ANNALES

DE

L'INSTITUT FOURIER

Anna ERSCHLER

Critical constants for recurrence of random walks on G-spaces

Tome 55, nº 2 (2005), p. 493-509.

<http://aif.cedram.org/item?id=AIF_2005__55_2_493_0>

Ann. Inst. Fourier, Grenoble
55, 2 (2005), 493–509

CRITICAL CONSTANTS FOR RECURRENCE OF RANDOM WALKS ON G-SPACES

by **Anna ERSCHLER**

1. Introduction.

Let G be a finitely generated group, H its subgroup and and let μ be a symmetric probability measure on G. Consider the induced random walk on G/H. We want to know whether this random walk is transient or recurrent.

First recall known facts about the case $\operatorname{supp}\mu < \infty$. It is known (see *e.g.* [27]) that in this case recurrence of the random walk does not depend on μ. Therefore, in this situation the recurrence of the random walk depends only on the unlabelled Schreier graph of G/H. If, moreover, H is a normal subgroup, the answer to the question under consideration is given by a theorem of Varopoulos ([25], see also [26] or [27]). He proved, that a finitely generated group admits a non-degenerate recurrent random walk if and only if it contains $\mathbb{Z}$, $\mathbb{Z}^2$ or the trivial group as a finite index subgroup. Therefore, if $\operatorname{supp}\mu$ is finite and H is a normal subgroup of G, then the induced random walk on G/H is recurrent if and only if G/H is a finite extension of $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$.

In the case when H is not normal, there are much more situations when the induced simple random walk on G/H is recurrent. In fact, any

Keywords: Growth of groups, Grigorchuk groups, branch groups, random walks, recurrence, drift.
Math. classification: 20F65, 20E08, 60B15.

connected regular graph without loops of even degree is a Schreier graph for some (G, H) (see *e.g.* Theorem 5.3 in [22], where this is stated for finite graphs, and the same proof works for infinite graphs as well) and there are many recurrent regular graphs (besides $\mathbb{Z}$, $\mathbb{Z}^2$ and finite ones).

In this paper we will be interested in the case when $\operatorname{supp}\mu$ is not necessarily finite. It is not difficult to check (see Lemma 7.1 in [12] and Lemma 3.1 below) that G always admits a symmetric measure (which can be taken non-degenerate) for which the random walk on G/H is transient, whenever H is of infinite index in G.

Definition 1 (critical constant for recurrence of (G, H)). — *Given a finitely generated group G and a subgroup H let $c_{\mathrm{rt}}(G, H) = \sup\beta$ where* sup *is over all $\beta \geqslant 0$ such that there exists a symmetric measure μ on G such that the β-moment of μ is finite, that is*

$$\sum_{g\in G} l(g)^\beta \mu(g) < \infty \tag{$\star$}$$

(here l denotes some word metric of G) and such that the induced random walk on G/H is transient. We say that $c_{\mathrm{rt}}(G, H)$ is the critical constant for recurrence of (G, H).

Note that in the definition we have not specified the word metric l on G, but it is clear that the condition $(\star)$ does not depend of the choice of such a metric.

We do not insist in the definition that μ is non-degenerate, but this is not important. For any transient measure there exists a non-degenerate transient measure with the same decay, as follows from [7].

Note also that the value of $c_{\mathrm{rt}}(G, H)$ does not change if one replaces the moment condition $(\star)$ by a tail condition

$$\mu(G \setminus B(e, R)) \leqslant \frac{K}{R^\alpha} \tag{$*$}$$

for some $k > 0$ and any $R \geqslant 1$ ($B(e, R)$ denotes the ball of radius R in some word metric l of G). In fact, suppose that μ satisfies $(*)$. Then for any $\beta' < \beta$ μ satisfies $(\star)$ of Definition 1. On the other hand, if μ satisfies $(\star)$ of Definition 1 for some $\beta > 0$, then μ satisfies $(*)$ for the same value of β.

The case when H is a normal subgroup is easy to describe: if G/H is not a finite extension of $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$, then $c_{\mathrm{rt}}(G, H) = \infty$ (as follows from a theorem of Varopoulos cited above) and it is not difficult to check that

$$c_{\mathrm{rt}}(G,H) = d, \text{ if } G/H \text{ is a finite extension of } \mathbb{Z}^d \text{ for } d = 0, 1 \text{ or } 2\ .$$

