P}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) + 2\,, \qquad \text{for} \;\; n \in \mathbb{N}\,. \tag{1.1}$$

In general it is has been shown [5] that for a uniformly recurrent word with $\limsup_{n\to\infty} \mathcal{P}(u) > 0$ the inequality

$$\mathcal{P}(n+1) + \mathcal{P}(n) \leqslant \mathcal{C}(n+1) - \mathcal{C}(n) + 2$$

holds for all $n \in \mathbb{N}$. Moreover, the authors proved the formula (1.1) to be valid for infinite words coding the r-interval exchange. Finally, it is known that the formula (1.1) holds also for Arnoux-Rauzy words [2] and for complementation-symmetric sequences [18].

We then give a complete description of the set of palindromes, its structure and properties. The exact palindromic complexity of the word u_β is given in Theorem 7.1.

Further on, we study the occurrence of palindromes of arbitrary length in the prefixes of the word u_β. It is known [19] that every word w of length n contains at most $n+1$ different palindromes. The value by which the number of palindromes differs from $n+1$ is called the *defect* of the word w. Infinite words whose every prefix has defect 0 are called *full.* We show that whenever $\limsup_{n\to\infty} \mathcal{P}(n) > 0$, the infinite word u_β is full.

2. Preliminaries

Let us first recall the basic notions which we work with, for more details reader is referred to [28]. An *alphabet* is a finite set whose elements are called letters. A *finite word* $w = w_1 w_2 \cdots w_n$ on the alphabet $\mathcal{A}$ is a concatenation of letters. The length n of the word w is denoted by $|w|$. The set of all finite words together with the empty word ε equipped with the operation of concatenation is a free monoid over the alphabet $\mathcal{A}$, denoted by $\mathcal{A}^*$.

An infinite sequence of letters of $\mathcal{A}$ of the form

$$u_0u_1u_2\cdots\,, \qquad \cdots u_2u_1u_0, \qquad \text{or} \qquad \cdots u_{-2}u_{-1}u_0u_1u_2\cdots$$

is called *right infinite* word, *left infinite* word, or *two-sided infinite* word, respectively. If for a two-sided infinite word the position of the letter indexed by 0 is important, we introduce *pointed* two-sided infinite words, $\cdots u_{-2}u_{-1}|u_0u_1u_2\cdots$.

A *factor* of a word v (finite or infinite) is a finite word w such that there exist words v_1, v_2 satisfying $v = v_1 w v_2$. If $v_1 = \varepsilon$, then w is called a *prefix* of v, if $v_2 = \varepsilon$, then w is a *suffix* of v. For a finite word $w = w_1 w_2 \cdots w_n$ with a prefix $v = w_1 \cdots w_k$, $k \leqslant n$, we define $v^{-1}w := w_{k+1} \cdots w_n$.

On the set $\mathcal{A}^*$ we can define the operation $\sim$ which to a finite word $w = w_1 \cdots w_n$ associates $\widetilde{w} = w_n \cdots w_1$. The word $\widetilde{w}$ is called the *reversal* of w. A finite word $w \in \mathcal{A}^*$ for which $w = \widetilde{w}$ is called a *palindrome*.

The set of all factors of an infinite word u is called the *language* of u and denoted by $\mathcal{L}(u)$. The set of all palindromes in $\mathcal{L}(u)$ is denoted by $\mathcal{P}al(u)$. The set of words of length n in $\mathcal{L}(u)$, respectively in $\mathcal{P}al(u)$ determines the *factor*, respectively *palindromic complexity* of the infinite word u. Formally, the functions $\mathcal{C} : \mathbb{N} \to \mathbb{N}$, $\mathcal{P} : \mathbb{N} \to \mathbb{N}$ are defined by

$$\begin{aligned}
\mathcal{C}(n) &:= \# \{ w \mid w \in \mathcal{L}(u),\ |w| = n \} , \\
\mathcal{P}(n) &:= \# \{ w \mid w \in \mathcal{P}al(u),\ |w| = n \} .
\end{aligned}$$

Obviously, we have $\mathcal{P}(n) \leqslant \mathcal{C}(n)$ for $n \in \mathbb{N}$. We have moreover

$$\mathcal{P}(n) \leqslant \frac{16}{n} \, \mathcal{C}\left(n + \left\lfloor \frac{n}{4} \right\rfloor \right) ,$$

as shown in [2].

For the determination of the factor complexity important is the notion of the so-called left or right special factors, introduced in [17]. The extension of a factor $w \in \mathcal{L}(u)$ by a letter to the left is called the *left extension* of w, analogously we define the *right extension* of a factor w. Formally, we have the sets

$$\begin{aligned}
\mathrm{Lext}(w) &:= \{ a \mid aw \in \mathcal{L}(u) \} , \\
\mathrm{Rext}(w) &:= \{ a \mid wa \in \mathcal{L}(u) \} .
\end{aligned}$$

If $\#\mathrm{Lext}(w) \geqslant 2$, we say that w is a *left special factor* of the infinite word u. Similarly, if $\#\mathrm{Rext}(w) \geqslant 2$, then w is a *right special factor* of u. For the first difference of complexity we have

$$\Delta \mathcal{C}(n) \ = \ \mathcal{C}(n+1) - \mathcal{C}(n) \ = \sum_{w \in \mathcal{L}(u),\ |w| = n} \left(\#\mathrm{Lext}(w) - 1 \right) .$$

In this formula we can exchange $\mathrm{Lext}(w)$ with $\mathrm{Rext}(w)$.

Infinite words which have for each n at most one left special factor and at most one right special factor are called *episturmian* words [26]. Arnoux-Rauzy words of order m are special cases of episturmian words; they are defined as words on a m-letter alphabet such that for every n there exist exactly one left special factor w_1 and exactly one right special factor w_2. Moreover, these special factors satisfy $\#\mathrm{Lext}(w_1) = \#\mathrm{Rext}(w_2) = m$.

Analogically to the case of factor complexity, for the palindromic complexity it is important to define the palindromic extension: If for a palindrome $p \in \mathcal{P}al(u)$ there exists a letter a such that $apa \in \mathcal{P}al(u)$, then we call the word apa the *palindromic extension* of p.

A mapping on a free monoid $\mathcal{A}^*$ is called a *morphism* if $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathcal{A}^*$. Obviously, for determining the morphism it is sufficient to define $\varphi(a)$ for all $a \in \mathcal{A}$. The action of a morphism can be naturally extended on right infinite words by the prescription

$$\varphi(u_0u_1u_2\cdots) := \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots .$$

A non-erasing[(2)] morphism φ, for which there exists a letter $a \in \mathcal{A}$ such that $\varphi(a) = aw$ for some non-empty word $w \in \mathcal{A}^*$, is called a *substitution*. An infinite word v such that $\varphi(v) = v$ is called a *fixed point* of the substitution φ. Obviously, any substitution has at least one fixed point, namely $\lim_{n\to\infty} \varphi^n(a)$. Assume that there exists an index $k \in \mathbb{N}$ such that for every pair of letters $i, j \in \mathcal{A}$ the word $\varphi^k(i)$ contains as a factor the letter j. Then the substitution φ is called *primitive*.

Similarly, one can extend the action of a morphism to left infinite words. For a pointed two-sided infinite word $u = \cdots u_{-3}u_{-2}u_{-1}|u_0u_1\cdots$ we define action of a morphism φ by $\varphi(u) = \cdots\varphi(u_{-3})\varphi(u_{-2})\varphi(u_{-1})|\varphi(u_0)\varphi(u_1)\cdots$. One can also define analogically the notion of a fixed point.

Right infinite words which will be studied in this paper, are connected with the Rényi β-expansion of real numbers [34]. For a real number $\beta > 1$ the transformation $T_\beta : [0,1] \to [0,1)$ is defined by the prescription

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor .$$

The sequence of non-negative integers $(t_n)_{n\geqslant 1}$ defined by $t_i = \lfloor \beta T^{i-1}(1) \rfloor$ satisfies $1 = \frac{t_1}{\beta} + \frac{t_2}{\beta 2} + \frac{t_3}{\beta 3} + \cdots$. It is called the *Rényi expansion* of 1 and denoted by

$$d_\beta(1) = t_1t_2t_3\cdots .$$

In order that the sequence $t_1t_2t_3\cdots$ be the Rényi expansion of 1 for some β, it must satisfy the so-called Parry condition [29]

$$t_it_{i+1}t_{i+2}\cdots \prec t_1t_2t_3\cdots \qquad \text{for all } \ i = 2, 3, \ldots ,$$

where the symbol $\prec$ stands for "lexicographically strictly smaller". A number $\beta > 1$ for which $d_\beta(1)$ is eventually periodic is called a *Parry number*. If moreover $d_\beta(1)$ has only finitely many non-zero elements, we say that β

(2) A morphism φ on an alphabet $\mathcal{A}$ is *non-erasing* if for any $a \in \mathcal{A}$ the image $\varphi(a)$ is a non-empty word.

is a *simple* Parry number and in the notation for d_β we omit the ending zeros, i.e. $d_\beta(1) = t_1 t_2 \cdots t_m$, where $t_m \neq 0$.

