Richard Sacksteder
Art J. Schwartz

Limit sets of foliations

Annales de l’institut Fourier, tome 15, no 2 (1965), p. 201-213

<http://www.numdam.org/item?id=AIF_1965__15_2_201_0>
LIMIT SETS OF FOLIATIONS
by Richard SACKSTEDER and Arthur J. SCHWARTZ

1. Introduction.

Let $V$ be an $n$-manifold with a foliated structure of co-dimension one. A leaf of the foliation is called proper if its topology as an $(n-1)$-manifold agrees with its topology as a subset of $V$. One type of theorem which is proved here asserts that proper leaves behave much like compact leaves with respect to certain stability properties. For example, Theorem 1 can be viewed as an extension of a theorem of Reeb on the behavior of leaves in a neighborhood of compact leaf to non-compact, proper leaves. Other theorems show that leaves whose holonomy is finite in a certain sense have properties like those of non-periodic solutions of differential equations on 2-manifolds. For instance, Theorems 2, 3, 4 are closely related to results concerning differential equations on 2-manifolds contained implicitly, at least, in the papers of Haas [1], [2] (however, see [8]).

Some examples are given in Section 9 which illustrate how the hypotheses of Theorem 1-4 can be satisfied. Finally, in Section 10 some sharper results, which hold only if $n = 2$, are obtained.

2. Statements of the theorems.

In all of the theorems stated in this section, $V$ denotes a (connected, paracompact, Hausdorff) $n$-manifold ($n \geq 2$) with a foliated structure of co-dimension one. If $x$ is a point of $V$, the leaf containing $x$ is denoted by $F_x$, its closure by $C_x$, $D_x$, $\ldots$.
called the limit set of $x$, is defined to be the intersection of the closures of the sets $F^a_x - K$ where $K$ is any compact subset of $F_x$. The concept of a locally infinite holonomy pseudogroup, which appears in the conclusion of the theorems is defined in Section 3. Its meaning is elucidated by Propositions 3.4 and 3.5.

**Theorem 1.** — Let $x$ be a point of $V$ such that $F_x$ is proper, $C_x$ is compact, and $F_x \subset D_y$ for some $y$ in $V$. Then $F_x$ has a locally infinite holonomy pseudogroup.

**Theorem 2.** — Let $y$ be a point of $V$ such that $C_y$ is compact, $C_y \neq V$, and $C_y$ contains an open subset of $V$. Then there is a leaf in the boundary of $C_y$ with a locally infinite holonomy pseudogroup.

**Theorem 3.** — Let $V$ contain a dense leaf, $F_y$, and a non-dense leaf $F_x$ such that $C_x$ is compact. Then $C_x$ contains a leaf with a locally infinite holonomy pseudogroup.

**Theorem 4.** — Let $S^1 \subset V$ be a closed curve intersecting each leaf transversally. Suppose that the subset $A$ of $V$ consisting of the leaves which intersect $S^1$ is relatively compact. Then either $A = V$ or there is a leaf, $F_x$, in the boundary of $A$ which has a locally infinite holonomy pseudogroup.

It will be clear from the proofs that very little smoothness need be assumed in these theorems in contrast to those obtained in [7] and [8]. A sufficient condition is that there exist a continuous vector field on $V$ which is never tangent to a leaf and is such that the solution curve through each point is unique. We shall always suppose that such a vector field has been given. The solution curve through a point $x$ of $V$ will be denoted by $T_x$ and called the transversal through $x$.

3. The holonomy group and pseudogroup.

The material in this section is, to a large extent, well known, but we include it here in order to make our presentation as self contained as possible. For the sake of simplicity we consider
only foliations of co-dimension one and we assume that there is a metric defined on each transversal curve.

We introduce first a notion that may be roughly described as lifting a path from one leaf to another, continuously, along transversals.

**Definition.** — Let \( P : [a, b] \times [0, S] \to V \) be a continuous function satisfying for each \((t, s)\):

1) \( P(t, s) \) is in \( F_{P(0, S)} \cap T_{P(0, 0)} \).
2) \( s \to P(0, s) \) is an isometry.

