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ON THE FINITE BLOCKING PROPERTY

by Thierry MONTEIL

Dedicated to the memory of Jean-Marie Exbrayat

Introduction.

When studying the motion of a point-mass in a polygonal billiard $\mathcal{P}$, we work on the phase space $X = \mathcal{P} \times \mathbb{S}^1$ suitably quotiented: we identify the points (p_1, θ_1) and (p_2, θ_2) if $p_1 = p_2$ is on the boundary of $\mathcal{P}$ and if the angles θ_1 and θ_2 are such that the Descartes law of reflection is respected (see Figure 1).

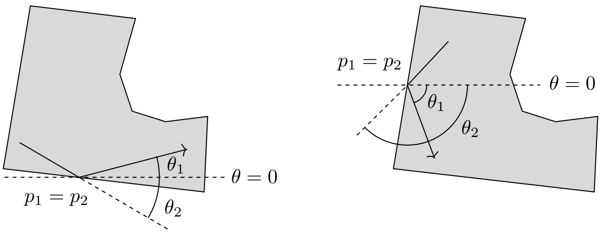

Figure 1. *The Descartes law*: the incidence angle equals the angle of reflection

The phase space enjoys also such a global decomposition in the study of the dynamics on a translation surface.

There are essentially two points of view depending on whether the variable is the first or the second projection:

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1) We can fix one (or a finite number of) particular direction: this corresponds for rational billiards to the study of the directional flow in a translation surface (we are interested by the ergodic properties depending on whether θ is a saddle connection direction or not) (see [KMS], [Ve], [MT], [Vo]). It is also useful for finding periodic trajectories in irrational billiards by starting perpendicularly to an edge (see [ST]). This point of view is the most studied.

2) We can also fix one (or a finite number of) point in $\mathcal{P}$ and look at which points we can reach when we let θ move. This class of problems is called "illumination problems". The first published question seems to appear in [Kl] (see [KW] for a more precise story). The first published result in this direction seems to be the paper of George W. Tokarski [To] who finds a polygon that is not illuminable from every point. Independently, Michael Boshernitzan [Bos] constructed such an example in a correspondence with Howard Masur.

We are interested here in an illumination problem called the finite blocking property.

A planar polygon (resp. translation surface) $\mathcal{P}$ is said to have the *finite blocking property* if for every pair (O,A) of points in $\mathcal{P}$, there exists a finite number of points $B_1,\dots,B_n$ (different from O and A) such that every billiard trajectory (resp. geodesic) from O to A meets one of the B_i's.

In this paper we will primarily focus on translation surfaces. The paper is organized as follows: in Section 1, we will give some definitions and prove that the finite blocking property is stable under branched covering, stable under the Zemljakov-Katok's construction, and stable under the action of $\mathrm{GL}(2,\mathbb{R})$. Section 2 is devoted to the study of Hiemer and Snurnikov's proof, leading to some comments about the finite property in the torus $\mathbb{R}^2/\mathbb{Z}^2$. In section 3, we prove a local sufficient condition for a translation surface to fail the finite blocking property (Lemma 1). The aim of the next two sections is to prove the following theorems:

Theorem 1. — *Let $n \geq 3$ be an integer. The following assertions are equivalent*:

• *The regular n-gon has the finite blocking property.*

• *The right-angled triangle with an angle equal to π/n has the finite blocking property.*

• $n \in \{3,4,6\}$.

THEOREM 2. — *A Veech surface has the finite blocking property if and only if it is a torus covering, branched over only one point.*

The last section is devoted to some other applications of Lemma 1, such as

PROPOSITION 10. — *Let a and b be two positive real numbers. Then the L-shaped surface $L(a,b)$ has the finite blocking property if and only if $(a,b) \in \mathbb{Q}^2$.*

THEOREM 3. — *In genus $g \geq 2$, the set of translation surfaces that fail the finite blocking property is dense in every stratum.*

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1. Definitions and first results.

1.1. Translation surfaces and geodesics.

A *translation surface* is a triple $(\mathcal{S}, \Sigma, \omega)$ such that $\mathcal{S}$ is a topological compact connected surface, Σ is a finite subset of $\mathcal{S}$ (whose elements are called *singularities*) and $\omega = (U_i, \phi_i)_{i\in I}$ is an atlas of $\mathcal{S} \setminus \Sigma$ (consistent with the topological structure on $\mathcal{S}$) such that the transition maps (i.e. the $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ for $(i,j) \in I^2$) are translations. This atlas gives to $\mathcal{S} \setminus \Sigma$ a Riemannian structure; we therefore have notions of length, angle, measure, geodesic... We assume moreover that $\mathcal{S}$ is the completion of $\mathcal{S} \setminus \Sigma$ for this metric. We will sometimes use the notation $(\mathcal{S}, \Sigma)$ or simply $\mathcal{S}$ to refer to $(\mathcal{S}, \Sigma, \omega)$. A singularity $\sigma \in \Sigma$ is said to be *removable* if there exists an atlas $\omega' \supset \omega$ such that $(\mathcal{S}, \Sigma \setminus \{\sigma\}, \omega')$ is a translation surface.

There are many conventions about what happens if a geodesic γ meets one singularity $\sigma \in \Sigma$. Some people want to stop γ here; some other people want to extend γ with a multi-geodesic path; other people want to extend γ after σ if and only if σ is an removable singularity. *The finite blocking property does not depend on the convention* since, for every pair (O, A) of points in $\mathcal{S}$, we can always add the set $\Sigma \setminus \{O, A\}$ (that is finite) to a blocking configuration. Therefore, if (S, Σ) is a translation surface, if $\mathcal{E}$ is a finite subset of $\mathcal{S}$ that we want to add in Σ as removable singularities, then (S,Σ) has the finite blocking property if and only if $(S,\Sigma\cup\mathcal{E})$ has it.

1.2. Branched coverings.

A *branched covering* between two translation surfaces is a mapping $\pi\colon(\mathcal{S},\Sigma)\to(\mathcal{S}',\Sigma')$ that is a topological branched covering that locally preserves the translation structure.

Proposition 1. — *Let $\pi:(\mathcal{S},\Sigma)\to(\mathcal{S}',\Sigma')$ be a covering of translation surfaces branched on a finite set $\mathcal{R}'\subset\mathcal{S}'$. Then $\mathcal{S}$ has the finite blocking property if and only if $\mathcal{S}'$ has.*

Proof. — Direction $\Rightarrow$. Suppose that $\mathcal{S}$ has the finite blocking property. Let (O',A') be a pair of points in $\mathcal{S}'$. Let O be a point chosen in $\pi^{-1}(\{O'\})$. If $A\in\pi^{-1}(\{A'\})$, there exists a finite set $\mathcal{B}_A$ of points in $\mathcal{S}\setminus\{O,A\}$ such that every geodesic in $\mathcal{S}$ from O to A meets $\mathcal{B}_A$. Let

$$\mathcal{B}'\stackrel{\text{def}}{=}\Big(\bigcup_{A\in\pi^{-1}(\{A'\})}\pi(\mathcal{B}_A)\cup\mathcal{R}'\Big)\setminus\{O',A'\}.$$

Let $\gamma'\colon[a,b]\to\mathcal{S}'$ be a geodesic from O' to A'. Up to a restriction, we can suppose that $\gamma'(]a,b[)\cap\{O',A'\}=\emptyset$. Suppose by contradiction that $\gamma'([a,b])\cap\mathcal{B}'=\emptyset$. In particular, $\gamma'(]a,b[)\cap\mathcal{R}'=\emptyset$. So, γ' can be lifted to a geodesic $\gamma\colon[a,b]\to\mathcal{S}'$ from O to some $A\in\pi^{-1}(\{A'\})$ such that $\pi\circ\gamma=\gamma'$. Then, there exists $t\in]a,b[$ such that $\gamma(t)\in\mathcal{B}_A$. Hence $\gamma'(t)\in\mathcal{B}'$, leading to a contradiction. So, $\mathcal{B}'$ is a finite blocking configuration and $\mathcal{S}'$ has the finite blocking property.

