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# PAINLEVÉ EQUATIONS AND COMPLEX REFLECTIONS

by Philip BOALCH

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## 1. Introduction.

One of the themes of this article is the study of monodromy actions on nonabelian cohomology. In [14] Katzarkov–Pantev–Simpson searched for fundamental group representations having dense orbits under the monodromy action, whereas we will consider the opposite extreme of finite orbits, in the simplest non-trivial case. Namely we will look for finite orbits of the monodromy action on the set of  $SL_2(\mathbb{C})$  representations of the fundamental group of the four-punctured sphere, or equivalently for finite branching solutions of the sixth Painlevé equation. Somewhat surprisingly these will arise from certain subgroups of  $GL_3(\mathbb{C})$ .

Another theme is that of extending results relating to real reflection groups (or Weyl groups) to complex reflection groups. Various instances of this trend have appeared recently in the literature (cf. e.g. Broué–Malle–Michel [3] or Totaro [16]), although the instance here has quite a different flavour.

The sixth Painlevé equation (PVI):

$$\eta'' = \left( \frac{1}{\eta} + \frac{1}{\eta-1} + \frac{1}{\eta-t} \right) \frac{(\eta')^2}{2} - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{\eta-t} \right) \eta' \\ + \frac{\eta(\eta-1)(\eta-t)}{t^2(t-1)^2} \left( \alpha + \frac{\beta t}{\eta^2} + \frac{\gamma(t-1)}{(\eta-1)^2} + \frac{\delta t(t-1)}{(\eta-t)^2} \right)$$

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(where  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$  are parameters) for a (local) function  $\eta(t) : \mathbb{P}^1 \setminus \{0, 1, \infty\} \rightarrow \mathbb{C}$  has the Painlevé property, and so any such local solution extends to a meromorphic function on the universal cover of the three-punctured sphere. Recently Dubrovin–Mazzocco [7] have classified all the algebraic solutions of the one-parameter family  $\text{PVI}_\mu$  of Painlevé six equations having parameters of the form  $(\alpha, \beta, \gamma, \delta) = ((2\mu - 1)^2, 0, 0, 1)/2$  for  $\mu \in \mathbb{C}$  with  $2\mu \notin \mathbb{Z}$ . They found that such algebraic solutions are precisely the finite branching solutions<sup>1</sup> and that, up to equivalence, they are in one-to-one correspondence with braid group orbits of generating triples of reflection groups in three dimensional Euclidean space; for the tetrahedral and octahedral groups there is just one such braid group orbit and for the icosahedral group there are three, so they obtain five solutions altogether. Some of these algebraic solutions have also been independently discovered by Hitchin [9], [11], and Hitchin’s approach may be summarised as follows:

It is well-known (cf. e.g. [13]) that PVI is equivalent to the equations for isomonodromic deformations of a linear system of Fuchsian differential equations of the form

$$(1) \quad \frac{d\Phi}{dz} = A(z)\Phi; \quad A(z) = \sum_{i=1}^3 \frac{A_i}{z - a_i}$$

(where the  $A_i$ ’s are  $2 \times 2$  matrices) as the collection of pole positions  $(a_1, a_2, a_3)$  is varied in  $\mathbb{C}^3 \setminus \text{diagonals}$ . Such isomonodromic deformations are governed by Schlesinger’s equations:

$$(2) \quad \frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \quad \text{if } i \neq j, \text{ and} \quad \frac{\partial A_i}{\partial a_i} = - \sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}$$

and it is shown in [13] how these are equivalent to PVI, where the time  $t$  is the cross-ratio of the four pole positions  $(a_1, a_2, a_3, \infty)$  of (1), and the four parameters  $\alpha, \beta, \gamma, \delta$  are essentially the differences of the eigenvalues of the four residues  $A_1, A_2, A_3, A_4 := -A_1 - A_2 - A_3$  of  $A(z)dz$ . By observing that Schlesinger’s equations preserve the adjoint orbit  $O_i$  containing each  $A_i$  and are invariant under overall conjugation of  $(A_1, A_2, A_3, A_4)$  one sees that Schlesinger’s equations amount to a flat connection on the trivial fibre bundle

$$(3) \quad \mathcal{M}^* := (O_1 \times O_2 \times O_3 \times O_4) // G \times B \longrightarrow B$$

