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Maximal Hamiltonian tori for polygon spaces

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1. Introduction.

Let $M$ be a symplectic manifold and let $\mathcal{S}(M)$ be the group of symplectomorphisms of $M$. A sub-torus of $\mathcal{S}(M)$ is called a symplectic torus; these tori are partially ordered by inclusions. In this paper, we study the maximal symplectic tori of polygon spaces with a particular emphasis on bending tori (see the definitions below). Since polygon spaces are simply connected, symplectic tori act on $M$ in a Hamiltonian fashion so we refer to them as Hamiltonian tori.

Let $E$ be a finite set together with a function $\lambda : E \to \mathbb{R}_+$. Define the space $\widetilde{\text{Pol}}(E, \lambda)$ by

$$\widetilde{\text{Pol}}(E, \lambda) := \left\{ \rho : E \to \mathbb{R}^3 \mid \sum_{e \in E} \rho(e) = 0 \text{ and } |\rho(e)| = \lambda(e) \forall e \in E \right\}.$$

The polygon space $\text{Pol}(E, \lambda)$ is the quotient $\text{Pol}(E, \lambda) := \widetilde{\text{Pol}}(E, \lambda) / SO_3$. By choosing a bijection between $E$ and $\{1, \ldots, m\}$, the space $\text{Pol}(E, \lambda)$ is regarded as the space of configurations in $\mathbb{R}^3$ of a polygon with $m$ edges.

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of length $\lambda_1, \ldots, \lambda_m$, modulo rotation, whence the name “polygon space”. Also, we call an element of $E$ an edge and $\lambda$ the length function.

A length function $\lambda$ is called generic if there is no map $\varepsilon : E \to \{\pm 1\}$ so that $\sum_{e \in E} \varepsilon(e)\lambda(e) = 0$. This guarantees that the polygon cannot collapse to a line. In this paper, we always assume that $\lambda$ is generic and that $\text{Pol}(E, \lambda)$ is not empty. In this case, $\text{Pol}(E, \lambda)$ is a closed smooth symplectic manifold of dimension $2(|E| - 3) \geq 0$. The polygon spaces are better known as the moduli spaces of (weighted) ordered points on $\mathbb{P}^1$, and also arise via other symplectic reductions (see [Kl], [KM], [HK1] and the proof of Proposition 2.4 below).

A subset $I$ of $E$ is called lopsided if there exists $e_0 \in I$ such that $\lambda(e_0) > \sum_{e \in I - \{e_0\}} \lambda(e)$. The empty set is not lopsided, while a singleton $\{e\}$ is always lopsided since the length function takes strictly positive values. The total set $E$ is not lopsided since $\text{Pol}(E, \lambda)$ is assumed to be non-empty.

For $I \subset E$ define $\rho_I : \widehat{\text{Pol}}(E, \lambda) \to \mathbb{R}^3$ by $\rho_I := \sum_{e \in I} \rho(e)$. The continuous function and $f_I : \widehat{\text{Pol}}(E, \lambda) \to \mathbb{R}$ by $f_I(\rho) := |\sum_{e \in I} \rho(e)|$ descends to a function on $\text{Pol}(E, \lambda)$, still called $f_I$. When $I$ is lopsided, this function does not vanish and is therefore smooth. Its Hamiltonian flow $\Phi_I^t$ is called the bending flow associated to $I$. Bending flows have been introduced in [Kl] and [KM]. They are periodic (see [Kl, §2.1] or [KM, Corollary 3.9]): $\Phi_I^t$ rotates at constant speed the set of vectors $\{\rho(e) \mid e \in I\}$ around the axis $\rho_I$.

A bending torus is a Hamiltonian torus in $S(\text{Pol}(E, \lambda))$ generated by bending flows. Since the dimension of $\text{Pol}(E, \lambda)$ is $2(|E| - 3)$, the dimension of any Hamiltonian torus is at most $|E| - 3$.

In this paper, we study the poset of bending tori and compare it with that of Hamiltonian ones. For instance, the following result is proved in Section 3 (see Corollary 3.2):

**Theorem A.** — Let $N(\lambda)$ be the minimal number of lopsided subsets which are necessary for a partition of $E$. Then the maximal dimension of a bending torus for $\text{Pol}(E, \lambda)$ is $|E| - \max\{3, N(\lambda)\}$.

We also give a more general statement that allows us to characterize maximal bending tori. In some cases, these coincide with maximal Hamiltonian tori:
THEOREM B. — Let $T$ be a bending torus of $\text{Pol}(E, \lambda)$ of dimension $\geq |E| - 5$. Then $T$ is a maximal Hamiltonian torus if and only if it is a maximal bending torus.

In Section 5, we give several examples where maximal Hamiltonian tori are not all of the same dimension. Using the work of Y. Karshon [Ka], we show the existence of Hamiltonian tori which are not conjugate to a bending torus (Proposition 5.5). Finally, the relationship with maximal tori in the contactomorphism group of pre-quantum circle bundles, due to E. Lerman [Le], is mentioned in 5.6.

2. Preliminaries - Bending sets.

LEMMA 2.1. — Let $\mathcal{I}$ be a family of lopsided subsets of $E$. The following conditions are equivalent:

a) The bending flows $\{\Phi^t_I \mid I \in \mathcal{I}\}$ generate a bending torus.

b) For each pair $A, B \subseteq \mathcal{I}$, either $A \cap B = \emptyset$ or one is contained into the other.

