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THE RING OF MULTISYMMETRIC FUNCTIONS

by Francesco VACCARINO

Introduction.

Let R be a commutative ring and let n, m be two positive integers. Let $A_R(n,m)$ be the polynomial ring in the commuting independent variables $x_i(j)$ with $i = 1, \dots, m\,; j = 1, \dots, n$ and coefficients in R. The symmetric group on n letters S_n acts on $A_R(n,m)$ by means of $\sigma(x_i(j)) = x_i(\sigma(j))$ for all $\sigma \in S_n$ and $i = 1, \dots, m\,; j = 1, \dots, n$. Let us denote by $A_R(n,m)^{S_n}$ the ring of invariants for this action: its elements are usually called multisymmetric functions and they are the usual symmetric functions when $m = 1$. In this case, $A_R(n,1) \cong R[x_1, x_2, \dots, x_n]$, and $R[x_1, x_2, \dots, x_n]^{S_n}$ is freely generated by the elementary symmetric functions $e_1, \dots, e_n$ given by the equality

$$(0.1) \qquad \sum_{k=0}^{n} t^k e_k := \prod_{i=1}^{n} (1 + t x_i).$$

Here $e_0 = 1$ and t is a commuting independent variable (see [M]). Furthermore one has

$$(0.2) \qquad e_k(x_1, \dots, x_n) = \sum_{i_1 < i_2 < \dots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

Unless otherwise stated, we now assume that $m > 1$. We first obtain generators of the ring $A_R(n,m)^{S_n}$.

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Let $A_R(m) := R[y_1, \ldots, y_m]$, where $y_1, \ldots, y_m$ are commuting independent variables, let $f = f(y_1, \ldots, y_m) \in A_R(m)$ and define

$$f(j) := f(x_1(j), \ldots, x_m(j)) \text{ for } 1 \leqslant j \leqslant n. \tag{0.3}$$

Notice that $f(j) \in A_R(n,m)$ for all $1 \leqslant j \leqslant n$ and that $\sigma(f(j)) = f(\sigma(j))$, for all $\sigma \in S_n$ and $j = 1, \ldots, n$.

Define $e_k(f) := e_k(f(1), f(2), \ldots, f(n))$ i.e.

$$\sum_{k=0}^{n} t^k e_k(f) := \prod_{i=1}^{n} (1 + t f(i)), \tag{0.4}$$

where t is a commuting independent variable. Then $e_k(f) \in A_R(n,m)^{S_n}$.

One may think about the y_i as diagonal matrices in the following sense: let $M_n(A_R(n,m))$ be the full ring of $n \times n$ matrices with coefficients in $A_R(n,m)$. Then there is an embedding

$$\rho_n : A_R(m) \hookrightarrow M_n(A_R(n,m)) \tag{0.5}$$

given by

$$\rho_n(y_i) := \begin{pmatrix} x_i(1) & 0 & \ldots & 0 \\ 0 & x_i(2) & \ldots & 0 \\ 0 & 0 & \ldots & x_i(n) \end{pmatrix} \quad \text{for } i = 1, \ldots, m. \tag{0.6}$$

Now (0.4) gives

$$\sum_{k=0}^{n} t^k e_k(f) = \prod_{j=1}^{n} (1 + t \rho_n(f)_{jj}) = \det(1 + t \rho_n(f)), \tag{0.7}$$

where $\det(-)$ is the usual determinant of $n \times n$ matrices.

Let $\mathcal{M}_m$ be the set of monomials in $A_R(m)$. For $\mu \in \mathcal{M}_m$ let $\partial_i(\mu)$ denote the degree of μ in y_i, for all $i = 1, \ldots, m$. We set

$$\partial(\mu) := (\partial_1(\mu), \ldots, \partial_m(\mu)) \tag{0.8}$$

for its multidegree. The total degree of μ is $\sum_i \partial_i(\mu)$. Let $\mathcal{M}_m^+$ be the set of monomials of positive degree. A monomial $\mu \in \mathcal{M}_m^+$ is called *primitive* it is not a power of another one. We denote by $\mathfrak{M}_m^+$ the set of primitive monomials. We define an S_n invariant multidegree on $A_R(n,m)$ by setting $\partial(x_i(j)) = \partial(y_i) \in \mathbb{N}^m$ for all $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant m$. If $f \in A_R(m)$ is homogeneous of total degree l, then $e_k(f)$ has total degree kl (for all k and n).

We are now in a position to state the first part of our result (recall that $m > 1$).

THEOREM 1 (generators). — *The ring of multisymmetric functions $A_R(n,m)^{S_n}$ is generated by the $e_k(\mu)$, where $\mu \in \mathfrak{M}_m^+$, $k = 1, \dots n$ and the total degree of $e_k(\mu)$ is less or equal than $m(n-1)$. If $n = p^s$ is a power of a prime and $R = \mathbb{Z}$ or $p \cdot 1_R = 0$, then at least one generator has degree equal to $m(n-1)$.*

If $R \supset \mathbb{Q}$ then $A_R(n,m)^{S_n}$ is generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$ and the degree of μ is less or equal than n.

