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MAXIMAL HAMILTONIAN TORI FOR POLYGON SPACES

by J.-C. HAUSMANN and S. TOLMAN

1. Introduction.

Let M be a symplectic manifold and let $\mathcal{S}(M)$ be the group of symplectomorphisms of M. A sub-torus of $\mathcal{S}(M)$ is called a *symplectic torus*; these tori are partially ordered by inclusions. In this paper, we study the maximal symplectic tori of polygon spaces with a particular emphasis on bending tori (see the definitions below). Since polygon spaces are simply connected, symplectic tori act on M in a Hamiltonian fashion so we refer to them as *Hamiltonian tori.*

Let E be a finite set together with a function $\lambda : E \to \mathbb{R}_+$. Define the space $\widetilde{\mathrm{Pol}}\,(E,\lambda)$ by

$$\widetilde{\mathrm{Pol}}\,(E,\lambda) := \left\{\rho : E \to \mathbb{R}^3 \;\middle|\; \sum_{e\in E} \rho(e) = 0 \text{ and } |\rho(e)| = \lambda(e)\ \forall e \in E\right\} .$$

The *polygon space* $\mathrm{Pol}\,(E,\lambda)$ is the quotient $\mathrm{Pol}\,(E,\lambda) := \widetilde{\mathrm{Pol}}\,(E,\lambda) \;/\; SO_3$. By choosing a bijection between E and $\{1,\dots,m\}$, the space $\mathrm{Pol}\,(E,\lambda)$ is regarded as the space of configurations in $\mathbb{R}^3$ of a polygon with m edges

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of length $\lambda_1, \ldots, \lambda_m$, modulo rotation, whence the name "polygon space". Also, we call an element of E an *edge* and λ the *length function.*

A length function λ is called *generic* if there is no map $\varepsilon : E \to \{\pm 1\}$ so that $\sum_{e\in E} \varepsilon(e)\lambda(e) = 0$. This guarantees that the polygon cannot collapse to a line. In this paper, we always assume that λ is generic and that $\mathrm{Pol}\,(E,\lambda)$ is not empty. In this case, $\mathrm{Pol}\,(E,\lambda)$ is a closed smooth symplectic manifold of dimension $2(|E|-3) \geqslant 0$. The polygon spaces are better known as the moduli spaces of (weighted) ordered points on $\mathbb{P}^1$, and also arise via other symplectic reductions (see [Kl], [KM], [HK1] and the proof of Proposition 2.4 below).

A subset I of E is called *lopsided* if there exists $e_0 \in I$ such that $\lambda(e_0) > \sum_{e\in I-\{e_0\}} \lambda(e)$. The empty set is not lopsided, while a singleton $\{e\}$ is always lopsided since the length function takes strictly positive values. The total set E is not lopsided since $\mathrm{Pol}\,(E,\lambda)$ is assumed to be non-empty.

For $I \subset E$ define $\rho_I : \widetilde{\mathrm{Pol}}\,(E,\lambda) \to \mathbb{R}^3$ by $\rho_I := \sum_{e\in I} \rho(e)$. The continuous function and $f_I : \widetilde{\mathrm{Pol}}\,(E,\lambda) \to \mathbb{R}$ by $f_I(\rho) := |\sum_{i\in I} \rho_i|$ descends to a function on $\mathrm{Pol}\,(E,\lambda)$, still called f_I. When I is lopsided, this function does not vanish and is therefore smooth. Its Hamiltonian flow Φ_I^t is called the *bending flow* associated to I. Bending flows have been introduced in [Kl] and [KM]. They are periodic (see [Kl, §2.1] or [KM, Corollary 3.9]): Φ_I^t rotates at constant speed the set of vectors $\{\rho(e) \mid e \in I\}$ around the axis ρ_I.

A *bending torus* is a Hamiltonian torus in $\mathcal{S}(\mathrm{Pol}\,(E,\lambda))$ generated by bending flows. Since the dimension of $\mathrm{Pol}\,(E,\lambda)$ is $2(|E|-3)$, the dimension of any Hamiltonian torus is at most $|E|-3$.

In this paper, we study the poset of bending tori and compare it with that of Hamiltonian ones. For instance, the following result is proved in Section 3 (see Corollary 3.2):

Theorem A. — *Let $N(\lambda)$ be the minimal number of lopsided subsets which are necessary for a partition of E. Then the maximal dimension of a bending torus for* $\mathrm{Pol}\,(E,\lambda)$ *is* $|E| - \max\{3, N(\lambda)\}$.

We also give a more general statement that allows us to characterize maximal bending tori. In some cases, these coincide with maximal Hamiltonian tori:

THEOREM B.— *Let T be a bending torus of* $\mathrm{Pol}\,(E,\lambda)$ *of dimension* $\geqslant |E|-5$. *Then T is a maximal Hamiltonian torus if and only if it is a maximal bending torus.*

In Section 5, we give several examples where maximal Hamiltonian tori are not all of the same dimension. Using the work of Y. Karshon [Ka], we show the existence of Hamiltonian tori which are not conjugate to a bending torus (Proposition 5.5). Finally, the relationship with maximal tori in the contactomorphism group of pre-quantum circle bundles, due to E. Lerman [Le], is mentioned in 5.6.