In fact, all symmetric measures on $\mathbb{Z}$ with finite first moment and all symmetric measures on $\mathbb{Z}^2$ with finite second moment are recurrent (see [24], Theorem 8.1). This shows that $c_{\mathrm{rt}}(\mathbb{Z}, \{e\}) \leqslant 1$ and that $c_{\mathrm{rt}}(\mathbb{Z}^2, \{e\}) \leqslant 2$. For any $1 > \epsilon > 0$ consider a symmetric measure on $\mathbb{Z}$ such that $\mu(n) = C/n^{2-\epsilon}$. This random walk is transient ([24], Example 8.2). Note that $\mu(\mathbb{Z} \setminus B(e,R)) \leqslant K/n^{1-\epsilon}$ for some $K > 0$, and hence $c_{\mathrm{rt}}(\mathbb{Z}, \{e\}) \geqslant 1$. Now consider a symmetric probability ν measure on $\mathbb{Z}$ such that $\nu(i) = c/|i|^{3-\epsilon}$, $\nu(0) = 0$ and the measure μ on $\mathbb{Z}^2$ such that

$$\mu(a,b) = \nu(a) \text{ if } b = 0, \nu(b) \text{ if } a = 0 \text{ and } 0 \text{ otherwise.}$$

Using the recurrence criterion [14], [24] it is easy to check that for any $0 < \epsilon < 1$ μ is transient, and hence $c_{\mathrm{rt}}(\mathbb{Z}^2, \{e\}) = 2$. (There is also another more general way to construct transient radial measures with a prescribed moment on a group of polynomial growth – see Lemma VI.4.2 in [26]).

So we will be mainly interested in the case when H is not normal. If G/H is finite, then $c_{\mathrm{rt}}(G,H)$ is obviously equal to 0, but for infinite index subgroups we have the following

Theorem 1. — *Let G/H be infinite.*

(1)

$$c_{rt}(G,H) \geqslant \frac{1}{2}.$$

(2) Suppose that for some symmetric non-degenerate finitely supported measure ν on G the drift of the corresponding random walk on G satisfies

$$L_{G,\nu}(k) \leqslant Ck^{\xi}$$

for some $C > 0$ and any $k \geqslant 0$. Then

$$c_{\mathrm{rt}}(G,H) \geqslant \frac{1}{2\xi}.$$

(Recall that the drift (or rate of escape) $L_{G,\nu} = L_{G,S,\nu}(k)$ of a random walk (G,ν) is the expectation $\mathbf{E}_{\nu^{*k}} l_S(g)$, where ν^{*k} denotes k-th convolution of ν and S is some finite generating set).

This does not look surprising since one could expect that the case $G/H = \mathbb{Z}$ is the smallest, that is that $c_{\mathrm{rt}}(G, H) \geqslant c_{\mathrm{rt}}(\mathbb{Z}, \{e\}) \geqslant 1$ for any infinite index subgroup H of G. However, this is not true.

THEOREM 2.— *Let G_1 be the first Grigorchuk group (see [15], [16] or Section 4 below for the definition of G_1 and its action on $(0,1]$) and $Stab(1)$ be the stabilizer of 1 for its action on $(0,1]$. Then*

$$c_{\mathrm{rt}}(G_1, Stab(1)) < 1.$$

We say that G/H is very small if $c_{\mathrm{rt}}(G,H) < 1$. Theorem 2 above says that $G_1/\mathrm{Stab}(1)$ is very small (but it is clear that this space is infinite). Here we stated Theorem 2 for the first Grigorchuk group G_1, but in fact it is valid for a larger class of groups acting on a segment, see the remark at the end of Section 4.

Recall that the growth function of G with respect to a generating set S is $v(n) = v_{G,S}(n) = \#\{g \in G : l_S(g) \leqslant n\}$, where l_S is the word metric in G. In the proof of Theorem 2 we will see that the critical constants for recurrence are related to growth of groups, namely to the limit of $\log\log v(n)/\log(n)$, where $v(n)$ is the growth function of a certain group. Thus the first part of Theorem 1 can be compared with the following (still unproven)

CONJECTURE [Grigorchuk].— *Let G be a group of not polynomial growth. Then the growth function of G satisfies*

$$v(n) \geqslant \exp\left(An^{1/2}\right)$$

for some $A > 0$ and any sufficiently large n.

It would be interesting to calculate $c_{\mathrm{rt}}(G, \mathrm{Stab}(1))$ for G_1 or other branch groups. In [12] it was shown that under certain restriction on the action of the branch group G of intermediate growth, the Schreier graph of $G/\mathrm{Stab}(1)$ is recurrent, so we may expect that for many among those groups $c_{\mathrm{rt}}(G, \mathrm{Stab}(1)) < \infty$.

The paper has the following structure. In Section 2 we prove some basic properties of c_{rt}. In Section 3 we prove Theorem 1 and in Section 4 we prove Theorem 2. In Section 4 we also show that $c_{\mathrm{rt}}(G,H)$ is not defined by the unlabelled Schreier graph of G/H. That is, there exist G, H

and G', H' such that the unlabelled Schreier graphs of G/H and G'/H' are the same, but $c_{\text{rt}}(G,H) \neq c_{\text{rt}}(G',H')$. Note that in a certain sense Schreier graphs provide less information than the Cayley graphs (of G/H in the case when H is a normal subgroup of G). Thus the critical constant provides an additional invariant of the G-space G/H for the case when H is not normal. In section 5 we show that the critical constant can be used to estimate the growth of groups and the drift for random walks on groups.