A *Pisot number* is an algebraic integer such that all its Galois conjugates are in modulus less than 1. A Pisot number is a Parry number [13]. It is known that a Parry number is a *Perron number*, i.e. an algebraic integer all of whose conjugates are in modulus less than β. Solomyak [35] has shown that all conjugates of a Parry number lie inside the disc of radius $\frac{1}{2}(1+\sqrt{5})$, i.e. the golden ratio.

With every Parry number one associates a canonical substitution φ_β, see [21]. For a simple Parry number β with $d_\beta(1) = t_1 t_2 \cdots t_m$ the substitution $\varphi = \varphi_\beta$ is defined on the alphabet $\mathcal{A} = \{0, 1, \dots, m-1\}$ by

$$\begin{array}{rcl} \varphi(0) & = & 0^{t_1}1 \\ \varphi(1) & = & 0^{t_2}2 \\ & \vdots & \\ \varphi(m-2) & = & 0^{t_{m-1}}(m-1) \\ \varphi(m-1) & = & 0^{t_m} \end{array} \tag{2.1}$$

The notation 0^k in the above stands for a concatenation of k zeros. The substitution φ has a unique fixed point, namely the word

$$u_\beta := \lim_{n\to\infty} \varphi^n(0)\,,$$

which is the subject of the study of this paper. The substitution (2.1) is primitive, and thus according to [32], the factor complexity of its fixed point is sublinear. The exact values of $\mathcal{C}(n)$ for u_β with $d_\beta = t_1 \cdots t_m$ satisfying $t_1 > \max\{t_2, \dots, t_{m-1}\}$ or $t_1 = t_2 = \cdots = t_{m-1}$ can be found in [23]. The determination of the palindromic complexity of u_β is the aim of this article.

A similar canonical substitution is defined for non-simple Parry numbers. Partial results about the factor and palindromic complexity of u_β for non-simple Parry numbers β can be found in [7, 24].

One can define the canonical substitution φ_β even if the Rényi expansion $d_\beta(1)$ is infinite non-periodic, i.e. β is not a Parry number. In this case, however, the substitution and its fixed point are defined over an infinite alphabet. The study of such words u_β is out of the scope of this paper.

3. Words u_β with bounded number of palindromes

The infinite word u_β associated with a Parry number β is a fixed point of a primitive substitution. This implies that the word u_β is uniformly recurrent [31]. Let us recall that an infinite word u is called *uniformly recurrent* if every factor w in $\mathcal{L}(u)$ occurs in u with bounded gaps.

LEMMA 3.1. — *If the language $\mathcal{L}(u)$ of a uniformly recurrent word u contains infinitely many palindromes, then $\mathcal{L}(u)$ is closed under reversal.*

Proof. — From the definition of a uniformly recurrent word u it follows that for every $n \in \mathbb{N}$ there exists an integer $R(n)$ such that every arbitrary factor of u of length $R(n)$ contains all factors of u of length n. Since we assume that $\mathcal{P}al(u)$ is an infinite set, it must contain a palindrome p of length $\geqslant R(n)$. Since p contains all factors of u of length n, and p is a palindrome, it contains with every w such that $|w| = n$ also its reversal $\widetilde{w}$. Thus $\widetilde{w} \in \mathcal{L}(u)$. This consideration if valid for all n and thus the statement of the lemma is proved. □

Note that this result was first stated, without proof, in [19].

The fact that the language is closed under reversal is thus a necessary condition so that a uniformly recurrent word has infinitely many palindromes. The converse is not true [11].

For infinite words u_β associated with simple Parry numbers β the invariance of $\mathcal{L}(u_\beta)$ under reversal is studied in [23].

PROPOSITION 3.2 ([23]). — *Let $\beta > 1$ be a simple Parry number such that $d_\beta(1) = t_1 t_2 \cdots t_m$.*

(1) *The language $\mathcal{L}(u_\beta)$ is closed under reversal, if and only if*

Condition (C) : $$t_1 = t_2 = \cdots = t_{m-1}\,.$$

(2) *The infinite word u_β is an Arnoux-Rauzy word if and only if Condition (C) is satisfied and $t_m = 1$.*

COROLLARY 3.3. — *Let β be a simple Parry number which does not satisfy Condition (C). Then there exists $n_0 \in \mathbb{N}$ such that $\mathcal{P}(n) = 0$ for $n \geqslant n_0$.*

Numbers β satisfying Condition (C) have been introduced and studied in [22] from the point of view of linear numeration systems. Confluent linear numeration systems are exactly those for which there is no propagation of the carry to the right in the process of normalization, which consists of transforming a non-admissible representation on the canonical alphabet of a number into the admissible β-expansion of that number. A number β satisfying Condition (C) is known to be a Pisot number, and will be called a *confluent* Pisot number.

Set
$$t := t_1 = t_2 = \cdots = t_{m-1} \qquad \text{and} \qquad s := t_m\,.$$

From the Parry condition for the Rényi expansion of 1 it follows that $t \geqslant s \geqslant 1$. Then the substitution φ is of the form

$$\begin{array}{rcl} \varphi(0) & = & 0^t1 \\ \varphi(1) & = & 0^t2 \\ & \vdots & \\ \varphi(m-2) & = & 0^t(m-1) \\ \varphi(m-1) & = & 0^s \end{array} \qquad t \geqslant s \geqslant 1\,. \tag{3.1}$$

Note that in the case $s = 1$, the number β is an algebraic unit, and the corresponding word u_β is an Arnoux-Rauzy word, for which the palindromic complexity is known. Therefore in the paper we often treat separately the cases $s \geqslant 2$ and $s = 1$.

4. Palindromic extensions in u_β

In the remaining part of the paper we study the palindromic structure of the words u_β for confluent Pisot numbers β.

For an Arnoux-Rauzy word u (and thus also for a Sturmian word) it has been shown that for every palindrome $p \in \mathcal{L}(u)$ there is exactly one letter a in the alphabet, such that $apa \in \mathcal{L}(u)$, i.e. any palindrome in an Arnoux-Rauzy word has exactly one palindromic extension [18]. Since the length of the palindromic extension apa of p is $|apa| = |p| + 2$, we have for Arnoux-Rauzy words $\mathcal{P}(n+2) = \mathcal{P}(n)$ and therefore

$$\mathcal{P}(2n) = \mathcal{P}(0) = 1 \qquad \text{and} \qquad \mathcal{P}(2n+1) = \mathcal{P}(1) = \#\mathcal{A}\,.$$

Determining the number of palindromic extensions for a given palindrome of u_β is essential also for our considerations here. However, let us first introduce the following notion.

Definition 4.1. — *We say that a palindrome p_1 is a* central factor *of a palindrome p_2 if there exists a finite word $w \in \mathcal{A}^*$ such that $p_2 = wp_1\widetilde{w}$.*

For example, a palindrome is a central factor of its palindromic extensions.

The following simple result can be easily obtained from the form of the substitution (3.1), and is a special case of a result given in [23].

Lemma 4.2 ([23]). — *All factors of u_β of the form $X0^nY$ for $X, Y \neq 0$ are the following*

$$X0^t1,\ 10^tX \ \textit{with}\ X \in \{1, 2, \ldots, m-1\}, \ \ \textit{and}\ \ 10^{t+s}1\,. \tag{4.1}$$

Remark 4.3. —

(1) Every pair of non-zero letters in u_β is separated by a block of at least t zeros. Therefore every palindrome $p \in \mathcal{L}(u_\beta)$ is a central factor of a palindrome with prefix and suffix 0^t.

(2) Since $\varphi(\mathcal{A})$ is a suffix code, the coding given by the substitution φ is uniquely decodable. In particular, if $w_1 \in \mathcal{L}(u_\beta)$ is a factor with the first and the last letter non-zero, then there exist a factor $w_2 \in \mathcal{L}(u_\beta)$ such that $0^t w_1 = \varphi(w_2)$.