2. Preliminaries - Bending sets.

LEMMA 2.1.— *Let $\mathcal{I}$ be a family of lopsided subsets of E. The following conditions are equivalent:*

a) *The bending flows $\{\Phi_I^t \mid I \in \mathcal{I}\}$ generate a bending torus.*

b) *For each pair $A, B \subseteq \mathcal{I}$, either $A \cap B = \emptyset$ or one is contained into the other.*

Proof.— By [Kl, §2.1] or [KM, Corollary 3.9], the bending flows are periodic. Therefore, a) is equivalent to the fact that $\{f_A, f_B\} = 0$ for all $A, B \in \mathcal{I}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket. Proposition 2.1.2 of [Kl] shows that $\{f_A^2, f_B^2\} = 0$ if and only if the pair A, B satisfies Condition b). Since f_A and f_B never vanish, the formula

$$\{f_A^2, f_B^2\} = 4\, f_A f_B\, \{f_A, f_B\}$$

implies that $\{f_A^2, f_B^2\} = 0$ if and only if $\{f_A, f_B\} = 0$. □

A set $\mathcal{I}$ of lopsided subsets of E is called a *bending set* if it contains every singleton $\{e\}$ and satisfies the following "absorption condition": *for each pair $A, B \subseteq \mathcal{I}$, either $A \cap B = \emptyset$ or one is contained in the other.*

Bending sets are technically convenient to parametrize bending tori. Indeed, let $\mathcal{I}$ be a bending set. By 2.1, the bending flows $\{\Phi_I^t \mid I \in \mathcal{I}\}$ generate a bending torus $T_{\mathcal{I}}$. Conversely, if T is a bending torus, there is at least one set $\mathcal{I}$ of lopsided subsets satisfying the absorption condition such that $T = T_{\mathcal{I}}$, and one can add singletons to $\mathcal{I}$ to make it a bending set.

The elements of $\mathcal{I}$ are partially ordered by inclusions, so one can associate to $\mathcal{I}$ the family $\mathcal{M}_{\mathcal{I}}$ of its maximal elements. A direct consequence of the definition is that $\mathcal{M}_{\mathcal{I}}$ is a partition of E.

A bending set $\mathcal{I}$ is called *full* if, for each $I \in \mathcal{I}$ which is not a singleton, there exist $I', I'' \in \mathcal{I}$ so that I is the disjoint union of I' and I''. It is easy to check that this condition is equivalent to either of the following:

a) Given I and I' in $\mathcal{I}$ such that $I' \subset I$, the union $\mathcal{I} \cup \{I'\}$ is not a bending set. This justifies the term "full": one can no longer add elements to $\mathcal{I}$ and keep the latter a bending set.

b) For all $I \in \mathcal{I}$ the set $\{I' \in \mathcal{I} : I' \subseteq I\}$ contains $2\,|I| - 1$ elements.

Remark. — Let $\mathcal{I}$ be a bending set. The reader might find it helpful to consider the graph of this poset. It is a union of disjoint trees, each of which contains a unique maximal element. The bending set $\mathcal{I}$ is full iff these trees are binary: each vertex has one edge leaving it (except the maximal ones which have none) and 2 edges pointing into it (except the singletons which have none).

LEMMA 2.2. — *Let $\mathcal{I}$ be a bending set. Then there exists a (non-unique) bending set $\hat{\mathcal{I}}$ such that the following conditions hold:*

1) *$\mathcal{I} \subset \hat{\mathcal{I}}$ (therefore $T_{\mathcal{I}} \subset T_{\hat{\mathcal{I}}}$).*

2) *$\hat{\mathcal{I}}$ is full.*

3) *$\mathcal{M}_{\hat{\mathcal{I}}} = \mathcal{M}_{\mathcal{I}}$.*

Proof. — If $\mathcal{I}$ is full we are done. Otherwise, we proceed by induction on the number of "non-full" elements of $\mathcal{I}$: those $I \in \mathcal{I}$ which are not singletons and are not the disjoint union of 2 elements of $\mathcal{I}$. Let $I \in \mathcal{I}$ be a minimal "non-full" element.

Let $I_1, \dots, I_r$ be the maximal proper subsets of I which are elements of $\mathcal{I}$. One of them, say I_1, contains the longest edge of I. For $i = 2, \dots, r-1$, define $R_i := I_1 \cup \cdots \cup I_i$ and let $\check{\mathcal{I}} := \mathcal{I} \cup \{R_2\} \cup \cdots \cup \{R_{r-1}\}$. One has $I = R_{r-1} \sqcup I_r$, $R_{r-1} = R_{r-2} \sqcup I_{r-1}$ etc. As I was minimal, it is no longer non-full in $\hat{\mathcal{I}}$. This gives the inductive step. □

We shall now compute the dimension of a bending tori. We need some knowledge about the critical points of the maps f_I and its symplectic reduction. The following lemma comes from [Ha, Theorem 3.2].