Direction $\Leftarrow$. Suppose that $\mathcal{S}'$ has the finite blocking property. Let (O,A) be a pair of points in $\mathcal{S}$. Let $O'\stackrel{\text{def}}{=}\pi(O)$ and $A'\stackrel{\text{def}}{=}\pi(A)$. There exists a finite set $\mathcal{B}'\subset\mathcal{S}'\setminus\{O',A'\}$ such that every geodesic in $\mathcal{S}'$ from O' to A' meets $\mathcal{B}'$. Let

$$\mathcal{B}\stackrel{\text{def}}{=}\pi^{-1}(\mathcal{B}')\subset\mathcal{S}\setminus\{O,A\}.$$

Let $\gamma\colon[a,b]\to\mathcal{S}$ be a geodesic from O to A. γ can be pushed to a geodesic $\gamma'\stackrel{\text{def}}{=}\pi\circ\gamma\colon[a,b]\to\mathcal{S}'$ from O' to A'. Then, there exists $t\in\,]a,b[$ such that $\gamma'(t)\in\mathcal{B}'$. Hence $\gamma(t)\in\mathcal{B}$. So, $\mathcal{B}$ is a finite blocking configuration and $\mathcal{S}$ has the finite blocking property. □

1.3. Rational billiards versus translation surfaces.

Let $\mathcal{P}$ denote a polygon in $\mathbb{R}^2$, whose set of vertices is denoted by V. Let $\Gamma\subset\mathrm{O}(2,\mathbb{R})$ be the group generated by the linear parts of the reflections in the sides of $\mathcal{P}$. When Γ is finite, we say that $\mathcal{P}$ is a *rational polygonal billiard*. When $\mathcal{P}$ is simply connected, $\mathcal{P}$ is rational if and only if all the angles between edges are rational multiples of π.

A classical construction due to Zemljakov and Katok (see [ZK], [MT]) allows us to associate to each rational billiard $\mathcal{P}$ a translation surface $\mathrm{ZK}(\mathcal{P})$ as follows:

Let $(P_\gamma)_{\gamma\in\Gamma}$ be a family of $|\Gamma|$ disjoint copies of $\mathcal{P}$, each $P_\gamma = \gamma(\mathcal{P})$ being rotated by the element $\gamma \in \Gamma$. If $\gamma \in \Gamma$, if e is an edge of P_γ, let $\delta \in \Gamma$ be the linear part of the reflection in e; we identify $e \in P_\gamma$ with $\delta(e) \in P_{\delta\gamma}$. We set

$$\mathrm{ZK}(\mathcal{P}) \overset{\text{def}}{=\!=} \bigsqcup_{\gamma\in\Gamma} P_\gamma \ /\!\sim$$

where $\sim$ is the relation above. The translation structure of each $\mathring{P}_\gamma \in \mathbb{R}^2$ can be extended to an atlas of $\bigcup_{\gamma\in\Gamma} P_\gamma \setminus \gamma(V)$, that gives to $\mathrm{ZK}(\mathcal{P})$ a translation structure whose set of singularities is $\Sigma = \bigcup_{\gamma\in\Gamma} \gamma(V)$.

In other terms, $\mathcal{P}$ tiles $\mathrm{ZK}(\mathcal{P})$ which can be written as $\mathrm{ZK}(\mathcal{P}) = \bigcup_{\gamma\in\Gamma} \psi_\gamma(\mathcal{P})$ where the ψ_γ's are isometries. Let

$$\psi \overset{\text{def}}{=\!=} \left(\begin{array}{c} \mathrm{ZK}(\mathcal{P}) \longrightarrow \mathcal{P} \\ x \longmapsto (\psi_{\gamma|\psi_\gamma(\mathcal{P})})^{-1}(x) \quad \text{if } x \in \psi_\gamma(\mathcal{P}) \end{array} \right).$$

The map ψ is well defined since if x belongs to $\psi_\gamma(\mathcal{P}) \cap \psi_\delta(\mathcal{P})$, then $(\psi_{\gamma|\psi_\gamma(\mathcal{P})})^{-1}(x) = (\psi_{\delta|\psi_\delta(\mathcal{P})})^{-1}(x)$ (this is just the compatibility with $\sim$). Moreover, ψ is a piecewise isometry.

PROPOSITION 2. — *Let $\mathcal{P}$ be a rational polygonal billiard. Then $\mathrm{ZK}(\mathcal{P})$ has the finite blocking property if and only if $\mathcal{P}$ has.*

Proof. — It is very similar to the proof given in Subsection 1.2 (ψ plays the role of π).

Indeed, for the direction $\Rightarrow$, if (O', A') is a pair of points in $\mathcal{P}$, if O is chosen in $\psi^{-1}(\{O'\})$, then for each A in $\psi^{-1}(\{A'\})$, there exists a finite set $\mathcal{B}_A$ of points in $\mathrm{ZK}(\mathcal{P}) \setminus \{O, A\}$ such that every geodesic in $\mathrm{ZK}(\mathcal{P})$ from O to A meets $\mathcal{B}_A$. Then

$$\mathcal{B}' \overset{\text{def}}{=\!=} \Big(\bigcup_{A\in\psi^{-1}(\{A'\})} \psi(\mathcal{B}_A) \cup V \Big) \setminus \{O', A'\} = \Big(\bigcup_{\gamma\in\Gamma} \psi(\mathcal{B}_{\psi_\gamma(A')}) \cup V \Big) \setminus \{O', A'\}$$

is a finite blocking configuration between O' and A', thus $\mathcal{P}$ has the finite blocking property.