To obtain the relations between these generators, we need more notation on (multi)symmetric functions.

The action of S_n on $A_R(n,1) \cong R[x_1, x_2, \dots, x_n]$ preserves the usual degree. We denote by $\Lambda^k_{R,n}$ the R-submodule of invariants of degree k.

Let $q_n : R[x_1, x_2, \dots, x_n] \longrightarrow R[x_1, x_2, \dots, x_{n-1}]$ be given by $x_n \mapsto 0$ and $x_i \mapsto x_i$, for $i = 1, \dots, n-1$. This map sends $\Lambda^k_{n,R}$ to $\Lambda^k_{n-1,R}$ and it is easy to see that $\Lambda^k_{n,R} \cong \Lambda^k_{k,R}$ for all $n \geqslant k$. Denote by Λ^k_R the limit of the inverse system obtained in this way.

The ring $\Lambda_R := \bigoplus_{k \geqslant 0} \Lambda^k_R$ is called the ring of *symmetric functions* (over R).

It can be shown [M] that Λ_R is a polynomial ring, freely generated by the (limits of the) e_k, that are given by

$$\sum_{k=0}^{\infty} t^k e_k := \prod_{i=1}^{\infty} (1 + t x_i). \tag{0.9}$$

Furthermore the kernel of the natural projection $\pi_n : \Lambda_R \longrightarrow \Lambda_{n,R}$ is generated by the e_{n+k}, where $k \geqslant 1$.

In a similar way we build a limit of multisymmetric functions. For any $a \in \mathbb{N}^m$ we set $A_R(n,m,a)$ for the linear span of the monomials of multidegree a. One has

$$A_R(n,m) = \bigoplus_{a \in \mathbb{N}^m} A_R(n,m,a). \tag{0.10}$$

Let $\pi_n : A_R(n,m) \longrightarrow A_R(n-1,m)$ be given by

$$\pi_n(x_i(j)) = \begin{cases} 0 & \text{if } j = n \\ x_i(j) & \text{if } j \leqslant n-1 \end{cases} \qquad \text{for all } i. \tag{0.11}$$

Then (see (3.5)) we prove that, for all $a \in \mathbb{N}^m$

$$\pi_n(A_R(n,m,a)^{S_n}) = A_R(n-1,m,a)^{S_{n-1}}. \tag{0.12}$$

For any $a \in \mathbb{N}^m$ set

$$A_R(\infty, m, a) := \varprojlim A_R(n, m, a)^{S_n}, \tag{0.13}$$

where the projective limit is taken with respect to n over the projective system $(A_R(n, m, a)^{S_n}, \pi_n)$.

Set

$$A_R(\infty, m) := \bigoplus_{a \in \mathbb{N}^m} A_R(\infty, m, a). \tag{0.14}$$

We set, by abuse of notation,

$$e_k(f) := \varprojlim e_k(f) \in A_R(\infty, m) \tag{0.15}$$

with $k \in \mathbb{N}$ and $f \in A(m)^+$, the augmentation ideal, i.e.

$$\sum_{k=0}^{\infty} t^k e_k(f) := \prod_{j=1}^{\infty} (1 + tf(j)). \tag{0.16}$$

Then e_k is a homogeneous polynomial of degree k. Now, if $f = \sum_{\mu \in \mathcal{M}_m^+} \lambda_\mu \mu$, we set

$$e_k(f) := \sum_{\alpha} \lambda^\alpha e_\alpha \tag{0.16}$$

where $\alpha := (\alpha_\mu)_{\mu \in \mathcal{M}_m^+}$ is such that $\alpha_\mu \in \mathbb{N}$, $\sum_{\mu \in \mathcal{M}_m^+} \alpha_\mu \leqslant k$ and $\lambda^\alpha := \prod_{\mu \in \mathcal{M}_m^+} \lambda^{\alpha_\mu}$.

We can now state the second part of our main result.

THEOREM 2 (relations). — (1) *The ring $A_R(\infty, m)$ is a polynomial ring, freely generated by the (limits of) the $e_k(\mu)$, where $\mu \in \mathfrak{M}_m^+$ and $k \in \mathbb{N}$.*

The kernel of the natural projection

$$A_R(\infty, m) \longrightarrow A_R(n, m)^{S_n}$$

is generated as R-module by the coefficients e_α of the elements

$$e_{n+k}(f), \text{ where } k \geqslant 1 \text{ and } f \in A_R(m)^+.$$

(2) *If $R \supset \mathbb{Q}$ then $A_R(\infty, m)$ is freely generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$.*

The kernel of the natural projection is generated as an ideal by the $e_{n+1}(f)$, where $f \in A_R(m)^+$.

In Dalbec's paper [D] generators and relations are found in the case where $R \supset \mathbb{Q}$. The relations found there are actually the same we find: indeed what Dalbec calls *monomial multisymmetric functions* are exactly those e_α we introduced in (0.17), so that his Proposition 1.9 is a special case of our Proposition 3.1(1) when $R \supset \mathbb{Q}$. Another paper on this theme, giving a minimal presentation when the base ring is a characteristic 2 field, is [A]. Again, its main results on multisymmetric functions are a corollary of ours when R is a characteristic 2 field.