LEMMA 2.3. — *Let I be a lopsided subset of E. An element $\rho \in$ $\mathrm{Pol}\,(E, \lambda)$ is a critical point for f_I if and only if either the set $\{\rho(e) \mid e \in I\}$ or the set $\{\rho(e) \mid e \notin I\}$ lies in a line.*

PROPOSITION 2.4. — *Let $A \subset E$. Define $\bar{A} := A \cup \{A\}$ and $\lambda^{A,t} : \bar{A} \to \mathbb{R}$ by $\lambda^{A,t}(e) := \lambda(e)$ for $e \in A$ and $\lambda^{A,t}(A) := t$. Then, if A is lopsided, the* *symplectic reduction of* Pol (E, λ) *at* t*, for the action of the bending circle* T_A*, is symplectomorphic to the product of the two polygon spaces*

$$\mathrm{Pol}\,(E,\lambda) \underset{t}{/\!/} T_A \;\cong\; \mathrm{Pol}\,(\bar{A},\lambda^{A,t}) \times \mathrm{Pol}\,(\overline{E-A},\lambda^{E-A,t}).$$

Remark 2.5.— a) Proposition 2.4 holds true even if t is not a regular value. If it is, the two right hand polygon spaces of the formula are generic by Lemma 2.3.

b) The following is clear from the proof below: if $T_{\mathcal{I}}$ is a bending torus and $A \in \mathcal{I}$, then the action of $T_{\mathcal{I}}$ descends to the reduced space, giving rise to a product of two bending tori: one for the bending set $\{I \in \mathcal{I} \mid I \subset A\}$ and the other for $\{I \in \mathcal{I} \mid I \not\subset A\}$

c) In this paper, Proposition 2.4 is used only for $|A| = 2$. In this case, the reduction of Pol (E, λ) at t is symplectomorphic to a polygon space with $|E| - 1$ edges, since Pol $(\bar{A}, \lambda^{A,t})$ is a point. However, the hypothesis $|A| = 2$ does not simplify the proof.

Proof of Proposition 2.4.— First recall the precise definition for the symplectic structure on Pol (E, λ) (for details, see [HK1, §1]). For $s \in \mathbb{R}$, let $\mathcal{O}(s)$ the coadjoint orbit of $SO(3)$ with symplectic volume $2s$. With the usual identification of $so(3)^*$ with $\mathbb{R}^3$, $\mathcal{O}(s)$ is the 2-sphere centered in 0 of radius r. For $A \subset E$, let $\mu_A : \prod_{e\in E} \mathcal{O}(\lambda(e)) \to \mathbb{R}^3$ be the partial sum $\mu_A((z_e)) := \sum_{e\in A} z_e$. This is the moment map for the diagonal action of $SO(3)$ on the component indexed by $e \in A$. The space Pol $(E, \lambda) = \mu_E^{-1}(0)/SO(3)$ is then the symplectic reduction

$$\mathrm{Pol}\,(E,\lambda) = \prod_{e\in E} \mathcal{O}(\lambda(e)) \underset{0}{/\!/} SO(3)$$

for the diagonal action of $SO(3)$. This determines the symplectic structure on Pol (E, λ).

The codimension 2-embedding

$$V_t := \mu_A^{-1}(\mathcal{O}(t)) \cap \mu_E^{-1}(0) \hookrightarrow \mu_A^{-1}(\mathcal{O}(t)) \times \mu_{E-A}^{-1}(\mathcal{O}(t)) \tag{1}$$

gives rise to a diffeomorphism

$$\begin{array}{ccc} \big[V_t/SO(3))\big]/T_A & \cong & \mu_A^{-1}(\mathcal{O}(t))/SO(3) \times \mu_{E-A}^{-1}(\mathcal{O}(t))/SO(3) \\ \| & & \| \\ \mathrm{Pol}\,(E,\lambda) \underset{t}{/\!/} T_A & & \mathrm{Pol}\,(\bar{A},\lambda^{A,t}) \times \mathrm{Pol}\,(\overline{E-A},\lambda^{E-A,t}). \end{array} \tag{2}$$

As the embedding (1) is the restriction of the obvious symplectomorphism

$$\prod_{e\in E}\mathcal{O}(\lambda(e)) \cong \prod_{e\in A}\mathcal{O}(\lambda(e)) \times \prod_{e\in E-A}\mathcal{O}(\lambda(e)) \tag{3}$$

and as all group actions preserve the symplectic forms, the diffeomorphism (2) is a symplectomorphism. □

Proposition 2.6. — *Let* $\mathcal{I}$ *be a bending set for* Pol (E,λ). *Then*

$$\dim T_{\mathcal{I}} \leqslant |E| - \max\{3, |\mathcal{M}_{\mathcal{I}}|\}$$

with equality if and only if $\mathcal{I}$ *is full.*

Proof. — By Lemma 2.2, it is enough to prove the formula when $\mathcal{I}$ is full. We proceed by induction on the number of elements of $\mathcal{I}$ which are not singletons. If there are none, then $\dim T_{\mathcal{I}} = 0 = |E| - |E|$ and the formula holds true (recall that $|E| \geqslant 3$ since we suppose that Pol $(E,\lambda) \neq \emptyset$). Otherwise, as $\mathcal{I}$ is full, there is $A \in \mathcal{I}$ with $|A| = 2$.

If $|E| = 3$, the formula holds true (the 0-torus, being a quotient of $\mathbb{R}^0$, is of dimension 0). We may then assume that $|E| \geqslant 4$.

The map f_A : Pol $(E,\lambda) \to \mathbb{R}$ is a moment map for the bending circle T_A. As $|E| \geqslant 4$, it is not constant. Let s be a regular value of f_A ($s > 0$ since A is lopsided). By Proposition 2.4, the symplectic reduction of Pol (E,λ) at s is a generic polygon space with $|E| - 1$ edges. By Part b) of Remark 2.5, the bending set $\mathcal{I}$ coinduces a bending set $\bar{\mathcal{I}}$ for $\bar{\lambda}$ which is full. The number of non-singletons elements of $\bar{\mathcal{I}}$ is one less than that of $\mathcal{I}$. By induction hypothesis, one has

$$\dim T_{\bar{\mathcal{I}}} = |E| - 1 - \max\{3, |\mathcal{M}_{\bar{\mathcal{I}}}|\}\ .$$

As $\dim T_{\mathcal{I}} = \dim T_{\bar{\mathcal{I}}} + 1$ and $\mathcal{M}_{\bar{\mathcal{I}}} = \mathcal{M}_{\mathcal{I}}$, one gets the required expression for $\dim T_{\mathcal{I}}$. □

3. Maximal bending tori.

In this section, we study the poset of bending tori. Let $\mathcal{K}$ and $\mathcal{L}$ be two partitions of E. We say that $\mathcal{L}$ is *coarser* than $\mathcal{K}$ if each element of $\mathcal{L}$ is a union of elements of $\mathcal{K}$.