Proposition 4.4. —

(i) *Let $p \in \mathcal{L}(u_\beta)$. Then $p \in \mathcal{P}al(u_\beta)$ if and only if $\varphi(p)0^t \in \mathcal{P}al(u_\beta)$.*

(ii) *Let $p \in \mathcal{P}al(u_\beta)$. The number of palindromic extensions of p and $\varphi(p)0^t$ is the same, i.e.*

$$\#\{a \in \mathcal{A} \mid apa \in \mathcal{P}al(u_\beta)\} = \#\{a \in \mathcal{A} \mid a\varphi(p)0^t a \in \mathcal{P}al(u_\beta)\}\,.$$

Proof. — (i) Let $p = w_0 w_1 \cdots w_{n-1} \in \mathcal{L}(u_\beta)$. Let us study under which conditions the word $\varphi(p)0^t$ is also a palindrome, i.e. when

$$\varphi(w_0)\varphi(w_1)\cdots\varphi(w_{n-1})0^t = 0^t\widetilde{\varphi(w_{n-1})}\cdots\widetilde{\varphi(w_1)}\widetilde{\varphi(w_0)}\,. \tag{4.2}$$

The substitution φ has the property that for each letter $a \in \mathcal{A}$ it satisfies $\widetilde{\varphi(a)} = 0^{-t}\varphi(a)0^t$. Using this property, the equality (4.2) can be equivalently written as

$$\varphi(p) = \varphi(w_0)\cdots\varphi(w_{n-1}) = \varphi(w_{n-1})\cdots\varphi(w_0) = \varphi(\widetilde{p})\,.$$

As a consequence of unique decodability of φ we obtain that (4.2) is valid if and only if $p = \widetilde{p}$.

(ii) We show that for a palindrome p it holds that

$$apa \in \mathcal{P}al(u_\beta) \quad \Longleftrightarrow \quad b\varphi(p)0^t b \in \mathcal{P}al(u_\beta)\,, \ \text{ where } b \equiv a+1 \pmod m\,,$$

which already implies the equality of the number of palindromic extensions of palindromes p and $\varphi(p)0^t$.

Let $apa \in \mathcal{P}al(u_\beta)$. Then

$$\varphi(a)\varphi(p)\varphi(a)0^t = \begin{cases} 0^t(a+1)\varphi(p)0^t(a+1)0^t\,, & \text{for } a \neq m-1\,, \\ 0^s\varphi(p)0^{t+s}\,, & \text{for } a = m-1\,, \end{cases}$$

is, according to (i) of this proposition, also a palindrome, which has a central factor $(a+1)\varphi(p)0^t(a+1)$ for $a \neq m-1$, and $0\varphi(p)0^t0$ for $a = m-1$.

On the other hand, assume that $b\varphi(p)0^t b \in \mathcal{P}al(u_\beta)$. If $b \neq 0$, then using 1. of Remark 4.3, we have $0^t b\varphi(p)0^t b0^t = \varphi\big((b-1)p(b-1)\big)0^t \in \mathcal{P}al(u_\beta)$. Point (i) implies that $(b-1)p(b-1) \in \mathcal{P}al(u_\beta)$ and thus $(b-1)p(b-1)$

is a palindromic extension of p. If $b = 0$, then Lemma 4.2 implies that $10^s\varphi(p)0^t0^s1 \in \mathcal{L}(u_\beta)$ and so $1\varphi\big((m-1)p(m-1)0\big) \in \mathcal{L}(u_\beta)$, which means that $(m-1)p(m-1)$ is a palindromic extension of p. □

Unlike Arnoux-Rauzy words, in the case of infinite words u_β with $d_\beta(1) = tt\cdots ts$, $t \geqslant s \geqslant 2$, it is not difficult to see using Lemma 4.2 that there exist palindromes which do not have any palindromic extension. Such a palindrome is for example the word 0^{t+s-1}.

Definition 4.5. — *A palindrome $p \in \mathcal{P}al(u_\beta)$ which has no palindromic extension is called a* maximal *palindrome.*

It is obvious that every palindrome is either a central factor of a maximal palindrome, or is a central factor of palindromes of arbitrary length.

Proposition 4.4 allows us to define a sequence of maximal palindromes starting from an initial maximal palindrome. Put

$$U^{(1)} := 0^{t+s-1}, \qquad U^{(n)} := \varphi(U^{(n-1)})0^t, \quad \text{for } n \geqslant 2\,. \tag{4.3}$$

Lemma 4.2 also implies that the palindrome 0^t has for $s \geqslant 2$ two palindromic extensions, namely 00^t0 and 10^t1. Using Proposition 4.4 we create a sequence of palindromes, all having two palindromic extensions. Put

$$V^{(1)} := 0^t, \qquad V^{(n)} := \varphi(V^{(n-1)})0^t, \quad \text{for } n \geqslant 2\,. \tag{4.4}$$

Remark 4.6. — It is necessary to mention that the factors $U^{(n)}$ and $V^{(n)}$ defined above play an important role in the description of factor complexity of the infinite word u_β. Let us cite several results for u_β invariant under the substitution (3.1) with $s \geqslant 2$, taken from [23], which will be used in the sequel.

(1) Any prefix w of u_β is a left special factor which can be extended to the left by any letter of the alphabet, i.e. $aw \in \mathcal{L}(u_\beta)$ for all $a \in \mathcal{A}$, or equivalently $\mathrm{Lext}(w) = \mathcal{A}$.

(2) Any left special factor w which is not a prefix of u_β is a prefix of $U^{(n)}$ for some $n \geqslant 1$ and such w can be extended to the left by exactly two letters.

(3) The words $U^{(n)}$, $n \geqslant 1$ are maximal left special factors of u_β, i.e. $U^{(n)}a$ is not a left special factor for any $a \in \mathcal{A}$. The infinite word u_β has no other maximal left special factors.

(4) The word $V^{(n)}$ is the longest common prefix of u_β and $U^{(n)}$, moreover, for every $n \geqslant 1$ we have

$$|V^{(n)}| < |U^{(n)}| < |V^{(n+1)}| \tag{4.5}$$

(5) For the first difference of factor complexity we have

$$\Delta\mathcal{C}(n) = \begin{cases} m & \text{if } \ |V^{(k)}| < n \leqslant |U^{(k)}| \ \text{ for some } k \geqslant 1\,, \\ m-1 & \text{otherwise}\,. \end{cases}$$

Now we are in position to describe the palindromic extensions in u_β. The main result is the following one.

PROPOSITION 4.7. — *Let u_β be the fixed point of the substitution φ given by* (3.1) *with parameters $t \geqslant s \geqslant 2$, and let p be a palindrome in u_β. Then*

(i) *p is a maximal palindrome if and only if $p = U^{(n)}$ for some $n \geqslant 1$;*
(ii) *p has two palindromic extensions in u_β if and only if $p = V^{(n)}$ for some $n \geqslant 1$;*
(iii) *p has a unique palindromic extension if and only if $p \neq U^{(n)}$, $p \neq V^{(n)}$ for all $n \geqslant 1$.*

Proof. — (i) Proposition 4.4, point (ii) and the construction of $U^{(n)}$ imply that $U^{(n)}$ is a maximal palindrome for every n. The proof that no other palindrome p is maximal will be done by induction on the length $|p|$ of the palindrome p.

Let p be a maximal palindrome. If p does not contain a non-zero letter, then using Lemma 4.2, obviously $p = U^{(1)}$. Assume therefore that p contains a non-zero letter. Point 1. of Remark 4.3 implies that $p = 0^t\hat{p}0^t$, where $\hat{p}$ is a palindrome. Since p is a maximal palindrome, $\hat{p}$ ends and starts in a non-zero letter. Otherwise, p would be extendable to a palindrome, which contradicts maximality. From 2. of Remark 4.3 we obtain that $p = 0^t\hat{p}0^t = \varphi(w)0^t$ for some factor w. Proposition 4.4, (i), implies that w is a palindrome. Point (ii) of the same proposition implies that w has no palindromic extension, i.e. w is a maximal palindrome, with clearly $|w| < |p|$. The induction hypothesis implies that $w = U^{(n)}$ for some $n \geqslant 1$ and $p = \varphi(U^{(n)})0^t = U^{(n+1)}$.

(ii) and (iii) From what we have just proved it follows that every palindrome $p \neq U^{(n)}$, $n \geqslant 1$, has at least one palindromic extension. Since we know that $V^{(n)}$ has exactly two palindromic extensions, for proving (ii) and (iii) it remains to show that if a palindrome p has more than one extension, then $p = V^{(n)}$, for some $n \geqslant 1$.

Assume that ipi and jpj are in $\mathcal{L}(u_\beta)$ for $i, j \in \mathcal{A}$, $i \neq j$. Obviously, p is a left special factor of u_β. We distinguish two cases, according to whether p is a prefix of u_β, or not.