Proof. — By [Kl, §2.1] or [KM, Corollary 3.9], the bending flows are periodic. Therefore, a) is equivalent to the fact that $\{f_A, f_B\} = 0$ for all $A, B \in \mathcal{I}$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. Proposition 2.1.2 of [Kl] shows that $\{f_A^2, f_B^2\} = 0$ if and only if the pair $A, B$ satisfies Condition b).

Since $f_A$ and $f_B$ never vanish, the formula

$$\{f_A^2, f_B^2\} = 4f_A f_B \{f_A, f_B\}$$

implies that $\{f_A^2, f_B^2\} = 0$ if and only if $\{f_A, f_B\} = 0$. □

A set $\mathcal{I}$ of lopsided subsets of $E$ is called a bending set if it contains every singleton $\{e\}$ and satisfies the following “absorption condition”: for each pair $A, B \subseteq \mathcal{I}$, either $A \cap B = \emptyset$ or one is contained in the other.

Bending sets are technically convenient to parametrize bending tori. Indeed, let $\mathcal{I}$ be a bending set. By 2.1, the bending flows $\{\Phi^t_I \mid I \in \mathcal{I}\}$ generate a bending torus $T_\mathcal{I}$. Conversely, if $T$ is a bending torus, there is at least one set $\mathcal{I}$ of lopsided subsets satisfying the absorption condition such that $T = T_\mathcal{I}$, and one can add singletons to $\mathcal{I}$ to make it a bending set.

The elements of $\mathcal{I}$ are partially ordered by inclusions, so one can associate to $\mathcal{I}$ the family $\mathcal{M}_\mathcal{I}$ of its maximal elements. A direct consequence of the definition is that $\mathcal{M}_\mathcal{I}$ is a partition of $E$.
A bending set $\mathcal{I}$ is called full if, for each $I \in \mathcal{I}$ which is not a singleton, there exist $I', I'' \in \mathcal{I}$ so that $I$ is the disjoint union of $I'$ and $I''$. It is easy to check that this condition is equivalent to either of the following:

a) Given $I$ and $I'$ in $\mathcal{I}$ such that $I' \subset I$, the union $\mathcal{I} \cup \{I'\}$ is not a bending set. This justifies the term “full”: one can no longer add elements to $\mathcal{I}$ and keep the latter a bending set.

b) For all $I \in \mathcal{I}$ the set $\{I' \in \mathcal{I} : I' \subset I\}$ contains $2|I| - 1$ elements.

Remark. — Let $\mathcal{I}$ be a bending set. The reader might find it helpful to consider the graph of this poset. It is a union of disjoint trees, each of which contains a unique maximal element. The bending set $\mathcal{I}$ is full iff these trees are binary: each vertex has one edge leaving it (except the maximal ones which have none) and 2 edges pointing into it (except the singletons which have none).

**Lemma 2.2.** — Let $\mathcal{I}$ be a bending set. Then there exists a (non-unique) bending set $\hat{\mathcal{I}}$ such that the following conditions hold:

1) $\mathcal{I} \subset \hat{\mathcal{I}}$ (therefore $T_\mathcal{I} \subset T_{\hat{\mathcal{I}}}$).

2) $\hat{\mathcal{I}}$ is full.

3) $\mathcal{M}_{\hat{\mathcal{I}}} = \mathcal{M}_\mathcal{I}$.

Proof. — If $\mathcal{I}$ is full we are done. Otherwise, we proceed by induction on the number of “non-full” elements of $\mathcal{I}$: those $I \in \mathcal{I}$ which are not singletons and are not the disjoint union of 2 elements of $\mathcal{I}$. Let $I \in \mathcal{I}$ be a minimal “non-full” element.

Let $I_1, \ldots, I_r$ be the maximal proper subsets of $I$ which are elements of $\mathcal{I}$. One of them, say $I_1$, contains the longest edge of $I$. For $i = 2, \ldots, r - 1$, define $R_i := I_1 \cup \cdots \cup I_i$ and let $\hat{\mathcal{I}} := \mathcal{I} \cup \{R_2\} \cup \cdots \cup \{R_{r-1}\}$. One has $I = R_{r-1} \cup I_r$, $R_{r-1} = R_{r-2} \cup I_{r-1}$ etc. As $I$ was minimal, it is no longer non-full in $\hat{\mathcal{I}}$. This gives the inductive step. □

We shall now compute the dimension of a bending tori. We need some knowledge about the critical points of the maps $f_I$ and its symplectic reduction. The following lemma comes from [Ha, Theorem 3.2].

**Lemma 2.3.** — Let $I$ be a lopsided subset of $E$. An element $\rho \in \text{Pol}(E, \lambda)$ is a critical point for $f_I$ if and only if either the set $\{\rho(e) \mid e \in I\}$ or the set $\{\rho(e) \mid e \notin I\}$ lies in a line.

