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THE KOWALEVSKI'S TOP AND THE METHOD OF SYZYGIES

by Franco MAGRI

1. The Kowalevski's top.

In 1889 Sophie Kowalevski, following Euler and Lagrange, found a third case of integrable top, and solved the corresponding equations of motion by the method of separation of variables. From a mathematical viewpoint the equations of the top define a Hamiltonian vector field on R^6 , endowed with the degenerate Poisson brackets

$$\{L_i, L_j\} = \sum_{k=1}^{k=3} \epsilon_{ijk} L_k, \quad \{L_i, y_j\} = \sum_{k=1}^{k=3} \epsilon_{ijk} y_k, \quad \{y_i, y_j\} = 0,$$

where the symbols (L_1, L_2, L_3) denote the components of the angular momentum of the top, and (y_1, y_2, y_3) are the components of its weight. These Poisson brackets have two Casimir's functions

$$c_1 = y_1^2 + y_2^2 + y_3^2 \\ c_2 = L_1 y_1 + L_2 y_2 + L_3 y_3,$$

and the top has two integrals of motion. They are the energy

$$h_1 = 1/4L_1^2 + 1/4L_2^2 + 1/2L_3^2 - y_1$$

and the famous quartic integral

$$h_2 = 1/8(L_1^2 - L_2^2 + 4y_1)^2 + 1/8(2L_1L_2 + 4y_2)^2$$

discovered by Kowalevski. By this property the top is an integrable system. To solve her equations of motion, Kowalevski had the astonishing idea of replacing the mechanical variables $(L_1, L_2, L_3, y_1, y_2, y_3)$ by the four integrals (h_1, h_2, c_1, c_2) and by the roots (λ_1, λ_2) of the second-order polynomial

$$S(\lambda) = (x_1 - x_2)^2(\lambda - 1/6h_1)^2 - R(x_1, x_2)(\lambda - 1/6h_1) - 1/4R_1(x_1, x_2),$$

where

$$2x_1 = L_1 + iL_2$$

$$2x_2 = L_1 - iL_2$$

and

$$\begin{aligned} R(x_1, x_2) &= -x_1^2x_2^2 + 2h_1x_1x_2 + c_2(x_1 + x_2) + c_1 - 1/2h_2 \\ R_1(x_1, x_2) &= -2h_1x_1^2x_2^2 - (c_1 - 1/2h_2)(x_1 + x_2)^2 - 2c_2(x_1 + x_2)x_1x_2 \\ &\quad + 2h_1(c_1 - 1/2h_2) - c_2^2. \end{aligned}$$

By this change of variables she was able to write the equations of motion in the Abel's form

$$\begin{aligned} \frac{ds_1}{\sqrt{P(s_1)}} + \frac{ds_2}{\sqrt{P(s_2)}} &= 0 \\ \frac{s_1 ds_1}{\sqrt{P(s_1)}} + \frac{s_2 ds_2}{\sqrt{P(s_2)}} &= dt \end{aligned}$$

where $P(s)$ is a fifth-order polynomial with distinct roots. This beautiful result allowed her to explicitly solve the equations of motion by means of hyperelliptic functions.

The discovery of the polynomial $S(\lambda)$ has always been a *vexata quaestio*. In her paper Kowalevski did not provide a convincing motivation for her choice, but only the evidence, *a posteriori*, that it actually works. Her choice therefore appears as the outcome of a magical intuition. The purpose of the present paper is to derive the polynomial $S(\lambda)$ directly from the equations of motion by the method of syzygies. This method is a way of implementing a new algorithm for the search of separation coordinates in the case of polynomial equations. The aim of the study is to try to identify the properties of the equations of motion which are responsible for the existence of the separation coordinates.