- Let p be a prefix of u_β. Then there exists a letter $k \in \mathcal{A}$ such that pk is a prefix of u_β and using (1) of Remark 4.6, the word $apk \in \mathcal{L}(u_\beta)$

for every letter $a \in \mathcal{A}$, in particular ipk and jpk belong to $\mathcal{L}(u_\beta)$. We have either $k \neq i$, or $k \neq j$; without loss of generality assume that $k \neq i$. Since $\mathcal{L}(u_\beta)$ is closed under reversal, we must have $kpi \in \mathcal{L}(u_\beta)$. Since ipi and kpi are in $\mathcal{L}(u_\beta)$, we obtain that pi is also a left special factor of u_β, and pi is not a prefix of u_β. By (2) of Remark 4.6, p is the longest common prefix of u_β and some maximal left special factor $U^{(n)}$, therefore using (4) of Remark 4.6 we have $p = V^{(n)}$.

- If p is a left special factor of u_β, which is not a prefix of u_β, then by (2) of Remark 4.6, p is a prefix of some $U^{(n)}$ and the letters i, j are the only possible left extensions of p. Since $p \neq U^{(n)}$, there exists a unique letter k such that pk is a left special factor of u_β and pk is a prefix of $U^{(n)}$, i.e. the possible left extensions of pk are the letters i, j. Since by symmetry $kp \in \mathcal{L}(u_\beta)$, we have $k = i$ or $k = j$, say $k = i$. Since $jpk = jpi \in \mathcal{L}(u_\beta)$, we have also $ipj \in \mathcal{L}(u_\beta)$. Since by assumption ipi and jpj are in $\mathcal{L}(u_\beta)$, both pi and pj are left special factors of u_β. Since p is not a prefix of u_β, neither pi nor pj are prefixes of u_β. This contradicts the fact that k is a unique letter such that pk is left special.

Thus we have shown that if a palindrome p has at least two palindromic extensions, then $p = V^{(n)}$. □

From the above result it follows that if $n \neq |V^{(k)}|$, $n \neq |U^{(k)}|$ for all $k \geqslant 1$, then every palindrome of length n has exactly one palindromic extension, and therefore $\mathcal{P}(n+2) = \mathcal{P}(n)$. Inequalities in (4) of Remark 4.6 further imply that $|V^{(i)}| \neq |U^{(k)}|$ for all $i, k \geqslant 1$. Therefore the statement of Proposition 4.7 can be reformulated in the following way:

$$\mathcal{P}(n+2) - \mathcal{P}(n) = \begin{cases} 1 & \text{if } \ n = |V^{(k)}|\,, \\ -1 & \text{if } \ n = |U^{(k)}|\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Point (5) of Remark 4.6 can be used for deriving for the second difference of factor complexity

$$\Delta 2\mathcal{C}(n) = \Delta\mathcal{C}(n+1) - \Delta\mathcal{C}(n) = \begin{cases} 1 & \text{if } \ n = |V^{(k)}|\,, \\ -1 & \text{if } \ n = |U^{(k)}|\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Therefore we have for $s \geqslant 2$ that $\mathcal{P}(n+2) - \mathcal{P}(n) = \Delta\mathcal{C}(n+1) - \Delta\mathcal{C}(n)$, for all $n \in \mathbb{N}$. We thus can derive the following theorem.

THEOREM 4.8. — *Let u_β be the fixed point of the substitution* (3.1). *Then*

$$\mathcal{P}(n+1)+\mathcal{P}(n)=\Delta\mathcal{C}(n)+2\,,\qquad \textit{for}\ \ n\in\mathbb{N}\,.$$

Proof. — Let the parameter in the substitution (3.1) be $s=1$. Then u_β is an Arnoux-Rauzy word, for which $\mathcal{P}(n+2)-\mathcal{P}(n)=0=\Delta\mathcal{C}(n+1)-\Delta\mathcal{C}(n)$. For $s\geqslant 2$ we use $\mathcal{P}(n+2)-\mathcal{P}(n)=\Delta\mathcal{C}(n+1)-\Delta\mathcal{C}(n)$ derived above.

We have

$$\begin{aligned}\mathcal{P}(n+1)+\mathcal{P}(n)=\mathcal{P}(0)+\mathcal{P}(1)+\sum_{i=1}^{n}\big(\mathcal{P}(i+1)-\mathcal{P}(i-1)\big)=&\\ =1+m+\sum_{i=1}^{n}\big(\Delta\mathcal{C}(i)-\Delta\mathcal{C}(i-1)\big)=&\\ =1+m+\Delta\mathcal{C}(n)-\Delta\mathcal{C}(0)=&\\ =1+m+\Delta\mathcal{C}(n)-\mathcal{C}(1)+\mathcal{C}(0)=&\ \Delta\mathcal{C}(n)+2\,,\end{aligned}$$

where we have used $\mathcal{P}(0)=\mathcal{C}(0)=1$ and $\mathcal{P}(1)=\mathcal{C}(1)=m=\#\mathcal{A}$. □

Remark 4.9. — According to (5) of Remark 4.6, we have $\Delta\mathcal{C}(n)\leqslant\#\mathcal{A}$. This implies $\mathcal{P}(n+1)+\mathcal{P}(n)\leqslant\#\mathcal{A}+2$, and thus the palindromic complexity is bounded.

5. Centers of palindromes

We have seen that the set of palindromes of u_β is closed under the mapping $p\mapsto\varphi(p)0^t$. We study the action of this mapping on the centers of the palindromes. Let us mention that the results of this section are valid for β a confluent Pisot number with $t\geqslant s\geqslant 1$, i.e. also for the Arnoux-Rauzy case.

DEFINITION 5.1. — *Let p be a palindrome of odd length. The* center *of p is a letter a such that $p=wa\widetilde{w}$ for some $w\in\mathcal{A}^*$. The center of a palindrome p of even length is the empty word.*

If palindromes p_1,p_2 have the same center, then also palindromes $\varphi(p_1)0^t$, $\varphi(p_2)0^t$ have the same center. This is a consequence of the following lemma.

LEMMA 5.2. — *Let $p,q\in\mathcal{P}al(u_\beta)$ and let q be a central factor of p. Then $\varphi(q)0^t$ is a central factor of $\varphi(p)0^t$.*

Note that the statement is valid also for q being the empty word.

Proof. — Since $p = wq\widetilde{w}$ for some $w \in \mathcal{A}^*$, we have

$$\varphi(p)0^t = \varphi(w)\varphi(q)\varphi(\widetilde{w})0^t ,$$

which is a palindrome by (i) of Proposition 4.4. It suffices to realize that 0^t is a prefix of $\varphi(\widetilde{w})0^t$. Therefore we can write $\varphi(p)0^t = \varphi(w)\varphi(q)0^t 0^{-t}\varphi(\widetilde{w})0^t$. Since $|\varphi(w)| = |0^{-t}\varphi(\widetilde{w})0^t|$, the word $\varphi(q)0^t$ is a central factor of $\varphi(p)0^t$. □

The following lemma describes the dependence of the center of the palindrome $\varphi(p)0^t$ on the center of the palindrome p. Its proof is a simple application of properties of the substitution φ, we will omit it here.

Lemma 5.3. — *Let $p_1 \in \mathcal{P}al(u_\beta)$ and let $p_2 = \varphi(p_1)0^t$.*

(i) *If $p_1 = w_1 a \widetilde{w}_1$, where $a \in \mathcal{A}$, $a \neq m-1$, then $p_2 = w_2(a+1)\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^t$.*
(ii) *If $p_1 = w_1(m-1)\widetilde{w}_1$ and $s+t$ is odd, then $p_2 = w_2 0 \widetilde{w}_2$, where $w_2 = \varphi(w_1)0^{\frac{s+t-1}{2}}$.*
(iii) *If $p_1 = w_1(m-1)\widetilde{w}_1$ and $s+t$ is even, then $p_2 = w_2\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^{\frac{s+t}{2}}$.*
(iv) *If $p_1 = w_1\widetilde{w}_1$ and t is even, then $p_2 = w_2\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^{\frac{t}{2}}$.*
(v) *If $p_1 = w_1\widetilde{w}_1$ and t is odd, then $p_2 = w_2 0\widetilde{w}_2$, where $w_2 = \varphi(w_1)0^{\frac{t-1}{2}}$.*

Lemmas 5.2 and 5.3 allow us to describe the centers of palindromes $V^{(n)}$ which are in case $s \geqslant 2$ characterized by having two palindromic extensions.