**Proposition 2.4.** — Let $A \subset E$. Define $\bar{A} := A \cup \{A\}$ and $\lambda^{A,t} : \bar{A} \to \mathbb{R}$ by $\lambda^{A,t}(e) := \lambda(e)$ for $e \in A$ and $\lambda^{A,t}(A) := t$. Then, if $A$ is lopsided, the
symplectic reduction of \( Pol(E, \lambda) \) at \( t \), for the action of the bending circle \( T_A \), is symplectomorphic to the product of the two polygon spaces

\[
Pol(E, \lambda) \sslash_T T_A \cong Pol(\bar{A}, \lambda^{A,t}) \times Pol(\bar{E} - \bar{A}, \lambda^{E-A,t}).
\]

Remark 2.5. — a) Proposition 2.4 holds true even if \( t \) is not a regular value. If it is, the two right hand polygon spaces of the formula are generic by Lemma 2.3.

b) The following is clear from the proof below: if \( T_T \) is a bending torus and \( A \in \mathcal{T} \), then the action of \( T_T \) descends to the reduced space, giving rise to a product of two bending tori: one for the bending set \( \{ I \in \mathcal{T} \mid I \subset A \} \) and the other for \( \{ I \in \mathcal{T} \mid I \not\subset A \} \)

c) In this paper, Proposition 2.4 is used only for \( |A| = 2 \). In this case, the reduction of \( Pol(E, \lambda) \) at \( t \) is symplectomorphic to a polygon space with \( |E| - 1 \) edges, since \( Pol(\bar{A}, \lambda^{A,t}) \) is a point. However, the hypothesis \( |A| = 2 \) does not simplify the proof.

Proof of Proposition 2.4. — First recall the precise definition for the symplectic structure on \( Pol(E, \lambda) \) (for details, see [HK1, §1]). For \( s \in \mathbb{R} \), let \( O(s) \) the coadjoint orbit of \( SO(3) \) with symplectic volume \( 2s \). With the usual identification of \( so(3)^* \) with \( \mathbb{R}^3 \), \( O(s) \) is the 2-sphere centered in 0 of radius \( r \). For \( A \subset E \), let \( \mu_A : \prod_{e \in E} O(\lambda(e)) \to \mathbb{R}^3 \) be the partial sum \( \mu_A((z_e)) := \sum_{e \in A} z_e \). This is the moment map for the diagonal action of \( SO(3) \) on the component indexed by \( e \in A \). The space \( Pol(E, \lambda) = \mu^{-1}_E(0)/SO(3) \) is then the symplectic reduction

\[
Pol(E, \lambda) = \prod_{e \in E} O(\lambda(e)) \sslash 0 SO(3)
\]

for the diagonal action of \( SO(3) \). This determines the symplectic structure on \( Pol(E, \lambda) \).

The codimension 2-embedding

\[
(1) \quad V_t := \mu_A^{-1}(O(t)) \cap \mu_E^{-1}(0) \hookrightarrow \mu_A^{-1}(O(t)) \times \mu_E^{-1}(O(t))
\]

gives rise to a diffeomorphism

\[
[V_t/\text{SO}(3)])/T_A \cong \mu_A^{-1}(O(t))/SO(3) \times \mu_E^{-1}(O(t))/SO(3)
\]

(2)

\[
Pol(E, \lambda) \sslash_T T_A \cong Pol(\bar{A}, \lambda^{A,t}) \times Pol(\bar{E} - \bar{A}, \lambda^{E-A,t}).
\]
As the embedding (1) is the restriction of the obvious symplectomorphism

\[(3) \prod_{e \in E} \mathcal{O}(\lambda(e)) \cong \prod_{e \in A} \mathcal{O}(\lambda(e)) \times \prod_{e \in E \setminus A} \mathcal{O}(\lambda(e))\]

and as all group actions preserve the symplectic forms, the diffeomorphism (2) is a symplectomorphism.

**Proposition 2.6.** — Let \( \mathcal{I} \) be a bending set for \( \text{Pol}(E, \lambda) \). Then

\[\dim T_\mathcal{I} \leq |E| - \max\{3, |\mathcal{M}_\mathcal{I}|\}\]

with equality if and only if \( \mathcal{I} \) is full.

**Proof.** — By Lemma 2.2, it is enough to prove the formula when \( \mathcal{I} \) is full. We proceed by induction on the number of elements of \( \mathcal{I} \) which are not singletons. If there are none, then \( \dim T_\mathcal{I} = 0 = |E| - |E| \) and the formula holds true (recall that \( |E| \geq 3 \) since we suppose that \( \text{Pol}(E, \lambda) \neq \emptyset \)). Otherwise, as \( \mathcal{I} \) is full, there is \( A \in \mathcal{I} \) with \( |A| = 2 \).

If \( |E| = 3 \), the formula holds true (the 0-torus, being a quotient of \( \mathbb{R}^0 \), is of dimension 0). We may then assume that \( |E| \geq 4 \).

The map \( f_A : \text{Pol}(E, \lambda) \rightarrow \mathbb{R} \) is a moment map for the bending circle \( T_A \). As \( |E| \geq 4 \), it is not constant. Let \( s \) be a regular value of \( f_A \) (\( s > 0 \) since \( A \) is lopsided). By Proposition 2.4, the symplectic reduction of \( \text{Pol}(E, \lambda) \) at \( s \) is a generic polygon space with \( |E| - 1 \) edges. By Part b) of Remark 2.5, the bending set \( \mathcal{I} \) coinduces a bending set \( \mathcal{I} \) for \( \lambda \) which is full. The number of non-singletons elements of \( \mathcal{I} \) is one less than that of \( \mathcal{I} \). By induction hypothesis, one has

\[\dim T_\mathcal{I} = |E| - 1 - \max\{3, |\mathcal{M}_\mathcal{I}|\}\]

As \( \dim T_\mathcal{I} = \dim T_\mathcal{I}' + 1 \) and \( \mathcal{M}_\mathcal{I} = \mathcal{M}_\mathcal{I}' \), one gets the required expression for \( \dim T_\mathcal{I} \).