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<sup>1</sup> Clearly the algebraic solutions are finite branching, but the converse is not obvious; one knows a finite branching solution lifts to a meromorphic function on a punctured algebraic curve (finite over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ ) and must show this function is algebraic.

over  $B := \mathbb{C}^3 \setminus \text{diagonals}$ , where the fibre  $(O_1 \times \cdots \times O_4) // G$  is the quotient of

$$\{(A_1, A_2, A_3, A_4) \in O_1 \times O_2 \times O_3 \times O_4 \mid \sum A_i = 0\}$$

by overall conjugation by  $G = GL_2(\mathbb{C})$ . (Generically this fibre is two dimensional, relating to the fact that PVI is second order, and has a natural complex symplectic structure.) Now for each point  $(a_1, a_2, a_3)$  of the base  $B$  one can also consider the set

$$\text{Hom}_{\mathbb{C}}(\pi_1(\mathbb{C} \setminus \{a_i\}), G) / G$$

of conjugacy classes of representations of the fundamental group of the four-punctured sphere, where representations are restricted to take the simple loop around  $a_i$  into the conjugacy class  $\mathcal{C}_i := \exp(2\pi\sqrt{-1}O_i) \subset G$  ( $i = 1, \dots, 4, a_4 = \infty$ ). These spaces of representations are also generically two dimensional and fit together into a fibre bundle

$$M \longrightarrow B.$$

Moreover this bundle  $M$  has a *complete* flat connection (the “isomonodromy connection”) defined locally by identifying representations taking the same values on a fixed set of fundamental group generators. The Schlesinger equations are the pullback of the isomonodromy connection along the natural bundle map

$$\nu : \mathcal{M}^* \longrightarrow M$$

defined by taking the systems (1) to their monodromy representations (cf. [8], [1]).

Thus one approach to finding algebraic solutions to PVI is to start by finding finite branching solutions to Schlesinger’s equation. In turn these correspond to finite branching sections of the isomonodromy connection. However the isomonodromy connection is complete and so its branching amounts to an action of the fundamental group of the base  $\pi_1(B)$  (the pure three-string braid group) on a fibre  $\text{Hom}_{\mathbb{C}}(\pi_1(\mathbb{P}^1 \setminus \{a_i\}), G) / G$ . This action extends to an action of the full braid group  $B_3$  on  $\text{Hom}(\pi_1(\mathbb{P}^1 \setminus \{a_i\}), G)$  which in turn comes from the standard action of  $B_3$  on triples of generators of the free group  $\mathbb{F}_3 \cong \pi_1(\mathbb{P}^1 \setminus \{a_i\})$  on three generators. In particular if one chooses a representation  $\rho \in \text{Hom}(\pi_1(\mathbb{P}^1 \setminus \{a_i\}), G)$  whose image  $\rho(\pi_1(\mathbb{P}^1 \setminus \{a_i\}))$  in  $G$  is a *finite* group, then one knows immediately that the braid group orbit containing  $\rho$  is finite. (Clearly choosing such a representation is equivalent to choosing a triple of elements of  $G$  which generate a finite subgroup.) The algebraic solutions of Hitchin were found in

this way, starting with finite subgroups of  $SU_2 \subset G = GL_2(\mathbb{C})$ . Dubrovin–Mazzocco use a different procedure to obtain finite orbits of the braid group but note ([7] Remark 0.2) that Hitchin’s solutions are related to theirs by a symmetry of PVI.

In this paper we will use a different representation of PVI as an isomonodromy equation, this time for certain rank *three* Fuchsian linear systems, and thereby obtain finite branching solutions of PVI from certain finite subgroups of  $GL_3(\mathbb{C})$ . The main result is the following:

**THEOREM.** — *For each triple of generators of a three-dimensional complex reflection group there is a finite branching solution of the sixth Painlevé equation.*

Thus we have found that the word “complex” may be added to the statement of Dubrovin–Mazzocco [7] and so more solutions are obtained.