THEOREM 3.1. — *Let $\mathcal{I}$ be a bending set for* Pol (E,λ). *Let $N(\lambda,\mathcal{I})$ be the minimal number of* lopsided *subsets which are necessary for a partition of E which is coarser than $\mathcal{M}_{\mathcal{I}}$. Then, the maximal dimension $n(\lambda,\mathcal{I})$ of a bending torus for* Pol (E,λ) *containing $T_{\mathcal{I}}$ is*

$$n(\lambda,\mathcal{I}) = |E| - \max\{3, N(\lambda,\mathcal{I})\}\ .$$

Proof. — Let T be a bending torus containing $T_{\mathcal{I}}$. By Section 2, $T = T_{\mathcal{J}}$ for a bending set $\mathcal{J}$. By Lemma 2.1, the partition $\mathcal{M}_{\mathcal{J}}$ is coarser than $\mathcal{M}_{\mathcal{I}}$. By 2.6, one has

$$\dim T_{\mathcal{J}} \leqslant |E| - \max\{3, |\mathcal{M}_{\mathcal{J}}|\} \leqslant |E| - \max\{3, N(\lambda,\mathcal{I})\}$$

and therefore

$$n(\lambda,\mathcal{I}) \leqslant |E| - \max\{3, N(\lambda,\mathcal{I})\}.$$

Conversely, let $\mathcal{J}_0$ be a partition of E into lopsided subsets, coarser than $\mathcal{M}_{\mathcal{I}}$, with $N(\lambda,\mathcal{I})$ elements. Let $\mathcal{J} := \mathcal{J}_0 \cup \mathcal{I}$. One check easily that $\mathcal{J}$ is a bending set. Let $\hat{\mathcal{J}}$ be a full bending set associated to $\mathcal{J}$ as in Lemma 2.2. One has $\mathcal{M}_{\hat{\mathcal{J}}} = \mathcal{J}_0$ and, by Proposition 2.6, one has,

$$n(\lambda,\mathcal{I}) \geqslant \dim T_{\hat{\mathcal{J}}} = |E| - \max\{3, N(\lambda,\mathcal{J})\}\ .$$

□

As a corollary, we obtain Theorem A of the introduction:

THEOREM 3.2 (Theorem A). — *Let $N(\lambda)$ be the minimal number of lopsided subsets which are necessary for a partition of E. Then the maximal dimension of a bending torus for* Pol (E,λ) *is* $|E| - \max\{3, N(\lambda)\}$.

Proof. — Set $\mathcal{I}$ be the sets of singletons of E in the statement of Theorem 3.1. □

We now give a characterization of the maximal bending tori which will be used later. We can restrict our attention to those $T_{\mathcal{I}}$, for $\mathcal{I}$ a full bending set, whose dimension is less than $|E| - 3$ (the maximal possible dimension of a Hamiltonian torus of Pol (E,λ)).

PROPOSITION 3.3. — *Let $\mathcal{I}$ be a full bending set so that* $\dim T_{\mathcal{I}} <$ $|E| - 3$. *Then, $T_{\mathcal{I}}$ is a maximal bending torus iff*

$$\bigcap_{J \in \mathcal{M}_{\mathcal{J}}} \mathrm{Image}(f_J) \neq \emptyset.$$

Proof. — Observe that $T_{\mathcal{I}}$ is a maximal bending torus if and only if for each pair $I, J \in \mathcal{M}_{\mathcal{I}}$, one has $\mathrm{Image}(f_I) \cap \mathrm{Image}(f_J) \neq \emptyset$ ($I \cup J$ is not lopsided). The condition of Proposition 3.1 is *a priori* stronger than that but in fact equivalent, thanks to the following lemma.

LEMMA 3.4. — *Let $A_0, \ldots, A_n$ be intervals of the real line. If $A_i \cap A_j \neq \emptyset$ for all i, j, then $A_1 \cap \cdots \cap A_n \neq \emptyset$.*

Proof. — By induction on n, starting with $n = 2$. The condition $A_i \cap A_j \neq \emptyset$ for all i, j implies that $A := A_1 \cup \cdots \cup A_n$ is connected and hence is an interval. The set $\mathcal{A} := \{A_0, \ldots, A_n\}$ is an acyclic covering of A and therefore its nerve $\mathcal{N}(\mathcal{A})$ can be used to compute the cohomology of A: $H^*(A) = H^*(\mathcal{N}(\mathcal{A}))$. By induction hypothesis, the simplicial set $\mathcal{N}(\mathcal{A})$ contains the $n-1$ skeleton of the simplex Δ^n. As $H^{n-1}(A) = 0$, $\mathcal{N}(\mathcal{A})$ must contain Δ^n which is to say $A_1 \cap \cdots \cap A_n \neq \emptyset$. □

4. Maximal Hamiltonian tori.

We start with an important special case which illustrate the technique: the almost regular pentagon. A function $\lambda : \{1, \ldots, 5\} \to \mathbb{R}_+$ is called the length function of an *almost regular pentagon* if $\lambda(i) = 1$ for $i = 1, \ldots, 4$ and $1 < \lambda(5) < 2$. In this case, $\dim \mathrm{Pol}\,(E, \lambda) = 4$.