**3. Maximal bending tori.**

In this section, we study the poset of bending tori. Let \( \mathcal{K} \) and \( \mathcal{L} \) be two partitions of \( E \). We say that \( \mathcal{L} \) is **coarser** than \( \mathcal{K} \) if each element of \( \mathcal{L} \) is a union of elements of \( \mathcal{K} \).
THEOREM 3.1. — Let $\mathcal{I}$ be a bending set for $\text{Pol} \ (E, \lambda)$. Let $N(\lambda, \mathcal{I})$ be the minimal number of lopsided subsets which are necessary for a partition of $E$ which is coarser than $\mathcal{M}_T$. Then, the maximal dimension $n(\lambda, \mathcal{I})$ of a bending torus for $\text{Pol} \ (E, \lambda)$ containing $T_\mathcal{I}$ is

$$n(\lambda, \mathcal{I}) = |E| - \max\{3, N(\lambda, \mathcal{I})\}.$$ 

Proof. — Let $T$ be a bending torus containing $T_\mathcal{I}$. By Section 2, $T = T_{\mathcal{J}}$ for a bending set $\mathcal{J}$. By Lemma 2.1, the partition $\mathcal{M}_\mathcal{J}$ is coarser than $\mathcal{M}_T$. By 2.6, one has

$$\dim T_\mathcal{J} \leq |E| - \max\{3, |\mathcal{M}_\mathcal{J}|\} \leq |E| - \max\{3, N(\lambda, \mathcal{I})\}$$

and therefore

$$n(\lambda, \mathcal{I}) \leq |E| - \max\{3, N(\lambda, \mathcal{I})\}.$$ 

Conversely, let $\mathcal{J}_0$ be a partition of $E$ into lopsided subsets, coarser than $\mathcal{M}_T$, with $N(\lambda, \mathcal{I})$ elements. Let $\mathcal{J} := \mathcal{J}_0 \cup \mathcal{I}$. One check easily that $\mathcal{J}$ is a bending set. Let $\tilde{\mathcal{J}}$ be a full bending set associated to $\mathcal{J}$ as in Lemma 2.2. One has $\mathcal{M}_{\tilde{\mathcal{J}}} = \mathcal{J}_0$ and, by Proposition 2.6, one has,

$$n(\lambda, \mathcal{I}) \geq \dim T_{\mathcal{J}} = |E| - \max\{3, N(\lambda, \mathcal{J})\}.$$ 

As a corollary, we obtain Theorem A of the introduction:

THEOREM 3.2 (Theorem A). — Let $N(\lambda)$ be the minimal number of lopsided subsets which are necessary for a partition of $E$. Then the maximal dimension of a bending torus for $\text{Pol} \ (E, \lambda)$ is $|E| - \max\{3, N(\lambda)\}$. 

Proof. — Set $\mathcal{I}$ be the sets of singletons of $E$ in the statement of Theorem 3.1.

We now give a characterization of the maximal bending tori which will be used later. We can restrict our attention to those $T_\mathcal{I}$, for $\mathcal{I}$ a full bending set, whose dimension is less than $|E| - 3$ (the maximal possible dimension of a Hamiltonian torus of $\text{Pol} \ (E, \lambda)$).

PROPOSITION 3.3. — Let $\mathcal{I}$ be a full bending set so that $\dim T_\mathcal{I} < |E| - 3$. Then, $T_\mathcal{I}$ is a maximal bending torus iff

$$\bigcap_{J \in \mathcal{M}_{\mathcal{J}}} \text{Image}(f_J) \neq \emptyset.$$
Proof. — Observe that $T_T$ is a maximal bending torus if and only if for each pair $I, J \in \mathcal{M}_T$, one has $\text{Image}(f_I) \cap \text{Image}(f_J) \neq \emptyset$ ($I \cup J$ is not lopsided). The condition of Proposition 3.1 is a priori stronger than that but in fact equivalent, thanks to the following lemma.

Lemma 3.4. — Let $A_0, \ldots, A_n$ be intervals of the real line. If $A_i \cap A_j \neq \emptyset$ for all $i, j$, then $A_1 \cap \cdots \cap A_n \neq \emptyset$.

Proof. — By induction on $n$, starting with $n = 2$. The condition $A_i \cap A_j \neq \emptyset$ for all $i, j$ implies that $A := \bigcup_{i=0}^{n} A_i$ is connected and hence is an interval. The set $A := \{A_0, \ldots, A_n\}$ is an acyclic covering of $A$ and therefore its nerve $\mathcal{N}(A)$ can be used to compute the cohomology of $A$: $H^*(A) = H^*(\mathcal{N}(A))$. By induction hypothesis, the simplicial set $\mathcal{N}(A)$ contains the $n - 1$ skeleton of the simplex $\Delta^n$. As $H^{n-1}(A) = 0$, $\mathcal{N}(A)$ must contain $\Delta^n$ which is to say $A_1 \cap \cdots \cap A_n \neq \emptyset$.

