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**SCATTERING ON STRATIFIED MEDIA:  
THE MICROLOCAL PROPERTIES  
OF THE SCATTERING MATRIX  
AND RECOVERING ASYMPTOTICS  
OF PERTURBATIONS**

by **T. CHRISTIANSEN** and **M. S. JOSHI**

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**1. Introduction.**

In this paper, we study the structure of the scattering matrix on a perturbed stratified medium. In particular, we show that its main part is a Fourier integral operator. En route to proving this theorem, we develop an improved limiting absorption principle for a large class of perturbations, using techniques from Fourier analysis and microlocal analysis. As an application of our results, we prove that the asymptotics of a perturbation can be recovered from the scattering matrix at one energy.

Recall that a stratified medium is a model space in which sound waves propagate with a variable sound speed that depends on only one coordinate. Thus, if we write the coordinates on  $\mathbb{R}^n$  as  $z = (x, y)$  with  $x \in \mathbb{R}^{n-1}$  and  $y \in \mathbb{R}$ , we take the wave speed to be of the form  $c_0(y)$  and study the wave equation

$$(1) \quad \left( -\frac{\partial^2}{\partial t^2} - c_0^2 \Delta \right) w = 0,$$

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where  $\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}$ . We assume that  $c_0$  is constant for  $|y|$  large and that it is piecewise smooth. Set

$$(2) \quad c_+ = \lim_{y \rightarrow \infty} c_0(y),$$

$$(3) \quad c_- = \lim_{y \rightarrow -\infty} c_0(y).$$

In general, we do not require that  $c_+$  be equal to  $c_-$ . However, some of our results are stronger when they are equal.

A perturbed stratified medium is a medium in which the variable sound speed,  $c$ , has the property that  $c - c_0$  is well-behaved at infinity. The case where the perturbation,  $c - c_0$ , is rapidly decaying has been studied in many previous papers. In particular, precise asymptotics for  $(c^2\Delta - (\lambda - i0)^2)^{-1}f$ , when  $f \in \mathcal{S}(\mathbb{R}^n)$ , were established in [5]. That  $c_0$  and the scattering matrix at fixed non-zero energy determine an exponentially decaying perturbation was proved in [20] ( $c_0(y) = c_{\pm}$  for  $\pm y > 0$ ), [15] ( $c_0(y) = c_{\pm}$  for  $\pm y > y_M$ ), and [33] (for  $c_0$  a Pekeris profile). In [1], [2] a similar result was proved under more relaxed requirements on  $c_0$ . In particular,  $c_0$  was required to exponentially approach constants as  $\pm y \rightarrow \infty$ , whilst still requiring that  $c - c_0$  exponentially decay.

Here we study the case where the perturbation,  $c - c_0$ , has an asymptotic expansion in homogeneous terms at infinity. Under certain conditions on  $c$  and  $c_0$  made more precise in Section 2, we show that the scattering matrix for  $c^2\Delta$  is a Fourier integral operator and describe its singular set. Moreover, we show that the asymptotics of the perturbation can be recovered from the scattering matrix at fixed energy. We also establish the leading term of the asymptotics for the limiting absorption principle.

Our results use techniques developed by Joshi and Sá Barreto, [21], [22], [23], [24], [25], to study inverse problems in other settings. These build on work by Melrose [26], and Melrose-Zworski [27], on the structure of the scattering matrix on asymptotically Euclidean spaces. As in those inverse results, the fundamental idea here is to compute the symbol of the scattering matrix by solving transport equations along geodesics on the sphere at infinity. These equations express the propagation of growth at infinity.

The analysis here is, however, considerably more involved as the unperturbed wave speed,  $c_0$ , is not smooth on the compactified space obtained by adding the sphere at infinity, even when  $c_0(y)$  is a smooth

function of  $y$ . This is because  $c_0$  does not have nice asymptotics in  $|z|$ . Therefore  $c_0$  is well-behaved on the compactified space only after the space has been blown-up on the equator at infinity. This manifests itself in our analysis by requiring the geodesic flow at infinity to be refracted and reflected by the equator. It was also seen in [5] that it makes the asymptotics in the limiting absorption principle much more complicated. There is a certain similarity here with many-body scattering, compare, e.g. [29]. There the scattering problem is complicated by the presence of a potential that does not decay in certain directions and thus appears as a spike on the sphere at infinity. This causes refractions and reflections of the geodesic flows [29]. Indeed, the case where  $c_+ = c_-$  bears much resemblance to the many-body case. However, when  $c_+ \neq c_-$ , there are effectively different energy levels in the two hemispheres, and this introduces new complications which are not present in the many-body setting, and much of this paper is dedicated to coping with these complications.