PROPOSITION 4.1. — *Let $\lambda : \{1, \ldots, 5\} \to \mathbb{R}_+$ be a length function of an almost regular pentagon. Then, the maximal bending tori of* $\mathrm{Pol}\,(E, \lambda)$, *which are 1-dimensional, are maximal Hamiltonian tori.*

Proof. — The maximal lopsided subset of E are of the form $\{k, 5\}$. Therefore, all maximal bending tori are of dimension 1. Since they are all of the same form, it is enough to prove Proposition 4.1 for one of them, say $T_{\mathcal{I}}$ with $\mathcal{I} := \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$. This gives a Hamiltonian circle action with moment map $f := f_{\{4,5\}} = |\rho(4) + \rho(5)|$. By Lemma 2.3, this map has three critical values:

a) The two extremals $z = \lambda(5) - 1$ and $z = \lambda(5) + 1$ are of course critical values. In both cases, the critical set is a 2-sphere, the configuration spaces of the quadrilateral with side length $(1, 1, 1, z)$.

b) The value 1 for which the critical set consists of three points, namely the configurations $\rho : \{1, \ldots, 5\} \to \mathbb{R}^3$ given by one of the line of equations below:

$$\begin{array}{rclcl} -\rho(1) & = & \rho(2) & = & \rho(3) = -\rho(4) - \rho(5), \\ \rho(1) & = & -\rho(2) & = & \rho(3) - \rho(4) - \rho(5) \text{ or} \\ \rho(1) & = & \rho(2) & = & -\rho(3) - \rho(4) - \rho(5). \end{array}$$

The proof then follows from the lemma below.

LEMMA 4.2.— *Let $\mu : M \to \mathbb{R}^{m-1}$ be the moment map for a Hamiltonian action of of T^{m-1} on a compact symplectic manifold M^{2m}. Denote by* $\operatorname{Crit}\mu \subset M$ *the set of critical points of μ. Suppose that there is a point δ in the interior of the moment polytope $\mu(M)$ such that $\mu^{-1}(\delta) \cap \operatorname{Crit}\mu$ has at least 3 connected components. Then the action does not extend to an effective Hamiltonian action of a m-torus.*

Proof.— Suppose that T extends to a Hamiltonian action of $T \times S^1$ with moment map $\Phi : \operatorname{Pol}(E, \lambda) \to \mathbb{R}^n$. Then the moment map f is the composition of Φ with the projection $\mathbb{R}^n \to \mathbb{R}$ onto the last coordinate. Additionally, this action, being effective, would make $\operatorname{Pol}(\lambda)$ a symplectic toric manifold. Thus, $\Phi(\rho)$ are distinct points on the boundary of the moment polytope $\phi(\operatorname{Pol}(E, \lambda))$ (see [De]), which all project to 1. As at most two points of this boundary can project onto one point of $\mathbb{R}$, we get a contradiction. □

The rest of this section is devoted to the proof of our second main result:

THEOREM 4.3 (Theorem B).— *Let T be a bending torus of* $\operatorname{Pol}(E, \lambda)$ *of dimension $\geqslant |E| - 5$. Then T is a maximal Hamiltonian torus if and only if it is a maximal bending torus.*

We only need to prove Theorem B in the cases $\dim T = |E| - 4$ and $|E| - 5$, since it is obvious for $\dim T = |E| - 3$.

Proof for $\dim T = |E| - 4$.— Let $\mathcal{I}$ be a bending set so that $T_{\mathcal{I}}$ is a maximal bending torus of dimension $|E| - 4$. We suppose that there is a Hamiltonian circle S^1 commuting with $T_{\mathcal{I}}$; we shall prove that the resulting action of $\widehat{T} := T_{\mathcal{I}} \times S^1$ is not effective.

Let $f_{\mathcal{I}} : \operatorname{Pol}(E, \lambda) \to \mathbb{R}^{\mathcal{I}}$ be the product map $f_{\mathcal{I}} := \prod_{A \in \mathcal{I}} f_A$. This is a moment map for the action of $T_{\mathcal{I}}$. Its image Δ is a convex polytope of dimension $|E| - 4$. Let μ be the composition of $f_{\mathcal{I}}$ with the projection to the affine space spaned by Δ (the "essential" moment map).

By Proposition 2.6, $\mathcal{I}$ is full and has 4 maximal elements: $\mathcal{M}_{\mathcal{I}} = \{I, J, K, L\}$. By Proposition 3.3, there exists a point c in the intersection of the images of f_I, f_J, f_K and f_L. The proof divides into 3 cases :

Case a) : Suppose that c is in the interior of each image. Then $\vec{c} := (c, c, c, c)$ belongs to the interior of the image of the product map $f := f_I \times f_J \times f_K \times f_L : \operatorname{Pol}(E, \lambda) \to \mathbb{R}^4$. This product map is the composition of μ with the projection to $\mathbb{R}^{\mathcal{M}_\mathcal{I}}$. Hence, there exists δ in the interior of Δ which projects to $\vec{c}$.