The results of this paper were presented in 1997 at a congress on algebraic groups representations in Ascona (CH) organized by H.P. Kraft. They are published only now for personal reasons.

1. Notations and basic facts.

The monomials of $A_R(n,m)$ form a R-basis, permuted by the action of S_n. Thus, the sums of monomials over the orbits form a R-basis of the ring of multisymmetric functions. We now introduce some notation and preliminary results concerning these functions and orbit sums.

Let $k \in \mathbb{N}$, we denote by $\mathbf{f}$ the sequence $(f_1 \ldots, f_k)$ in $A_R(m)$ and by α the element $(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, where $\sum \alpha_j \leqslant n$. Let $t_1, \ldots, t_k$ be commuting independent variables, we set as usual $t^\alpha := \prod_i t_i^{\alpha_i}$. We define elements $e_\alpha(\mathbf{f}) \in A_R(n,m)^{S_n}$ by

$$(1.1) \qquad \sum_\alpha t^\alpha e_\alpha(\mathbf{f}) := \det\Big(1 + \sum_h t_h \rho_n(f_h)\Big) = \prod_{i=1}^{n} \Big(1 + \sum_h t_h f_h(i)\Big).$$

Example 1.1. — Let $n = 3$ and $f, g \in A_R(m)$ then

$$e_{(2,1)}(f,g) = f(1)f(2)g(3) + f(1)g(2)f(3) + g(1)f(2)f(3).$$

If $n = 4$ then

$$\begin{aligned} e_{(2,1)}(f,g) = {} & f(1)f(2)g(3) + f(1)g(2)f(3) + g(1)f(2)f(3) \\ & + f(1)f(2)g(4) + f(1)g(2)f(4) + g(1)f(2)f(4) \\ & + f(1)f(3)g(4) + f(1)g(3)f(4) + g(1)f(3)f(4) \\ & + f(2)f(3)g(4) + f(2)g(3)f(4) + g(2)f(3)f(4) \end{aligned}$$

Let $k = m$ and $f_j = y_j$ for $j = 1, \ldots, m$, then the $e_\alpha(\mathbf{y}) = e_{(\alpha_1,\ldots,\alpha_m)}(y_1, \ldots, y_m)$ where $\sum \alpha_j \leqslant n$ are the well–known *elementary* *multisymmetric functions*. These generate $A_R(n,m)^{S_n}$ when $R \supset \mathbb{Q}$ (see [G] or [W]), and satisfy

$$(1.2)\qquad \sum_{\alpha} t^{\alpha} e_{\alpha}(\mathbf{y}) = \det\Big(1+\sum_j t_j \rho_n(y_j)\Big) = \prod_{i=1}^{n}\Big(1+\sum_{j=1}^{m} t_j x_j(i)\Big).$$

LEMMA 1.2. — *The multisymmetric function $e_{(\alpha_1,\ldots,\alpha_k)}(f_1,\ldots,f_k)$ is the orbit sum (under the considered action of S_n) of*

$$f_1(1)f_1(2)\cdots f_1(\alpha_1)f_2(\alpha_1+1)\cdots f_2(\alpha_1+\alpha_2)\cdots f_k\big(\sum_h \alpha_h\big).$$

Proof. — Let E be the set of mappings $\phi\colon\{1,\ldots,n\}\to\{1,\ldots,k+1\}$. We define a mapping $\phi\mapsto\phi^*$ of E into $\mathbb{N}^{k+1}$ by putting $\phi^*(i)$ equal to the cardinality of $\phi^{-1}(i)$. For two elements ϕ_1,ϕ_2 of E, to satisfy $\phi_1^*=\phi_2^*$ it is necessary and sufficient that there should exist $\sigma\in S_n$ such that $\phi_2=\phi_1\circ\sigma$. Set $f_{k+1}:=1_R$ and $E(\alpha):=\{\phi\in E \mid \phi^*=(\alpha_1,\ldots,\alpha_k,n-\sum_i\alpha_i)\}$, then we have

$$(1.3)\qquad e_{\alpha}(\mathbf{f}) = \sum_{\phi\in E(\alpha)} f_{\phi(1)}(1)f_{\phi(2)}(2)\cdots f_{\phi(n)}(n)$$

and the lemma is proved. □

It is clear that $e_{(\alpha_1,\ldots,\alpha_k)}(f_1,\ldots,f_k)=e_{(\alpha_{\tau(1)},\ldots,\alpha_{\tau(k)})}(f_{\tau(1)},\ldots,f_{\tau(k)})$ for all $\tau\in S_k$. If two entries are equal, say $f_1=f_2$, then, by (1.1)

$$(1.4)\qquad e_{(\alpha_1,\ldots,\alpha_k)}(f_1,\ldots,f_k)=\frac{(\alpha_1+\alpha_2)!}{\alpha_1!\alpha_2!}\,e_{(\alpha_1+\alpha_2,\ldots,\alpha_k)}(f_1,f_3\ldots,f_k).$$