Thus the maps \( g_s \), defined by \( g_s(t) = P(t, s) \) determine a family of paths, each contained in a single leaf, with initial points in \( T_{P(0, 0)} \). On the other hand, the maps \( P_t \), defined by \( P_t(s) = P(t, s) \) determine a family of transversals with initial points in \( g_0([a, b]) \).

We call \( P \) a projector. \( P \) is said to project \( g_0 \) onto \( g_s \), and \( g_s \) is called a projection of \( g_0 \) by \( P \) along \( P_0([0, S]) \).

**Proposition 3.1.** — If \( P, Q : [a, b] \times [0, S] \to V \) are projectors and \( P(t, s) = Q(t, s) \) for \((t, s)\) in \( 0 \times [0, S] \cup [a, b] \times 0 \), then \( P = Q \).

The proof is straightforward.

Thus, if \( g_0 : [a, b] \to V \) is a path contained in a single leaf, and \( P_0 \) is a transversal interval with \( g_0(a) \) as one end point, there exists at most one projection of \( g_0 \) along \( P_0 \).

**Proposition 3.2.** — Suppose that \( 0 < \lambda < 1 \). — If \( g : [a, b] \to V \) is a path contained in a single leaf, there exists a projector \( P : [a, b] \times [0, S] \to V \), with \( S > 0 \), such that \( g(t) = P(t, \lambda S) \).

The proof is straightforward.

**Proposition 3.3.** — Let \( g : [a, b] \to V \) be a path contained in a single leaf such that \( g(t) \) is the midpoint of a transversal interval of length \( 2L > 0 \), for each \( t \) in \([a, b]\). Let \( J \) be a transversal interval with length \( M < L \) with one end point at \( g(a) \). Let \( c \) be the largest number in \([a, b]\) such that there exists a projector \( P : [a, c] \times [0, M] \to V \) satisfying.

1) \( P(t, 0) = g(t) \).
2) $P(0 \times [0, M]) = J.$
3) $P(t \times [0, M])$ has length less than or equal to $L.$ Then either $c = b$ or $P(c \times [0, M])$ is of length $L.$

Proof. — If $c < b$ and $P(c \times [0, M]) < L$ it would be possible to extend $P$ to $P: [a, c + \varepsilon] \times [0, M] \to V$ for some positive $\varepsilon.$

Let $x$ be a point of $V$ and $g: [0, 1] \to V$ a path in $F_x$ such that $g(0) = g(1) = x.$ Let $P: [0, 1] \times [0, S] \to V$ be the projector satisfying the conclusion of Proposition 3.2. Define the map $H_g: [-\lambda S, (1 - \lambda)S] \to \mathbb{R}^3$ by $H_g(u) = h(P_1(u + \lambda S)),$ where $P_1(s) = P(1, s)$ and $h: P_1([0, S]) \to \mathbb{R}^3$ is the isometry such that $h(P(0, s)) = s - \lambda S$ for $s$ near $\lambda S.$ Then $H_g$ is a diffeomorphism of neighborhoods of zero in $\mathbb{R}^3.$ The pseudogroup generated (cf. [7] sect. 3) by the diffeomorphisms $H_g$ for any path $g$ in $F_x$ with $g(0) = g(1) = x$ is called the holonomy pseudogroup of $F_x$ at $x.$ This pseudogroup depends, of course, on the metric on $T_x,$ but only up to an inner automorphism. The set of germs of elements of the holonomy pseudogroup at $x$ forms a group called the holonomy group of $F_x$ at $x.$ It depends similarly on the metric on $T_x.$ The element of the holonomy group of $F_x$ corresponding to a path $g$ depends only on the homotopy class of $g$ in $F_x,$ hence, there is a homomorphism from the fundamental group of $F_x$ onto the holonomy group.