For the direction $\Leftarrow$, if (O, A) is a pair of points in $\mathrm{ZK}(\mathcal{P})$, there exists a finite set $\mathcal{B}'$ of points in $\mathcal{P} \setminus \{\psi(O), \psi(A)\}$ such that every billiard path in $\mathcal{P}$ from $\psi(O)$ to $\psi(A)$ meets $\mathcal{B}'$. Then

$$\mathcal{B} \stackrel{\text{def}}{=\!=} \psi^{-1}(\mathcal{B}') \subset \mathrm{ZK}(\mathcal{P}) \setminus \{O, A\}$$

is a finite blocking configuration between O and A, thus $\mathrm{ZK}(\mathcal{P})$ has the finite blocking property. □

Unfolding a billiard table. — Combining Propositions 1 and 2, we have (see [MT]) the

Proposition 3. — *Let $\mathcal{P}$ and $\mathcal{P}'$ be two rational polygonal billiards such that $\mathcal{P}$ is obtained by reflecting $\mathcal{P}'$ at its edges finitely many times, without overlapping* (*we allow some barriers along parts of some sides of copies of $\mathcal{P}'$ inside $\mathcal{P}$*)*. Since there exists a branched covering from* $\mathrm{ZK}(\mathcal{P})$ *to* $\mathrm{ZK}(\mathcal{P}')$*, then $\mathcal{P}$ has the finite blocking property if and only if $\mathcal{P}'$ has.*

1.4. Action of $\mathbf{GL}(\mathbf{2}, \mathbb{R})$.

If $A \in \mathrm{GL}(2, \mathbb{R})$, we can define the translation surface

$$A \cdot \big(\mathcal{S}, \Sigma, (U_i, \phi_i)_{i \in I}\big) \stackrel{\text{def}}{=\!=} \big(\mathcal{S}, \Sigma, (U_i, A \circ \phi_i)_{i \in I}\big);$$

hence we have an action of $\mathrm{GL}(2, \mathbb{R})$ on the class of translation surfaces. We classically consider only elements of $\mathrm{SL}(2, \mathbb{R})$ (see [MT]), but we do not need to preserve area here.

Proposition 4. — *Let $\mathcal{S}$ be a translation surface and A be in* $\mathrm{GL}(2,\mathbb{R})$*. Then $\mathcal{S}$ has the finite blocking property if and only if $A \cdot \mathcal{S}$ has.*

Proof. — Such an action sends geodesic to geodesic. □

To summarize this section, we can say that the finite blocking property enjoys many properties of *stability*. As an illustration, it suffices to apply successively Propositions 2, 1, 4, 2, 3 and to follow a construction of [Mc], to reduce the problem of the finite blocking property for the billiard in the regular pentagon to the problem in the $\perp$-shaped billiard studied in [Mo] with parameters $\alpha = 1 + 2\cos(\frac{2}{5}\pi)$, $L_1 = 1$, $L_2 = 2\cos(\frac{2}{5}\pi)$. It is proved there that this billiard table fails the finite blocking property ($\alpha \in \mathbb{R} \setminus \mathbb{Q}$).

2. Some remarks around Hiemer and Snurnikov's proof.

In their article [HS], Philipp Hiemer and Vadim Snurnikov tried to prove that any rational billiard $\mathcal{P}$ has the finite blocking property.

For this, they use the subgroup $G_{\mathcal{P}}$ of $\mathrm{Isom}(\mathbb{R}^2)$ generated by the reflections at the edges of the polygon $\mathcal{P}$. In the proof of their Theorem 5, they construct a finite number of points in $\mathbb{R}^2$ called the $P_{i,\lambda}$'s and *choose* a blocking point arbitrarily in each orbit of the $P_{i,\lambda}$'s under the action of the group $G_{\mathcal{P}}$ on $\mathbb{R}^2$ (the (i,λ)'s belong to $G_{\mathcal{P}}/T_{\mathcal{P}} \times \{0, 1/2\}^{\dim_{\mathbb{Q}} T_{\mathcal{P}}}$).

The polygon $\mathcal{P}$ drawn on Figure 2 is a rational one but the subgroup $T_{\mathcal{P}}$ of $G_{\mathcal{P}}$ consisting of translations is dense in $\mathbb{R}^2$ (identified with the group of all translations of $\mathbb{R}^2$), since $\mathbb{Z} + \sqrt{2}\,\mathbb{Z}$ is dense in $\mathbb{R}$. Hence the orbit of any point of the plane under $G_{\mathcal{P}}$ is dense in the plane and therefore in $\mathcal{P}$ (which is the closure of an open set).

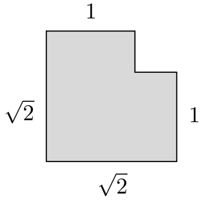

Figure 2. A rational polygonal billiard whose translation group is dense in $\mathbb{R}^2$

Now, if we take two points O and A in $\mathcal{P}$ such that the segment $[O, A]$ is included in $\mathcal{P}$, we can *choose* all the blocking points in a small open set $U \subset \mathcal{P}$ which does not intersect $[O, A]$. Such points cannot block the direct path from O to A (see Figure 3).

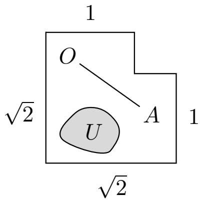

Figure 3. The direct path from O *to* A *does not meet the points in* U

Hence the proof in [HS] does not work. In fact, we have shown in [Mo] that the billiard table drawn in Figure 2 fails the finite blocking property. Meanwhile, the proof given in [HS] (Theorem 5) works for rational polygons $\mathcal{P}$ such that $T_{\mathcal{P}}$ is discrete (hence lattice) (such billiards are called *almost integrable*). Indeed, instead of choosing one point in each orbit of the $P_{i,\lambda}$'s in $\mathcal{P}$, it suffices to take *all* the points of the orbits of the $P_{i,\lambda}$'s that lie in $\mathcal{P}$ (and that are distinct from O and A).

We can bound (badly but uniformly) the number of such points as follows:

Proposition 5. — *If $\mathcal{P}$ is an almost integrable rational polygonal billiard with angles of the form $\pi p_i/q$ $(1 \leq i \leq n)$, if the diameter of $\mathcal{P}$ is D, if (v_1, v_2) is a basis of $T_{\mathcal{P}}$ (that is a lattice), then for every pair (O, A) in $\mathcal{P}$, there exist a set of at most*

$$8q\left[\frac{\pi\big(D/\sqrt{3}+\frac{1}{2}(\|v_1\|+\|v_2\|)\big)^2}{\det(v_1,v_2)}\right]$$

blocking points.

Proof. — Include $\mathcal{P}$ in a disk of radius $D/\sqrt{3}$, enlarge this disk by $\frac{1}{2}(\|v_1\|+\|v_2\|)$ and look at the area ($8q$ is larger than $\mathrm{card}(G_{\mathcal{P}}/T_{\mathcal{P}} \times \{0, \frac{1}{2}\}^{\{v_1,v_2\}})$). □

In fact, the conditions of this proposition (the fact of being an almost integrable billiard) work only for the polygons constructed by reflecting $\mathcal{C}$ at its edges finitely many times (we can add barriers along interior edges), where $\mathcal{C}$ is one of the following elementary polygons:

- the right-angled isosceles triangle with angles $(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{4}\pi)$;
- the half-equilateral triangle with angles $(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{6}\pi)$;
- any rectangle.

This fact is a direct consequence of the theory of reflection groups and chamber systems (see [Bou] for an extensive study or [Car] for a brief introduction). The three elementary polygons correspond to the types $\widetilde{C}_2$, $\widetilde{G}_2$ and $\widetilde{A}_1 \times \lambda\widetilde{A}_1$ $(\lambda > 0)$.

For example, the equilateral triangle, the square and the regular hexagon have the finite blocking property.