For any $\rho \in \widetilde{\mathrm{Pol}}\,(E,\lambda)$ such that $\mu(\rho) = \delta$, there exist $R_I, R_J, R_K, R_L \in SO(3)$ such that

$$R_I(\rho_I) = R_J(\rho_J) = -R_K(\rho_K) = -R_L(\rho_L).$$

Then the configuration ρ' defined by

$$\rho'(e) := R_I(\rho(e)) \text{ if } e \in I \ , \ \rho'(e) := R_J(\rho(e)) \text{ if } e \in J, \text{ etc.}$$

also satisfies $\mu(\rho') = \delta$ and moreover $\rho'_I = \rho'_J = -\rho'_K = -\rho'_L$. This implies that ρ' is a critical point for the function $h := f_I + f_J - f_K - f_L$ and hence for μ. Indeed, the Hamiltonian flow of h would be a global rotation around the axis ρ_I, and therefore induces the identity on $\mathrm{Pol}\,(E,\lambda)$.

Similarly, one constructs critical configurations in $\mu^{-1}(\delta)$ with $\rho_I = -\rho_J = \rho_K = -\rho_L$ and $\rho_I = -\rho_J = -\rho_K = \rho_L$. By Lemma 4.2, this completes the first case.

Case b) : the argument of Case a) works as well if c is in the interior of the image f_A for each $A \in \mathcal{M}_\mathcal{I}$ which is not a singleton (by genericity of λ, there exists at least one such element).

Case c) : in the general case, there may be some set $A \in \mathcal{M}_\mathcal{I}$, such that c is in the boundary of the image of f_A. Let $\mathcal{M}' \subset \mathcal{M}_\mathcal{I}$ be the set of such A's and let $\bar{\mathcal{M}}'$ be the partition of E generated by $\mathcal{M}'$ (formed by the elements of $\mathcal{M}'$ and the singletons). Call $\mathcal{I}'$ the largest sub-poset of $\mathcal{I}$ so that $\mathcal{M}_{\mathcal{I}'} = \bar{\mathcal{M}}'$; this is a full bending set.

In this case, $\bar{P} := f^{-1}(\vec{c})$ is a symplectic submanifold of $\mathrm{Pol}\,(E,\lambda)$ on which $T_{\mathcal{I}'}$ acts trivially. As $\bar{P}$ coincides with the result of successive symplectic reductions at c for the various f_A with $A \in \mathcal{M}'$, it is, by Proposition 2.4, symplectomorphic to the polygon space $\mathrm{Pol}\,(\bar{\mathcal{M}}', \bar{\lambda})$, where

$$\bar{\lambda}(\{e\}) = \lambda(e) \ \text{ and } \ \bar{\lambda}(A) = c \ \text{ if } A \in \mathcal{M}'.$$

The bending torus $T_\mathcal{I}$ acts on $\bar{P}$, giving rise to a bending torus $T_{\bar{I}}$ isomorphic to $T_\mathcal{I}/T_{\mathcal{I}'}$. Observe that $\bar{I}$ has 4 maximal elements and that we are in Case b). Therefore, $T_{\bar{I}}$ is a maximal Hamiltonian torus and the induced action of $\widehat{T}$ on $\bar{P}$ has a kernel of dimension strictly larger than that of $T_{\mathcal{I}'}$. Therefore, as

$$\dim \mathrm{Pol}\,(E,\lambda) - \dim \bar{P} = 2\Big(\sum_{A\in\mathcal{M}'} |A| - |\mathcal{M}'| \Big) = 2 \dim T_{\mathcal{I}'},$$

there is a circle in $\widehat{T}$ acting trivially on a tubular neighborhood of $\bar{P}$. Hence, by the generic orbit type theorem [Au, §2.2], the action of $\widehat{T}$ on $\mathrm{Pol}\,(E,\lambda)$ is not effective. □

Proof for $\dim T = |E| - 5$.— Let $\mathcal{I}$ be a bending set so that $T_{\mathcal{I}}$ is a maximal bending torus of dimension $|E|-5$. We suppose that there is a Hamiltonian circle S^1 commuting with $T_{\mathcal{I}}$ and we shall prove that the resulting action of $\widehat{T} := T_{\mathcal{I}} \times S^1$ is not effective.

Let $\mu : \mathrm{Pol}\,(E,\lambda) \to \mathbb{R}^{|E|-5}$ be the essential moment map, defined as in the proof for $\dim T = |E| - 4$, and let and Δ be the image of μ. Let $\widehat{\mu} : \mathrm{Pol}\,(E,\lambda) \to \Delta \times \mathbb{R}$ be a moment map for the action of $\widehat{T}$ with first component equal to μ and let $\widehat{\Delta}$ be the image of $\widehat{\mu}$.

By Proposition 2.6, $\mathcal{M}_{\mathcal{I}}$ has 5 elements. By Proposition 3.3, there exists a point c in the intersection of the images of f_A for $A \in \mathcal{M}_{\mathcal{I}}$. The proof divides into several cases :

Case 1) : Suppose that $|E| = 5$. Then T_I is of dimension 0 and we have to know that a maximal Hamiltonian torus for a regular pentagon space is also of dimension 0. This is the contents of [HK2, Theorem 3.2].