Let $\mathbb{N}^{(\mathcal{M}_m^+)}$ be the set of functions $\mathcal{M}_m^+\longrightarrow\mathbb{N}$ with finite support. We set

$$(1.5)\qquad |\,\alpha\,| := \sum_{\mu\in\mathcal{M}_m^+}\alpha(\mu).$$

Let $\alpha\in\mathbb{N}^{(\mathcal{M}_m^+)}$, then there exist $k\in\mathbb{N}$ and $\mu_1,\ldots,\mu_k\in\mathcal{M}_m^+$ such that $\alpha(\mu_i)=\alpha_i\neq 0$ for $i=1,\ldots,k$ and $\alpha(\mu)=0$ when $\mu\neq\mu_1,\ldots,\mu_k$. We set

$$(1.6)\qquad e_{\alpha}:=e_{(\alpha_1,\ldots,\alpha_k)}(\mu_1,\ldots,\mu_k),$$

i.e. we substitute $(\mu_1,\ldots,\mu_k)$ to variables in the elementary multisymmetric function $e_{(\alpha_1,\ldots,\alpha_k)}(y_1,\ldots,y_k)$.

Then

$$\sum_{|\alpha|\leqslant n} t^{\alpha} e_{\alpha} = \prod_{i=1}^{n} \Big(1 + \sum_{\mu\in\mathcal{M}_m^+} t_\mu \mu(i)\Big), \tag{1.7}$$

where t_μ are commuting independent variables indexed by monomials and

$$t^{\alpha} := \prod_{\mu\in\mathcal{M}_m^+} t_\mu^{\alpha(\mu)} \tag{1.8}$$

for all $\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}$.

If $\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}$ is such that $\alpha(\mu) = k$ for some $\mu \in \mathcal{M}_m^+$ and $\alpha(\nu) = 0$ for all $\nu \in \mathcal{M}_m^+$ with $\nu \neq \mu$, we see that $e_\alpha = e_k(\mu)$, the k-th elementary symmetric function evaluated at $(\mu(1), \mu(2), \ldots, \mu(n))$.

Lemma 1.3. — *Given a monomial $\mu \in A_R(n,m)$, there exist $\mu_1, \ldots, \mu_n \in A_R(m)$ such that $\mu = \mu_1(1) \cdots \mu_n(n)$.*

Proof. — Let $\mu = \prod_{ij} x_i(j)^{a_{ij}}$ then $\mu_j = \prod_i y_i^{a_{ij}}$ for $j = 1, \ldots, n$. □

Proposition 1.4. — *The set*

$$\mathcal{B}_{n,m,R} := \{e_\alpha : |\, \alpha \,| \leqslant n\}$$

is a R-basis of $A_R(n,m)^{S_n}$.

The set

$$\mathcal{B}_{n,m,a,R} := \{e_\alpha \ : |\, \alpha \,| \leqslant n \text{ and } \partial(e_\alpha) = a\}$$

is a R-basis of $A_R(n,m,a)^{S_n}$, for all $a \in \mathbb{N}^m$.

Proof. — By Lemma 1.2 and (1.6), the e_α are a complete system of representatives (for the action of S_n) of the orbit sums of the products

$$\{\mu_1(1)\mu_2(2)\cdots\mu_n(n) \,:\, \mu_i \in \mathcal{M}_m\,,\, i = 1, \ldots, n\}.$$

So the first statement follows by Lemma 1.3.

Notice that $\partial(e_\alpha) = \sum_{\mu\in\mathcal{M}_m^+} \alpha_\mu \partial(\mu)$ to prove the second statement. □

2. Generators.

Let us calculate the product between two elements $e_\alpha, e_\beta \in \mathcal{B}_{n,m,R}$ of the basis $\mathcal{B}_{n,m,R}$.

THEOREM 2.1 (Product Formula). — *Let $k, h \in \mathbb{N}$, $f_1 \dots, f_k$, $g_1, \dots, g_h \in A_R(m)$ and $t_1, \dots, t_k, s_1, \dots, s_h$ be commuting independent variables. Set as in (1.1)*

$$e_\alpha(\mathbf{f}) := e_{(\alpha_1,\dots,\alpha_k)}(f_1, \dots, f_k) \text{ and } e_\beta(\mathbf{g}) := e_{(\beta_1,\dots,\beta_h)}(g_1, \dots, g_h).$$

Then

$$e_\alpha(\mathbf{f})e_\beta(\mathbf{g}) = \sum_\gamma e_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{fg}),$$

where $\mathbf{fg} := (f_1g_1, f_1g_2, \dots, f_1g_h, f_2g_1, \dots, f_2g_h, \dots, f_kg_h)$ and $\gamma := (\gamma_{10}, \dots, \gamma_{k0}, \gamma_{01}, \dots, \gamma_{0h}, \gamma_{11}, \gamma_{12}, \dots, \gamma_{kh})$ are such that

$$\begin{cases} \gamma_{ij} \in \mathbb{N} \\ \mid \gamma \mid \leqslant n \\ \sum_{j=0}^h \gamma_{ij} = \alpha_i \ \text{ for } \ i = 1, \dots, k \\ \sum_{i=0}^k \gamma_{ij} = \beta_j \ \text{ for } \ j = 1, \dots, h. \end{cases}$$