We recall that Shephard–Todd [15] have classified the finite groups generated by complex reflections and showed that in three-dimensions, apart from the real reflection groups, there are four irreducible complex reflection groups generated by triples of reflections, of orders 336, 648, 1296 and 2160 respectively, as well as two infinite families  $G(m, p, 3)$ ,  $m \geq 3$ ,  $p = 1, m$  of groups of orders  $6m^3/p$ . For  $m = 2$  and  $p = 1, 2$  these would be the symmetry groups of the octahedron and tetrahedron respectively. (In general, for other  $p$  dividing  $m$ ,  $G(m, p, 3)$  is not generated by a triple of reflections and so does not satisfy the hypotheses of the theorem.)

Note that we will not address here the further problem of finding explicit formulae for these solutions. (Also we will not prove the algebraicity of these solutions here but remark that, in the example we consider in Section 4, this may be proved directly using Jimbo’s work [12] on the asymptotics of PVI. In general some modification of [12] would be necessary to prove the algebraicity of all the other solutions coming from complex reflection groups.)

The organisation of this paper is as follows. In Section 2 we will describe the three-dimensional Fuchsian systems we are interested in and explain how to relate the Schlesinger equations for their isomonodromic deformations to the full four-parameter family of PVI equations. Section 3 will then describe the braid group action on the corresponding space of fundamental group representations. Finally Section 4 describes an example of the finite branching solutions of PVI that arise from triples of generators of three-dimensional complex reflection groups.

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### 2. The rank three systems.

Let  $V = \mathbb{C}^3$ , define  $G = GL(V)$  now and let  $\mathfrak{g} = \text{End}(V)$  denote its Lie algebra. Choose three non-integral complex numbers  $\lambda_i \in \mathbb{C} \setminus \mathbb{Z}$  and let  $O_i \subset \mathfrak{g}$  be the adjoint orbit of rank-one matrices having trace  $\lambda_i$ , for  $i = 1, 2, 3$ . Thus  $O_i$  is four dimensional and consists of matrices conjugate to  $\text{diag}(\lambda_i, 0, 0)$  and any element  $B_i \in O_i$  maybe written in the form

$$B_i = e_i \otimes \alpha_i \quad \text{where } e_i \in V, \alpha_i \in V^* \text{ and } \alpha_i(e_i) = \lambda_i.$$

Also choose a generic adjoint orbit  $O_4 \subset \mathfrak{g}$  (which has dimension six) and consider the space

$$\mathcal{F} := (O_1 \times O_2 \times O_3 \times O_4) // G$$

which is defined as the quotient of

$$(4) \quad \{(B_1, B_2, B_3, B_4) \in O_1 \times \dots \times O_4 \mid \sum B_i = 0\}$$

by overall conjugation by  $G$ . Observe that  $\mathcal{F}$  is of dimension two and so heuristically we would expect the equations for isomonodromic deformations of the Fuchsian systems

$$(5) \quad \frac{d\Phi}{dz} = B(z)\Phi; \quad B(z) = \sum_{i=1}^3 \frac{B_i}{z - a_i}$$

(with  $B_i \in O_i$ ) to be equivalent to a Painlevé equation (i.e., to a second order equation with the Painlevé property); this is indeed the case and we will now show we in fact get PVI.

Without loss of generality let us restrict the pole positions of (5) to be  $(a_1, a_2, a_3) = (0, 1, t)$  with  $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ . Then the Schlesinger equations for isomonodromic deformations of (5) take the form

$$(6) \quad \frac{dB_1}{dt} = \frac{[B_3, B_1]}{t}, \quad \frac{dB_2}{dt} = \frac{[B_3, B_2]}{t-1}, \quad \frac{dB_3}{dt} = \frac{[B_1, B_3]}{t} + \frac{[B_2, B_3]}{t-1}$$

<sup>2</sup> In particular the relation to Stokes multipliers and one of the original motivations (to better understand the braid group actions of [4], [2] for  $GL_n$ ) are no longer apparent, but will be elucidated elsewhere.

where the third equation follows from the first two and the fact that  $B_1 + B_2 + B_3 = -B_4$  is constant. As before these equations descend to define a connection on the trivial bundle

$$\mathcal{N}^* := \mathcal{F} \times \mathbb{P}^1 \setminus \{0, 1, \infty\} \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

with fibre  $\mathcal{F}$ . Let us choose some coordinates on  $\mathcal{F}$  and then rewrite Schlesinger's equations in terms of them.