2. Basic properties of c_{rt}.

Below we always suppose that G, G_1 and G_2 are finitely generated groups.

LEMMA 2.1. —

(1) *If $H \subset G \subset G_1$, then*

$$c_{\text{rt}}(G,H) \leqslant c_{\text{rt}}(G_1,H).$$

(2) *If $H \subset G \subset G_1$ and $[G : G_1] < \infty$, then*

$$c_{\text{rt}}(G,H) = c_{\text{rt}}(G_1,H).$$

(3) *If $H \subset H_1 \subset G$, then*

$$c_{\text{rt}}(G,H) \geqslant c_{\text{rt}}(G,H_1).$$

(4) *If $H \subset H_1 \subset G$ and $[H_1 : H] < \infty$, then*

$$c_{\text{rt}}(G,H) = c_{\text{rt}}(G,H_1).$$

(5) *Let $G_2 = G/N$ where N is a normal subgroup of G and $\phi_N : G \to G_2$ be the canonical projection. Suppose that H_2 is a subgroup of G_2 and put $H = \phi_N^{-1}(H_2)$. Then*

$$c_{\text{rt}}(G,H) = c_{\text{rt}}(G_2,H_2).$$

(6) *If there exist $g \in G$ such that for any $k \geqslant 1$ $g^k \notin H$, then* $c_{\text{rt}}(G,H) \geqslant 1$.

(7) *If the normal closure of H is of infinite index in G, then* $c_{\mathrm{rt}}(G,H) \geqslant 1$.

(8) *For any $H \subset G$ there exists a subgroup A of the free group F_2 such that*

$$c_{\mathrm{rt}}(G,H) = c_{\mathrm{rt}}(F_2, A).$$

Proof. —

(1) Consider a system of generators S and S_1 of G and G_1 respectively such that $S \subset S_1$. It is clear that $B_{G,S}(e,R) \subset B_{G_1,S_1}(e,R)$ Take any symmetric measure μ on G such that

$$\mu(G \setminus B_{G,S}(e,R)) \leqslant \frac{K}{R^\alpha}.$$

We can consider the same measure as a measure on G_1 and it satisfies

$$\mu(G \setminus B_{G_1,S_1}(e,R)) \leqslant \frac{K}{R^\alpha}.$$

(2) In view of the previous statement it is sufficient to check that $c_{\mathrm{rt}}(G,H) \geqslant c_{\mathrm{rt}}(G_1,H)$. Take some symmetric measure μ on G_1. It is clear that G is a recurrent set for the corresponding random walk on G_1. Let ν be the measure on G such that $\nu(g)$ is equal to the probability that g is the first element of G hit by the random walk (G_1,μ). It is clear that if $(\mu, G_1/H)$ is transient, then $(\nu, G/H)$ is also transient. It is not difficult to check that

$$\nu(G \setminus B_{G,S}(e,R)) \leqslant K_1 \mu(G_1 \setminus B_{G_1,S_1}(e, K_2 R))$$

for some positive K_1 and K_2. Hence if for some positive K, we have

$$\mu(G_1 \setminus B_{G_1,S_1}(e,R)) \leqslant \frac{K}{R^\alpha},$$

then there exists $K_3 > 0$ such that

$$\nu(G \setminus B_{G,S}(e,R)) \leqslant \frac{K_3}{R^\alpha}.$$

(3) Take any measure μ such that H_1 is transient for the corresponding random walk. Obviously, H is also transient for this random walk.

(4) It is sufficient to check that $c_{\mathrm{rt}}(G, H) \leqslant c_{\mathrm{rt}}(G, H_1)$. Suppose that H is transient for some random walk on G. Then H_1 is also transient. In fact, since H is has finite index in H_1, then if the random walk visits H_1 infinitely many times with positive probability, then it visits H also infinitely many times with positive probability.

(5) Note that if some measure on G satisfies $(*)$, then its projection onto G_2 also satisfies $(*)$. On the other hand, if μ is a measure on G_2 satisfying $(*)$, for any $g_2 \in G_2$ chose $g \in G$ such that $l_G(g) = l_{G_2}(g_2)$ and consider the measure ν on G such that for any g_2 $\nu(g) = \nu(g_2)$. It is clear, that ν satisfies $(*)$ and that H_2 is transient for μ if and only if H is transient for ν.

(6) This follows from 1 and from the fact, that $c_{\mathrm{rt}}(\mathbb{Z}, \{e\}) = 1$.

(7) This follows from 5 and from the theorem of Varopoulos, cited in the introduction.

(8) This follows from 5 and 2. □

Remark. — Note, however, that under the assumption of 4 of the lemma above it is possible that the Schreier graphs of G/H and H/H_1 are not quasi-isometric. In fact, let $G = \langle a, b : a^2 = b^2 = e \rangle$ be the infinite dihedral group, $H = e$ and $H_1 = \langle a \rangle$. Then the Schreier graph of G/H is quasi-isometric to $\mathbb{Z}$, but the Schreier graph of G/H_1 is quasi-isometric to a ray $\mathbb{Z}_+$.