In Section 5 we define the scattering matrix, and its “main part.” When the operator  $c_0^2(D_y^2 + \kappa^2)$  has no eigenvalues as an operator on  $L^2(\mathbb{R}, c_0^{-2} dy)$  for all real  $\kappa$ , then the main part of the scattering matrix is the same as the scattering matrix.

Below we refer to the hypotheses (H1) and (H2). For full details, see Section 2, but roughly speaking hypothesis (H1) is that  $c_+ = c_-$ ,  $c$  and  $c_0$  are smooth, and  $|c - c_0| \leq C(1 + |z|)^{-2}$ . Hypothesis (H2) is that  $|c - c_0| \leq C(1 + |z|)^{-4}$ .

Our first main result is

**THEOREM 1.1.** — *Suppose  $c, c_0$  satisfy the general assumptions of Section 2, and either hypothesis (H1) or (H2). Then, if  $c_+ = c_-$ , the main part of the scattering matrix is a zeroth order Fourier integral operator associated with broken geodesic flow at time  $\pi$ . If  $c_- > c_+$ , then the main part of the scattering matrix is a sum of Fourier integral operators associated with the mapping*

$$(\bar{\omega}, \omega_n) \mapsto (-\bar{\omega}, \omega_n)$$

and the mapping

$$(\bar{\omega}, \omega_n) \mapsto (-c_- \bar{\omega} / c_+, -\sqrt{1 - c_-^2 |\bar{\omega}|^2 / c_+^2}) \quad \text{if } \sqrt{1 - c_+^2 / c_-^2} < \omega_n$$

and

$$(\bar{\omega}, \omega_n) \mapsto (-c_+ \bar{\omega} / c_-, \sqrt{1 - c_+^2 |\bar{\omega}|^2 / c_-^2}) \quad \text{if } \omega_n < 0.$$

Here, when  $c_+ = c_-$ , the geodesic flow is broken at the equator  $(\bar{\omega}, 0) \subset \mathbb{S}^{n-1}$ . This can be compared to the situation for the Laplacian [27], or a perturbation of the Laplacian to an integral power [6], on a manifold with asymptotically Euclidean ends, where the scattering matrix is a zeroth order Fourier integral operator associated to geodesic flow at time  $\pi$  on the the boundary “at infinity.” An additional analogy is to 3-body scattering, where the three-cluster to three-cluster part of the scattering matrix is a sum of Fourier integral operators associated to broken geodesic flow at time  $\pi$  [29]. Other results on the structure of the scattering matrix in  $n$ -body scattering may be found in [30].

Further results on the structure of the scattering matrix are given in Proposition 5.2.

Our central inverse result is

**THEOREM 1.2.** — *Suppose  $c$  and  $c_0$  satisfy the general assumptions of Section 2, as well as either hypothesis (H1) or (H2), and  $n \geq 3$ . Then, if  $c_+ = c_-$ , the asymptotic expansion at infinity of  $c - c_0$  is uniquely determined by  $c_0$  and the transmitted singularities of the main part of the scattering matrix at fixed non-zero energy. If  $c_+ < c_-$ , then the asymptotic expansion is uniquely determined by  $c_0$  and the reflected singularities of the main part of the scattering matrix at fixed non-zero energy.*

The reflected singularities are those associated to the mapping  $(\bar{\omega}, \omega_n) \mapsto (-\bar{\omega}, \omega_n)$ , and, for  $c_+ = c_-$  the transmitted singularities are those associated to the mapping  $\omega \mapsto -\omega$ . Corollary 8.1 shows that knowledge of  $c_+$ ,  $c_-$ ,  $\lambda$  and the singularities of the (absolute) scattering matrix for any fixed non-zero energy determine the asymptotics of  $c$ .