LEMMA 1. — *The functions  $x := \text{Tr}(B_1 B_3)$  and  $y := \text{Tr}(B_2 B_3)$  are local coordinates near a generic point of  $\mathcal{F}$ .*

*Proof.* — Let  $e_1, e_2, e_3$  be a basis of  $V$ . If  $O_4$  is generic then each  $G$  orbit in (4) contains a point having  $B_i = e_i \otimes \alpha_i$  for some  $\alpha_i \in V^*$ , for  $i = 1, 2, 3$ , and any other such point of the  $G$ -orbit is of the form  $(uB_1u^{-1}, uB_2u^{-1}, uB_3u^{-1}, uB_4u^{-1})$  for some diagonal matrix  $u$ . Thus if  $b_{ij} = \alpha_i(e_j) = (-B_4)_{ij}$  then we see

$$\mathcal{F} \cong \{B_4 \in O_4 \mid (B_4)_{ii} = -\lambda_i \text{ for } i = 1, 2, 3\} / T$$

where the diagonal torus  $T \subset G$  acts by conjugation. Thus we just have to examine the action of  $T$  on the off-diagonal entries of  $B_4$ . Now, the  $T$ -invariant functions of the off-diagonal entries of  $B_4$  are generated by the five functions

$$w = b_{12}b_{21}, \quad x = b_{13}b_{31}, \quad y = b_{23}b_{32}, \quad p = b_{12}b_{23}b_{31}, \quad q = b_{13}b_{32}b_{21},$$

and they satisfy the relation  $wxy = pq$ , so that  $\mathcal{F}$  is embedded in the subvariety of  $\mathbb{C}^5$  cut out by this equation. The further equations determining  $\mathcal{F}$  (corresponding to fixing the eigenvalues  $B_4$ ) may be written in terms of  $w, x, y, p, q$  as follows. The sum of the eigenvalues is fixed since the diagonal part of  $B_4$  is fixed. The sum of the squares of the eigenvalues is

$$\text{Tr}(B_4^2) = \text{Tr}(B_1 + B_2 + B_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(w + x + y)$$

and so fixing this amounts to a constraint of the form  $w = c - x - y$  for a constant  $c$ . Similarly fixing the sum of the cubes of the eigenvalues leads to a constraint of the form  $q = -p + ax + by + k$  for constants  $a, b, k$ . Thus eliminating  $w, q$  we see  $\mathcal{F}$  is locally identified with the variety in  $\mathbb{C}^3 \ni (x, y, p)$  with equation

$$xy(x + y - c) = p(p - ax - by - k).$$

Thus fixing some generic values of  $(x, y)$  determines  $p$  up to a sign, and so  $(x, y)$  are indeed generically good coordinates.  $\square$

Now we immediately see Schlesinger's equations (6) become:

$$(7) \quad \frac{dx}{dt} = \frac{f(x, y)}{t - 1}, \quad \frac{dy}{dt} = \frac{-f(x, y)}{t}$$

where  $f(x, y) = \text{Tr}(B_1[B_2, B_3])$ . To write this in terms of  $x, y$  we note  $f = p - q$  so that

$$f^2 = (p + q)^2 - 4pq = (ax + by + k)^2 + 4xy(x + y - c)$$

and therefore  $f$  is the square root of a cubic polynomial. The following is then immediate:

LEMMA 2. — *By translating  $x, y$  by constants, any cubic polynomial of the form*

$$4x^2y + 4y^2x + \text{quadratic terms}$$

*maybe put in the form*

$$4x^2y + 4y^2x + Axy + Bx + Cy + D$$

*for constants  $A, B, C, D$ .*

Now in [10] Hitchin has carried out the analogous procedure for the rank two (trace-free) Schlesinger equations (2) (which we know are equivalent to PVI) using local coordinates  $x := \text{Tr}(A_1A_3)$  and  $y := \text{Tr}(A_2A_3)$  on the fibre of (3), and he found that they become equation (7) but with  $f$  replaced by the function  $f_{\text{Hitchin}}$  which satisfies

$$(8) \quad f_{\text{Hitchin}}^2 = -2 \det \begin{pmatrix} \epsilon_1 & \epsilon - x - y & x \\ \epsilon - x - y & \epsilon_2 & y \\ x & y & \epsilon_3 \end{pmatrix}$$

where  $\epsilon_i := \text{Tr}(A_i^2)$  and  $2\epsilon := \epsilon_4 - \epsilon_1 - \epsilon_2 - \epsilon_3$  are constants. One easily sees this is again a cubic polynomial of the form appearing in Lemma 2; this shows that the system of equations (7) is equivalent to PVI, and that the four constants  $A, B, C, D$  parameterising the cubic polynomials correspond to the four parameters  $(\alpha, \beta, \gamma, \delta)$  of PVI. In turn we deduce the desired result that the rank three Schlesinger equations (6) are also equivalent to PVI.

Finally let us relate the parameters of the  $3 \times 3$  systems to those of the corresponding  $2 \times 2$  systems and in turn to the parameters  $\alpha, \beta, \gamma, \delta$  of PVI. Define  $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$  such that the eigenvalues of  $B_4 \in O_4$  are  $\{-\mu_1, -\mu_2, -\mu_3\}$ . Then the parameters of the  $3 \times 3$  system are  $\{\lambda_i\}, \{\mu_i\}$  for  $i = 1, 2, 3$  and (by taking the trace of  $\sum B_i = 0$ ) we see they are constrained by  $\sum \lambda_i = \sum \mu_i$ . On the other hand if we define  $\theta_i$  to be the

difference between the eigenvalues (in some order) of  $A_i$  for  $i = 1, 2, 3, 4$  then the parameters for the  $2 \times 2$  system are  $\{\theta_i\}$  and the parameters used by Hitchin are  $\epsilon_i = \text{Tr}(A_i^2) = \theta_i^2/2$ . From [13] the relation between  $(\alpha, \beta, \gamma, \delta)$  and  $\{\theta_i\}$  is

$$(9) \quad \alpha = (\theta_4 - 1)^2/2, \beta = -\theta_1^2/2, \gamma = \theta_2^2/2, \delta = (1 - \theta_3^2)/2.$$

To obtain  $\{\theta_i\}$  from  $\{\lambda_i\}, \{\mu_i\}$  we should go via the parameters  $A, B, C, D$  (using  $f_{\text{Hitchin}}$  and  $f$  respectively). This appears to be difficult since in either case  $A, B, C, D$  are complicated degree six polynomials in  $\{\theta_i\}$  or  $\{\lambda_i, \mu_i\}$  respectively. However in fact there are *linear* maps from  $\{\lambda_i, \mu_i\}$  to  $\{\theta_i\}$  leading to corresponding parameters  $A, B, C, D$ :

LEMMA 3. — *If we define*

$$(10) \quad \theta_i := \lambda_i - \mu_1 \quad (i = 1, 2, 3), \quad \theta_4 := \mu_2 - \mu_3$$

and then define  $f, f_{\text{Hitchin}}$  as above in terms of  $\{\lambda_i, \mu_i\}, \{\theta_i\}$  respectively then

$$f^2(x, y) = f_{\text{Hitchin}}^2(x - \theta_1\theta_3/2, y - \theta_2\theta_3/2).$$

Moreover the same result holds if  $\mu_1, \mu_2, \mu_3$  are permuted arbitrarily.

*Proof.* — This may be proved by direct calculation. □

Thus the parameters of PVI corresponding to  $\{\lambda_i, \mu_i\}$  are given immediately by combining (9) and (10). (The other 5 permutations of  $\{\mu_1, \mu_2, \mu_3\}$  give equivalent parameters.)