4. Maximal Hamiltonian tori.

We start with an important special case which illustrate the technique: the almost regular pentagon. A function $\lambda : \{1, \ldots, 5\} \rightarrow \mathbb{R}_+$ is called the length function of an almost regular pentagon if $\lambda(i) = 1$ for $i = 1, \ldots, 4$ and $1 < \lambda(5) < 2$. In this case, $\dim \text{Pol}(E, \lambda) = 4$.

Proposition 4.1. — Let $\lambda : \{1, \ldots, 5\} \rightarrow \mathbb{R}_+$ be a length function of an almost regular pentagon. Then, the maximal bending tori of $\text{Pol}(E, \lambda)$, which are 1-dimensional, are maximal Hamiltonian tori.

Proof. — The maximal lopsided subset of $E$ are of the form $\{k, 5\}$. Therefore, all maximal bending tori are of dimension 1. Since they are all of the same form, it is enough to prove Proposition 4.1 for one of them, say $T_T$ with $\mathcal{I} := \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$. This gives a Hamiltonian circle action with moment map $f := f_{\{4,5\}} = [\rho(4) + \rho(5)]$. By Lemma 2.3, this map has three critical values:

a) The two extremals $z = \lambda(5) - 1$ and $z = \lambda(5) + 1$ are of course critical values. In both cases, the critical set is a 2-sphere, the configuration spaces of the quadrilateral with side length $(1, 1, 1, z)$.

b) The value 1 for which the critical set consists of three points, namely the configurations $\rho : \{1, \ldots, 5\} \rightarrow \mathbb{R}^3$ given by one of the line of equations below:

$$
\begin{align*}
-\rho(1) &= \rho(2) &= \rho(3) = -\rho(4) - \rho(5), \\
\rho(1) &= -\rho(2) &= \rho(3) - \rho(4) - \rho(5), \\
\rho(1) &= \rho(2) &= -\rho(3) - \rho(4) - \rho(5).
\end{align*}
$$
The proof then follows from the lemma below.

**Lemma 4.2.** — Let \( \mu : M \to \mathbb{R}^{m-1} \) be the moment map for a Hamiltonian action of of \( T^{m-1} \) on a compact symplectic manifold \( M^{2m} \). Denote by Crit \( \mu \subset M \) the set of critical points of \( \mu \). Suppose that there is a point \( \delta \) in the interior of the moment polytope \( \mu(M) \) such that \( \mu^{-1}(\delta) \cap \text{Crit } \mu \) has at least 3 connected components. Then the action does not extend to an effective Hamiltonian action of a \( m \)-torus.

**Proof.** — Suppose that \( T \) extends to a Hamiltonian action of \( T \times S^1 \) with moment map \( \Phi : \text{Pol}(E,\lambda) \to \mathbb{R}^n \). Then the moment map \( f \) is the composition of \( \Phi \) with the projection \( \mathbb{R}^n \to \mathbb{R} \) onto the last coordinate. Additionally, this action, being effective, would make \( \text{Pol}(\lambda) \) a symplectic toric manifold. Thus, \( \Phi(\rho) \) are distinct points on the boundary of the moment polytope \( \phi(\text{Pol}(E,\lambda)) \) (see [De]), which all project to 1. As at most two points of this boundary can project onto one point of \( \mathbb{R} \), we get a contradiction. \( \square \)

The rest of this section is devoted to the proof of our second main result:

**Theorem 4.3 (Theorem B).** — Let \( T \) be a bending torus of \( \text{Pol}(E,\lambda) \) of dimension \( \geq |E|-5 \). Then \( T \) is a maximal Hamiltonian torus if and only if it is a maximal bending torus.

We only need to prove Theorem B in the cases \( \dim T = |E|-4 \) and \( |E|-5 \), since it is obvious for \( \dim T = |E|-3 \).

**Proof for dim \( T = |E|-4 \).** — Let \( \mathcal{I} \) be a bending set so that \( T_{\mathcal{I}} \) is a maximal bending torus of dimension \( |E|-4 \). We suppose that there is a Hamiltonian circle \( S^1 \) commuting with \( T_{\mathcal{I}} \); we shall prove that the resulting action of \( \hat{T} := T_{\mathcal{I}} \times S^1 \) is not effective.

Let \( f_{T} : \text{Pol}(E,\lambda) \to \mathbb{R}^{\mathcal{I}} \) be the product map \( f_{T} := \prod_{A \in \mathcal{I}} f_{A} \). This is a moment map for the action of \( T_{\mathcal{I}} \). Its image \( \Delta \) is a convex polytope of dimension \( |E|-4 \). Let \( \mu \) be the composition of \( f_{T} \) with the projection to the affine space spanned by \( \Delta \) (the “essential” moment map).

By Proposition 2.6, \( \mathcal{I} \) is full and has 4 maximal elements: \( \mathcal{M}_{\mathcal{I}} = \{ I, J, K, L \} \). By Proposition 3.3, there exists a point \( c \) in the intersection of the images of \( f_{I}, f_{J}, f_{K} \) and \( f_{L} \). The proof divides into 3 cases:

**Case a:** Suppose that \( c \) is in the interior of each image. Then \( \tilde{c} := (c,c,c,c) \) belongs to the interior of the image of the product map \( f := f_{I} \times f_{J} \times f_{K} \times f_{L} : \text{Pol}(E,\lambda) \to \mathbb{R}^{4} \). This product map is the composition of \( \mu \) with

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the projection to $\mathbb{R}^{\mathcal{M}_T}$. Hence, there exists $\delta$ in the interior of $\Delta$ which projects to $\bar{c}$.