2. An algorithm for the search of separation coordinates.

Motivated by the example of Kowalevski, let us consider an integrable Hamiltonian system which is defined by n independent and involutive functions (h_1, h_2, \dots, h_n) on a Poisson manifold M of dimension $(2n + r)$, which is endowed with a degenerate Poisson bracket possessing r Casimir's functions (c_1, c_2, \dots, c_r) . Let us call F the Lagrangian foliation defined by both the Hamiltonians and the Casimir's functions, and let us say that the polynomial

$$S(\lambda) = \lambda^n - (s_1 \lambda^{n-1} + \dots + s_n),$$

with coefficients defined on M , is *admissible* for F if it verifies three conditions. The first is that almost everywhere on M

$$(1) \quad ds_1 \wedge \dots \wedge ds_n \wedge dh_1 \wedge \dots \wedge dh_n \wedge dc_1 \wedge \dots \wedge dc_r \neq 0,$$

so that the coefficients of $S(\lambda)$ may be used as local coordinates on the leaves of F . The second is that almost everywhere on M

$$(2) \quad \text{discr}_\lambda S(\lambda) \neq 0,$$

so that the roots of $S(\lambda)$ may be used as local coordinates on F as well. The third is that

$$(3) \quad \{s_i, s_j\} = 0,$$

so that the roots of $S(\lambda)$ may be regarded as the first-half of a set of canonical coordinates $(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n, c_1, \dots, c_r)$ adapted to F . In these coordinates the parametric equations of the leaves of F have the form

$$\mu_i = \frac{\partial W}{\partial \lambda_i},$$

where W is a function of the roots, of the Casimir's functions, and of a set of parameters labelling the different leaves of F . If the function W has the form

$$W(\lambda_1, \dots, \lambda_n, c_1, \dots, c_r, e_1, \dots, e_n) = \sum_{j=1}^{j=n} W_j(\lambda_j, c_1, \dots, c_r, e_1, \dots, e_n),$$

the foliation F is said to be *separable* and $S(\lambda)$ is called its *separation polynomial*.

To find $S(\lambda)$ we resort to an algorithm which is based on the idea that the separation polynomial is always the characteristic polynomial of a suitable matrix \mathbf{S} . The selection of \mathbf{S} proceeds in three stages. The only

data which are used are the Poisson brackets and the Hamiltonian vector fields X_{h_k} associated with the functions h_k . To simplify the notation, the j -th row of the matrix \mathbf{S} will be simply denoted by \mathbf{S}_j .

ALGORITHM. — *The first stage is to find a matrix \mathbf{S} verifying the linear equation*

$$(4) \quad X_{h_j}(\mathbf{S}_k) - X_{h_k}(\mathbf{S}_j) = 0.$$

Among the solutions of this equation, if any, one must then select those verifying the quadratic equation

$$(5) \quad X_{h_j}((\mathbf{S}^2)_k) - X_{h_k}((\mathbf{S}^2)_j) = 0.$$

If also this second stage is passed, one finally computes the characteristic polynomial of \mathbf{S} and checks if it is admissible for the given foliation F . In the affirmative case one is enabled to claim that the roots of

$$(6) \quad S(\lambda) = \det(\lambda - \mathbf{S})$$

are the separation coordinates of F .

The *Algorithm* is proved in a paper which will appear after this one [2]. I apologize for the inconvenience, but I hope that the lack of a proof should not undermine the purposes of the present paper, which are simply to show the existence of an algorithm leading in a systematic way to the result of Kowalevski. The algorithm may be used and its results may be appreciated even if one has not seen the proof. In substitution, I present the ideas which are behind the proof. The main idea is to use *recursion operators* to characterize separable systems. By this name it is conventionally meant a class of tensor fields of type $(1, 1)$ on a manifold, which are diagonalizable and have vanishing Nijenhuis torsion. Some minor additional assumptions on the minimal polynomial of these operators are also needed, but they are uninteresting in the present context. Suitably combined with symplectic 2-forms, the recursion operators give rise to the concept of ωN -manifold.

DEFINITION 1. — *A ωN -manifold is a symplectic manifold (M, ω) endowed with a recursion operator N such that $\omega \circ N$ is a closed 2-form.*

On a ωN -manifold each Hamiltonian vector field is the generator a distribution D_h .

DEFINITION 2. — *The Levi Civita distribution of X_h is the minimal invariant distribution containing X_h .*

The distribution D_h allows to characterize the separable Hamiltonian systems.