Proof. — The result follows from

$$\left(\sum_{\sum \alpha_j \leqslant n} \prod_{j=1}^k t_j^{\alpha_j} e_\alpha(\mathbf{f})\right)\left(\sum_{\sum \beta_l \leqslant n} \prod_{l=1}^h s_l^{\beta_l} e_\beta(\mathbf{g})\right)$$

$$= \left(\sum_\alpha t^\alpha e_\alpha(\mathbf{f})\right)\left(\sum_\beta s^\beta e_\beta(\mathbf{g})\right)$$

$$= \prod_{i=1}^n \left(1 + \sum_{j=1}^k t_j f_j(i)\right) \prod_{i=1}^n \left(1 + \sum_{l=1}^h s_l g_l(i)\right)$$

$$= \prod_{i=1}^n \left(1 + \sum_{j=1}^k t_j f_j(i) + \sum_{l=1}^h s_l g_l(i) + \sum_{j,l} t_j s_l f_j(i) g_l(i)\right).$$

Introduce the new variables u_{jl} with $j = 1, \dots, k$ and $l = 1, \dots, h$, then

$$\prod_{i=1}^n \left(1 + \sum_{j=1}^k t_j f_j(i) + \sum_{l=1}^h s_l g_l(i) + \sum_{j,l} t_j s_l f_j(i) g_l(i)\right)$$

$$= \prod_{i=1}^n \left(1 + \sum_{j=1}^k t_j f_j(i) + \sum_{l=1}^h s_l g_l(i) + \sum_{j,l} u_{jl}(i) g_l(i)\right)$$

$$= \sum_\gamma v^\gamma e_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{fg})$$

where v is the cumulative variable t, s, u. Then substitute $u_{jl} = t_j s_l$ to obtain

$$\begin{aligned}\sum_\gamma v^\gamma e_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{fg}) \\ = \sum_\gamma \left(\prod_{a=1}^{k} t_a^{\gamma_{a0}} \prod_{b=1}^{h} s_b^{\gamma_{0b}} \prod_{a=1}^{k} \prod_{b=1}^{h} (t_a s_b)^{\gamma_{ab}} e_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{fg}) \right),\end{aligned}$$

where $\mathbf{fg} = (f_1 g_1, f_1 g_2, \ldots, f_k g_1, \ldots, f_k g_h)$ and γ satisfy the condition of the theorem.

Example 2.2. — Let us calculate in $A_R(2,3)^{S_2}$

$$e_{(1,1)}(a,b)e_2(c) = \sum_{0 \leqslant k,h \leqslant 1} e_{(1-k,1-h,2-k-h,h,k)}(a,b,c,ac,bc) = e_{(1,1)}(ac,bc),$$

since $1-k+1-h+2-k-h+h+k = 4-k-h \leqslant 2$.

Corollary 2.3. — *Let $k \in \mathbb{N}$, $a_1, \ldots, a_k \in A_R(m)$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ with $\sum \alpha_j \leqslant n$. Then $e_{(\alpha_1,\ldots,\alpha_k)}(a_1, \ldots, a_k)$ belongs to the subring of $A_R(n,m)^{S_n}$ generated by the $e_i(\mu)$, where $i = 1, \ldots, n$ and μ is a monomial in the $a_1, \ldots, a_k$.*

Proof. — We prove the claim by induction on $\sum_j \alpha_j$ (notice that $1 \leqslant k \leqslant \sum_j \alpha_j$) assuming that $\alpha_i > 0$ for all i. If $\sum_j \alpha_j = 1$ then $k = 1$ and $e_{(\alpha_1,\ldots,\alpha_k)}(a_1, \ldots, a_k) = e_1(a_1)$. Suppose the claim true for all $e_{(\beta_1,\ldots,\beta_h)}(b_1, \ldots, b_h)$ with $b_1, \ldots, b_h \in A_R(m)$ and $\sum_i \beta_i < \sum_j \alpha_j$. Let $k, a_1, \ldots, a_k, \alpha$ be as in the statement, then we have by Theorem 2.1

$$\begin{aligned}e_{\alpha_1}(a_1)e_{(\alpha_2,\ldots,\alpha_k)}(a_2, \ldots, a_k) &= e_{(\alpha_1,\ldots,\alpha_k)}(a_1, \ldots, a_k) \\ &\quad + \sum e_\gamma(a_1, \ldots, a_k, a_1a_2, \ldots, a_1a_k),\end{aligned}$$

where

$$\gamma = (\gamma_{10}, \gamma_{01}, \ldots, \gamma_{0h}, \gamma_{11}, \gamma_{12}, \ldots, \gamma_{1h})$$

with $h = k-1$, $\sum_{j=0}^{h} \gamma_{1j} = \alpha_1$ with $\sum_{j=1}^{h} \gamma_{1j} > 0$, and $\gamma_{0j} + \gamma_{1j} = \alpha_j$ for $j = 1, \ldots, h$. Thus

$$\gamma_{10} + \gamma_{01} + \ldots + \gamma_{0h} + \gamma_{11} + \ldots + \gamma_{1h} = \sum_j \alpha_j - \sum_{j=1}^{h} \gamma_{1j} < \sum_j \alpha_j.$$