3. Proof of Theorem 1.

Let ν be a symmetric non-degenerate measure on G with finite support. For any $0 < \epsilon < 1/2$ define the measure μ_ϵ by

$$\mu_\epsilon = \frac{1}{C} \sum_{i=1}^{\infty} \frac{\nu^{*i}}{i^{3/2-\epsilon}},$$

where ν^{*i} denotes the i-th convolution of ν and $C = \sum_{i=1}^{\infty} \frac{1}{i^{3/2-\epsilon}}$.

It is clear that μ_ϵ is a symmetric probability measure on G.

Lemma 3.1. — *For any infinite index subgroup H in G and any $0 < \epsilon < 1/2$ the induced random walk $(G/H, \mu_\epsilon)$ is transient.*

Proof. — The proof of this lemma is based on the following lemma. □

LEMMA 3.2. — *Let* Γ *be an infinite regular graph and* v *be a vertex of* Γ*. The return probability for the simple random walk on* Γ *satisfies*

$$p_n^\Gamma(v,v) = O\left(\frac{1}{n^{1/2}}\right)$$

Proof. — See *e.g.* Corollary 14.6 in [27] □

Now we return to the proof of Lemma 3.1.

Since $[G:H]=\infty$, the Schreier graph of G/H is infinite, and hence by the previous lemma we know that

$$\nu^{*n}(H) = O\left(\frac{1}{n^{\frac{1}{2}}}\right).$$

Consider the measure $\bar{\mu}_\epsilon$ on $\mathbb{Z}^+$ such that $\bar{\mu}_\epsilon(n) = \dfrac{1}{Cn^{3/2-\epsilon}}$.

Note that by the construction of μ_ϵ

$$\mu_\epsilon^{*n}(H) = C_1 \sum_{i\in\mathbb{N}} \bar{\mu}_\epsilon^{*n}(i)\nu^{*i}(H) \leqslant C_2 \sum_{i\in\mathbb{N}} \frac{\bar{\mu}_\epsilon^{*n}(i)}{i^{1/2}}$$

for some $C_1, C_2 > 0$.

By the Stable Law for $\bar{\mu}_\epsilon$ (see *e.g.* [14]) this is not greater than

$$C_3 \frac{1}{n^{\frac{2}{1-2\epsilon}}} \sum_{i=1}^{n^{\frac{2}{1-2\epsilon}}} \frac{1}{i^{1/2}} = C_4 \frac{1}{n^{\frac{1}{1-2\epsilon}}}$$

for $C_3, C_4 > 0$.

Hence for any $\epsilon > 0$ there exists $C_5 > 0$ such that

$$\sum_{i+1}^{\infty} \mu_\epsilon^{*n}(H) \leqslant C_5 \sum_{i=1}^{\infty} \frac{1}{i^{1+\epsilon}} < \infty.$$

This implies that μ_ϵ induces a transient random walk on G/H. □

LEMMA 3.3. —

(1)

$$\mu_\epsilon(G \setminus B(e,R)) \leqslant \text{Const}\frac{1}{R^{1/2-\epsilon}}$$

(2) *Suppose, moreover, the drift of the random walk G,ν satisfies*

$$L_{G,\nu}(k) \leqslant C_1 k^\xi$$

for any $k \geqslant 0$.

Then for any $\beta \leqslant 1$ such that $\beta < (1/2-\epsilon)/\xi$ the β-moment of μ_ϵ is finite.

Proof. —

(1) Note that $\operatorname{supp}\nu^{*r} \subset B(e,R)$ for any $r < R$ (here ν is as in the definition of μ_ϵ and we consider the word metric with respect to the support of ν), and hence there exists Const > 0 such that

$$\mu_\epsilon(G \setminus B(e,R)) \leqslant \sum_{i=R}^{\infty} \frac{1}{n^{3/2-\epsilon}} \leqslant \text{Const}\frac{1}{R^{1/2-\epsilon}}.$$

(2) We know that

$$\sum_{g\in G} \nu^{*i}(g) l(g) \leqslant C_1 i^\xi.$$

Since $\beta \leqslant 1$ this implies that

$$\sum_{g\in G} \nu^{*i}(g) l(g)^\beta \leqslant C_2 i^{\xi\beta}.$$

Therefore, by definition of μ_ϵ

$$\sum_{g\in G} \mu_\epsilon(g) l(g)^\beta = \frac{1}{C}\sum_{i=1}^{\infty} \frac{1}{i^{3/2-\epsilon}} \sum_{g\in G} \nu^{*i}(g) l(g)^\beta \leqslant C_3 \sum_{i=1}^{\infty} \frac{i^{\xi\beta}}{i^{3/2-\epsilon}}.$$

Since $\beta < (1/2-\epsilon)/\xi$, $3/2-\epsilon-\xi\beta > 1$, and hence the last sum is finite. □