PROPOSITION 1. — *The separable systems are the generators of the integrable Levi Civita distributions.*

This result presents the theory of separable systems as the theory of a special class of foliations. In this perspective the following problem becomes important. Given a Lagrangian foliation F on a symplectic manifold (M, ω) , one is required to find a recursion operator N transforming M into a ωN -manifold and leaving F invariant. If this operator exists, F is a Levi Civita foliation. The mechanical implications of this problem are clear. Since a Lagrangian foliation represents an integrable system, and since a Levi Civita foliation represents a separable system, to transform a Lagrangian foliation into a Levi Civita foliation is equivalent to discover that an integrable system is separable. Hence, solving the inverse problem of the theory of Levi Civita foliations one simultaneously solves the problem of Kowalevski.

The role of the matrices \mathbf{S} selected by the *Algorithm* is to parametrize the solutions of the inverse problem. Each matrix \mathbf{S} defines a tensor field L on F according to

$$LX_{h_j} = \sum_{k=1}^{k=n} S_{jk} X_{h_k}.$$

If \mathbf{S} obeys the conditions of the *Algorithm*, the torsion of L vanishes, and L can be prolonged into a recursion operator N on M , in such a way that $\omega \circ N$ is a closed 2-form. The latter tensor field solves the inverse problem, and the roots of its minimal polynomial provide the separation coordinates. In this way the matrices selected by the *Algorithm* solve the Kowalevski's problem.

In the case of Kowalevski's top $n = r = 2$, and therefore the matrix \mathbf{S} has four entries:

$$\mathbf{S} = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}.$$

They must verify a system of five partial differential equations. The first four are:

$$\begin{aligned} \{f_3, h_1\} &= \{f_1, h_2\} \\ \{f_4, h_1\} &= \{f_2, h_2\} \\ \{f_1 f_3 + f_3 f_4, h_1\} &= \{f_1^2 + f_2 f_3, h_2\} \\ \{f_2 f_3 + f_4^2, h_1\} &= \{f_1 f_2 + f_2 f_4, h_2\}. \end{aligned}$$

The fifth equation is

$$\{p, q\} = 0,$$

where

$$\begin{aligned} p &= f_1 + f_4 \\ q &= f_2 f_3 - f_1 f_4 \end{aligned}$$

are the coefficients of the characteristic polynomial of \mathbf{S} . The problem is to construct a solution of these equations when the Poisson brackets and the functions h_1 and h_2 are those associated with the Kowalevski's top.

3. The Kowalevski's endomorphism.

In view of solving the previous equations, it is suitable to make a change of variables partially decoupling the separability conditions. The transformation is suggested by geometric considerations. Let us consider on F a second endomorphism $K : TF \rightarrow TF$ which has the property of mapping X_{h_1} into X_{h_2} . Its defining equations are

$$\begin{aligned} KX_{h_1} &= X_{h_2} \\ KX_{h_2} &= -sX_{h_1} + rX_{h_2}. \end{aligned}$$

Let us furthermore relate K to L by demanding that K has the same eigenvectors of L . This requirement entails the existence of a pair of functions (l, m) such that

$$L = lK + m.$$

One may use this equation to define a transformation between (r, s, l, m) and (f_1, f_2, f_3, f_4) . Since

$$\begin{aligned} LX_{h_1} &= f_1 X_{h_1} + f_2 X_{h_2} \\ LX_{h_2} &= f_3 X_{h_1} + f_4 X_{h_2}, \end{aligned}$$

the transformation is

$$\begin{aligned} f_1 &= m \\ f_2 &= l \\ f_3 &= -ls \\ f_4 &= m + lr. \end{aligned}$$

Let us choose (r, s, f_2, f_4) as principal unknowns (under the assumption that $f_2 \neq 0$), and let us write anew the separability conditions on these

functions. The first four separability conditions become

- (7) $\{s, h_1\} = \{r, h_2\}$
- (8) $\{s, h_2\} = r\{r, h_2\} - s\{r, h_1\}$
- (9) $\{f_4, h_1\} = \{f_2, h_2\}$
- (10) $\{f_4, h_2\} = r\{f_2, h_2\} - s\{f_2, h_1\}$.