For any $\rho \in \widetilde{\text{Pol}}(E, \lambda)$ such that $\mu(\rho) = \delta$, there exist $R_I, R_J, R_K, R_L \in SO(3)$ such that

$$R_I(\rho_I) = R_J(\rho_J) = -R_K(\rho_K) = -R_L(\rho_L).$$

Then the configuration $\rho'$ defined by

$$\rho'(e) := R_I(\rho(e)) \text{ if } e \in I, \rho'(e) := R_J(\rho(e)) \text{ if } e \in J,$$

also satisfies $\mu(\rho') = \delta$ and moreover $\rho'_I = -\rho'_K = -\rho'_L$. This implies that $\rho'$ is a critical point for the function $h := f_I + f_J - f_K - f_L$ and hence for $\mu$. Indeed, the Hamiltonian flow of $h$ would be a global rotation around the axis $\rho_I$, and therefore induces the identity on $\text{Pol}(E, \lambda)$.

Similarly, one constructs critical configurations in $\mu^{-1}(\delta)$ with $\rho_I = -\rho_J = \rho_K = -\rho_L$ and $\rho_I = -\rho_J = -\rho_K = \rho_L$. By Lemma 4.2, this completes the first case.

Case b) : the argument of Case a) works as well if $c$ is in the interior of the image $f_A$ for each $A \in \mathcal{M}_T$ which is not a singleton (by genericity of $\lambda$, there exists at least one such element).

Case c) : in the general case, there may be some set $A \in \mathcal{M}_T$, such that $c$ is in the boundary of the image of $f_A$. Let $\mathcal{M}' \subset \mathcal{M}_T$ be the set of such $A$'s and let $\tilde{\mathcal{M}}'$ be the partition of $E$ generated by $\mathcal{M}'$ (formed by the elements of $\mathcal{M}'$ and the singletons). Call $I'$ the largest sub-poset of $I$ so that $A \in \mathcal{M}_T$, this is a full bending set.

In this case, $\tilde{P} := f^{-1}(\tilde{c})$ is a symplectic submanifold of $\text{Pol}(E, \lambda)$ on which $T_{I'}$ acts trivially. As $\tilde{P}$ coincides with the result of successive symplectic reductions at $c$ for the various $f_A$ with $A \in \mathcal{M}'$, it is, by Proposition 2.4, symplectomorphic to the polygon space $\text{Pol}(\tilde{\mathcal{M}}', \tilde{\lambda})$, where

$$\tilde{\lambda}(-\{e\}) = \lambda(e) \quad \text{and} \quad \tilde{\lambda}(A) = c \text{ if } A \in \mathcal{M}'.
$$

The bending torus $T_{I'}$ acts on $\tilde{P}$, giving rise to a bending torus $T_I$ isomorphic to $T_{I'}/T_{I'}$. Observe that $I$ has 4 maximal elements and that we are in Case b). Therefore, $T_I$ is a maximal Hamiltonian torus and the induced action of $\hat{T}$ on $\tilde{P}$ has a kernel of dimension strictly larger than that.
of $T_T$. Therefore, as

$$\dim \text{Pol} (E, \lambda) - \dim \bar{P} = 2 \left( \sum_{A \in M'} |A| - |M'| \right) = 2 \dim T_T,$$

there is a circle in $\tilde{T}$ acting trivially on a tubular neighborhood of $\bar{P}$. Hence, by the generic orbit type theorem [Au, §2.2], the action of $\tilde{T}$ on $\text{Pol} (E, \lambda)$ is not effective.

Proof for $\dim T = |E| - 5$. — Let $\mathcal{I}$ be a bending set so that $T_{\mathcal{I}}$ is a maximal bending torus of dimension $|E| - 5$. We suppose that there is a Hamiltonian circle $S^1$ commuting with $T_{\mathcal{I}}$ and we shall prove that the resulting action of $\tilde{T} := T_{\mathcal{I}} \times S^1$ is not effective.

Let $\mu : \text{Pol} (E, \lambda) \to \mathbb{R}^{|E| - 5}$ be the essential moment map, defined as in the proof for $\dim T = |E| - 4$, and let $\Delta$ be the image of $\mu$. Let $\hat{\mu} : \text{Pol} (E, \lambda) \to \Delta \times \mathbb{R}$ be a moment map for the action of $\tilde{T}$ with first component equal to $\mu$ and let $\hat{\Delta}$ be the image of $\hat{\mu}$.

By Proposition 2.6, $M_{\mathcal{I}}$ has 5 elements. By Proposition 3.3, there exists a point $c$ in the intersection of the images of $f_A$ for $A \in M_{\mathcal{I}}$. The proof divides into several cases:

**Case 1**): Suppose that $|E| = 5$. Then $T_{\mathcal{I}}$ is of dimension 0 and we have to know that a maximal Hamiltonian torus for a regular pentagon space is also of dimension 0. This is the contents of [HK2, Theorem 3.2].