The first part of Theorem 1 follows from the first part of the previous lemma, from Lemma 3.1 and definition of the critical constant. The second part of Theorem 1 follows from Lemma 3.1 and the second part of the previous lemma. □

4. Proof of Theorem 2.

First we recall the definition of certain Grigorchuk groups. We start with introducing some notation. We consider transformations of the interval $(0,1]$. Let a be the cyclic permutation of the half-intervals of $(0,1]$. That is,

$$a(x) = x + \frac{1}{2} \text{ for } x \in (0, \tfrac{1}{2}] \text{ and } a(x) = x - \frac{1}{2} \text{ for } x \in (\tfrac{1}{2}, 1].$$

Given for any $i \geqslant 1$ a bijective map $m_i : (0,1] \to (0,1]$, we consider an element g that acts on $(0,1]$ as follows. On $(0, \frac{1}{2}]$ it acts as m_1 on $(0,1]$, on $(\frac{1}{2}, \frac{3}{4}]$ it acts as m_2 on $(0,1]$, on $(\frac{3}{4}, \frac{7}{8}]$ it acts as m_3 on $(0,1]$ and so on.

More precisely, take $r \geqslant 1$ and put $\Delta_r = \left(1 - \frac{1}{2^{(r-1)}}, 1 - \frac{1}{2^r}\right]$.

Consider the linear map α_r from Δ_r onto $(0,1]$. Note that $(0,1]$ is a disjoint union of Δ_r $(r \geqslant 1)$. The map $g : (0,1] \to (0,1]$ is defined by

$$g(x) = \alpha_r^{-1}(m_r(\alpha_r(x)))$$

for any $x \in \Delta_r$.

In this situation we write

$$g = m_1 m_2 m_3 \dots$$

Let $b = aaIaaIaaI\dots$, $c = aIaaIaaIaaIa\dots$, $d = IaaIaaIaaIaa\dots$ (here I denotes the identity map on $(0,1]$. By definition [15] the first Grigorchuk group G_1 is the group generated by a, b, c and d.

We denote by Stab(1) the subgroup of G_1 which stabilize 1 with respect to the defined action of G_1 on $(0,1]$.

We will consider also a group G_2, generated by $a, b, c, d, b_1, c_1, d_1$, where a, b, c, d are as above and $b_1 = aaIaIaaaIaIa\dots$, $c_1 = aIaIaaaI$

$aIaa\ldots$ and $d_1 = IaaaaIIaaaaI\ldots$ (b_1, c_1 and d_1 are periodic with period 6). The group G_2 was already considered in [12].

Analogously to the case of the first Grigorchuk group [16] one can show that there exists $\gamma < 1$ such that

$$v_{G_2}(n) \leqslant \exp\left(Cn^{\gamma}\right)$$

for some $C > 0$ and any sufficiently large n.

Hence for the proof of Theorem 2 it is sufficient to prove the following proposition. Before stating this proposition recall that if G acts on $(0, 1]$ we can consider $g \in G$ as functions on this interval and in particular for any $x \in (0, 1]$ we can speak about the germ of g in the left neighbourhood of x, this germ is denoted by $\mathrm{germ}_x g$. Note that for any x the germ $\mathrm{germ}_x a$ is constant. For $g = b, c$ or d in the definition of the first Grigorchuk group $\mathrm{germ}_x g$ is constant for $x \neq 1$ and non-constant for $x = 1$.

PROPOSITION 4.1. — *Let Γ_1 and Γ_2 be two Grigorchuk groups generated by finite symmetric sets T_1 and T_2 respectively such that $a \in T_1, T_2$ and for any $g, h \in T_1$ (or respectively T_2) either $gh = e$ or $gh \in T_1$ (respectively T_2). Suppose that for any $g \in T_i$ $(i = 1, 2)$ and any $x \neq 1$ the germ of g in the left neighbourhood of x is non-constant. Suppose also that T_1 is a proper subset of T_2 (and hence Γ_1 is a subgroup of Γ_2) and that for some $g \in T_2$ $\mathrm{germ}_1 g \neq \mathrm{germ}_1 h$ for any $h \in T_1$.*

Suppose also that the growth function of the group Γ_2 satisfies

$$v_{\Gamma_2,T_2}(n) \leqslant \exp\left(Cn^{\gamma}\right)$$

for some $\gamma < 1$, $C > 0$ and infinitely many $n \geqslant 1$. Then $c_{\mathrm{rt}}(\Gamma_1/Stab(1)) \leqslant \gamma$.

Proof. — Suppose not. Then there exists $\beta > \gamma$ and a symmetric measure μ on Γ_1 satisfying $(\star)$ of Definition 1 and such that $\mathrm{Stab}\,(1)$ is transient for μ. Consider a symmetric finitary measure μ' on Γ_2 such that its support generates Γ_2 and put $\nu = (\mu + \mu')/2$. Then $\mathrm{Stab}\,(1)$ is transient for ν, as follows from [7]. Obviously, ν satisfies $(\star)$ of Definition 1.