Case 2) : Suppose that each $A \in \mathcal{M}_{\mathcal{I}}$ contains exactly 2 elements (hence $|E| = 10$) and c is in the interior of the image of f_A. This implies that $\vec{c} := (c,c,c,c,c)$ is a regular value of μ. The reduction Q of $\mathrm{Pol}\,(E,\lambda)$ at $\vec{c}$ is then symplectomorphic to a regular pentagon space (apply Proposition 2.4 five times). The induced Hamiltonian action of $\widehat{T}$ on Q is then trivial by Case 1). This implies that the image of the differential $D\widehat{\mu}$ at any point of $\mu^{-1}(\vec{c})$ is parallel to $\Delta \times \{0\}$. By convexity, we deduce that $\widehat{\Delta}$ and Δ have the same dimension and therefore the action of $\widehat{T}$ is not effective.

Case 3) : The argument of Case 2) works as well if each $A \in \mathcal{M}_{\mathcal{I}}$ has $\leqslant 2$ elements and c is in the interior of the image of f_A when $|A| = 2$. Also, if there are sets $A \in \mathcal{M}_{\mathcal{I}}$ with $|A| = 2$ and c is in the boundary of the image of f_A, one proceeds as in Case c) of the proof for $\dim T_{\mathcal{I}} = |E| - 4$ to deduce that the action of $\widehat{T}$ is not effective. Thus, we are able to prove our result when all the elements of $\mathcal{M}_{\mathcal{I}}$ are either singletons or doubletons.

General case) : For $A \in \mathcal{M}_{\mathcal{I}}$, let $k_A := \max\{0, |A| - 2\}$ and $k := \sum_{A \in \mathcal{M}_{\mathcal{I}}} k_A$. The proof goes by induction on k, the case $k = 0$ being established in Case 3). If $k > 0$, let $A \in \mathcal{M}_{\mathcal{I}}$ such that $|A| \geqslant 3$. If c lies in the boundary of the image of f_A, one proceeds as in Case c) of the proof for $\dim T_{\mathcal{I}} = |E| - 4$ to deduce that the action of $\widehat{T}$ is not effective (using the induction hypothesis). Otherwise, as $\mathcal{I}$ is full, there exists $B \in \mathcal{I}$ such that $|B| = 2$, $B \subset A$ and $f_B(f_A^{-1}(c))$ is an interval of positive length. It contains an open interval J of regular values of f_B. For $t \in J$, the reduction of $\mathrm{Pol}\,(E, \lambda)$ for the action of the Hamiltonian circle with moment map f_B is, by Proposition 2.4, symplectomorphic to an $(|E| - 1)$-gon space $\bar{P}$. The bending torus $T_{\mathcal{I}}$ descends to a bending torus $T_{\bar{\mathcal{I}}}$ for $\bar{P}$. One has $\mathcal{M}_{\bar{\mathcal{I}}} = \mathcal{M}_{\mathcal{I}}$ and $\bar{k} = k - 1$. By induction hypothesis, $T_{\bar{\mathcal{I}}}$ is a maximal Hamiltonian torus. This implies that each point of $f_B^{-1}(t)$ has a stabilizer of positive dimension for the action of $\widehat{T}$. This holds true for all $t \in J$, therefore for an open set of $\mathrm{Pol}\,(E, \lambda)$. By the generic orbit type theorem [Au, §2.2], this implies that the action of $\widehat{T}$ on $\mathrm{Pol}\,(E, \lambda)$ is not effective. □

5. Examples.

NOTATIONS : When $E = \{1, \dots, n\}$, we describe $\mathrm{Pol}\,(E, \lambda)$ by writing the values of λ. For instance, $\mathrm{Pol}\,(1,1,1,2)$ stands for $\mathrm{Pol}\,(\{1,2,3,4\}, \lambda)$ with $\lambda(1) = \lambda(2) = \lambda(3) = 1$ and $\lambda(4) = 2$. A bending set is described by listing its elements which are not singletons and labeling the edges by their length.

5.1. — *The "two long edge" case* : Suppose that the set of edges E contains two elements a, b such that

$$\lambda(a) + \lambda(b) > \sum_{e \in E - \{a,b\}} \lambda(e).$$

Then E is the disjoint union of E_a and E_b so that E_a is lopsided with longest edge a and E_b is lopsided with longest edge b. One then has $N(\lambda) = 2$ and, by Theorem 3.1, $\mathrm{Pol}\,(E, \lambda)$ admits a bending torus of dimension $|E| - 3$. In particular, $\mathrm{Pol}\,(E, \lambda)$ is a toric manifold.

5.2. — *Almost regular pentagon* : The almost regular pentagon $\mathrm{Pol}\,(1,1,1,1,a)$ with $1 < a < 2$ (or $0 < a < 1$) is a very important special case, already used in Proposition 4.1. Notice $\mathrm{Pol}\,(E, \lambda)$ is diffeomorphic to $\mathbb{C}P^2 \sharp 4\, \overline{\mathbb{C}P^2}$ (see [HK1, Example 10.4]).

We used the result of [HK2] that the regular pentagon space admits no non-trivial circle action. This is not known for regular polygon spaces with more edges. Nor it is known whether an almost regular neptagon space is diffeomorphic to a toric manifold.

5.3. — *Hamiltonian tori of different dimensions* : Consider a generic pentagon space of the form $P_{a,b} := \mathrm{Pol}(1,1,1,a,b)$ with $a \neq 1 \neq b$ and $0 < a - b < 1 < a + b$. The bending circle $\{a,b\}$ is a maximal Hamiltonian torus by Proposition 3.3 and 4.3. However, $\mathrm{Pol}(1,1,1,a,b)$ is a toric manifold by the bending tori $T_{\mathcal{I}}$ of the form $\mathcal{I} := \{\{1,a\},\{1,b\}\}$. In this example, one sees that maximal bending tori, as well as maximal Hamiltonian tori, are not all of the same dimension.