**Case 2**): Suppose that each $A \in M_{\mathcal{I}}$ contains exactly 2 elements (hence $|E| = 10$) and $c$ is in the interior of the image of $f_A$. This implies that $\bar{c} := (c, c, c, c, c)$ is a regular value of $\mu$. The reduction $Q$ of $\text{Pol} (E, \lambda)$ at $\bar{c}$ is then symplectomorphic to a regular pentagon space (apply Proposition 2.4 five times). The induced Hamiltonian action of $\tilde{T}$ on $Q$ is then trivial by Case 1). This implies that the image of the differential $D\hat{\mu}$ at any point of $\mu^{-1}(\bar{c})$ is parallel to $\Delta \times \{0\}$. By convexity, we deduce that $\hat{\Delta}$ and $\Delta$ have the same dimension and therefore the action of $\tilde{T}$ is not effective.

**Case 3**): The argument of Case 2) works as well if each $A \in M_{\mathcal{I}}$ has $\leq 2$ elements and $c$ is in the interior of the image of $f_A$ when $|A| = 2$. Also, if there are sets $A \in M_{\mathcal{I}}$ with $|A| = 2$ and $c$ is in the boundary of the image of $f_A$, one proceeds as in Case c) of the proof for $\dim T_{\mathcal{I}} = |E| - 4$ to deduce that the action of $\tilde{T}$ is not effective. Thus, we are able to prove our result when all the elements of $M_{\mathcal{I}}$ are either singletons or doubletons.
General case) : For $A \in \mathcal{M}_\mathcal{I}$, let $k_A := \max\{0, |A| - 2\}$ and $k := \sum_{A \in \mathcal{M}_\mathcal{I}} k_A$. The proof goes by induction on $k$, the case $k = 0$ being established in Case 3). If $k > 0$, let $A \in \mathcal{M}_\mathcal{I}$ such that $|A| \geq 3$. If $c$ lies in the boundary of the image of $f_A$, one proceeds as in Case c) of the proof for $\dim T_{\mathcal{I}} = |E| - 4$ to deduce that the action of $T$ is not effective (using the induction hypothesis). Otherwise, as $\mathcal{I}$ is full, there exists $B \in \mathcal{I}$ such that $|B| = 2$, $B \subset A$ and $f_B(f_A^{-1}(c))$ is an interval of positive length. It contains an open interval $J$ of regular values of $f_B$. For $t \in J$, the reduction of $\text{Pol}(E, \lambda)$ for the action of the Hamiltonian circle with moment map $f_B$ is, by Proposition 2.4, symplectomorphic to an $(|E| - 1)$-gon space $\tilde{P}$. The bending torus $T_{\mathcal{I}}$ descends to a bending torus $T_\mathcal{I}$ for $\tilde{P}$. One has $\mathcal{M}_\mathcal{I} = \mathcal{M}_{\mathcal{I}}$ and $\tilde{k} = k - 1$. By induction hypothesis, $T_\mathcal{I}$ is a maximal Hamiltonian torus. This implies that each point of $f_B^{-1}(t)$ has a stabilizer of positive dimension for the action of $T$. This holds true for all $t \in J$, therefore for an open set of $\text{Pol}(E, \lambda)$. By the generic orbit type theorem [Au, §2.2], this implies that the action of $T$ on $\text{Pol}(E, \lambda)$ is not effective. 

5. Examples.

Notations : When $E = \{1, \ldots, n\}$, we describe $\text{Pol}(E, \lambda)$ by writing the values of $\lambda$. For instance, $\text{Pol}(1, 1, 1, 2)$ stands for $\text{Pol}(\{1, 2, 3, 4\}, \lambda)$ with $\lambda(1) = \lambda(2) = \lambda(3) = 1$ and $\lambda(4) = 2$. A bending set is described by listing its elements which are not singletons and labeling the edges by their length.

5.1. — The “two long edge” case : Suppose that the set of edges $E$ contains two elements $a, b$ such that

$$\lambda(a) + \lambda(b) > \sum_{e \in E \setminus \{a, b\}} \lambda(e).$$

Then $E$ is the disjoint union of $E_a$ and $E_b$ so that $E_a$ is lopsided with longest edge $a$ and $E_b$ is lopsided with longest edge $b$. One then has $N(\lambda) = 2$ and, by Theorem 3.1, $\text{Pol}(E, \lambda)$ admits a bending torus of dimension $|E| - 3$. In particular, $\text{Pol}(E, \lambda)$ is a toric manifold.

5.2. — Almost regular pentagon : The almost regular pentagon $\text{Pol}(1, 1, 1, 1, a)$ with $1 < a < 2$ (or $0 < a < 1$) is a very important special case, already used in Proposition 4.1. Notice $\text{Pol}(E, \lambda)$ is diffeomorphic to $\mathbb{C}P^2 \# 4 \overline{\mathbb{C}P^2}$ (see [HK1, Example 10.4]).
We used the result of [HK2] that the regular pentagon space admits no non-trivial circle action. This is not known for regular polygon spaces with more edges. Nor it is known whether an almost regular neptagon space is diffeomorphic to a toric manifold.