Note that the translation surface associated to the square is a (flat) torus. We can notice that the translation surface associated to an almost integrable polygon is a torus branched covering (this is easy for one of the elementary polygons, the rest follows by reflecting). As we have seen in Subsection 1.2, the finite blocking property is preserved by branched covering and, since four points suffice to block every geodesic between two fixed points in the torus, we have another bound for the number of blocking points for such billiards that depends on the degree of the covering.

With this remark, we can see that Proposition 5 can be seen as a consequence of the only fact that the square billiard (or equivalently the translation surface $\mathbb{R}^2/\mathbb{Z}^2$) has the finite blocking property. This last result seems to appear for the first time in the Leningrad's Olympiad in 1989 selection round, 9th form (see [Fo]). The problem was the following:

"Professor Smith is standing in the squared hall with mirror walls. Professor Jones wants to place in the hall several students in such a way that professor Smith could not see from his place his own mirror images. Is it possible? (Both professors and students are points, the students can be placed in corners and walls)."

The author of this problem was Dmitrij Fomin. None of the school students solved it. The booklet of the olympiad contains an answer: 16 students, and an example of this arrangement in coordinates.

Another consequence of the fact that the torus $\mathbb{R}^2/\mathbb{Z}^2$ has the finite blocking property is the answer to a question that Anton Zorich reported to me.

Proposition 6. — *There exists a translation surface with the property that every geodesic going from a singularity to itself has to meet first another singularity. In other words, this surface does not have any saddle connection going from a singularity to itself.*

Proof. — Let $\mathcal{S}$ be the translation surface drawn in Figure 4.

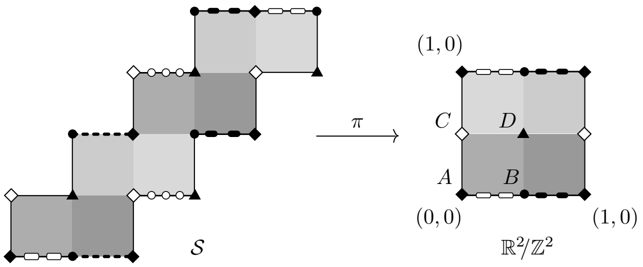

Figure 4. Covering between $\mathcal{S}$ and $\mathbb{R}^2/\mathbb{Z}^2$ (the opposite vertical lines are identified by horizontal translation, the horizontal lines with the same style are identified)

The coloring allows us to see how to construct a covering $\pi\colon \mathcal{S} \to \mathbb{R}^2/\mathbb{Z}^2$ of degree 2, branched at the points $A = (0,0)$, $B = (\frac{1}{2},0)$, $C = (0,\frac{1}{2})$, $D = (\frac{1}{2},\frac{1}{2})$. The singularities of $\mathcal{S}$ are located at the preimages of those four points.

The finite blocking property in the square says that three points in $\{A, B, C, D\}$ block every geodesic from the fourth point to itself. Hence, by lifting, every geodesic from the singularity $\pi^{-1}(A)$ to itself in $\mathcal{S}$ has to meet one of the other singularities $\pi^{-1}(B)$, $\pi^{-1}(C)$ or $\pi^{-1}(D)$, and by symmetry it works for the three other singularities. □

3. A local lemma.

A *subcylinder* $\mathcal{C}$ is an isometric copy of $\mathbb{R}/w\mathbb{Z} \times]0, h[$ in a translation surface $\mathcal{S}$ ($w > 0$, $h > 0$). The parameters w and h are unique and called the *width* and the *height* of $\mathcal{C}$.

The images in $\mathcal{S}$ of $\mathbb{R}/w\mathbb{Z}\times\{0\}$ and $\mathbb{R}/w\mathbb{Z}\times\{h\}$ (which are well-defined if we extend the isometry, which is uniformly continuous, by continuity in $\mathcal{S}$) are called the *sides* of $\mathcal{C}$.

The *direction* of $\mathcal{C}$ is the direction of the image of $\mathbb{R}/w\mathbb{Z} \times \{\frac{1}{2}h\}$ (which is a closed geodesic).

A *cylinder* is a maximal subcylinder (for the inclusion). By maximality, each side of a cylinder must contain at least one singularity and is a finite union of saddle connections.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two cylinders with width w_1 (resp. w_2) and height h_1 (resp. h_2) in a translation surface $\mathcal{S}$. The cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be

- *parallel* if their directions are parallel;
- *commensurable* if the ratios w_1/h_1 and w_2/h_2 are commensurable.

We will study the case where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two different parallel cylinders whose closures have a nontrivial (i.e. not reduced to a finite set) intersection. The situation can be described as in Figure 5.

Lemma 1. — *Let $\mathcal{S}$ be a translation surface that contains two different parallel cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ whose closures have a nontrivial intersection. If their widths are uncommensurable, then $\mathcal{S}$ fails the finite blocking property.*

Proof. — Up to a vertical dilatation, we can assume that h_1 and h_2 are greater than 1. In the system of coordinates given in Figure 5, we set $O = (w_1 - \frac{1}{2}\ell, -1)$ and $A = (w_1 - \frac{1}{2}\ell, 1)$.

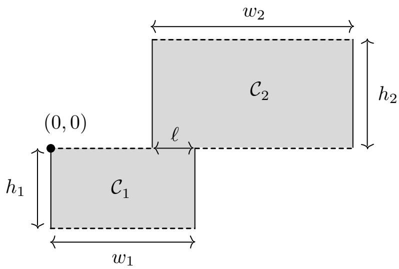

Figure 5. The cylinder $\mathcal{C}$: the opposite vertical sides are identified by translation while the dotted horizontal ones are glued with the rest of the surface

Since w_1/w_2 is a positive irrational number, $\mathbb{N}^* - w_1/w_2\mathbb{N}^*$ is dense in $\mathbb{R}$ so there exists two positive integer sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ such that:

- q_n is strictly increasing;
- $p_n w_2 - q_n w_1 \in\,]-\ell,\ell[$.

For $n \in \mathbb{N}$, let γ_n be the geodesic starting form O with slope

$$\frac{2}{q_n w_1 + p_n w_2} = \frac{1}{q_n w_1 + \lambda_n} = \frac{1}{p_n w_2 - \lambda_n}$$

where $\lambda_n = \frac{1}{2}(p_n w_2 - q_n w_1) \in\,]-\frac{1}{2}\ell, \frac{1}{2}\ell[$.

So, we can check by unfolding the trajectory in the universal cover of $\mathcal{S}$ (see Figure 6) that γ_n passes q_n times through the line $\{0\} \times]-h_1, 0[$, passes through $(w_1 - \frac{1}{2}\ell + \lambda_n, 0) \in]w_1 - \ell, w_1[\times\{0\}$, passes p_n times through the line $\{w_1 - \ell\} \times]0, h_2[$ and then passes through A. So, γ_n lies completely in $\mathcal{C}_1 \cup \mathcal{C}_2 \cup]w_1 - \ell, w_1[\times \{0\}$.