Hence

$$\begin{aligned}e_{(\alpha_1,\ldots,\alpha_k)}(a_1, \ldots, a_k) &= e_{\alpha_1}(a_1)e_{(\alpha_2,\ldots,\alpha_k)}(a_2, \ldots, a_k) \\ &\quad - \sum e_\gamma(a_1, \ldots, a_k, a_1a_2, a_1a_3, \ldots, a_1a_k),\end{aligned}$$

where $\sum_{r,s}\gamma_{rs} < \sum_j \alpha_j$. So the claim follows by induction hypothesis.

Example 2.4. — Consider $e_{(2,1)}(a,b)$ in $A_R(3,m)$ as in Example 1.2, then

$$e_{(2,1)}(a,b) = e_2(a)e_1(b) - e_{(1,1)}(a,ab) = e_2(a)e_1(b) - e_1(a)e_1(ab) + e_1(a^2b).$$

We now recall some basic facts about classical symmetric functions, for further reading on this topic see [M].

We have another distinguished kind of functions in Λ_R beside the elementary symmetric ones: the *power sums*.

For any $r \in \mathbb{N}$ the r-th power sum is

$$p_r := \sum_{i\geqslant 1} x_i^r.$$

Let $g \in \Lambda_R$, set $g \cdot p_r = g(x_1^r, x_2^r, \ldots, x_k^r, \ldots)$, this is again a symmetric function. Since the e_i generate Λ_R we have that $g \cdot p_r$ can be expressed as a polynomial in the e_i. In particular,

$$P_{h,k} := e_h \cdot p_k$$

is a polynomial in the e_i.

Proposition 2.5. — *For all $f \in A_R(m)$, and $k, h \in \mathbb{N}$, $e_h(f^k)$ belongs to the subring of $A_R(n,m)^{S_n}$ generated by the $e_j(f)$.*

Proof. — Let $f \in A_R(m)$ and consider $e_h(f^k) \in A_R(n,m)^{S_n}$, we have (see Introduction)

$$e_h(f^k) = e_h(f(1)^k, \ldots, f(n)^k) = P_{h,k}(e_1(f(1), \ldots, f(n)), \ldots, e_n(f(1), \ldots, f(n)))$$

and the result is proved.

We are now ready to prove Theorem 1 stated in the introduction.

Proof of Theorem 1. — Recall that a monomial $\mu \in \mathcal{M}_m^+$ is called *primitive* if it is not a power of another one and we denote by $\mathfrak{M}_m^+$ the set of primitive monomials. The elements $e_\alpha \in \mathcal{B}_{n,m,R}$, that form a R-basis by Proposition 1.4, can be expressed as polynomials in $e_i(\mu)$ with $i = 1, \ldots, n$ and $\mu \in \mathcal{M}_m^+$, by Corollary 2.3. If $\mu = \nu^k$ with $\nu \in \mathfrak{M}_m^+$, then $e_i(\mu)$ can be expressed as a polynomial in the $e_j(\nu)$, by Proposition 2.5. Since for all $\mu \in \mathcal{M}_m^+$ there exist $k \in \mathbb{N}$ and $\nu \in \mathfrak{M}_m^+$ such that $\mu = \nu^k$, we have that

$A(n,m)^{S_n}$ is generated as a commutative ring by the $e_j(\nu)$, where $\nu \in \mathfrak{M}_m^+$ and $j = 1, \dots, n$.

The theorem then follows by the following result due to Fleischmann [F]: the ring $A_R(n,m)^{S_n}$ is generated by elements of total degree $\ell \leqslant m(n-1)$, for any commutative ring R, with sharp bound if $n = p^s$ a power of a prime and $R = \mathbb{Z}$ or $p \cdot 1_R = 0$. If $R \supset \mathbb{Q}$ then the result follows from Newton's Formulas and a well–known result of H.Weyl (see [G], [W]). □

3. Relations.

We write a generating series for the orbits of monomials

$$G_n(t) := \prod_{i=1}^{n} \left(1 + \sum_{\mathcal{M}_m^+} t_\mu \mu(i)\right) = \sum_{\alpha, |\alpha| \leqslant n} t^\alpha e_\alpha(n), \tag{3.1}$$

where $\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}$ and $t^\alpha e_\alpha(n) = 0$ when $\alpha = 0$.