The first two equations have an interesting geometrical interpretation. They mean that $K : TF \rightarrow TF$ obeys the condition

$$(11) \quad K^t d(\text{tr } K) = d(\det K).$$

This condition is the long shadow, on K , of the vanishing of the torsion of L . Let us call *endomorphisms of Kowalevski* the linear operators which map X_{h_1} into X_{h_2} and verify the previous relation. The point of the discussion is thus captured by the following claim.

PROPOSITION 2. — *Any integrable system with two degrees of freedom which is also separable has a Kowalevski's endomorphism.*

The search of separation coordinates must accordingly start from the search of this endomorphism.

4. The method of syzygies.

The problem to be discussed in this section is to understand what property of the equations of motion of the Kowalevski's top

$$\begin{aligned} \dot{L}_1 &= +1/2L_2L_3 \\ \dot{L}_2 &= -1/2L_1L_3 - y_3 \\ \dot{L}_3 &= +y_2 \\ y_1 &= +L_3y_2 - 1/2L_2y_3 \\ y_2 &= +1/2L_1y_3 - L_3y_1 \\ y_3 &= +1/2L_2y_1 - 1/2L_1y_2, \end{aligned}$$

and of the equations of motion associated with the quadratic integral

$$\begin{aligned} L'_1 &= k_1(1/2L_2L_3) - k_2(1/2L_1L_3 + y_3) \\ L'_2 &= k_1(1/2L_1L_3 + y_3) + k_2((1/2L_2L_3) \\ L'_3 &= -k_1L_1L_2 + 1/2k_2(L_1^2 - L_2^2) + (k_2y_1 - k_1y_2) \\ y'_1 &= 1/2(k_1L_2 - k_2L_1)y_3 \\ y'_2 &= 1/2(k_1L_1 + k_2L_2)y_3 \\ y'_3 &= -1/2k_1(L_1y_2 + L_2y_1) + 1/2k_2(L_1y_1 - L_2y_2), \end{aligned}$$

where the quantities k_1 and k_2 are defined by

$$\begin{aligned}k_1 &= L_1^2 - L_2^2 + 4y_1 \\ k_2 &= 2L_1L_2 + 4y_2,\end{aligned}$$

is responsible for the existence of the separation coordinates. The complex mechanism relating the equations of motion to the separation coordinates may be broken in four units. Each unit describes an elementary process. The processes are:

- The equations produce the syzygies
- The syzygies define the Kowalevski's endomorphism
- The endomorphism K generates the endomorphism L
- The endomorphism L defines the separation coordinates.

They are presently investigated separately.

2.1. The analysis of nonlinearities.

As a first step in the analysis of the equations, let us choose coordinates adapted to the integrals of motion, so to reduce the number of equations from twelve to four. Following Kowalevski, let us replace the dynamical coordinates $(L_1, L_2, L_3, y_1, y_2, y_3)$ by the adapted coordinates $(x_1, x_2, h_1, h_2, c_1, c_2)$. If one tries to explicitly write the reduced equations

$$\begin{aligned}\dot{x}_1 &= Z_1(x_1, x_2, h_1, h_2, c_1, c_2) \\ \dot{x}_2 &= Z_2(x_1, x_2, h_1, h_2, c_1, c_2) \\ x'_1 &= Z_3(x_1, x_2, h_1, h_2, c_1, c_2) \\ x'_2 &= Z_4(x_1, x_2, h_1, h_2, c_1, c_2),\end{aligned}$$

one immediately realizes the loss of the polynomial form of the equations. The nonlinearity of the integrals of motion forces the equations to become irrational. This unpleasant occurrence may be used as the key for the search of separation coordinates of the Kowalevski's top. The reason is connected to a peculiar balance of the irrational expressions appearing into the different reduced equations, allowing to get rid of them simply by forming linear combinations of the equations of motion with *rational coefficients*. In other terms, the irrationality of the equations does not forbid the existence of syzygies with rational coefficients. The existence of these

syzygies reflects a deep property of the structure of the equations, and the aim of this section is to convince the reader that the separation coordinates follow from this property.