We say that a leaf $F_x$ has a locally infinite holonomy pseudogroup if for every neighborhood $N$ of $x$ on $T_x$ there is an orientation-preserving element of the holonomy pseudogroup of $F_x$ which is not a restriction of the identity and whose domain corresponds to a subset of $N.$ We do not know if a leaf can have a locally infinite holonomy pseudogroup if its holonomy group is finite; however, the following propositions show a relationship between a locally infinite holonomy pseudogroup and the holonomy group.

Proposition 3.4. — Let $F_x$ have a locally infinite holonomy pseudogroup. Then given any neighborhood $N$ of $x,$ there is a leaf which intersects $N$ and has an infinite holonomy group.

Proof. — Let $S > 0$ be arbitrarily small and suppose that $g: [0, 1] \to F_x$ satisfies $g(0) = g(1) = x$ and that the element
The hypothesis implies that $g$ and $H_g$ with these properties exist for any $S > 0$. Let $H_g^p$ denote $p$-fold composition and define $c = \lim_{p \to \infty} H_g^p(t) \geq 0$ and $y = P_0(c)$. One easily verifies that $H_g(c) = c$ and if $c < u < t$, $c < H_g(u) < u < t$. From this it easily follows that $H_g$ represents a germ of infinite order in the holonomy group of $F_x$. This proves Proposition 3.4.

Another implication of the condition that a leaf have a locally infinite holonomy group is given by:

**Proposition 3.5.** Suppose that $n = 2$ and $F_x$ is homeomorphic to $\mathbb{R}^1$. Then the holonomy pseudogroup of $F_x$ consists entirely of maps which are restrictions of the identity map. In particular, it is not locally infinite.

The idea of the proof is simple, but the details are tedious, hence we only give a rough indication of the proof. Let $g: [0, 1] \to F_x$ be a path with $g(0) = g(1) = x$ and $P$ a projector as in Proposition 3.2. If the image of $P$ lies entirely within a distinguished neighborhood the conclusion is almost obvious. In general, the image of $P$ is covered by a finite number of distinguished neighborhoods and the proposition can be proved by induction on the number of them.

### 4. The main Lemma.

The following lemma is used in the proof of Theorems 1-4.

**Lemma 4.1.** Let $x$ in $V$ be such that $C_x$ is compact. Let $P_i: [0, b_i] \times [0, S_i] \to V$ be a sequence of projectors such that $P_i(0, 0) = x$ for all $i$, $P_i(0 \times (0, S_i)) \cap F_x = \emptyset$, $S_i$ tends to $0$, and the length of $P_i(b_i \times [0, S_i])$ tends to $L > 0$.

Then $F_x$ has a locally infinite holonomy pseudogroup.

**Proof.** Since $C_x$ is compact, we may assume that $P_i(b_i, 0) \to z$ and $P(b_i \times [0, S_i]) \to T$, a transversal interval with end points $z$ and $y$. Let $N$ be a distinguished neighborhood
containing $T$ such that $N = \varphi\left((-\varepsilon, L + \varepsilon) \times (-\varepsilon, \varepsilon)^{n-1}\right)$ for some $\varepsilon$, $0 < \varepsilon < \frac{L}{3}$ where $\varphi$ satisfies.

1) $\varphi(0, \ldots, 0) = z$.
2) $\varphi(x_1, x_2, \ldots, x_n)$ is in $T_{\varphi(0, x_2, \ldots, x_n)}$.
3) $\varphi(x_1, x_2, \ldots, x_n)$ is in the plaque through $\varphi(x_1, 0, \ldots, 0)$.
4) $\varphi(L, 0, \ldots, 0) = y$.

For sufficiently large $K$, we may modify $P_i$ without changing $P_i$ at any point whose image is outside $N$ so that $P_i(b_i, [0, S_i])$ belongs to $\varphi((-\varepsilon, L + \varepsilon), 0, \ldots, 0)$ if $i \geq K$.