The assumption of the Proposition implies that the group of germs of Γ_1 (as defined in [12]) is stricly smaller than the group of germs of Γ_2. Hence Γ_1, Γ_2 and ν satisfy the assumption of Proposition 2 of [12], and hence the Poisson boundary of (Γ_2, ν) is non-trivial.

Recall that the entropy $H(\nu)$ the entropy of a probability measure ν on a countable space X is $H(\nu) = -\sum_x \nu(x)\ln(\nu(x))$.

Recall also that the *entropy of the random walk* [avez] (G,μ) (G is a finitely generated group and μ is a probability measure on G) is

$$h(\mu) = \lim_{n\to\infty} H(\mu^{*n})/n.$$

Note that the entropy of ν is finite, and hence by the entropy criterion [20], [8] non-triviality of the Poisson boundary implies that the entropy of the random walk is positive. The condition $(\star)$ says that

$$C_1 = \sum \nu(y) l(y)^\beta < \infty.$$

Note that since x^β is concave and since $l(g_1 g_2) \leqslant l(g_1) + l(g_2)$ for any $g_1, g_2 \in G$ this implies that

$$\sum \nu^{*k}(y) l(y)^\beta \leqslant C_1 k.$$

Therefore,

$$\nu^{*k}(G \setminus B_{\Gamma_2}(e, (2C_1)^{1/\beta} k^{1/\beta}) \leqslant \frac{1}{2}$$

and hence for some $C > 0$

$$\nu^{*k}(B_{\Gamma_2}(e, Ck^{1/\beta})) \geqslant \frac{1}{2}.$$

Since the entropy is positive, by Shannon type theorem [20] this implies that

$$\# B_{G_2}(e, Ck^{1/\beta}) \geqslant \exp(K_1 k)$$

for some $K_1 > 0$. Hence

$$v_{\Gamma_2}(n) \geqslant \exp(C n^\beta)$$

for some $C > 0$ and any sufficiently large n.

But this is in contradiction with the assumption of the proposition. □

Claim .— *The critical constant of (G,H) is not defined by the unlabelled Schreier graph of G/H. That is, there exist G, H and G', H' such that the unlabelled Schreier graphs of G/H and G'/H' are the same, but $c_{\mathrm{rt}}(G,H) \neq c_{\mathrm{rt}}(G',H')$.*

Proof. — Note that the Schreier graph of $G_1/\mathrm{Stab}\,(1)$ with respect to the generating set $S = \{a, b, c, d\}$ is isometric to a ray. On the other hand, consider a group G_3 which is generated by a and $d = aaaaaaaa\ldots$. The Schreier graph of $G_3/\mathrm{Stab}\,(1)$ is also isometric to a ray, but by 6 of Lemma 2.1 $c_{\mathrm{rt}}(G_3, \mathrm{Stab}\,(1)) \geqslant 1$, since $(af)^k \notin \mathrm{Stab}\,(1)$ for any $k \neq 1$. (the latter property of af is implicitly stated in [16], see also [12]).

In the example above the Schreier graph are isometric, but not the same (that is, the multiplicity of its edges and loops differ for G and G').

Now consider G' that is generated by a, $b' = aIaIaI\ldots$ and $c' = IaIaIa\ldots$ and let $S' = \{a, b, c, d = bc\}$. Then the Schreier graphs of $G/\mathrm{Stab}\,(1)$ and $G'/\mathrm{Stab}\,(1)$ with respect to S and S' are exactly the same, but $c_{\mathrm{rt}}(G', \mathrm{Stab}\,(1)) \geqslant 1$ and $c_{\mathrm{rt}}(G, \mathrm{Stab}\,(1)) < 1$. □

Remark. — Let now a_p be the cyclic permutation of p-th subintervals of $(0, 1]$ and G be a group generated by $a_p, g_1, \ldots g_k$ such that $g_i = a^{k_{1,i}}, I, I, \ldots a^{k_{2,i}}, I, I \ldots a^{k_{3,i}} \ldots$ (each time there are $p-1$ I's between $a^{k_{j,i}}$ and $a^{k_{j+1,i}}$. Suppose that there exist $M > 0$ such that for any g lying in the subgroup generated by $g_1, \ldots g_k$, $g = a^{k_1}, I, I, \ldots a^{k_2}, I, I \ldots a^{k_3} \ldots$, for any N at least one of the numbers $k_N, k_{N+1}, \ldots k_{N+M}$ is equal to 0. (This condition ensures that a finite index subgroup of G admits a contractive up to additive constant map to a direct sum of several copies of groups, similar to G, see [17]). Then $G/\mathrm{Stab}\,(1)$ is very small. The proof is analogous to the proof of Theorem 2.

5. Applications.

The corollary below follows immediately from Theorem 1.1 and Theorem 1.2.