Proposition 5.4. — *Let $V^{(n)}$ be palindromes defined by* (4.4)*.*

(i) *If t is even, then for every $n \geqslant 1$, $V^{(n)}$ has the empty word ε for center and $V^{(n)}$ is a central factor of $V^{(n+1)}$.*
(ii) *If t is odd and s is even, then for every $n \geqslant 1$, $V^{(n)}$ has the letter $i \equiv n-1 \pmod{m}$ for center, and $V^{(n)}$ is a central factor of $V^{(n+m)}$.*
(iii) *If t is odd and s is odd, then for every $n \geqslant 1$, $V^{(n)}$ has the empty word ε for center if $n \equiv 0 \pmod{(m+1)}$, otherwise it has for center the letter $i \equiv n-1 \pmod{(m+1)}$. Moreover, $V^{(n)}$ is a central factor of $V^{(m+n+1)}$.*

Proof. — If t is even, then the empty word ε is the center of $V^{(1)} = 0^t$. Using Lemma 5.2 we have that $\varphi(\varepsilon)0^t = V^{(1)}$ is a central factor of $\varphi(V^{(1)})0^t = V^{(2)}$. Repeating Lemma 5.2 we obtain that $V^{(n)}$ is a central factor of $V^{(n+1)}$. Since ε is the center of $V^{(1)}$, it is also the center of $V^{(n)}$ for all $n \geqslant 1$.

It t is odd, the palindrome $V^{(1)}$ has center 0 and using Lemma 5.3, $V^{(2)}$ has center 1, $V^{(3)}$ has center 2, …, $V^{(m)}$ has center $m-1$. If moreover s

is even, then $V^{(m+1)}$ has again center 0. Moreover, from (ii) of Lemma 5.3 we see that 0^{s+t} is a central factor of $V^{(m+1)}$, which implies that $V^{(1)} = 0^t$ is a central factor of $V^{(m+1)}$. In case that s is odd, then $V^{(m)}$ having center $m-1$ implies that $V^{(m+1)}$ has center ε and $V^{(m+2)}$ has center 0. Moreover, using (v) of Lemma 5.3 we see that $V^{(1)} = 0^t$ is a central factor of $V^{(m+2)}$. Repeated application of Lemma 5.2 implies the statement of the proposition. □

As we have said, every palindrome p is either a central factor of a maximal palindrome $U^{(n)}$, for some $n \geqslant 1$, or p is a central factor of palindromes with increasing length. An example of such a palindrome is $V^{(n)}$, for $n \geqslant 1$, which is according to Proposition 5.4 central factor of palindromes of arbitrary length. According to the notation introduced by Cassaigne in [17] for left and right special factors extendable to arbitrary length special factors, we introduce the notion of infinite palindromic branch. We will study infinite palindromic branches in the next section.

6. Infinite palindromic branches

Definition 6.1. — *Let $v = \cdots v_3 v_2 v_1$ be a left infinite word in the alphabet $\mathcal{A}$. Denote by $\widetilde{v}$ the right infinite word $\widetilde{v} = v_1 v_2 v_3 \cdots$.*

- *Let $a \in \mathcal{A}$. If for every index $n \geqslant 1$, the word*

$$p = v_n v_{n-1} \cdots v_1 a v_1 v_2 \cdots v_n \in \mathcal{P}al(u_\beta)\,,$$

  then the two-sided infinite word $va\widetilde{v}$ is called an infinite palindromic branch of u_β with center a, and the palindrome p is called a central factor of the infinite palindromic branch $va\widetilde{v}$.
- *If for every index $n \geqslant 1$, the word $p = v_n v_{n-1} \cdots v_1 v_1 v_2 \cdots v_n \in \mathcal{P}al(u_\beta)$, then the two-sided infinite word $v\widetilde{v}$ is called an infinite palindromic branch of u_β with center ε, and the palindrome p is called a central factor of the infinite palindromic branch $v\widetilde{v}$.*

Since for Arnoux-Rauzy words every palindrome has exactly one palindromic extension, we obtain for every letter $a \in \mathcal{A}$ exactly one infinite palindromic branch with center a; there is also one infinite palindromic branch with center ε.

Obviously, every infinite word with bounded palindromic complexity $\mathcal{P}(n)$ has only a finite number of infinite palindromic branches. This is therefore valid also for u_β.

PROPOSITION 6.2. — *The infinite word u_β invariant under the substitution* (3.1) *has for each center $c \in \mathcal{A} \cup \{\varepsilon\}$ at most one infinite palindromic branch with center c.*

Proof. — Lemma 5.3 allows us to create from one infinite palindromic branch another infinite palindromic branch. For example, if $va\widetilde{v}$ is an infinite palindromic branch with center $a \neq m-1$, then using (i) of Lemma 5.3, the two-sided word $\varphi(v)0^t(a+1)0^t\widetilde{\varphi(v)}$ is an infinite palindromic branch with center $(a+1)$. Similarly for the center $m-1$ or ε. Obviously, this procedure creates from distinct palindromic branches with the same center $c \in \mathcal{A} \cup \{\varepsilon\}$ again distinct palindromic branches, for which the length of the maximal common central factor is longer than the length of the maximal common central factor of the original infinite palindromic branches. This would imply that u_β has infinitely many infinite palindromic branches, which is in contradiction with the boundedness of the palindromic complexity of u_β, see Remark 4.9. □

Remark 6.3. — Examples of infinite palindromic branches can be easily obtained from Proposition 5.4 as a centered limit of palindromes $V^{(k_n)}$ for a suitably chosen subsequence $(k_n)_{n\in\mathbb{N}}$ and n going to infinity, namely

- If t is even, then the centered limit of palindromes $V^{(n)}$ is an infinite palindromic branch with center ε.
- If t is odd and s even, then the centered limit of palindromes

  $$V^{(k+mn)}$$

  for $k = 1, 2, \dots, m$ is an infinite palindromic branch with center $k-1$.
- If t is odd and s odd, then the centered limit of palindromes

  $$V^{(k+(m+1)n)}$$

  for $k = 1, 2, \dots, m$ is an infinite palindromic branch with center $k-1$, and for $k = m+1$ it is an infinite palindromic branch with center ε.

COROLLARY 6.4. —

(i) *If s is odd, then u_β has exactly one infinite palindromic branch with center c for every $c \in \mathcal{A} \cup \{\varepsilon\}$.*

(ii) *If s is even and t is odd, then u_β has exactly one infinite palindromic branch with center c for every $c \in \mathcal{A}$, and u_β has no infinite palindromic branch with center ε.*

(iii) *If s is even and t is even, then u_β has exactly one infinite palindromic branch with center ε, and u_β has no infinite palindromic branch with center $a \in \mathcal{A}$.*

Proof. — According to Proposition 6.2, u_β may have at most one infinite palindromic branch for each center $c \in \mathcal{A} \cup \{\varepsilon\}$. Therefore it suffices to show existence/non-existence of such a palindromic branch. We distinguish four cases:

- Let s be odd and t odd. Then an infinite palindromic branch with center c exists for every $c \in \mathcal{A} \cup \{\varepsilon\}$, by Remark 6.3.
- Let s be odd and t even. The existence of an infinite palindromic branch with center ε is ensured again by Remark 6.3. For determining the infinite palindromic branches with other centers, we define a sequence of words

  $$W^{(1)} = 0\,, \qquad\qquad W^{(n+1)} = \varphi(W^{(n)})0^t\,, \quad n \in \mathbb{N}, n \geqslant 1\,.$$

  Since $s+t$ is odd, using (i) and (ii) of Lemma 5.3, we know that $W^{(n)}$ is a palindrome with center $i \equiv n–1 \pmod{m}$. In particular, we have that $0 = W^{(1)}$ is a central factor of $W^{(m+1)}$. Using Lemma 5.2, also $W^{(n)}$ is a central factor of $W^{(m+n)}$ for all $n \geqslant 1$. Therefore we can construct the centered limit of palindromes $W^{(k+mn)}$ for n going to infinity, to obtain an infinite palindromic branch with center $k-1$ for all $k = 1, 2, \dots, m$.
- Let s be even and t be odd. Then an infinite palindromic branch with center c exists for every $c \in \mathcal{A}$, by Remark 6.3. A palindromic branch with center ε does not exist, since using Lemma 4.2 two non-zero letters in the word u_β are separated by a block of 0's of odd length, which implies that palindromes of even length must be shorter than $t+s$.
- Let s and t be even. The existence of an infinite palindromic branch with center ε is ensured again by Remark 6.3. Infinite palindromic branches with other centers do not exist. The reason is that in this case the maximal palindrome $U^{(1)} = 0^{t+s-1}$ has center 0 and using Lemma 5.3 the palindromes $U^{(2)}, U^{(3)}, \dots, U^{(m)}$ have centers $1, 2, \dots, m-1$, respectively. For all $n > m$ the center of $U^{(n)}$ is the empty word ε. If there existed an infinite palindromic branch $va\widetilde{v}$, then the maximal common central factor p of $va\widetilde{v}$ and $U^{(a+1)}$ would be a palindrome with center a and with two palindromic extensions. Using Proposition 4.7, $p = V^{(k)}$ for some k. Proposition 5.4 however

implies that for t even the center of $V^{(k)}$ is the empty word ε, which is a contradiction.