Recall the map $\pi_n : A_R(n,m) \longrightarrow A_R(n-1,m)$ defined by

$$\pi_n(x_i(j)) = \begin{cases} 0 & \text{if } j = n \\ x_i(j) & \text{if } j \leqslant n-1 \end{cases} \qquad \text{for all } i. \tag{3.2}$$

Then we have of course that $\pi_n(G_n(t)) = G_{n-1}(t)$, so that

$$\pi_n((e_\alpha)) = \begin{cases} e_\alpha & \text{if } \mid \alpha \mid < n \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Thus, by Proposition 1.4, for all $a \in \mathbb{N}^m$ the restriction

$$\pi_{n,a} : A_R(n,m,a) \longrightarrow A_R(n-1,m,a) \tag{3.4}$$

is such that

$$\pi_{n,a}(A_R(n,m,a)^{S_n}) = A_R(n-1,m,a)^{S_{n-1}} \tag{3.5}$$

and then $(A_R(n,m,a)^{S_n}, \pi_{n,a})$ is a projective sytem.

For any $a \in \mathbb{N}^m$ set

$$A_R(\infty,m,a) := \varprojlim A_R(n,m,a)^{S_n}, \tag{3.6}$$

where the projective limit is taken with respect to n over the above projective system and set

$$\tilde{\pi}_{n,a} : A_R(\infty,m,a) \longrightarrow A_R(n,m,a)^{S_n} \tag{3.7}$$

for the natural projection.

Set

$$A_R(\infty, m) := \bigoplus_{a \in \mathbb{N}^m} A_R(\infty, m, a) \tag{3.8}$$

and

$$\tilde{\pi}_n := \bigoplus_{a \in \mathbb{N}^m} \tilde{\pi}_{n,a}. \tag{3.9}$$

Similarly to the classical case ($m = 1$) and recalling (3.1), (3.3) we make an abuse of notation and set

$$e_\alpha := \lim_{\leftarrow} \ e_\alpha(n),$$

for any $\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}$. In the same way we set $e_j(f) := \lim\limits_{\leftarrow} \ e_j(f)$ with $j \in \mathbb{N}$, where $f \in A_R(m)^+$ is homogeneous of positive multidegree, so that $j\, \partial(f) = a$.

Proposition 3.1. — *Let $a \in \mathbb{N}^m$.*

(1) *The R-module $\ker \tilde{\pi}_{n,a}$ is the linear span of*

$$\{e_\alpha \in A_R(\infty, m, a) \ : \mid \alpha \mid > n\}.$$

(2) *The R-module homomorphisms $\tilde{\pi}_{n,a} : A_R(\infty, m, a) \to A_R(n, m, a)^{S_n}$ are onto for all $n \in \mathbb{N}$ and $A_R(\infty, m, a) \cong A_R(n, m, a)^{S_n}$ for all $n \geqslant \mid a \mid$.*

(3) *The R-module $A_R(\infty, m, a)$ is free with basis*

$$\{e_\alpha \ : \ \partial(e_\alpha) = a\},$$

(4) *The R-module $A_R(\infty, m)$ is free with basis*

$$\{e_\alpha \ : \ \alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}\}.$$

Proof. — (1) By (3.3) and (3.5), for all $a \in \mathbb{N}^m$, the following is a split exact sequence of R-modules

$$0 \longrightarrow \ker \pi_{n,a} \longrightarrow A(n, m, a)^{S_n} \overset{\pi_{n,a}}{\longrightarrow} A(n-1, m, a)^{S_{n-1}} \longrightarrow 0,$$

and the claim follows.

(2) If $\sum_{j=1}^m a_j < n$, then $\ker \tilde{\pi}_{n,a} = 0$, indeed

$$\partial(e_\alpha) = \sum_{\mu \in \mathcal{M}_m^+} \alpha_\mu \ \partial(\mu) = a \Longrightarrow \mid \alpha \mid \leqslant \sum_{j=1}^m a_j < n.$$

Hence $A(h,m,a)^{S_h} \cong A(b,m,a)^{S_b}$ where $b := \sum_{j=1}^m a_j$, for all $h \geqslant \sum_{j=1}^m a_j$ and the claim follows by (3.5).

(3) follows from (1) and (2).

(4) follows from (3) and (3.8) □

Remark 3.2. — Notice that $A_R(m)^{\otimes n} \cong A_R(n,m)$ as multigraded S_n-algebras by means of

$$f_1 \otimes \cdots \otimes f_n \leftrightarrow f_1(1)f_2(2)\cdots f_n(n) \tag{3.10}$$

for all $f_1,\ldots,f_n \in A_R(m)$. Hence $A_R(n,m)^{S_n} \cong TS^n(A_R(m))$, where $TS^n(\,-\,)$ denotes the symmetric tensors functor. Since $TS^n(A_R(m)) \cong R\bigotimes TS^n(A_{\mathbb{Z}}(m))$ (see [B]), we have

$$A_R(n,m)^{S_n} \cong R \otimes A_{\mathbb{Z}}(n,m)^{S_n} \tag{3.11}$$

for any commutative ring R.

We then work with $R = \mathbb{Z}$ and we suppress the $\mathbb{Z}$ subscript for the sake of simplicity.

Remark 3.3. — The $\mathbb{Z}$-module $A(\infty,m)$ can be endowed with a structure of $\mathbb{N}^m$-graded ring such that the π_n are $\mathbb{N}^m$-graded ring homomorphisms: the product $e_\alpha e_\beta$, where $\alpha,\beta \in \mathbb{N}^{(\mathcal{M}_m^+)}$, is defined by using the product formula of Theorem 2.1 with no upper bound on $\mid \gamma \mid$, where γ appears in the summation.