The analysis of the irrational expressions ensuing from the change of variables has been performed by Kowalevski. She wrote the integrals of motion in the form

$$\begin{aligned} c_1 &= 1/16(k_1 + L_2^2 - L_1^2)^2 + 1/16(k_2 - 2L_1L_2)^2 + y_3^2 \\ c_2 &= 1/4(k_1 + L_2^2 - L_1^2)L_1 + 1/4(k_2 - 2L_1L_2)L_2 + y_3L_3 \\ h_1 &= 1/4(L_1^2 + L_2^2) + 1/2L_3^2 - 1/4(k_1 + L_2^2 - L_1^2) \\ h_2 &= 1/8(k_1^2 + k_2^2), \end{aligned}$$

by replacing the coordinates y_1 and y_2 by the basic quantities k_1 and k_2 , and she noticed that the quantities y_3^2 , y_3L_3 , and L_3^2 depend linearly on k_1 and k_2 , on account of the fourth equation. The equations

$$\begin{aligned} y_3^2 &= \frac{1}{8} ((L_1^2 - L_2^2)k_1 + 2L_1L_2k_2) - \frac{1}{16}(L_1^2 + L_2^2)^2 + c_1 - \frac{1}{2}h_2 \\ y_3L_3 &= -\frac{1}{4}(L_1k_1 + L_2k_2) + \frac{1}{4}(L_1^2 + L_2^2)L_1 + c_2 \\ L_3^2 &= \frac{1}{2}k_1 - L_1^2 + 2h_1 \end{aligned}$$

serve to compute the second constraint on k_1 and k_2 . Pursuing the computation of the identity $L_3^2y_3^2 - (L_3y_3)^2 = 0$, she discovered the second unexpected property that this constraint is again *linear* in k_1 and k_2 . So, k_1 and k_2 are the roots of a second-order equation. For these roots she found the representation:

$$\begin{aligned} \xi_1 &= -\frac{R_1(x_1, x_2) + 1/2h_2(x_1 - x_2)^2 - W}{2R(x_2, x_2)} \\ \xi_2 &= -\frac{R_1(x_1, x_2) + 1/2h_2(x_1 - x_2)^2 + W}{2R(x_1, x_1)} \end{aligned}$$

using the complex coordinates

$$\begin{aligned} \xi_1 &= 1/4(k_1 + ik_2) \\ \xi_2 &= 1/4(k_1 - ik_2), \end{aligned}$$

and the function W defined by the equation

$$W^2 = (R_1(x_1, x_2) + 1/2h_2(x_1 - x_2)^2)^2 - 2h_2R(x_1, x_1)R(x_2, x_2).$$

In the language of field extension, the result of Kowalevski may be stated by saying that y_3^2 , y_3L_3 , L_3^2 , k_1 , k_2 , ξ_1 , and ξ_2 belong to the extension F of field E of rational functions in $(x_1, x_2, h_1, h_2, c_1, c_2)$, whose elements are first-order polynomials in W with coefficients in E .

2.2. The syzygies.

The above property has important consequences on the form of the reduced equations. These equations do not belong either to E or to F, but the following property is true.

LEMMA 1. — *The product $Z_a Z_b$ of any pair of reduced equations is in F.*

This Lemma was partially known to Kowalevski, who actually computed the three products

$$(12) \quad -4\dot{x}_1\dot{x}_1 = R(x_1, x_1) + (x_1 - x_2)^2 \xi_1$$

$$(13) \quad 4\dot{x}_1\dot{x}_2 = R(x_1, x_2)$$

$$(14) \quad -4\dot{x}_2\dot{x}_2 = R(x_2, x_2) + (x_1 - x_2)^2 \xi_2.$$

Unfortunately she missed the other two

$$(15) \quad \dot{x}_1\dot{x}'_1 = R(x_1, x_2)\xi_1$$

$$(16) \quad \dot{x}_2\dot{x}'_2 = R(x_1, x_2)\xi_2$$

which are needed to form the syzygies. These syzygies arise as *compatibility conditions* of the following two linear systems. Consider the equations