COROLLARY 1. — *For the drift $L(n)$ of any simple random walk on G_1 there exists $\kappa > 1/2$ such that*

$$L(n) \geqslant Cn^{\kappa}$$

for some $C > 0$ and infinitely many $n \geqslant 1$.

Note, however, that there are groups of exponential growth (for example, the solvable Baumslag Solitar group $\langle a, b : b^{-1}ab = b^k\rangle$ (for $k \geqslant 2$)

or the lamplighter group $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$) for which the drift of simple random walk is asymptotically equal to $\sqrt{n}$ (for further examples of the drift see [9], [10]).

Now we state a general lemma, connecting the growth and the drift of the random walk (which in particular gives an upper bound for the drift considered in Corollary 1 above).

Lemma 5.1. — *Suppose that the growth function of some group satisfies*

$$v_G(n) \leqslant \exp{(CR^\gamma)}$$

for some $C > 0$ and any $n \geqslant 0$. Then the drift of any simple random walk (G, μ) satisfies

$$L_{G,\mu}(n) \leqslant Dn^{1/(2-\gamma)},$$

where in the definition of the drift we consider the word metric, corresponding to $\operatorname{supp} \mu$.

Proof. — In [13] (Lemma 7) it is shown that $L_{G,\mu}(n) \leqslant B\sqrt{nH(n)}$ for some $B > 0$ and any $n \geqslant 0$.

Let $a_i^{(n)} = \Pr_{\mu^{*n}}[l(g) = i]$. Then by definition of the drift

$$L(n) = \sum_{i=0}^{n} i a_i^{(n)}.$$

Comparing μ^{*n} with the measure which is equidistributed on every sphere in the group we get

$$H(n) \leqslant \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)/a_i^{(n)}) =$$

$$\sum_{i=1}^{n} a_i^{(n)} \ln(v(i)) + \sum_{i=1}^{n} a_i^{(n)} (-\ln(a_i^{(n)})) \leqslant \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)) + \ln(n).$$

Hence the assumption of the lemma implies that

$$H(n) \leqslant A\, \mathbf{E}\, l(g)^\gamma + \ln(n) \leqslant L(n)^\gamma + \ln(n)$$

for some $A > 0$.

Therefore, for some $D > 0$ and any $n \geqslant 0$

$$L(n) \leqslant D\sqrt{n}L(n)^{\gamma/2} + \ln(n).$$

This implies the statement of the lemma. □

Now let $G(m)$ be the group generated by the first Grigorchuk group G_1 and by

$$g = \underbrace{aaIaaIaaI\ldots}_{3m} III \underbrace{aaIaaIaaI\ldots}_{3m} III\ldots$$

Corollary 2. — *Let $G = G_2$ (where G_2 is the group defined in the previous section) or $G = G(m)$ (defined above) for some $m > 1$*

Then the growth function of the group G satisfies

$$v_G(n) \geqslant \exp\left(C_\delta n^{2/3-\delta}\right)$$

for any $\delta > 0$, some $C_\delta > 0$ and infinitely many n.

Proof. — Suppose not. Then for some $x < 2/3$ and $K > 0$

$$v_G(n) \leqslant \exp\left(Kn^x\right)$$

for any $n \geqslant 0$. By Proposition 4.1 this shows that $c_{\mathrm{rt}}(G_1, \mathrm{Stab}\,(1)) \leqslant x$. By Theorem 1.1 this implies that the drift of any simple random walk on G_1 satisfies

$$L_{G_1}(k) \geqslant C_1 k^{1/2x}$$

for infinitely many k. On the other hand, $v_{G_1}(n) \leqslant v_G(n) \leqslant \exp\left(Kn^x\right)$, and hence by the previous lemma

$$L_{G_1}(k) \leqslant C_2 k^{1/(2-x)}.$$

But since $x < 2/3$, $1/(2-x) < 1/2x$, and this contradiction completes the proof of the corollary. □

Remark. — As it was already mentioned, if $G = G_1$, G_2 or $G(i)$, then the growth function of G satisfies $v_G(n) \leqslant \exp\left(n^\alpha\right)$ for some $\alpha < 1$ and any sufficiently large n ([16]). It is known ([5]) that in the case $G = G_1$ one can take $\alpha = \log(2)/\log(2/X)$, where X is the positive solution of the equation $X^3 + X^2 + X = 2$ (see also [23] for certain generalizations of this type of estimates). One can check that for any $\epsilon > 0$ and for any i large enough (depending on ϵ), the growth function of the group $G(i)$ satisfies $v_{G(i)}(n) \leqslant \exp(n^{\alpha+\epsilon})$ for any sufficiently large n. Let $v_G^{\sup} = \limsup log \log v_{G,S}(n)/\log n$. (Clearly, this constant does not depend on the generating set S).

Thus (in view of the previous corollary) for any $\epsilon > 0$ and for any i large enough

$$\frac{2}{3} \leqslant v_{G(i)}^{\sup} \leqslant \alpha + \epsilon.$$

(Combining this with Proposition 4.1 we obtain that $c_{\mathrm{rt}}(G_1, \mathrm{Stab}\,(1)) \leqslant \alpha$).