Now, we assume by contradiction that there is a point $B(x,y)$ in $\mathcal{S}$ distinct from O and A such that infinitely many γ_n pass through B. Hence, there is a subsequence such that for all n in $\mathbb{N}$, γ_{i_n} passes through B. There are two cases to consider:

First case: $y \in\,]-1,0]$. — By looking at the unfolded version of the trajectory (Figure 6), we see that, if k_{i_n} denotes the number of times that γ_{i_n} pass through the line $\{0\}\times]-h_1,0[$ before hitting B, then (by calculating the slope of γ_{i_n} from O to B)

$$\frac{y+1}{(x + k_{i_n} w_1) - (w_1 - \frac{1}{2}\ell)} = \frac{2}{q_n w_1 + p_n w_2}.$$

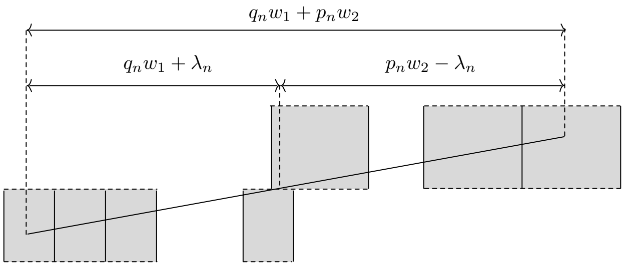

Figure 6. The unfolding procedure

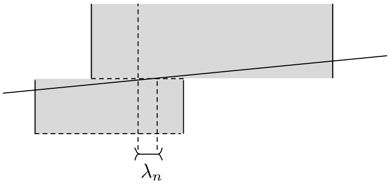

Figure 7. Zoom on Figure 6

So, $x - w_1 + \frac{1}{2}\ell = \frac{1}{2}(q_{i_n}w_1 + p_{i_n}w_2)(y+1) - k_{i_n}w_1$. In particular,

$$\begin{aligned} x - w_1 + \frac{1}{2}\ell &= \frac{1}{2}(q_{i_0}w_1 + p_{i_0}w_2)(y+1) - k_{i_0}w_1 \\ &= \frac{1}{2}(q_{i_1}w_1 + p_{i_1}w_2)(y+1) - k_{i_1}w_1. \end{aligned}$$

Hence,

$$(p_{i_1} - p_{i_0})w_2 + (q_{i_1} - q_{i_0})w_1 = \frac{2w_1}{y+1}(k_{i_1} - k_{i_0}) \neq 0.$$

So, $2w_1/(y+1)$ can be written as $rw_2 + sw_1$ where r and s are rational numbers.

Now, if $n \geq 1$, we still have

$$(p_{i_n} - p_{i_0})w_2 + (q_{i_n} - q_{i_0})w_1 = (rw_2 + sw_1)(k_{i_n} - k_{i_0}).$$

Because (w_2, w_1) is free over $\mathbb{Q}$, we have

- $(p_{i_n} - p_{i_0}) = r(k_{i_n} - k_{i_0})$,
- $(q_{i_n} - q_{i_0}) = s(k_{i_n} - k_{i_0}) \neq 0$ (remember that q_n is strictly increasing).

Thus, by dividing,

$$\frac{r}{s} = \frac{p_{i_n} - p_{i_0}}{q_{i_n} - q_{i_0}} = \frac{p_{i_n}}{q_{i_n}}\left(1 - \frac{p_{i_0}}{p_{i_n}}\right)\left(\frac{1}{1 - q_{i_0}/q_{i_n}}\right) \xrightarrow[n\to\infty]{} \frac{w_1}{w_2} \in \mathbb{R} \setminus \mathbb{Q}$$

leading to a contradiction.

For the second case, if $y \in [0,1[$, it is exactly the same (just reverse Figure 6).

Thus, (O, A) is not finitely blockable, and $\mathcal{S}$ fails the finite blocking property. □

In the proof of Lemma 1, we have chosen the points $O \in \mathcal{C}_1$ and $A \in \mathcal{C}_2$ in such a way that the calculus was easy. In fact, *almost all* pair $(O, A) \in \mathcal{C}_1 \times \mathcal{C}_2$ is not finitely blockable. More precisely, we have the

Proposition 7. — *In the system of coordinates given in Figure* 5, *if* $\alpha \stackrel{\text{def}}{=} w_1/w_2 \in \mathbb{R} \setminus \mathbb{Q}$, *if* $A_i(x_i, y_i) \in \mathcal{C}_i$ $(i \in \{1,2\})$, *if* $\beta \stackrel{\text{def}}{=} -y_1/y_2$ *is uncommensurable to* α, *then* (A_1, A_2) *is not finitely blockable.*

Proof. — It is very similar to the proof of Lemma 1; the introduction of many parameters just makes it heavier to read. Let

$$R \stackrel{\text{def}}{=} -y_2\ell + y_2 w_1 - y_2 x_1 + x_2 y_1 - w_1 y_1 + y_1 \ell \quad \text{(it is just an offset)},$$

$$I \stackrel{\text{def}}{=} \left] \frac{-R}{w_2 y_2}, \frac{\ell(y_2 - y_1) - R}{w_2 y_2} \right[.$$

I is a non empty open set since $\ell(y_2 - y_1) > 0$.

Since α/β is a positive irrational number, $\mathbb{N} - \alpha/\beta\,\mathbb{N}$ is dense in $\mathbb{R}$ so there exists two positive integer sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ such that:

- q_n is strictly increasing;
- $\alpha q_n - \beta p_n \in I$,

Let

$$\lambda_n \stackrel{\text{def}}{=} \frac{y_2 w_1 q_n + y_1 w_2 p_n + R}{y_2 - y_1} \in]0, \ell[\quad (n \in \mathbb{N}).$$

For $n \in \mathbb{N}$, let γ_n be the geodesic starting form A_1 with slope

$$\begin{aligned} \frac{y_2 - y_1}{w_1 q_n + w_2 p_n + x_2 - x_1} &= \frac{-y_1}{w_1 q_n + w_1 - x_1 - \ell + \lambda_n} \\ &= \frac{y_2}{w_2 p_n - w_1 + x_2 + \ell - \lambda_n}. \end{aligned}$$

So, we can check that γ_n passes q_n times through the line $\{0\} \times]y_1, 0[$, passes through $(x_1 - \ell + \lambda_n, 0) \in]x_1 - \ell, x_1[\times \{0\}$, passes p_n times through the line $\{x_1 - l\} \times]0, y_1[$ and passes through A_2. Stop γ_n here (see Figure 8). So, γ_n lies completely in $\mathcal{C}_1 \cup \mathcal{C}_2 \cup]x_1 - \ell, x_1[\times \{0\}$.

Now, assume by contradiction that there is a point $B(x, y)$ in $\mathcal{S}$ distinct from A_1 and A_2 such that infinitely many γ_n pass through B.