$$\begin{pmatrix} \dot{x}_1\dot{x}_1 \\ \dot{x}_1\dot{x}_2 \\ \dot{x}_1\dot{x}'_1 \end{pmatrix} = \begin{pmatrix} -1/4R(x_1, x_1) & -1/4(x_1 - x_2)^2 & 0 \\ 1/4R(x_1, x_2) & 0 & 0 \\ 0 & R(x_1, x_2) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \xi_1 \\ \xi_2 \end{pmatrix},$$

and also the equations

$$\begin{pmatrix} \dot{x}_2\dot{x}_1 \\ \dot{x}_2\dot{x}_2 \\ \dot{x}_2\dot{x}'_2 \end{pmatrix} = \begin{pmatrix} 1/4R(x_1, x_1) & 0 & 0 \\ -1/4R(x_2, x_2) & 0 & -1/4(x_1 - x_2)^2 \\ 0 & 0 & R(x_1, x_2) \end{pmatrix} \begin{pmatrix} 1 \\ \xi_1 \\ \xi_2 \end{pmatrix}.$$

The matrices of the coefficients are singular and therefore have a kernel. Let us call $(U, V, -1)$ and $(W, Z, -1)$ two normalized row vectors spanning the left kernel of the first and of the second matrix respectively. For the existence of the solution it is then required that

$$(17) \quad \dot{x}_1\dot{x}_1U + \dot{x}_1\dot{x}_2V - \dot{x}_1\dot{x}'_1 = 0$$

$$(18) \quad \dot{x}_2\dot{x}_1W + \dot{x}_2\dot{x}_2Z - \dot{x}_2\dot{x}'_2 = 0.$$

These equations are the syzygies among the equations previously referred to. They exist since the nonlinearities of the equations of motion are sufficiently mild to entail the validity of the *Lemma*. The point is now to exploit these syzygies to find the separation coordinates.

2.3. The endomorphism K .

The link is provided by the concept of Kowalevski's endomorphism. Simplifying the first syzygy by \dot{x}_1 , and the second by \dot{x}_2 , one easily transforms the syzygies into a system of two linear equations

$$(19) \quad \dot{x}'_1 = U\dot{x}_1 + V\dot{x}_2$$

$$(20) \quad \dot{x}'_2 = W\dot{x}_1 + Z\dot{x}_2,$$

which may be read as a recursion relation between the reduced equations. The coefficients U, V, W, Z are given by

$$(21) \quad U = -4 \frac{R(x_1, x_2)}{(x_1 - x_2)^2}$$

$$(22) \quad V = -4 \frac{R(x_1, x_1)}{(x_1 - x_2)^2}$$

$$(23) \quad W = -4 \frac{R(x_2, x_2)}{(x_1 - x_2)^2}$$

$$(24) \quad Z = -4 \frac{R(x_1, x_2)}{(x_1 - x_2)^2}.$$

These coefficients are used to define the linear endomorphism K according to

$$(25) \quad K \frac{\partial}{\partial x_1} = U \frac{\partial}{\partial x_1} + W \frac{\partial}{\partial x_2}$$

$$(26) \quad K \frac{\partial}{\partial x_2} = V \frac{\partial}{\partial x_1} + Z \frac{\partial}{\partial x_2}.$$

By construction it automatically verifies the recursion relation $KX_{h_1} = X_{h_2}$. Furthermore its trace and determinant are given by

$$(27) \quad r = -8 \frac{R(x_1, x_2)}{(x_1 - x_2)^2}$$

$$(28) \quad s = -16 \frac{R_1(x_1, x_2)}{(x_1 - x_2)^2}.$$

So it easy to check if it has the property of Kowalevski. The answer is affirmative, and the conclusion is that K is the Kowalevski's endomorphism of the Kowalevski's top. The remarkable fact is that this endomorphism has been provided by the equations of motion themselves, through the analysis of their nonlinearities. This result is encouraging and means that we are on the good track . Nevertheless one has still to find the functions f_2 and f_4 , which allows to deform K into the endomorphism L which has the properties required by the *Algorithm*. The solution of this problem is made easy by the property of *homogeneity* exhibited by the equations.