□

Remark 6.5. — The proof of the previous corollary implies:

(i) In case t odd, s even, u_β has only finitely many palindromes of even length, all of them being central factors of $U^{(1)} = 0^{t+s-1}$.

(ii) In case t and s are even, u_β has only finitely many palindromes of odd length and all of them are central factors of one of the palindromes $U^{(1)}$, $U^{(2)}$, …, $U^{(m)}$, with center $0, 1, \dots, m-1$, respectively.

7. Palindromic complexity of u_β

The aim of this section is to give explicit values of the palindromic complexity of u_β. We shall derive them from Theorem 4.8, which expresses $\mathcal{P}(n) + \mathcal{P}(n+1)$ using the first difference of factor complexity; and from (5) of Remark 4.6, which recalls the results about $\mathcal{C}(n)$ of [23].

Theorem 7.1. — *Let u_β be the fixed point of the substitution* (3.1), *with parameters $t \geqslant s \geqslant 2$.*

(i) *Let s be odd and let t be even. Then*

$$\mathcal{P}(2n+1) = m$$

$$\mathcal{P}(2n) = \begin{cases} 2, & \textit{if } |V^{(k)}| < 2n \leqslant |U^{(k)}| \textit{ for some } k, \\ 1, & \textit{otherwise.} \end{cases}$$

(ii) *Let s and t be odd. Then*

$$\mathcal{P}(2n+1) = \begin{cases} m+1, & \textit{if } |V^{(k)}| < 2n+1 \leqslant |U^{(k)}| \textit{ for some } k \\ & \textit{with } k \not\equiv 0 \pmod{(m+1)}, \\ m, & \textit{otherwise.} \end{cases}$$

$$\mathcal{P}(2n) = \begin{cases} 2, & \textit{if } |V^{(k)}| < 2n \leqslant |U^{(k)}| \textit{ for some } k \\ & \textit{with } k \equiv 0 \pmod{(m+1)}, \\ 1, & \textit{otherwise.} \end{cases}$$

(iii) *Let s be even and t be odd. Then*

$$\mathcal{P}(2n+1) = \begin{cases} m+2, & \text{if } |V^{(k)}| < 2n+1 \leqslant |U^{(k)}| \text{ for some } k \geqslant 2\,, \\ m, & \text{if } 2n+1 \leqslant |V^{(1)}|\,, \\ m+1, & \text{otherwise.} \end{cases}$$

$$\mathcal{P}(2n) = \begin{cases} 1, & \text{if } 2n \leqslant |U^{(1)}|\,, \\ 0, & \text{otherwise.} \end{cases}$$

(iv) *Let s and t be even. Then*

$$\mathcal{P}(2n+1) = \begin{cases} \#\big\{k \leqslant m \;\big|\; 2n+1 \leqslant |U^{(k)}|\big\}, & \text{if } 2n+1 \leqslant |U^{(m)}|\,, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{P}(2n) = \begin{cases} m+2, & \text{if } |V^{(k)}| < 2n \leqslant |U^{(k)}| \\ & \text{for some } k \geqslant m+1\,, \\ \#\big\{k \leqslant m \;\big|\; 2n > |V^{(k)}|\big\} + 1, & \text{if } 2n \leqslant |V^{(m+1)}|\,, \\ m+1, & \text{otherwise.} \end{cases}$$

Proof. — We prove the statement by cases:

(i) Let s be odd and t be even. It is enough to show that $\mathcal{P}(2n+1) = m$ for all $n \in \mathbb{N}$. The value of $\mathcal{P}(2n)$ can then be easily calculated from Theorem 4.8 and (5) of Remark 4.6.

From (i) of Corollary 6.4 we know that there exists an infinite palindromic branch with center c for all $c \in \mathcal{A}$. This implies that $\mathcal{P}(2n+1) \geqslant m$. In order to show the equality, it suffices to show that all maximal palindromes $U^{(k)}$ are of even length, or equivalently, have ε for center. Since both t and $t+s-1$ are even, $0^t = V^{(1)}$ is a central factor of $0^{t+s-1} = U^{(1)}$. Using Lemma 5.2, $V^{(k)}$ is a central factor of $U^{(k)}$ for all $k \geqslant 1$. According to (i) of Proposition 5.4, $V^{(k)}$ are palindromes of even length, and thus also the maximal palindromes $U^{(k)}$ are of even length. Therefore they do not contribute to $\mathcal{P}(2n+1)$.

(ii) Let s and t be odd. We shall determine $\mathcal{P}(2n)$ and the values of $\mathcal{P}(2n+1)$ can be deduced from Theorem 4.8 and (5) of Remark 4.6.

From (i) of Corollary 6.4 we know that there exists an infinite palindromic branch with center ε. Thus $\mathcal{P}(2n) \geqslant 1$ for all $n \in \mathbb{N}$. Again, $V^{(1)} = 0^t$ is a central factor of $U^{(1)} = 0^{t+s-1}$, and thus $V^{(k)}$ is a central factor of $U^{(k)}$ for all $k \geqslant 1$. A palindrome of even length, which is not a central factor of an infinite palindromic branch must be a central factor of $U^{(k)}$ for some k, and longer than $|V^{(k)}|$. Since

$|U^{(k)}| < |V^{(k+1)}| < |U^{(k+1)}|$ (cf. (5) of Remark 4.6), at most one such palindrome exists for each length. We have $\mathcal{P}(2n) \leqslant 2$. It suffices to determine for which k, the maximal palindrome $U^{(k)}$ is of even length, which happens exactly when its central factor $V^{(k)}$ is of even length and that is, using (iii) of Proposition 5.4, for $k \equiv 0 \pmod{(m+1)}$.

(iii) Let s be even and t be odd. According to (i) of Remark 6.5, all palindromes of even length are central factors of $U^{(1)} = 0^{t+s-1}$. Therefore $\mathcal{P}(2n) = 1$ if $2n \leqslant |U^{(1)}|$ and 0 otherwise. The value of $\mathcal{P}(2n+1)$ can be calculated from Theorem 4.8 and (5) of Remark 4.6.

(iv) Let s and t be even. Using (ii) of Remark 6.5, the only palindromes of odd length are central factors of $U^{(k)}$ for $k = 1, 2, \ldots, m$. Therefore $\mathcal{P}(2n+1) = 0$ for $2n+1 > |U^{(m)}|$. If $2n+1 \leqslant |U^{(m)}|$, the number of palindromes of odd length is equal to the number of maximal palindromes longer than $2n+1$. The value of $\mathcal{P}(2n)$ can be calculated from Theorem 4.8 and (5) of Remark 4.6.

□

For the determination of the value $\mathcal{P}(n)$ for a given n, we have to know $|V^{(k)}|$, $|U^{(k)}|$. In [23] it is shown that

$$|V^{(k)}| = t\sum_{i=0}^{k-1} G_i\,, \qquad \text{and} \qquad |U^{(k)}| = |V^{(k)}| + (s-1)G_{k-1}\,,$$

where G_n is a sequence of integers defined by the recurrence

$$G_0 = 1,\ G_n = t(G_{n-1} + \cdots + G_0) + 1\,,\ \text{for } 1 \leqslant n \leqslant m-1\,,$$
$$G_n = t\,(G_{n-1} + \cdots + G_{n-m+1}) + sG_{n-m}\,,\ \text{for } n \geqslant m\,.$$

The sequence $(G_n)_{n\in\mathbb{N}}$ defines the canonical linear numeration system associated with the number β, see [14] for general results on these numeration systems. In this particular case, $(G_n)_{n\in\mathbb{N}}$ defines a confluent linear numeration system, see [22] for its properties.

8. Substitution invariance of palindromic branches

Infinite words u_β are invariant under the substitution (3.1). One can ask whether also their infinite palindromic branches are invariant under a substitution. In case that an infinite palindromic branch has as its center the empty word ε, we can use the notion of invariance under substitution as defined for pointed two-sided infinite words. We restrict our attention to infinite palindromic branches of such type.

Recall that an infinite palindromic branch of u_β with center ε exists, (according to Corollary 6.4), only if in the Rényi expansion $d_\beta(1) = tt\cdots ts$, t is even, or both t and s are odd. Therefore we shall study only such parameters.