Now observe that for each $i$ and $t$, $P_i(t, 0)$ does not belong to $F_x$ unless $s = 0$, since $P_i(0 \times [0, S_i]) \cap F_x = \emptyset$ by assumption. Using this fact we shall show that there exists $M$ sufficiently large so that $P_i(b_i, 0) = z$ if $i \geq M$. In fact, choose $M > K$ such that for $i > M$, $P_i(b_i, 0)$ is in $\varphi((-\varepsilon, L + \varepsilon), 0, \ldots, 0)$. If for some $i, j > M$, $P_i(b_i, 0) = P_j(b_j, 0)$ and $P_i(b_i, 0)$ is in $T_{\varphi((-\varepsilon, L + \varepsilon), 0, \ldots, 0)}$, this contradicts the assumption that $P_i((0 \times [0, S_i]) \cap F_x = \emptyset$; thus $P_i(b_i, 0) = P_j(b_j, 0) = z$ if $i, j \geq M$. Thus $z$ is in $F_x$.

Choose $s_0$ so that $P_M(b_m, s_0) = \varphi\left(\frac{1}{3}L, 0, \ldots, 0\right)$.

Choose $R > M$ so that $P_R(0, S_R)$ is in $P_M(0, [0, S_0])$.

Choose $s_1$ so that $P_R(b_r, s_1) = \varphi\left(\frac{1}{2}L, 0, \ldots, 0\right)$.

Let $g : [0, b_r + b_m] \rightarrow V$ be defined by

$$g(t) = \begin{cases} 
P_M(t, 0) & \text{if } 0 \leq t \leq b_m, \\
P_R(b_m + b_r - t, 0) & \text{if } b_m \leq t \leq b_r + b_m.
\end{cases}$$

Let $P : [0, b_r + b_m] \times [0, s_0] \rightarrow V$ be the projector such that $P(0, s) = P_M(0, s)$ and $P(t, 0) = g(t)$, which exists by Proposition 3.3. According to Proposition 3.1, $P(t, s) = P_M(t, s)$
for $t \leq b_M$ and $P(t, s) = P_R(b_M + b_R - t, h(s))$ for $t \geq b_M$ where $P_R(b_R, h(s)) = P_M(b_M, s)$ defines $h$.

Thus $H_x(s_0) = s_1 < s_0$, which proves that the holonomy pseudogroup of $F_x$ contains an orientation-preserving element which is not a restriction of the identity. Such an element can be chosen to have an arbitrarily small domain by the condition $S_i \to 0$. This proves the Lemma.

5. Proof of Theorem 1.

Let $L > 0$ be so small that no point on the subinterval of $T_x$ of length $2L$ centered at $x$ is in $F_x$ other than $x$ itself. Such an $L$ exists because $F_x$ is proper. Moreover, it can be assumed that $L$ is so small that each point of $C_x$ is at the center of a subinterval of an orthogonal trajectory of length $2L$.

Let $x_1, x_2, \ldots$ be a sequence of points in $F_x \cap T_x$ such that the distance between $x_i$ and $x$ along $T_x$ is less than $L$, and decreases to zero in a strictly monotone fashion.

Let $g_i : [0, 1] \to F_x (i = 1, 2, \ldots)$ be a sequence of paths satisfying $g_i(0) = x_i, g_i(1) = x_{i+1}$. According to Proposition 3.3, there are projectors $P_1, P_2, \ldots$ such that:

(5.1) $P_i : [0, t_i] \times [0, s_i] \to V, \quad (0 \leq t_i < 1),$

(5.2) $P_i(t, s_i) = g_i(t) \quad \text{for} \quad 0 \leq t \leq t_i,$

(5.3) $P_i(0, 0) = x,$

(5.4) $d_i(t) < L \quad \text{for} \quad 0 \leq t < t_i,$

(5.5) $d_i(t_i) = L \quad \text{if} \quad t_i < 1,$

where $d_i(t)$ denotes the length of the transversal $P_i(t \times [0, s_i]).$

The following two cases exhaust all possibilities:

1) For infinitely many $i, t_i = 1$; 2) for all but a finite number of $i, t_i < 1$. In the case 1), it is clear that, since $0 < s_{i+1} < s_i$, there is an element in holonomy pseudogroup of $F_x$ at $x$ corresponding to the path $P_i(t, 0) (0 \leq t \leq 1)$ which is orientation preserving and not a restriction of the identity map and since $x_i \to x$ it can be chosen to have an arbitrarily small domain. Thus $F_x$ has a locally infinite holonomy pseudogroup and Theorem 1 is valid in this case. In the case 2), the hypotheses of Lemma 4.1 apply to the sequence $P_1, P_2, \ldots$ Thus Theorem 1 is proved.
6. Proof of Theorem 2.