*Remark 4.* — If we choose  $B_4$  to be diagonal then each of the six off-diagonal entries of the matrix

$$z(z-1)(z-t)B(z)$$

of polynomials in  $z$ , is linear, and so has a unique zero on the complex plane. One may then prove (analogously to the  $2 \times 2$  case) that the positions  $\eta_{ij}$  of each of these zeros (as functions of  $t$ ) are solutions of PVI. (The parameters of PVI for each of the six solutions  $\eta_{ij}$  correspond to one of the six permutations of  $\{\mu_1, \mu_2, \mu_3\}$  in Lemma 3).

### 3. Braid group actions.

Now we consider the spaces of monodromy data corresponding to the above rank three systems and describe the natural braid group actions on them.

Let  $\mathcal{C}_i := \exp(2\pi\sqrt{-1}O_i) \subset GL(V)$  be the conjugacy class associated to the fixed adjoint orbit  $O_i$ . For  $i = 1, 2, 3$  let  $t_i := \exp(2\pi\sqrt{-1}\lambda_i)$  so that  $\mathcal{C}_i$  is four dimensional and contains pseudo-reflections of the form

$$r_i = 1 + e_i \otimes \alpha_i \quad \text{where } e_i \in V, \alpha_i \in V^* \text{ and } 1 + \alpha_i(e_i) = t_i \neq 0.$$

We suppose that  $O_4$  is sufficiently generic that  $\mathcal{C}_4$  is a generic conjugacy class (i.e., that the difference between any two eigenvalues of any element of  $O_4$  is not an integer).

Now the monodromy representation of the Fuchsian system (5) is an element

$$\rho \in \text{Hom}(\pi_1(\mathbb{C} \setminus \{a_i\}, p), G)$$

where  $p$  is a base point. If  $B_i \in O_i$  then  $\rho$  maps a simple positive loop around  $a_i$  into  $\mathcal{C}_i$ . Clearly  $\rho$  is determined by its values

$$(r_1, r_2, r_3) := (\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3))$$

on any set of loops  $\{\gamma_i\}$  generating  $\pi_1(\mathbb{C} \setminus \{a_i\})$ . By sliding the points  $a_1, a_2, a_3$  around the complex plane one obtains an action of the three-string braid group  $B_3$  on the set of such triples: Two generators of this action are:

$$\beta_1(r_1, r_2, r_3) = (r_2, r_2^{-1}r_1r_2, r_3), \quad \beta_2(r_1, r_2, r_3) = (r_1, r_3, r_3^{-1}r_2r_3).$$

(One may think of this action as choosing different generators of  $\pi_1(\mathbb{C} \setminus \{a_i\})$  as the loops are dragged around when the points  $a_i$  are permuted in the plane.)

Now we wish to descend this action to the space of conjugacy classes of fundamental group representations. Let  $\{e_i\}$  be the standard basis of  $V$ . If each  $r_i$  is pseudo-reflection and  $\rho$  is sufficiently generic then the triple  $(r_1, r_2, r_3)$  is conjugate to a triple with  $r_i = 1 + e_i \otimes \alpha_i$  for some  $\alpha_i \in V^*$ . If we let  $u_{ij} = \alpha_i(e_j)$  then the corresponding matrix  $U$  (whose rows represent the linear forms  $\alpha_i$ ) is determined by the conjugacy class of  $\rho$  up to conjugation by a diagonal matrix. Thus we have the following five functions on the space  $\text{Hom}_{\mathbb{P}\text{-}\Gamma}(\pi_1(\mathbb{C} \setminus \{a_i\}), G)/G$  of conjugacy classes of representations having pseudo-reflection monodromy around each  $a_i$ :

$$w = u_{12}u_{21}, \quad x = u_{13}u_{31}, \quad y = u_{23}u_{32}, \quad p = u_{12}u_{23}u_{31}, \quad q = u_{32}u_{21}u_{13},$$

which are the “multiplicative analogues” of the functions with the same labels in the previous section (and similarly we see they generate the ring of  $G$  invariant functions on  $\text{Hom}_{\mathbb{P}\text{-}\Gamma}(\pi_1(\mathbb{C} \setminus \{a_i\}), G)$ ). The functions  $w, x, y, p, q$  may be expressed directly in terms of the  $r_i$  using the formulae

$$\text{Tr}(r_i r_j) = 1 + t_i + t_j + u_{ij}u_{ji},$$

$$\mathrm{Tr}(r_1 r_2 r_3) = t_1 + t_2 + t_3 + w + x + y + p, \quad \mathrm{Tr}(r_3 r_2 r_1) = t_1 + t_2 + t_3 + w + x + y + q.$$