Let us first study the most simple case, $d_\beta(1) = t1$ for $t \geqslant 1$. Here β is a quadratic unit, and the infinite word u_β is a Sturmian word, expressible in the form of the mechanical word $\mu_{\alpha,\varrho}$,

$$\mu_{\alpha,\varrho}(n) = \lfloor (n+1)\alpha + \varrho \rfloor - \lfloor n\alpha + \varrho \rfloor , \quad n \in \mathbb{N} ,$$

where the irrational slope α and the intercept ρ satisfy $\alpha = \rho = \frac{\beta}{\beta+1}$. The infinite palindromic branch with center ε of the above word $u_\beta = \mu_{\alpha,\varrho}$ is a two-sided Sturmian word with the same slope $\alpha = \frac{\beta}{\beta+1}$, but intercept $\frac{1}{2}$. Indeed, two mechanical words with the same slope have the same set of factors independently on their intercepts, and moreover the Sturmian word $\mu_{\alpha,\frac{1}{2}}$ is an infinite palindromic branch of itself, since

$$\mu_{\alpha,\frac{1}{2}}(n) = \mu_{\alpha,\frac{1}{2}}(-n-1) , \quad \text{for all } n \in \mathbb{Z} .$$

Therefore if $v = \mu_{\alpha,\frac{1}{2}}(0)\mu_{\alpha,\frac{1}{2}}(1)\mu_{\alpha,\frac{1}{2}}(2)\cdots$, then $\widetilde{v}v$ is the infinite palindromic branch of u_β with the center ε.

Since the Sturmian word $\mu_{\alpha,\varrho}$ coincides with u_β, it is invariant under the substitution φ. As a consequence of [30], the slope α is a Sturm number, i.e. a quadratic number in $(0,1)$ such that its conjugate α' satisfies $\alpha' \notin (0,1)$, (using the equivalent definition of Sturm numbers given in [1]).

The question about the substitution invariance of the infinite palindromic branch $\widetilde{v}v$ is answered using the result of [4] (or also [36, 12]). It says that a Sturmian word whose slope is a Sturm number, and whose intercept is equal to $\frac{1}{2}$, is substitution invariant as a two-sided pointed word, i.e. there exists a substitution ψ such that $\widetilde{v}|v = \psi(\widetilde{v})|\psi(v)$.

Example 8.1. — The Fibonacci word u_β for $d_\beta(1) = 11$ is a fixed point of the substitution

$$\varphi(0) = 01, \quad \varphi(1) = 0 .$$

Its infinite palindromic branch with center ε is

$$\widetilde{v}v \qquad \text{for} \qquad v = 0101001001010010010 10\cdots$$

which is the fixed point $\lim_{n\to\infty} \psi^n(0)|\psi^n(0)$ of the substitution

$$\psi(0) = 01010, \quad \psi(1) = 010 .$$

Let us now study the question whether infinite palindromic branches in u_β for general $d_\beta(1) = tt\cdots ts$ with t even, or t and s odd, are also substitution invariant. It turns out that the answer is positive. For construction of a substitution ψ under which a given palindromic branch is invariant, we need the following lemma.

LEMMA 8.2. — *Let $v\widetilde{v}$ be an infinite palindromic branch with center ε. Then the left infinite word $v = \cdots v_3v_2v_1$ satisfies*

$$\begin{aligned} v &= \varphi(v)0^{\frac{t}{2}} && \text{for } t \text{ even,}\\ v &= \varphi^{m+1}(v)\varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}} && \text{for } t \text{ and } s \text{ odd.} \end{aligned}$$

Proof. — Let t be even and let $v\widetilde{v}$ be the unique infinite palindromic branch with center ε. Recall that $v\widetilde{v}$ is a centered limit of $V^{(n)}$. Consider arbitrary suffix v_{suf} of v, i.e. $v_{\text{suf}}\widetilde{v}_{\text{suf}}$ is a palindrome of u_β with center ε. Denote $w := \varphi(v_{\text{suf}})0^{\frac{t}{2}}$. Using (iv) of Lemma 5.3 the word $p = w\widetilde{w}$ is a palindrome of u_β with center ε. We show by contradiction that w is a suffix of v.

Suppose that $p = w\widetilde{w}$ is not a central factor of $v\widetilde{v}$, then there exists a unique n such that p is a central factor of $U^{(n)}$. Then according to Proposition 4.7, p is uniquely extendable into a maximal palindrome. In that case we take a longer suffix v'_{suf} of v, so that the length of the palindrome $p' = w'\widetilde{w}'$, $w' := \varphi(v'_{\text{suf}})0^{\frac{t}{2}}$ satisfies $|p'| > |U^{(n)}|$. However, p' (since it contains p as its central factor) is a palindromic extension of p, and therefore p' is a central factor of $U^{(n)}$, which is a contradiction. Thus $\varphi(v_{\text{suf}})0^{\frac{t}{2}}$ is a suffix of v for all suffixes v_{suf} of v, therefore $v = \varphi(v)0^{\frac{t}{2}}$.

Let now s and t be odd. If v_{suf} is a suffix of the word v, then $v_{\text{suf}}\widetilde{v}_{\text{suf}}$ is a palindrome of u_β with center ε. Using Lemma 5.3, the following holds true.

$$\begin{array}{rcl c c l} w_0 &=& \varphi(v_{\text{suf}})0^{\frac{t-1}{2}} & \Longrightarrow & w_0 0 \widetilde{w}_0 & \in \mathcal{P}al(u_\beta)\\ w_1 &=& \varphi(w_0)0^t & \Longrightarrow & w_1 1 \widetilde{w}_1 & \in \mathcal{P}al(u_\beta)\\ w_2 &=& \varphi(w_1)0^t & \Longrightarrow & w_2 2 \widetilde{w}_2 & \in \mathcal{P}al(u_\beta)\\ && & \vdots & & \\ w_{m-1} &=& \varphi(w_{m-2})0^t & \Longrightarrow & w_{m-1}(m-1)\widetilde{w}_{m-1} & \in \mathcal{P}al(u_\beta)\\ w_\varepsilon &=& \varphi(w_{m-1})0^{\frac{s+t}{2}} & \Longrightarrow & w_\varepsilon \widetilde{w}_\varepsilon & \in \mathcal{P}al(u_\beta) \end{array}$$

Together we obtain

$$w_\varepsilon = \varphi^{m+1}(v_{\text{suf}})\varphi^m(0^{\frac{t-1}{2}})\varphi^{m-1}(0^t)\cdots\varphi 2(0^t)\varphi(0^t)0^{\frac{s+t}{2}}\,.$$

Since $\varphi^m(0) = \varphi^{m-1}(0^t)\varphi^{m-2}(0^t)\cdots\varphi(0^t)0^s$, the word w_ε can be rewritten in a simpler form

$$w_\varepsilon = \varphi^{m+1}(v_{\mathrm{suf}})\varphi^m(0^{\frac{t-1}{2}})\varphi^m(0)0^{\frac{t-s}{2}} = \varphi^{m+1}(v_{\mathrm{suf}})\varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}}$$

Since w_ε is again a suffix of v, the statement of the lemma for s and t odd holds true. □

Theorem 8.3. — *Let u_β be the fixed point of the substitution φ given by* (3.1), *and let $v\widetilde{v}$ be the infinite palindromic branch of u_β with center ε. Then the left-sided infinite word v is invariant under the substitution ψ defined for all letters $a \in \{0,1,\dots,m-1\}$ by*

$$\psi(a) = \begin{cases} w^{-1}\varphi(a)w\,, & \text{where } \ w = 0^{\frac{t}{2}}\,, \qquad\qquad \text{for } t \text{ even,} \\ w^{-1}\varphi^{m+1}(a)w\,, & \text{where } \ w = \varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}}\,, \quad \text{for } t \text{ and } s \text{ odd.} \end{cases}$$

Moreover, if t is even, then $\psi(a)$ is a palindrome for all $a \in \mathcal{A}$ and $v\widetilde{v}$ as a pointed sequence is invariant under the same substitution ψ.

Proof. —

First let us show that the substitution ψ is well defined.