Let \( x \) be a point on the boundary of the interior of \( C_y \) such that for some \( L > 0 \), \( T_x \) contains an open interval of length \( 2L \) centered at \( x \) and such that one half of it, say \( I_1 \), is contained in \( C_y \), while the other half, \( I_2 \), contains points of \( V - C_y \) arbitrarily close to \( x \). Such a point is easily seen to exist. It can be supposed that \( L \) is so small that each point of \( C_x \) is the midpoint of a subinterval of an orthogonal trajectory of length \( 2L \). Let \( x_1, x_2, \ldots \) be a sequence of points in \( I_1 \cap F_y \) such that \( x_{i+1} \) lies between \( x_i \) and \( x \) and \( x_i \to x \). Let \( g_i : [0, 1] \to F_y \) be a sequence of curves satisfying \( g_i(0) = x_i \), \( g_i(1) = x_{i+1} \neq x_i \). Proposition 3.3 implies that there exists a sequence \( P_1, P_2, \ldots \) of projectors satisfying (5.1)-(5.5). One considers two cases exactly as in the proof of Theorem 1, and the treatment of case 1) here is exactly the same as for Theorem 1. In case 2) Lemma 4.1 applies as for Theorem 1, with the minor difference that here \( P_i(0 \times (0, s_i)) \cap F_x = \emptyset \) holds because \( I_1 \subset C_y \) implies that \( I_1 \cap F_x = \{ x \} \) since every point of \( F_x \) is on the boundary of \( C_y \). This proves Theorem 2.

7. Proof of Theorem 3.

Let \( x \) be a point of \( C_x \) which is such that for some \( L > 0 \), a subinterval \( I_1 \) of \( T_x \) with one endpoint at \( x \) contains no points of \( F_x \) other than \( x \). It can also be supposed that \( L \) is so small that every point of \( C_x \) is the midpoint of a subinterval of an orthogonal trajectory of length \( 2L \), and there are points of \( V \) at a distance greater than \( L \) from \( C_x \). Let \( x_1, x_2, \ldots \) be a sequence of points of \( I_1 \cap F_y \) which approach \( x \). Let \( g_i : [0, 1] \to F_y \) be a sequence of curves such that \( g_i(0) = x_i \) and \( g_i(1) \) is a point of \( V \) at a distance greater than \( L \) from \( C_x \). Such a sequence exists by the choice of \( L \) and the assumption that \( C_y = V \). Let \( P_1, P_2, \ldots \) be a corresponding sequence of projectors which exists and satisfies (5.1)-(5.5) by Proposition 3.3. Note that the condition on \( g_i(1) \) implies that \( t_i < 1 \). Now applying Lemma 4.1 gives the desired conclusion.

If $A \neq V$, there is a leaf $F_x$ in the boundary of the interior of $A$ such that $C_x$ is at a distance greater than $L > 0$ from $S^1$ and an open subinterval $I_1$ of $T_x$ of length $L$ with one end point at $x$ is in the interior of $A$. Again, it can be supposed that $L$ is so small that each point of $C_x$ is the midpoint of a subinterval of an orthogonal trajectory of length $2L$. Let $x_1, x_2, \ldots$ be a sequence of points of $A \cap I_1$ which approach $x$ and are at a distance less than $L$ from $x$. Denote the leaf containing $x_i$ by $F_i$. Let $g_i : [0, 1] \to F_i$ be a sequence of curves such that $g_i(0) = x_i$ and $g_i(1) \in S^1$. Let $P_1, P_2$ be the corresponding sequence of projectors satisfying (5.1)-(5.5) which exists by Proposition 3.3. The condition on $g_i(1)$ implies that $t_i < 1$, and $I_1 \subset \text{Int } A$ implies $P_i(0 \times [0, s_i]) \cap F_x = \emptyset$. Therefore, Lemma 4.1 again gives the desired conclusion.