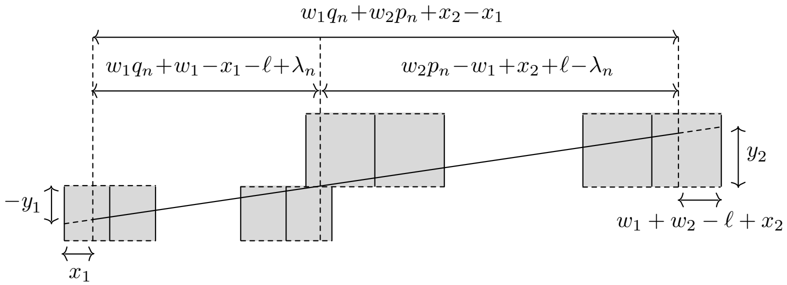

Figure 8. *The unfolding procedure*

Let $i \in \uparrow(\mathbb{N},\mathbb{N})$ be an extraction such that, for all n in $\mathbb{N}$, γ_{i_n} pass through B. There are two cases to consider:

First case: $y \in]y_1,0]$. — By looking at the unfolded version of the trajectory (Figure 8), we see that if k_{i_n} denotes the number of times that γ_{i_n} pass through the line $\{0\}\times]y_1,0[$ before hitting B, then

$$\frac{y_1-y}{x_1-(x+k_{i_n}w_1)}=\frac{y_2-y_1}{w_1q_n+w_2p_n+x_2-x_1}.$$

So, $x-x_1=(w_1q_{i_n}+w_2p_{i_n}+x_2-x_1)(y-y_1)/(y_2-y_1)-k_{i_n}w_1$. In particular,

$$\begin{aligned} x-x_1&=(w_1q_{i_0}+w_2p_{i_0}+x_2-x_1)(y-y_1)/(y_2-y_1)-k_{i_0}w_1\\ &=(w_1q_{i_1}+w_2p_{i_1}+x_2-x_1)(y-y_1)/(y_2-y_1)-k_{i_1}w_1. \end{aligned}$$

Hence, $(p_{i_1}-p_{i_0})+(q_{i_1}-q_{i_0})\alpha=\alpha(y_2-y_1)/(y-y_1)(k_{i_1}-k_{i_0})\neq 0$.

So, $\alpha(y_2-y_1)/(y-y_1)$ can be written as $r+s\alpha$ where r and s are rational numbers.

Now, if $n\geq 1$, we still have $(p_{i_n}-p_{i_0})+(q_{i_n}-q_{i_0})\alpha=(r+s\alpha)(k_{i_n}-k_{i_0})$. Because $(1,\alpha)$ is free over $\mathbb{Q}$, we have

- $(p_{i_n}-p_{i_0})=r(k_{i_n}-k_{i_0})$,
- $(q_{i_n}-q_{i_0})=s(k_{i_n}-k_{i_0})\neq 0$ (remember that q_n is strictly increasing).

Thus, by dividing,

$$\frac{r}{s}=\frac{p_{i_n}-p_{i_0}}{q_{i_n}-q_{i_0}}=\frac{p_{i_n}}{q_{i_n}}\Big(1-\frac{p_{i_0}}{p_{i_n}}\Big)\Big(\frac{1}{1-q_{i_0}/q_{i_n}}\Big)\xrightarrow[n\to\infty]{}\frac{\alpha}{\beta}\in\mathbb{R}\setminus\mathbb{Q}$$

leading to a contradiction.

For the second case, if $y\in[0,y_2[$, it is exactly the same (just reverse Figure 8). Thus, (A_1,A_2) is not finitely blockable. □

4. Finite blocking property in the regular polygons.

THEOREM 4. — *Let $n \geq 3$ be an integer. The following assertions are equivalent*:

• *the regular n-gon has the finite blocking property*;

• *the right-angled triangle $\mathcal{T}_n$ with an angle equal to π/n has the finite blocking property*;

• $n \in \{3,4,6\}$.

Proof. — According to Propositions 3 and 5, it suffices to prove that the second assertion implies the third one.

Suppose first that n is odd. $\mathrm{ZK}(\mathcal{T}_n)$ can be described as two regular n-gons P and P' symmetric one to each other, with identifications along the sides (see Figure 9).

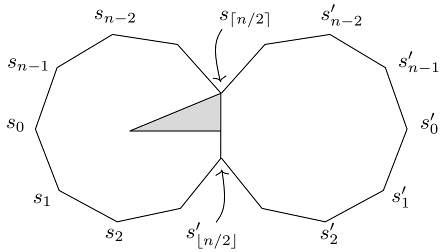

Figure 9. *The translation surface associated with the right-angled triangle with an angle equal to* $\frac{1}{9}\pi$ (*identify the opposite sides*)

We number the vertices of P (resp. P') counterclockwise (resp. clockwise) from s_0 to s_{n-1} (resp. from s'_0 to s'_{n-1}) (see Figure 9).

Now, look at Figure 10.

In $P \cup P'$, $(s_0, s_{\lceil n/2 \rceil})$ is parallel to $(s_1, s_{\lceil n/2 \rceil} - 1)$, to $(s_2, s_{\lceil n/2 \rceil} - 2)$, $\ldots$, to $(s_{n-1}, s_{\lceil n/2 \rceil} + 1)$, to $(s_{n-2}, s_{\lceil n/2 \rceil} + 2)$, $\ldots$, and by axial symmetry with respect to $(s_{\lfloor n/2 \rfloor}, s_{\lceil n/2 \rceil})$, it is parallel to $(s'_0, s'_{\lceil n/2 \rceil})$, to $(s'_1, s'_{\lceil n/2 \rceil} - 1)$, to $(s'_2, s'_{\lceil n/2 \rceil} - 2)$, $\ldots$, to $(s'_{n-1}, s'_{\lceil n/2 \rceil} + 1)$, to $(s'_{n-2}, s'_{\lceil n/2 \rceil} + 2), \ldots$

Let us call this common direction the dashed one.

Do the same for the direction $(s_0, s_{\lfloor n/2 \rfloor})$ and call it the dotted one.

This leads to a triangulation of the surface, each triangle having an edge dashed, an edge dotted and an edge that is an edge of P or P'.

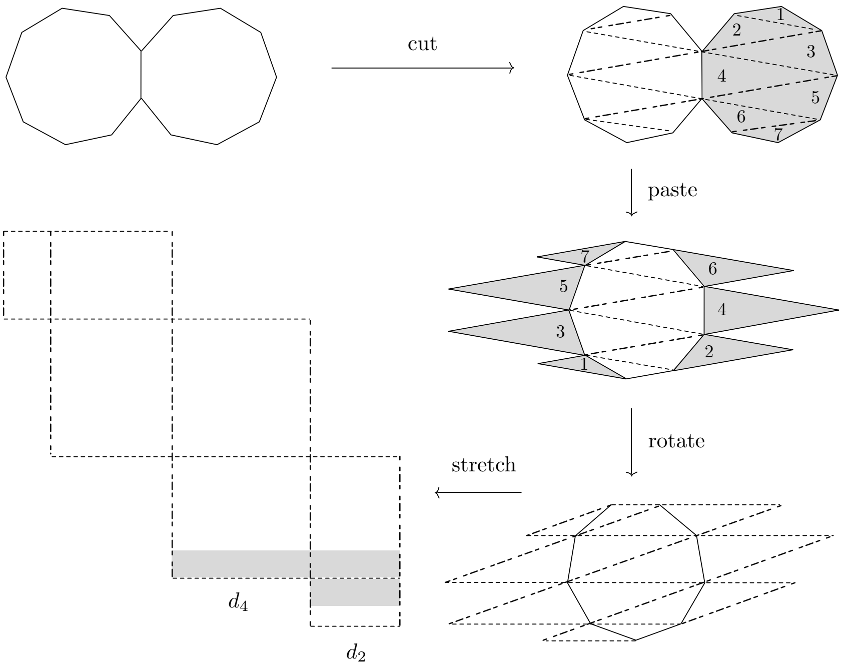

Figure 10. *From the surface associated to $\mathcal{T}_n$ to a more exploitable one in the same orbit under* $\mathrm{GL}(2,\mathbb{R})$ (n *odd*)

Now, take P as the base and glue all the triangles that lie in P' to the ones that lie in P, thanks to the identification between the edges of P and the edges of P' (this can be done in an unique way) (this is the cut and paste operation).