- Let t be even. Since $0^{\frac{t}{2}}$ is a prefix of $\varphi(a)$ for all $a \in \{0,1,\dots,m-2\}$ and $\varphi(m-1) = 0^s$, therefore $0^{\frac{t}{2}}$ is a prefix of $\varphi(m-1)0^{\frac{t}{2}} = 0^{s+\frac{t}{2}}$.
- Let t and s be odd. Let us verify that w is a prefix of $\varphi^{m+1}(a)w$.
  - If $a \neq m-1$, we show that $w = \varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}}$ is a prefix of

  $$\varphi^{m+1}(a) = \varphi^m\big(0^t(a+1)\big) = \varphi^m(0^{\frac{t+1}{2}})\varphi^m(0^{\frac{t-1}{2}})\varphi^m(a+1)\,.$$

  It suffices to show that $0^{\frac{t-s}{2}}$ is a prefix of $\varphi^m(0^{\frac{t-1}{2}})$. For $t = s$ it is obvious. For $t > s \geqslant 1$ we obtain $t \geqslant 3$ and so $\varphi^m(0^{\frac{t-1}{2}}) = \varphi^m(0)\varphi^m(0^{\frac{t-3}{2}})$ and clearly $0^{\frac{t-s}{2}}$ is a prefix of $\varphi^m(0)$.
  - If $a = m-1$, then

  $$\varphi^{m+1}(m-1)w = \varphi^m(0^s)\varphi^m(0^{\frac{t+1}{2}})0^{\frac{t-s}{2}} = \varphi^m(0^{\frac{t+1}{2}})\varphi^m(0)\varphi^m(0^{s-1})0^{\frac{t-s}{2}}\,.$$

  Since $0^{\frac{t-s}{2}}$ is a prefix of $\varphi^m(0)$, the correctness of the definition of the substitution ψ is proven.

Now it is enough to prove that $\psi(v) = v$. Lemma 8.2 says that in the case that t is even the left infinite word $v = \cdots v_3v_2v_1$ satisfies $v = \varphi(v)w$. Thus we have

$$\begin{aligned}\psi(v) = \cdots\psi(v_3)\psi(v_2)\psi(v_1) &= \cdots w^{-1}\varphi(v_3)ww^{-1}\varphi(v_2)ww^{-1}\varphi(v_2)w = \\ &= \cdots\varphi(v_3)\varphi(v_2)\varphi(v_1)w \ = \ \varphi(v)w \ = \ v\,.\end{aligned}$$

In case that t and s are odd, the proof is the same, using φ^{m+1} instead of φ.

If t is even, it is clear from the prescription for ψ, that $\psi(a)$ is a palindrome for any letter a, which implies the invariance of the word $v\widetilde{v}$ under ψ. $\square$

Let us mention that for t, s odd the words $\psi(a)$, $a \in \mathcal{A}$, may not be palindromes. In that case the right-sided word $\widetilde{v}$ is invariant under another substitution, namely $a \mapsto \widetilde{\psi(a)}$. Nevertheless even for t, s odd it may happen that $\psi(a)$ is a palindrome for all letters. Then the two-sided word $v\widetilde{v}$ is invariant under ψ. This situation is illustrated on the following example.

Example 8.4. — Consider the Tribonacci word, i.e. the word u_β for $d_\beta(1) = 111$. It is the fixed point of the substitution

$$\varphi(0) = 01, \quad \varphi(1) = 02, \quad \varphi(2) = 0\,,$$

which is in the form (3.1) for $t = s = 1$ and $m = 3$. Therefore $w = \varphi^3(0) = 0102010$. The substitution ψ, under which the infinite palindromic branch $v\widetilde{v}$ of the Tribonacci word is invariant, is therefore given as

$$\begin{array}{rcccl}
\psi(0) & := & w^{-1}\varphi^4(0)w & = & 0102010102010, \\
\psi(1) & := & w^{-1}\varphi^4(1)w & = & 01020102010, \\
\psi(2) & := & w^{-1}\varphi^4(2)w & = & 0102010.
\end{array}$$

Note that the substitution ψ has the following property: the word $\psi(a)$ is a palindrome for every $a \in \mathcal{A}$.

9. Number of palindromes in the prefixes of u_β

In [19] the authors obtain an interesting result which says that every finite word w contains at most $|w| + 1$ different palindromes. (The empty word is considered as a palindrome contained in every word.) Denote by $P(w)$ the number of palindromes contained in the finite word w. Formally, we have

$$P(w) \leqslant |w| + 1 \qquad \text{for every finite word } w.$$

The finite words w for which the equality is reached are called *full* (as suggested in [15]). An infinite word u is called full, if all its prefixes are full. In [19] the authors have shown that every Sturmian word is full. They have shown the same property for episturmian words.

The infinite word u_β can be full only if its language is closed under reversal, i.e. in the simple Parry case for $d_\beta(1) = tt \cdots ts$, $t \geqslant s \geqslant 1$. For $s \geqslant 2$ such words are not episturmian, nevertheless, we shall show that they are full.

We shall use the notions and results introduced in [19].

Definition 9.1. — *A finite word w satisfies property Ju, if there exists a palindromic suffix of w which is unioccurrent in w.*

Clearly, if w satisfies Ju, then it has exactly one palindromic suffix which is unioccurrent, namely the longest palindromic suffix of w.

Proposition 9.2 ([19]). — *Let w be a finite word. Then $P(w) = |w|+1$ if and only if all the prefixes $\hat{w}$ of w satisfy Ju, i.e. have a palindrome suffix which is unioccurrent in $\hat{w}$.*

Theorem 9.3. — *The infinite word u_β invariant under the substitution* (3.1) *is full.*

Proof. — We show the statement using Proposition 9.2 by contradiction. Let w be a prefix of u_β of minimal length which does not satisfy Ju, and let $X0^k$ be a suffix of w with $X \neq 0$.

First we show that $k \in \{0, t+1\}$. For, if $1 \leqslant k \leqslant t$ or $t+2 \leqslant k$, then q is the maximal palindromic suffix of $w0^{-1}$ if and only if $0q0$ is the maximal palindromic suffix of w. Since $0q0$ occurs at least twice in w, then also q occurs at least twice in $w0^{-1}$, which is a contradiction with the minimality of w.

Define

$$w_1 = \begin{cases} w0^t & \text{if } w \text{ has suffix } X \neq 0\,, \\ w0^{s-1} & \text{if } w \text{ has suffix } X0^{t+1},\ X \neq 0\,. \end{cases}$$

For the maximal palindromic suffix p of w denote

$$p_1 = \begin{cases} 0^t p 0^t & \text{if } w \text{ has suffix } X \neq 0\,, \\ 0^{s-1} p 0^{s-1} & \text{if } w \text{ has suffix } X0^{t+1},\ X \neq 0\,. \end{cases}$$

Since in u_β every two non-zero letters are separated by the word 0^t or 0^{t+s}, we obtain that

(i) p_1 is the maximal palindromic suffix of w_1.

(ii) the position of centers of palindromes p and p_1 coincide in all occurrences in u_β.

Since p occurs in w at least twice, also the palindromic suffix p_1 occurs at least twice in w_1, i.e. the word w_1 is a prefix of u_β which does not satisfy Ju.

From the definition of w_1 it follows that

$$w_1 = \varphi(\hat{w})0^t$$

for some prefix $\hat{w}$ of u_β. Thus the maximal palindromic suffix p_1 of w_1 is of the form $p_1 = \varphi(\hat{p})0^t$, where $\hat{p}$ is a factor of $\hat{w}$. According to (i) of

Proposition 4.4, $\hat{p}$ is a palindrome, and the same proposition implies that $\hat{p}$ is the maximal palindromic suffix of $\hat{w}$. Since p_1 occurs at least twice in w_1, also $\hat{p}$ occurs at least twice in $\hat{w}$. Therefore $\hat{w}$ does not satisfy the property Ju. As

$$|\hat{w}| < |\varphi(\hat{w})| < |w|,$$

we have a contradiction with the minimality of w. □

10. Conclusions

The study of palindromic complexity of an uniformly recurrent infinite word is interesting in the case that its language is closed under reversal. Infinite words u_β associated to Parry numbers β are uniformly recurrent. If β is a simple Parry number, the language of u_β is invariant under reversal if the Rényi expansion of 1 satisfies $d_\beta(1) = tt\cdots ts$, i.e. is a confluent Parry number, and the corresponding palindromic complexity is the subject of this paper.

For non-simple Parry number β, the condition under which the language of the infinite word u_β is closed under reversal has been stated by Bernat [9]. He has shown that the language of u_β is closed under reversal if and only if β is a quadratic number, i.e. a root of minimal polynomial $X^2 - aX + b$, with $a \geqslant b+2$ and $b \geqslant 1$. In this case $d_\beta(1) = (a-1)(a-b-1)^\omega$. The palindromic complexity of the corresponding infinite words u_β is described in [7].

Infinite words u_β for non-simple Parry numbers β are thus another example for which the equality

$$\mathcal{P}(n) + \mathcal{P}(n+1) = \Delta\mathcal{C}(n) + 2$$

is satisfied for all $n \in \mathbb{N}$. According to our knowledge, among all examples of infinite words satisfying this equality, the words u_β (for both simple and non-simple Parry number β) are exceptional in that they have the second difference $\Delta^2\mathcal{C}(n) \neq 0$.

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