The examples 1-4 below illustrate, respectively, Theorems 1-4. Similar examples have been given in articles by G. Reeb in the Annales de l'Institut Fourier, vol. 6 and vol. 11.

Example 1. — It is very easy to construct examples in which a compact (hence proper) leaf is in the limit set of another leaf. It is a little more difficult to construct an example in which a non-compact proper leaf is in the limit set of another leaf. To construct such an example, let $V = S^1 \times S^1 \times S^1$. Let $(x, y, z)$ denote a typical point of $V$, where $x, y,$ and $z$ are real numbers mod $\pi$. Let $f : (0, \pi) \to \mathbb{R}$ be a $C^\infty$ diffeomorphism. Then $\omega = df(x) + y + \sin^2(f(x) + y)dz$ is a completely integrable 1-form on $V_0 = (0, \pi) \times S^1 \times S^1$. The desired foliation of $V$ is obtained by completing this foliation of $V_0 \subset V$ by adding the leaf $F_0 = \{0\} \times S^1 \times S^1$. Then one easily sees that the leaf passing through any point $(x, y, z) \in V_0$ such that $f(x) + y = 0 \mod \pi$ is non-compact, proper, and in the limit set of every leaf through a point $(x', y', z')$ in $V_0$ such that $f(x') + y' \neq 0 \mod \pi$. The leaf $F_0$ is in the limit set of every other leaf.
It is perhaps also worth noting that $V_0$ can be imbedded in $S^3$ in such a way that its complement consists of two disjoint solid tori, which can be given a foliation which fits together with the foliation of $V_0$ to give a foliation of $S^3$.

**Example 2.** Let $V_0$ be as in example 1 and let the form $\omega = dx - (\sin x)(dy + \alpha dz)$, where $\alpha$ is irrational define a foliation of $V_0$. As above, $V_0$ can be imbedded in $S^3$ in such a way that its foliation can be extended to a foliation of all of $S^3$. The only leaves of this foliation with non-trivial holonomy groups will be two disjoint tori $S^1 \times S^1$ which are the boundary of $V_0$ as a subset of $S^3$. One easily verifies that the closure of every leaf containing a point of $V_0$ is the closure of $V_0$ in $S^3$. Thus there are leaves which are dense in an open subset of $S^3$, but not in $S^3$ itself.

**Example 3.** Let $V_0$ be foliated as in example 2, but imbed it in $V = S^1 \times S^1 \times S^1$, as in example 1. Complete the foliation induced on $V_0 \subset V$ be adding the leaf $F_0 = \{0\} \times S^1 \times S^1$. This leaf will be compact, hence nowhere dense, but every other leaf will be dense in $V$. The leaf $F_0$ is the only leaf with a non-trivial holonomy group.

**Example 4.** It is easy to see that the Reeb foliation of $S^3$, cf. [3], [7], admits a transversal curve $S^1$ which does not intersect every leaf. The torus which bounds the set of leaves which intersect $S^1$ has, of course, a non-trivial holonomy group.

10. Application to differential equations on surfaces.

Let $V$ be a surface and $\alpha : \mathbb{R}^1 \times V \rightarrow V$ a flow on $V$ determined by a continuous vector field $X$ on $V$. If $V_0 \subset V$ is the submanifold of $V$ consisting of those points where $X$ does not vanish, $X$ determines a foliation of $V_0$ in which the leaves are the integral curves of $X$. If $x$ is a point of $V_0$, we define the sets $A_x = \bigcap_{N \leq 0} \text{closure} \{\alpha(t, x) : t \leq N\}$, $\Omega_x = \bigcap_{N \geq 0} \text{closure} \{\alpha(t, x) : t \geq N\}$ called, respectively, the *alpha* and *omega* limit sets of $x$. We use the notation, $D_x$, $C_x$, $F_x$, etc., for the foliated manifold $V_0$ just as in previous sections.
The main goal of this section and the following one is to prove:

**Theorem 5.** — Let \( V \) be a surface (\( = \) 2-manifold) and \( \alpha : \mathbb{R}^1 \times V \rightarrow V \) a flow generated by a continuous vector field, \( X \), on \( V \). Let \( V_0 \subset V \) be the foliated manifold consisting of all points of \( V \) where \( X \) does not vanish. Suppose that \( x \) is in \( V_0 \) and \( C_x \) is compact. Then every leaf in \( \Lambda_x \) (or \( \Omega_x \)) is everywhere dense in \( \Lambda_x \) (or \( \Omega_x \)) and either

a) \( C_x = \Lambda_x = \Omega_x = V \),

b) \( C_x \) is nowhere dense.

This can be viewed as a strengthening of Theorem 1 for the case \( n = 2 \) in the sense that, in the terminology of topological dynamics, Theorem 1 (or more precisely, Lemma 10.4) is easily seen to imply that \( \Lambda_x \) and \( \Omega_x \) are Poisson stable, while Theorem 5 asserts the stronger condition that these sets are minimal.

We need to recall some preliminary results and definitions before proceeding to the proof of Theorem 5. An orbit (\( = \) leaf) \( F_x \subset V_0 \subset V \) is called periodic if \( \alpha(t, x) = x \) for some \( t > 0 \) and \( x \) is then called a periodic point. \( F_x \) is compact if and only if \( F_x \) is periodic, in which case \( F_x \) is proper, \( \Delta_x = \emptyset \), and \( F_x = \Omega_x = \Lambda_x \). If \( F_x \) is not periodic, \( \Delta_x = \Lambda_x \cup \Omega_x \), \( F_x \) is homeomorphic to \( \mathbb{R}^1 \), and if \( C_x \) is compact, both \( \Lambda_x \) and \( \Omega_x \) are non-empty.

**Lemma 10.1.** — If \( \Lambda_x \) (resp. \( \Omega_x \)) contains a periodic point, \( y \), then \( \Lambda_x = F_y \) (resp. \( \Omega_x = F_y \)).

*Proof.* — See [1].

The next two lemmas are partially due to Haas [1].

**Lemma 10.2.** — If for some \( x \) in \( V_0 \), \( C_x = V_0 \), is compact, then \( V = V_0 = S^1 \times S^1 \) and every leaf is dense in \( V \).

*Proof.* — Since \( F_x \) is dense in \( V = V_0 \), \( F_x \) is not proper and \( F_x \subset C_x = \Lambda_x \cup \Omega_x \).

We now claim that \( V \) contains no periodic leaf. Suppose, on the contrary, that \( V \) contains a periodic leaf, \( F_y \). Then \( F_y \subset \Lambda_x \) or \( F_y \subset \Omega_x \). Say \( F_y \subset \Lambda_x \), then according to Lemma 10.1, \( F_y = \Lambda_x \). But then \( \Omega_x = V \), \( F_y \subset \Omega_x \) and \( F_y = \Omega_x \) which yields \( V = F_y \) which is absurd.
Thus every leaf has a trivial holonomy group and it follows from Theorem 4 and Proposition 3.4 that every leaf is dense.

Now, $C_x = V$ is a compact surface. It carries a non-vanishing vector field and therefore the Euler characteristic of $V$ must vanish. Thus it follows that $V$ must be homeomorphic to $S^1 \times S^1$ or a Klein bottle. $V$ cannot be a Klein bottle for, according to Kneser [4], $V$ would then contain a periodic leaf.

**Lemma 10.3.** — Suppose that for some $x$ in $V_0$, $C_x$ is compact. Then either $V$ is homeomorphic to $S^1 \times S^1$ and each leaf is dense in $V$, or $C_x$ is nowhere dense.