We have now a new representation of the surface associated to $\mathcal{T}_n$, by a planar polygon with identifications along its sides that are only dashed or dotted sides.

Since those two directions are not parallel, it is easy to find an element A of $\mathrm{GL}(2,\mathbb{R})$ that put the dotted direction to the horizontal, the dashed one to the vertical and that preserves the lengths in those two directions (this is the rotate and stretch operation).

The surface $\mathcal{S}$ obtained by the action of A to the surface associated to $\mathcal{T}_n$ is more exploitable for our purpose.

For $1 \leq i \leq n-1$, we denote $d_i \stackrel{\text{def}}{=} ||s_0 - s_i||_2$.

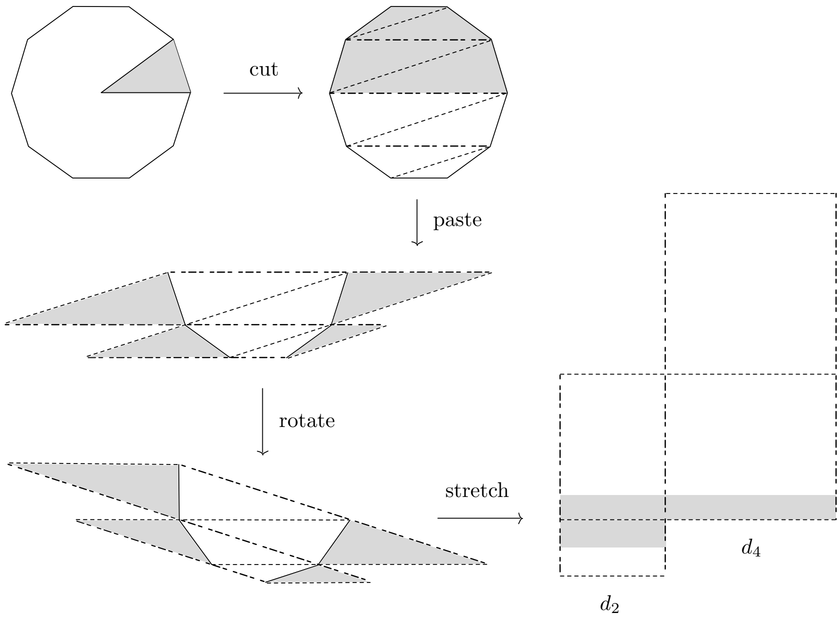

Figure 11. *The even case* $(n = 10)$

We can remark that exactly one edge of P is dashed. Starting from this edge in $\mathcal{S}$, it is easy to recognize the shape of the Figure 5 with $w_1 = \ell = d_2$, $w_2 = d_2 + d_4$, $h_1 = d_1$ and $h_2 = d_3$.

We have $w_2/w_1 = 1 + d_4/d_2 = 1 + 2\sin(4\pi/n)/2\sin(2\pi/n) = 1 + 2\cos(2\pi/n)$ which is irrational if $n \geq 5$. Hence, $\mathcal{S}$ and therefore $\mathcal{T}_n$ lacks the finite blocking property if $n \neq 3$.

For the even case, the translation surface $\mathrm{ZK}(\mathcal{T}_n)$ is constituted by only one regular n-gon with opposite sides identified. The construction is very similar, but the role of P is played by the lower-half of the polygon, and the role of P' is played by its upper-half (see Figure 11). In this case, $w_2/w_1 = 1 + 2\cos(2\pi/n)$ is irrational if $n \notin \{4, 6\}$. □

Nevertheless, we can notice that the situation is *not homogenous* among different pairs of points:

Proposition 8. — *For every regular n-gon (n even), there exists a finite set of points that blocks every billiard path from the center O to itself.*

Proof. — The set $\mathcal{B}$ consisting in the centers of the edges and the vertices is a (finite) blocking configuration. Indeed, suppose by contradiction that γ is a billiard path from O to O that does not meet $\mathcal{B}$: it can be folded into a billiard path in the triangle $\mathcal{T}_n$ from the vertex with angle π/n to himself. This contradicts [To], Lemma 4.1, page 871 (a nice argument on angles, measured modulo $2\pi/n$ makes this fact impossible). □

This proposition is not so clear if n is odd, since a billiard path γ in $\mathcal{T}_n$ starting from O with angle $\pi/2n$ is coming back to O after n bounces (see Figure 12).

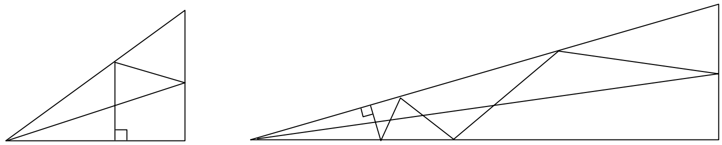

Figure 12. By starting with an angle $\pi/2n$, we are coming back in O ($n \in \{5, 11\} \subset 2\mathbb{N}+3$)

It would be interesting to find a rational polygonal billiard table $\mathcal{P}$ with the property that no pair of points in $\mathcal{P}$ can be finitely blocked.

5. Finite blocking property on Veech surfaces.

Proposition 9. — *Suppose that a translation surface $\mathcal{S}$ is decomposable into commensurable parallel cylinders, in at least two different directions. Then $\mathcal{S}$ has the finite blocking property if and only if $\mathcal{S}$ is a torus covering, branched over only one point.*

Proof. — Let $(\mathcal{C}_i)_{i=0}^n$ be a decomposition of $\mathcal{S}$ into parallel commensurable cylinders, with heights h_i and weights w_i ($i \le n$).

Starting from $\mathcal{C}_0$, by applying Lemma 1 step by step, we can see that all the w_i's are commensurable (recall that $\mathcal{S}$ is assumed to be connected).

Since, for all $i \le n$, w_i/h_i is a rational number, we can deduce that all the h_i's are commensurable. So, there exists $h > 0$ and $(k_i)_{i=0}^n \in \mathbb{N}^{n+1}$ such that for $i \le n$, $h_i = k_i h$. Then, each $\mathcal{C}_i$ is decomposed into k_i subcylinders $(\mathcal{C}_{i,j})_{j=1}^{k_i}$ of height h ($\mathcal{C}_{i,j}$ is the image of $\mathbb{R}/w_i\mathbb{Z} \times](j-1)h, jh[$). Note that the singularities of $\mathcal{S}$ lie in the sides of the $\mathcal{C}_{i,j}$'s.

By hypothesis, we have the same kind of decomposition of $\mathcal{S}$ into parallel subcylinders $(\mathcal{C}'_{i,j})_{i \le n', j \le k'_i}$ of height h' in another direction, with the property that each singularity of $\mathcal{S}$ lies in a side of $\mathcal{C}'_{i,j}$.

$$(\mathcal{P}_i)_{i \le \ell} \stackrel{\text{def}}{=} (\mathcal{C}_{i,j})_{i \le n, j \le k_i} \vee (\mathcal{C}'_{i,j})_{i \le n', j \le k'_i}$$

is a decomposition of $\mathcal{S}$ into parallel isometric parallelograms glued edge to edge.