PROPOSITION 3.4. — *Consider the free polynomial ring*

$$C(m) := \bigoplus_{a\in\mathbb{N}^m} C(m,a) := \mathbb{Z}[e_{i,\mu}]_{i\in\mathbb{N},\mu\in\mathfrak{M}_m^+}$$

with multidegree given by $\partial(e_{i,\mu}) = \partial(\mu)i$.

Then the multigraded ring homomorphism

$$\sigma_m : \mathbb{Z}[e_{i,\mu}]_{i\in\mathbb{N},\mu\in\mathfrak{M}_m^+} \longrightarrow A(\infty,m)$$

given by

$$\sigma_m : e_{i,\mu} \mapsto e_i(\mu),\ \textit{for all } i\in\mathbb{N},\ \mu\in\mathfrak{M}_m^+$$

is an isomorphism, i.e. $A(\infty,m)$ *is freely generated as a commutative ring by the* $e_i(\mu)$*, where* $i\in\mathbb{N}$ *and* $\mu\in\mathfrak{M}_m^+$.

Proof. — Since we defined the product in $A(\infty,m)$ as in Theorem 2.1, it is easy to verify, repeating the reasoning of the previous section, that $A(\infty, m)$ is generated as a commutative ring by the $e_i(\mu)$, where $i \in \mathbb{N}$, $\mu \in \mathfrak{M}_m^+$. Hence σ_m is onto for all $m \in \mathbb{N}$.

Let $a \in \mathbb{N}^m$ and consider the restriction $\sigma_{m,a} : C(m,a) \longrightarrow A(\infty, m, a)$. It is onto as we have just seen. A $\mathbb{Z}$-basis of $C(m,a)$ is

$$\left\{ \prod_{i\in\mathbb{N},k\in\mathbb{N},\mu\in\mathfrak{M}_m^+} e_{i,\mu} \ : \ \sum_{i\in\mathbb{N},k\in\mathbb{N},\mu\in\mathfrak{M}_m^+} i\, k\, \partial(\mu) \ = \ a \right\}.$$

On the other hand, a $\mathbb{Z}$-basis of $A(\infty, m, a)$ is

$$\left\{ e_\alpha \ : \ \sum_{\alpha_\mu\in\mathbb{N},\mu\in\mathcal{M}_m^+} \alpha_\mu\, \partial(\mu) \ = \ a \right\}.$$

Let $\mu \in \mathcal{M}_m^+$, then there are an unique $k \in \mathbb{N}$ and an unique $\nu \in \mathfrak{M}_m^+$ such that $\mu = \nu^k$. Hence

$$\sum_{\alpha_\mu\in\mathbb{N},\mu\in\mathcal{M}_m^+} \alpha_\mu\, \partial(\mu) = \sum_{k\in\mathbb{N},\alpha_\mu\in\mathbb{N},\nu\in\mathfrak{M}_m^+} \alpha_\mu\, k\, \partial(\nu),$$

so that $C(m,a)$ and $A(\infty, m, a)$ have the same (finite) $\mathbb{Z}$-rank and thus are isomorphic via $\sigma_{m,a}$. □

Corollary 3.5. — *Let $R \supset \mathbb{Q}$ then $A_R(\infty, m)$ is a polynomial ring freely generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$.*

Proof. — By Proposition 3.4 and Theorem 1. □

Proof of Theorem 2. — (1) As before we set $R = \mathbb{Z}$ and the result follows by Remark 3.2, Proposition 3.4. and Proposition 3.1.

(2) By Proposition 3.1 the kernel of

$$A(\infty, m) \xrightarrow{\tilde{\pi}_n} A(n,m)^{S_n}$$

has basis $\{e_\alpha :\mid \alpha \mid > n\}$. Let V_k be the submodule of $A(\infty, m)$ with basis $\{e_\alpha :\mid \alpha \mid = k\}$. Let A_k be the sub-$\mathbb{Z}$-module of $\mathbb{Q} \otimes V_k$ generated by the $e_k(f)$ with $f \in A(m)^+$. Let $g : \mathbb{Q} \otimes V_k \longrightarrow \mathbb{Q}$ be a linear form identically zero on A_k. Then

$$0 = g(e_k(f)) = g\left(e_k\Big(\sum_{\mu\in\mathcal{M}_m^+} \lambda_\mu\, \mu \Big) \right) = \left(\sum_{|\alpha|=k} \Big(\prod_{\mu\in\mathcal{M}_m^+} \lambda_\mu^{\alpha_\mu} \Big)\, g(e_\alpha) \right),$$

for all $\sum_{\mu\in\mathcal{M}_m^+} \lambda_\mu\, \mu \in A(m)^+$. Hence $g(e_\alpha) = 0$ for all e_α with $\mid \alpha \mid = k$; thus $g = 0$. If $R \supset \mathbb{Q}$ the result then follows from Newton's formulas and Corollary 3.5. □

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