**Proof.** — If $C_x = V$, the desired result follows from Lemma 10.2. Suppose that $C_x \neq V$, but $C_x$ contains an open subset of $V$. Then $F_x$ is not proper, hence $D_x = A_x \cup \Omega_x = C_x$. $D_x$ does not contain a periodic point $y$ in this case, because if $F_y \subset D_x$ is periodic, $F_y = A_x$ (or $\Omega_x$) by Lemma 10.1. But then $D_x = \Omega_x$ (or $A_x$) and $F_y = \Omega_x$ (or $A_x$), which is impossible if $D_x$ contains an open set. This shows that $D_x$ contains no periodic points. Now Theorem 2 implies that there is a leaf with a locally infinite holonomy pseudogroup in $C_x$. Proposition 3.5 then implies that this leaf is periodic. But this contradicts what has been proved and completes the proof of the lemma.

**Lemma 10.4.** — Let $C_x$ be compact. Then if $x \in \Omega_y$ (or $A_y$) then $x \in \Omega_x$.

The proof is clear if $x$ is a periodic point. Otherwise the argument goes almost exactly like proof of Theorem 1. We omit the details.

**11. Proof of Theorem 5.**

Lemma 10.3 gives all of the assertions of Theorem 5 except that « if $C_x$ is nowhere dense, every leaf in $A_x$ (or $\Omega_x$) is dense in $A_x$ (or $\Omega_x$) ». To verify this, suppose that $C_x$ is nowhere dense.

Let $y$ be a point in $\Omega_x$. Let $\varphi : [-\varepsilon, \varepsilon] \to V$ be an isometry into $T_y$ such that $\varphi(0) = y$, $\varphi(-\varepsilon)$ and $\varphi(\varepsilon)$ are not in $\Omega_x$. For each $z = \varphi(s)$, there exists $h_z > 0$ such that $\alpha(t, z)$ is not in $\varphi([-\varepsilon, \varepsilon])$ if $0 < t < h_z$. Thus according to Lemma 10.4
for each \( z \) in \( J = \varphi([-\varepsilon, \varepsilon]) \cap \Omega_z \), there exists a smallest positive number \( t_z \) such that \( \alpha(t_z, z) \) is in \( J \). Since \( J \) is compact there exists a number, \( M \), such that \( t_z \leq M \) for all \( z \) in \( J \), hence there is an \( M' \) such that if \( t \geq 0 \), \( \alpha([t, t + M'], x) \cap J \neq \emptyset \). Let \( N = \{ \alpha(t, w) : \omega \in \varphi([-\varepsilon, \varepsilon]), |t| < 1 \} \). Let \( d > 0 \) be such that \( q \) is in \( N \) if dist \((q, z) < d \) for some \( z \) in \( J \). Let \( \delta > 0 \) be sufficiently small so that dist \((\alpha(t, q), \alpha(t, p)) < d \) if

\[
0 \leq t \leq M' + 1,
\]

\( q \) is in \( C_x \), and dist \((q, p) \leq \delta \).

Now, for any \( p \) in \( \Omega_z \), there exists \( t_0 \geq 0 \) such that dist \((\alpha(t_0, x), p) < \delta \) and for some \( m \), \( 1 \leq m \leq M' + 1 \), \( \alpha(t_0 + m, x) \) is in \( J \). Then \( \alpha(m, p) \) is in \( N \), hence \( \alpha(m', p) \) is in \( \varphi((–\varepsilon, \varepsilon)) \) for some \( m' \), where \( 0 \leq m - 1 \leq m' \).

Thus for every \( p \in \Omega_z \), \( \alpha([0, \infty), p) \) intersects \( \varphi((–\varepsilon, \varepsilon)) \).

Since \( y \) and \( p \) were arbitrary points in \( \Omega_x \) and \( \varepsilon \) may be arbitrarily small, the theorem is proved.

**BIBLIOGRAPHY**


Manuscrit reçu en novembre 1964.

R. Sacksteder and A. J. Schwartz, Department of Mathematics, Columbia University in the City of New-York, New-York 27, N.Y. (U.S.A.)