This leads to a covering from $\mathcal{S}$ to $\mathcal{P}_0$ whose opposite edges are identified, i.e. from $\mathcal{S}$ to a torus. Note that all the singularities of $\mathcal{S}$ lie in a vertex of some $\mathcal{P}_i$, so they are sent to a common point in the torus (the image of a vertex of $\mathcal{P}_0$). $\square$

THEOREM 2. — *A Veech surface has the finite blocking property if and only if it is a torus covering, branched over only one point.*

Proof. — If the surface is a torus, there is nothing to prove (both statements are true). Otherwise, the genus of the surface is greater than 2, and then the surface has at least one singularity and therefore many saddle connection directions. In each saddle connection direction, a Veech surface admits a decomposition into commensurable cylinders. $\square$

6. Further results.

The aim of this section is to see how Lemma 1 can be used in different contexts. First, recall that Lemma 1 can be applied in the non-Veech context since:

• $\mathcal{C}_1$ and $\mathcal{C}_2$ do not need to be commensurable (there is no condition on the heights);

• the result is *local*: if the configuration of Figure 5 appears somewhere in a surface $\mathcal{S}$ with $w_1/w_2 \in \mathbb{R}\setminus\mathbb{Q}$, then $\mathcal{S}$ lacks the finite blocking property (see example on Figure 13).

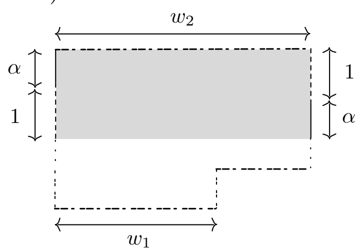

Figure 13. The surface cannot be fully decomposed into cylinders in the horizontal direction (the grey zone is a minimal component for the horizontal flow); identify the lines with the same style, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $w_1/w_2 \in \mathbb{R}\setminus\mathbb{Q}$

6.1. *L*-shaped surfaces.

Let $\mathcal{S}$ be a L-shaped translation surface; it is in the same $\mathrm{GL}(2,\mathbb{R})$-orbit than some $L(a,b)$ (see Figure 14). Lemma 1 allows us to decide wether $\mathcal{S}$ has the finite blocking property (we do not need $\mathcal{S}$ to be a Veech surface (such surfaces were characterized in [Mc] and [Cal])).

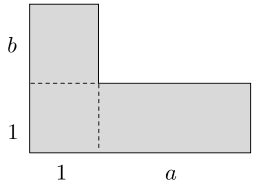

Figure 14. *The L-shaped translation surface $L(a,b)$ (identify the opposite sides by translation)*

Proposition 10. — *Let a and b be two positive real numbers. Then $L(a,b)$ has the finite blocking property if and only if $(a,b) \in \mathbb{Q}^2$.*

Proof. — If $L(a,b)$ has the finite blocking property, applying Lemma 1 in both horizontal and vertical direction leads to $(1+a)/1 \in \mathbb{Q}$ and $(1+b)/1 \in \mathbb{Q}$, so $(a,b) \in \mathbb{Q}^2$. □

6.2. Irrational billiards.

The construction of Zemljakov and Katok is also possible when the angles of a polygon $\mathcal{P}$ are not rational multiples of π; in this case, the group Γ is infinite and the surface is not compact. However, periodic trajectories can appear; we can even meet the situation of Lemma 1:

Proposition 11. — *There exists a non rational polygonal billiard that fails the finite blocking property.*

Proof. — Consider the billiard drawn in Figure 15 and apply Lemma 1 on the pair of cylinders defined by the grey zone. □

6.3. A density result.

A singularity $\sigma \in \Sigma$ has a conical angle of the form $2k\pi$, with $k \geq 1$; we say that σ is of *multiplicity* $k-1$. If $1 \leq k_1 \leq k_2 \leq \cdots \leq k_n$ is a sequence of integers whose sum is even, we denote $\mathcal{H}(k_1, k_2, \ldots, k_n)$ the *stratum* of translation surfaces with exactly n singularities whose multiplicities

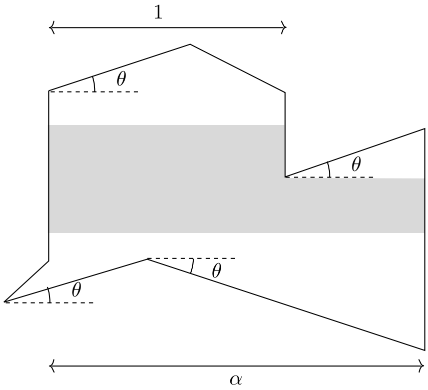

Figure 15. *A non rational billiard whose associated surface contains the configuration of Lemma* 1*;* $\theta \in \mathbb{R} \setminus \mathbb{Q}\pi$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

are $k_1, k_2, \dots, k_n$. A translation surface in $\mathcal{H}(k_1, k_2, \dots, k_n)$ has genus $g = 1 + \frac{1}{2}(k_1 + k_2 + \cdots + k_n)$.

Each stratum carries a natural topology that is for example defined in [Ko].

THEOREM 3. — *In genus* $g \geq 2$, *the set of translation surfaces that fail the finite blocking property is dense in every stratum.*

Sketch of proof. — The proof requires some material that is too long to describe here (like the precise definition of the topology on such a stratum — for this we have to see each translation surface as a Riemann surface with an abelian differential). It suffices to prove that the translation surfaces that satisfy the hypothesis of Lemma 1 are dense in each stratum. For this, we begin to prove that translation surfaces that admit a cylinder decomposition in the horizontal direction, with at least two non homologous horizontal cylinders, are dense (see [EO], [KZ], [Zo]). Using the local coordinates given by the period map, we have the possibility to perturb the perimeters of two such "consecutive" non homologous cylinders in order to let them non commensurable. We postpone the precise proof for a further paper (see [Mo2]). □

In fact, we will prove in [Mo2] that the translation surfaces that fail the finite blocking property is of full measure in each stratum. Moreover, we will prove that finite blocking property implies complete periodicity; we will also give the classification of the surfaces that have the finite blocking property in genus 2.

Of course, the notion of finite blocking property and the results presented in this paper can be translated in the vocabulary of quadratic differentials.

Conclusion.

One can define a stronger property: a planar polygonal billiard or a translation surface $\mathcal{P}$ is said to have the *bounded blocking property* if the number of blocking points can be chosen independently of the pair (O, A). Does it exist polygonal billiard tables with the finite blocking property but without the bounded blocking property?

Is it true that for general translation surfaces, the fact of being a torus covering is a necessary and sufficient condition to have the finite blocking property (resp. bounded blocking property)? Is it true that for rational billiards, the fact of being almost integrable is a necessary and sufficient condition to have the finite blocking property (resp. bounded blocking property)?

Note that for piecewise smooth billiard tables, the study of the finite blocking property seems very difficult, since for an ellipse, even a countable number of points do not suffice to block every path from a focus to the other one.

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