It is now straightforward to calculate the induced  $B_3$  action on the matrix  $U$  and in turn on the quintuple of functions  $w, x, y, p, q$ : If we assume that each  $r_i$  is an order two complex reflection (i.e.,  $t_i = -1$ ) then the formula simplifies to

$$\beta_1(w, x, y, p, q) = (w, y + p + q + wx, x, -q - wx, -p - wx)$$

$$\beta_2(w, x, y, p, q) = (x + p + q + wy, w, y, -q - wy, -p - wy)$$

(and is not much more complicated in general, but this case is sufficient for the example we will consider here).

Now if we consider the map

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} \rightarrow \mathbb{C}^3 \setminus \text{diagonals}, \quad t \mapsto (a_1, a_2, a_3) = (0, t, 1)$$

then loops around 0, 1 generating the fundamental group of the three-punctured sphere map to the generators  $\beta_1^2, \beta_2^2$  of the pure braid group  $P_3 = \pi_1(\mathbb{C}^3 \setminus \text{diags})$  (i.e., to the squares of the chosen generators of  $B_3$ ). From the picture sketched in the introduction it follows that the branching of solutions to the sixth Painlevé equation PVI are given by this action of  $P_3$ . Therefore finite orbits of the  $P_3$  action correspond to finite branching solutions to PVI. In particular if  $(r_1, r_2, r_3)$  are a triple of complex reflections (finite order pseudo-reflections) that generate a finite group then we can be sure to obtain a finite  $P_3$  orbit. Thus each triple of generators of each three-dimensional complex reflection group gives a finite branching solution to PVI.

*Remark 5.* — One may determine the parameters  $\{\lambda_i, \mu_i\}$  from the triple  $(r_1, r_2, r_3)$  (and so by Lemma 3 the parameters of PVI) using the fact that  $r_i$  has eigenvalues  $\{1, 1, e^{2\pi\sqrt{-1}\lambda_i}\}$  and that the product  $r_1 r_2 r_3$  has eigenvalues  $\{e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, e^{2\pi i\mu_3}\}$ . In particular by 5.4 of [15] if  $(r_1, r_2, r_3)$  are one of the standard triples of generators of a finite complex reflection group  $G$ , then  $\mu_i$  are related to the exponents  $m_1 \leq m_2 \leq m_3$  of  $G$  as follows:

$$\mu_i = m_i/h, \quad (i = 1, 2, 3) \quad h := m_3 + 1.$$

Group	Degrees $m_i + 1$	$(\theta_1, \theta_2, \theta_3, \theta_4)$
$G(m, m, 3)$	$3, m, 2m$	$(m - 2, m - 2, m - 2, m)/2m$
$G(m, 1, 3)$	$m, 2m, 3m$	$(m - 2, m - 2, 2m - 4, 4m)/6m$
Icosahedral	$2, 6, 10$	$(0, 0, 0, 4/5)$
$G_{336}$	$4, 6, 14$	$(2, 2, 2, 4)/7$
$G_{648}$	$6, 9, 12$	$(0, 0, 0, 1/2)$
$G_{1296}$	$6, 12, 18$	$(4, 7, 7, 12)/18$
$G_{2160}$	$6, 12, 30$	$(5, 5, 5, 9)/15$

TABLE 1. Parameters for solutions from standard generating triples.

This enables us to compile Table 1 of parameters for solutions from standard generating triples, where a suitable permutation of  $\{\mu_i\}$  has been used in each case and we have taken each  $\theta_i$  to be positive (since negating any  $\theta_i$  leads to equivalent PVI parameters).