2.4. The endomorphism L .

Among the solutions of the pair of conditions

$$\begin{aligned} \{f_4, h_1\} &= \{f_2, h_2\} \\ \{f_4, h_2\} &= r\{f_2, h_2\} - s\{f_2, h_1\}. \end{aligned}$$

one must find two functions f_2 and f_4 such that the related functions

$$(29) \quad p = -f_2 r + 2f_4$$

$$(30) \quad q = -f_2^2 s + f_2 f_4 r - f_4^2$$

obey the last separability condition

$$(31) \quad \{p, q\} = 0.$$

The form of the conditions on f_2 and f_4 suggests to take f_2 constant, and f_4 function of the integrals of motion. To fix the latter function, let us notice that the integrals of motion, the Casimir's functions, the equations of motion, and the functions r and s are all *homogeneous functions* of the mechanical coordinates $(L_1, L_2, L_3, y_1, y_2, y_3)$, of different degrees if one gives degree 1 to the components of the angular momentum and degree 2 to the components of the weight. It is then natural to seek for a function f_4 homogeneous. Since f_2 is of order zero, f_4 must be homogeneous of degree 2. Notice that h_1 is of degree 2, c_2 is of degree 3, and h_2 and c_1 are of degree 4. Thus the number of ways of building a function of the integrals of motion and of the Casimir's functions which is of degree 2 is limited. The simplest choice is to consider a constant multiple of the energy. These considerations suggest to set

$$(32) \quad f_2 = a, \quad a \in \mathbb{R}$$

$$(33) \quad f_4 = b h_1, \quad b \in \mathbb{R}.$$

The fifth and last condition splits then into a system of 43 algebraic equations for the indeterminate coefficients a and b , admitting the solutions $(a = 0, b \in \mathbb{R})$ and $(a \in \mathbb{R}, b = 4a)$. The first solution must be rejected since f_2 cannot vanish. Therefore, one is left with the one parameter family of matrices

$$L = f_2 K + (f_4 - f_2 r)$$

which are all admissible. One is thus allowed to claim that the roots of the characteristic polynomial

$$\det(\nu - L) = (\nu - f_4)^2 + f_2 r(\nu - f_4) + f_2^2 s$$

are separation coordinates of the Kowalevski's top. Inserting the values of (f_2, f_4, r, s) and setting $\nu = c\sigma$ leads to the equation

$$(34) \quad (x_1 - x_2)^2 \left(\sigma - \frac{4a}{c} h_1 \right)^2 - \frac{8a}{c} R(x_1, x_2) \left(\sigma - \frac{4a}{c} h_1 \right) - \left(\frac{4a}{c} \right)^2 R_1(x_1, x_2) = 0.$$

The choice $c = 8a$ reduces this equation to the final form

$$(35) \quad (x_1 - x_2)^2 \left(\sigma - \frac{h_1}{2} \right)^2 - R(x_1, x_2) \left(\sigma - \frac{h_1}{2} \right) - \frac{1}{4} R_1(x_1, x_2) = 0,$$

which coincides with that of Kowalevski up to the shift $\sigma = \lambda + \frac{1}{3}h_1$ of the independent variable. This means that the Kowalevski's coordinates (λ_1, λ_2) are related to the separation coordinates (σ_1, σ_2) by the formulas

$$(36) \quad \sigma_1 = \lambda_1 + 1/3h_1$$

$$(37) \quad \sigma_2 = \lambda_2 + 1/3h_2.$$

This remark ends our discussion. It simply remains to notice that the method of syzygies is not the unique method allowing to solve the Kowalevski's problem (see [2]), but it has a particular charm. It seems to bring forth the separation coordinates from the depth of the equations of motion.

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