The author is grateful to the referee for useful comments on the paper.

BIBLIOGRAPHY

[1] A. Avez, Entropie des groupes de type fini, C. R. Acad. Sc. Paris, 275A, 1972, 1363-1366.

[2] A. Avez, Théorème de Choquet-Deny pour les groupes à croissance non exponentielle, C. R. Acad. Sci. Paris Sér. A 279 (1974), 25–28.

[3] A. Avez, Croissance des groupes de type fini et fonctions harmoniques, Théorie ergodique (Actes Journées Ergodiques, Rennes, 1973/1974), pp. 35–49, Lecture Notes in Math., Vol. 532, Springer, Berlin, 1976.

[4] A. Avez, Harmonic functions on groups, Differential geometry and relativity, pp. 27–32, Mathematical Phys. and Appl. Math., Vol. 3, Reidel, Dordrecht, 1976.

[5] L. Bartholdi, The growth of Grigorchuk's torsion group, Internat. Math. Res. Notices 1998, no. 20, 1049–1054.

[6] L. Bartholdi, Lower bounds on the growth of a group acting on the binary rooted tree, Internat. J. Algebra Comput. 11 (2001), no. 1, 73–88.

[7] P. Baldi, N. Lohoué, J. Peyrière, Sur la classification des groupes récurrents, C. R. Acad. Sci. Paris Sér. A-B 285 (1977), no. 16, A1103–A1104.

[8] Y. Derriennic, Quelques applications du théorème ergodique sous-additif, Asterisque 74, 1980, 183-201.

[9] A. Dyubina, An example of growth rate for random walk on group, Russian Math. Surveys, 1999, T.54, no 5, 159-160.

[10] A. Erschler (Dyubina), On the asymptotics of drift, Zapiski Sem. POMI, 2001, V.283, 251-257.

[11] A. Erschler (Dyubina), Drift and entropy growth for random walk on groups, Russian Math. Surveys, 2001, T.56, no 3, 179-180.

[12] A. Erschler, Boundary behavior for groups of subexponetial growth, to appear in Ann. of Math.

[13] A. Erschler, Drift and entropy growth for random walks on groups, Annals of Probability, 31, 2003, no 3, 1193-1204.

[14] W. Feller, An introduction to probability theory and its applications. Vol II. 2nd ed. Wiley Series in Probability and Mathematical Statistics. New York etc.: John Wiley and Sons, (1971).

[15] R. I. Grigorchuk, On Burnside's problem on periodic groups, Funct. Anal. Appl. 14, 41-43 (1980).

[16] R. I. Grigorchuk, Degrees of growth of finitely generated groups, and the theory of invariant mean, Math USSR Izv, 25:2, 1985, 259-300.

[17] R. I. Grigorchuk, Groups with intermediate growth function and their applications, Doctoral Thesis, 1985.

[18] Y. Guivarch, Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire, Conference on Random Walks (Kleebach, 1979), pp. 47–98, 3, Asterisque, 74, Soc. Math. France, Paris, 1980.

[19] Y. Guivarch, Marches aléatoires sur les groupes, In: Development of Mathematics 1950-2000, Ed. J.P. Pier, Birkhauser.

[20] V. A. Kaimanovich, A. M. Vershik, Random walks on discrete groups: boundary and entropy, The Annals of Probability, 1983, vol.11, no 3, 457-490.

[21] Yu. G. Leonov, Yu. G. On a lower bound for the growth of a 3-generator 2-group, Mat. Sb. 192 (2001), no. 11, 77–92.

[22] A. Lubotzky, Cayley graphs: eigenvalues, expanders and random walks, in Surveys in Combinatorics, 1995, P .Rowlenson, Editor, Cambridge Univ. Press, 1995, 155-189.

[23] R. Muchnik, I. Pak, On growth of Grigorchuk groups, Internat. J. Algebra Comput. 11 (2001), no. 1, 1–17.

[24] F. Spitzer, Principles of random walk, Van Nostrand, Princeton, 1964.

[25] N. Th. Varoupoulos, Théorie du potentiel sur des groupes et variétés, C. R. Acad. Sci. Paris, Série I, 302 (1986), 203-205.

[26] N. Th. Varopoulos, L. Saloff-coste, T. Coulhon, Analysis and geometry on groups, Cambridge Tracts in Mathematics, 100. Cambridge University Press, Cambridge, 1992.

[27] W. Woess, Random walks on infinite graphs and groups, Cambr. Univ. Press, 2000.

Manuscrit reçu le 2 septembre 2003,
révisé le 13 décembre 2003,
accepté le 9 novembre 2004.

Anna ERSCHLER,
Université Lille 1, CNRS
UFR de Mathématiques
59655 Villeneuve d'Ascq Cedex (France)

erschler@math.univ-lille1.fr
erschler@pdmi.ras.ru