#### 4. An example.

As an example let us consider the smallest exceptional three dimensional non-real complex reflection group  $K \subset G$  of order 336. (The associated collineation group in  $PGL_3(\mathbb{C})$  is Klein’s simple group of order 168.) To classify the finite branching solutions of PVI associated to  $K$  we must classify up to conjugacy the braid group orbits of generating triples of reflections in  $K$ . (Each such triple will give a solution but conjugate triples, and those in the same braid group orbit, will give the same solution.) Let  $(r_1, r_2, r_3)$  denote the standard triple of generators of  $K$  (given as explicit  $3 \times 3$  matrices on p. 295 of [15]). Now  $K$  contains precisely 21 complex reflections all of which have order 2 and are all conjugate in  $K$ . Thus it is sufficient to consider braid orbits of elements of the form

$$(r_1, a, b)$$

for reflections  $a, b$ . There are  $441 = 21^2$  such triples but it turns out (using Maple) that these constitute just 45 distinct conjugacy classes of triples (i.e., the quintuple of functions  $(w, x, y, p, q)$  takes only 45 distinct values on these 441 triples). Thus there are 45 conjugacy classes of triples of reflections in  $K$ , since every reflection is conjugate to  $r_1$ . Then it is quite

manageable to calculate the braid group orbits on these conjugacy classes of triples: one finds the 45 classes are partitioned into orbits of size

$$1, 1, 3, 3, 4, 4, 6, 7, 7, 9.$$

Then we find that, except for the orbits of size 7, all the corresponding triples generate proper subgroups of  $K$ . On the other hand the two orbits of size seven come from the triples

$$(r_1, r_2, r_3), \quad \text{and} \quad (r_1, r_3, r_2)$$

of generators of  $K$ . Let us focus on the orbit of the first triple  $(r_1, r_2, r_3)$  (the other triple gives a solution to PVI with equivalent parameters). One finds this orbit does not break up into smaller orbits up when we restrict to the pure braid group; the  $P_3$  orbit still has size seven and so by the remarks of the preceding sections this implies the existence of a solution to PVI with seven branches. From Table 1 this solution has  $\theta$ -parameters  $(2, 2, 2, 4)/7$ . By examining the permutation of the branches at each of the three branch-points (i.e., the cycle decomposition of the action of the two generators of  $P_3$  and of their product) we find this solution is single-valued on a genus zero covering (at each branch-point one finds a 3-cycle and two 2-cycles). To the authors knowledge such a solution does not appear in the existing literature.

## 5. Conclusion/Outlook.

In summary we have shown how the general PVI equation governs isomonodromic deformations of rank three systems having four singularities on the sphere with rank one residue at three of the singularities. The corresponding space of monodromy data consists of representations of the fundamental group of the four-punctured sphere such that three of the local monodromies are pseudo-reflections. It follows that the branching of the solutions to PVI (i.e., the “nonlinear monodromy of PVI”) is governed by the action of the pure three-string braid group on the set of conjugacy classes of such representations. Thus for each finite subgroup of  $GL_3(\mathbb{C})$  generated by three pseudo-reflections (i.e., each triply-generated three-dimensional complex reflection group) we obtain a finite braid group orbit and thus a finite branching solution to PVI. Finally we have started to describe some of the new solutions that arise in this way.

Some remaining questions are:

1) Are all these solutions algebraic? (It is hard to believe they are not, and a proof should be possible as in [7] using/adapting Jimbo's study [12] of the asymptotics of PVI.)<sup>3</sup>

2) What are the explicit formulae for the solutions? (This could be amenable to brute force methods on a computer, at least for the smaller braid group orbits, once the asymptotics in 1) are understood. However, the fact that the icosahedral solution with 18 branches took 9 pages to write down implicitly in the preprint version of [7], leads us to question how valuable such explicit formulae are.)

3) Is there a geometrical or physical interpretation of these solutions? (This appears to be a deeper question: for example i) Dubrovin (see [6]) has shown how solutions to  $PVI_\mu$  may be used to construct three-dimensional semisimple Frobenius manifolds (i.e., certain approximations to 2D topological quantum field theories), and ii) Hitchin [9], [11] and Doran [5] have constructed some of the algebraic solutions purely geometrically relating them to previously solved, often classical, algebro-geometric problems.)

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<sup>3</sup> By finding the corresponding  $2 \times 2$  triples of monodromy data we have recently found that in fact Jimbo's work may be used directly to prove that the solutions found in Section 4 are indeed algebraic. Details will appear elsewhere.

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