The moment polytope for $T_{\mathcal{I}}$ shows that $P_{a,b}$ is diffeomorphic to $\mathbb{C}P^2 \sharp 4 \overline{\mathbb{C}P^2}$ if $a+b<3$ and to $\mathbb{C}P^2 \sharp 3 \overline{\mathbb{C}P^2}$ if $a+b>3$ (the case $a+b=3$ is not generic). It is known that the other pentagon spaces are 4-manifolds with second Betti number < 3. For them, any Hamiltonian circle action extends to a toric action by [Ka, Th. 1].

An example with maximal Hamiltonian tori of 3 different dimensions is provided by the heptagon spaces $\mathrm{Pol}(1,1,2,2,3,3,3)$ (it is generic since lengths are integral and the perimeter is odd). The 3 bending sets with maximal (non-singleton) elements of the form

$$\{\{2,1\},\{2,1\}\} \quad , \quad \{\{2,1\},\{3,1\},\{3,2\}\} \quad , \quad \{\{3,1,1\},\{3,2\},\{3,2\}\}$$

determine maximal Hamiltonian tori of dimension respectively $2,3$ and 4. Observe that the bending circle $\{3,2\}$ is contained in two maximal tori of different dimension.

Examples in higher dimension can be constructed by adding "little edges" to the previous one, for instance the $(7+m)$-gon space

$$\mathrm{Pol}(1,1,2,2,3,3,3,1/2,1/4,\ldots,1/2^m).$$

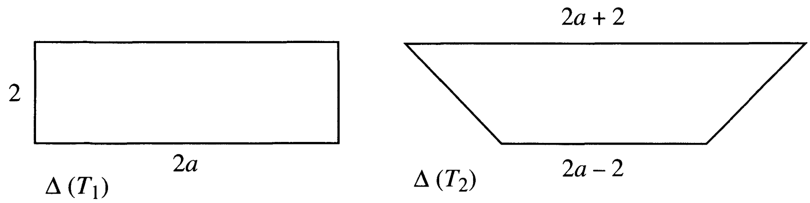
It admit full bending sets with maximal (non-singleton) elements of the form

- $\{\{2,1\},\{2,1\},\{3,1/2,1/4,\ldots,1/2^m\}\}$
- $\{\{2,1\},\{3,1\},\{3,2\},\{3,1/2,1/4,\ldots,1/2^m\}\}$
- $\{\{3,1,1\},\{3,2\},\{3,2\},\{3,1/2,1/4,\ldots,1/2^m\}\}$

which determine maximal Hamiltonian tori of dimension respectively $m+2$,$m+3$ and $m+4$.

5.4. — Let T_1 and T_2 be two Hamiltonian tori of dimension n for a symplectic manifold M^{2n}. Choose isomorphisms $\mathrm{Lie}(T_1)^* \approx \mathbb{R}^n \approx \mathrm{Lie}(T_2)^*$. The moment polytopes Δ_1 and Δ_2 of the two actions are in $\mathbb{R}^n$. By Delzant's theorem, T_1 is conjugate to T_2 in the group $\mathcal{S}(M)$ of sympectomorphism of M if and only if the moment polytopes $\Delta(T_i)$ satisfy $\Delta(T_2) = \psi(\Delta(T_1))$ where ψ is a composition of translations and transformations in $GL(\mathbb{Z}^n)$.

Consider the pentagon space $P := \mathrm{Pol}\,(1, a, c, c, c)$, with $c > a+1 > 2$. The two bending tori $T_1 = \{\{c, 1\}, \{c, a\}\}$ and $T_2 = \{\{c, 1\}, \{c, a, 1\}\}$ have moment polytopes



Therefore, T_1 and T_2 are not conjugate in the group $\mathcal{S}(P)$. One can check that any other bending torus is conjugate to either T_1 or T_2.

On the other hand, the polytope $\Delta(T_1)$ shows that P is symplectomorphic to $(S^2 \times S^2, \omega_1 + a\omega_2)$, where ω_1 and ω_2 are the pull back of the standard area form on S^2 via the two projection maps. By [Ka, Th. 2], the number of conjugacy classes of maximal Hamiltonian tori is equal to $\lceil a \rceil$, the smallest integer greater than or equal to a. This proves the following

PROPOSITION 5.5. — *If* $c > a + 1 > 3$, *then* $\mathrm{Pol}\,(1, a, c, c, c)$ *admits Hamiltonian tori which are not conjugate to a bending torus.*

5.6. — Let (M, ω) be a simply connected symplectic manifold such that $[\omega] \in H^2(M; \mathbb{R})$ is integral. Then there exists a principal circle bundle $S^1 \to Q \to M$ with Euler class $[\omega]$ and Q carries a natural contact distribution by a theorem of Boothby and Wang [BW, Th. 3]. In [Le, Th. 1], E. Lerman recently proved that maximal Hamiltonian tori in M (of dimension k) give rise to maximal tori (of dimension $k+1$) in the group of diffeomorphism of Q preserving the contact distribution.

By [HK1, Prop. 6.5], the symplectic form on $\mathrm{Pol}\,(E, \lambda)$ is integral when, for example, λ takes integral values. Then, our examples in 5.3 give rise to contact manifolds with maximal tori of different dimensions in their group of contactomorphisms (see [Le, Example 2]).

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