5.3. — Hamiltonian tori of different dimensions : Consider a generic pentagon space of the form $P_{a,b} := \text{Pol}(1,1,1,a,b)$ with $a \neq 1 \neq b$ and $0 < a-b < 1 < a+b$. The bending circle $\{a,b\}$ is a maximal Hamiltonian torus by Proposition 3.3 and 4.3. However, $\text{Pol}(1,1,1,a,b)$ is a toric manifold by the bending tori $T_\mathcal{I}$ of the form $\mathcal{I} := \{(1,a),\{1,b\}\}$. In this example, one sees that maximal bending tori, as well as maximal Hamiltonian tori, are not all of the same dimension.

The moment polytope for $T_\mathcal{I}$ shows that $P_{a,b}$ is diffeomorphic to $\mathbb{C}P^2 \# 4 \mathbb{C}P^2$ if $a+b < 3$ and to $\mathbb{C}P^2 \# 3 \mathbb{C}P^2$ if $a+b > 3$ (the case $a+b = 3$ is not generic). It is known that the other pentagon spaces are 4-manifolds with second Betti number $< 3$. For them, any Hamiltonian circle action extends to a toric action by [Ka, Th. 1].

An example with maximal Hamiltonian tori of 3 different dimensions is provided by the heptagon spaces $\text{Pol}(1,1,2,2,3,3,3)$ (it is generic since lengths are integral and the perimeter is odd). The 3 bending sets with maximal (non-singleton) elements of the form

$$\{(2,1),\{2,1\}\}, \{(2,1),\{3,1\},\{3,2\}\}, \{(3,1,1),\{3,2\},\{3,2\}\}$$

determine maximal Hamiltonian tori of dimension respectively 2, 3 and 4. Observe that the bending circle $\{3,2\}$ is contained in two maximal tori of different dimension.

Examples in higher dimension can be constructed by adding “little edges” to the previous one, for instance the $(7+m)$-gon space

$$\text{Pol}(1,1,2,2,3,3,1/2,1/4,\ldots,1/2^m).$$

It admit full bending sets with maximal (non-singleton) elements of the form

- $\{(2,1),\{2,1\},\{3,1/2,1/4,\ldots,1/2^m\}\}$
- $\{(2,1),\{3,1\},\{3,2\},\{3,1/2,1/4,\ldots,1/2^m\}\}$
- $\{(3,1,1),\{3,2\},\{3,2\},\{3,1/2,1/4,\ldots,1/2^m\}\}$

which determine maximal Hamiltonian tori of dimension respectively $m + 2, m + 3$ and $m + 4$. 

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5.4. — Let $T_1$ and $T_2$ be two Hamiltonian tori of dimension $n$ for a symplectic manifold $M^{2n}$. Choose isomorphisms $\text{Lie}(T_1)^* \approx \mathbb{R}^n \approx \text{Lie}(T_2)^*$. The moment polytopes $\Delta_1$ and $\Delta_2$ of the two actions are in $\mathbb{R}^n$. By Delzant’s theorem, $T_1$ is conjugate to $T_2$ in the group $S(M)$ of symplectomorphism of $M$ if and only if the moment polytopes $\Delta(T_i)$ satisfy $\Delta(T_2) = \psi(\Delta(T_1))$ where $\psi$ is a composition of translations and transformations in $GL(\mathbb{Z}^n)$.

Consider the pentagon space $P := \text{Pol}(1, a, c, c, c)$, with $c > a + 1 > 2$. The two bending tori $T_1 = \{(c, 1), (c, a)\}$ and $T_2 = \{(c, 1), (c, a, 1)\}$ have moment polytopes

$$
\begin{align*}
\Delta(T_1) & \quad \Delta(T_2) \\
2a & \quad \Delta(T_2) \\
2a + 2 & \quad 2a - 2
\end{align*}
$$

Therefore, $T_1$ and $T_2$ are not conjugate in the group $S(P)$. One can check that any other bending torus is conjugate to either $T_1$ or $T_2$.

On the other hand, the polytope $\Delta(T_1)$ shows that $P$ is symplectomorphic to $(S^2 \times S^2, \omega_1 + a \omega_2)$, where $\omega_1$ and $\omega_2$ are the pull back of the standard area form on $S^2$ via the two projection maps. By [Ka, Th. 2], the number of conjugacy classes of maximal Hamiltonian tori is equal to $[a]$, the smallest integer greater than or equal to $a$. This proves the following

**Proposition 5.5.** — If $c > a + 1 > 3$, then $\text{Pol}(1, a, c, c, c)$ admits Hamiltonian tori which are not conjugate to a bending torus.

5.6. — Let $(M, \omega)$ be a simply connected symplectic manifold such that $[\omega] \in H^2(M; \mathbb{R})$ is integral. Then there exists a principal circle bundle $S^1 \to Q \to M$ with Euler class $[\omega]$ and $Q$ carries a natural contact distribution by a theorem of Boothby and Wang [BW, Th. 3]. In [Le, Th. 1], E. Lerman recently proved that maximal Hamiltonian tori in $M$ (of dimension $k$) give rise to maximal tori (of dimension $k + 1$) in the group of diffeomorphism of $Q$ preserving the contact distribution.

By [HK1, Prop. 6.5], the symplectic form on $\text{Pol}(E, \lambda)$ is integral when, for example, $\lambda$ takes integral values. Then, our examples in 5.3 give rise to contact manifolds with maximal tori of different dimensions in their group of contactomorphisms (see [Le, Example 2]).

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