Following the approach to studying the scattering matrix introduced in [27], in Section 7 we construct a parametrix for the Poisson operator. This is a key part of our proofs, as it facilitates an understanding of the singularities of the scattering matrix. We work particularly by adapting the techniques of [22] which are essentially a concretization of the approach introduced in [27]. However, the different behaviour of the unperturbed operator  $c_0^2 \Delta$  in different regions at infinity means that the analysis is considerably more involved.

To pass from a parametrix to the actual Poisson operator, we need a good understanding of the behaviour of  $(\Delta - (\lambda - i0)^2 c^{-2})^{-1} f$  at infinity, when  $f \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$  and  $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$  for some  $\phi \in C_c^\infty(\mathbb{R})$ . In practice, we shall apply this understanding to the error arising from

the parametrix of the Poisson operator. When  $c_+ = c_-$ , we can obtain the necessary results by modifying some  $n$ -body results of [14] and [28]. However, when  $c_+ < c_-$  these results no longer apply, and we develop new techniques. The essential idea of these techniques is to repeatedly develop better approximations with improving smoothness properties. Thus Sections 9 and 10 are devoted to understanding  $(\Delta - (\lambda - i0)^2 c^{-2})^{-1}$ , allowing us to finish the proof of Theorem 1.1. In particular, we prove the following limiting absorption principle:

**THEOREM 1.3.** — *Let  $c$  and  $c_0$  satisfy the hypotheses of Section 2 and hypothesis (H1) or (H2). For any  $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$ , and  $f \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ , such that  $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$  for some  $\phi \in C_c^\infty(\mathbb{R})$ , we have*

$$(\chi(z/|z|)(\Delta - c^{-2}(\lambda - i0)^2)^{-1}f)|_{|z|>1} = e^{-i\lambda|z|/c}|z|^{-(n-1)/2}a_0(z/|z|) + u_1$$

where  $a_0 \in C^\infty(\mathbb{S}_c^{n-1})$  and  $u_1 \in \langle z \rangle^\epsilon L^2(\mathbb{R}^n)$  for all  $\epsilon > 0$ .

Here  $C_c^\infty(\mathbb{S}_c^{n-1})$  is the space of smooth functions vanishing in a neighbourhood of the equator and in a neighbourhood of  $\{(\bar{\omega}, \omega_n) \in \mathbb{S}^{n-1} : \omega_n = \sqrt{1 - c_+^2/c_-^2}\}$ .

An announcement of some of these results and an outline of part of the proof can be found in the lecture notes [7].

A note on organization: We need Theorem 1.3 to prove Theorems 1.1 and 1.2. However, since the proof of Theorem 1.3 is rather involved and uses different techniques from much of the remainder of the paper, we defer its proof to Section 10. Sections 2 and 3 contain preliminary information, stating assumptions, fixing notation, and recalling some results of other papers. In Section 4 we define the Poisson operator, which we use in Section 5 to define the (absolute) scattering matrix. Also in Section 4 we prove the existence of the Poisson operator and prove Theorem 1.1, though we use results proved later in the paper. The proof that the Poisson operator (and thus the scattering matrix) is well defined is deferred to Section 6. In Section 7 we construct  $\tilde{P}$ , an approximation of the Poisson operator  $P$ , proving Proposition 4.2. Theorem 1.2 is proved in Section 8, using the construction of  $\tilde{P}$  and Theorem 1.3. Finally, Sections 9 and 10 contain results about  $\lim_{\epsilon \downarrow 0} (\Delta - c^{-2}(\lambda - i\epsilon)^2)^{-1}$ , culminating in the proof of Theorem 1.3.

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### 2. Assumptions and notation.

Throughout,  $z = (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ .

Both sound speeds  $c$  and  $c_0$  satisfy  $0 < c_m \leq c, c_0 \leq c_M < \infty$ . Moreover,  $c_0(y)$  is piecewise smooth and there exists a finite  $y_M$  so that  $c_0(y) = c_{\pm}$  when  $\pm y > y_M$ , with  $c_- \geq c_+$ . In addition, all derivatives of  $c_0$  are bounded except at finitely many values of  $y$ . This includes the case where  $c_0$  is piecewise constant.

We require that, away from the hypersurface  $\{y = 0\}$ ,  $c - c_0$  be smooth outside of a compact set  $K$ , and for simplicity we choose  $y_M$  so that  $K \subset \mathbb{R}^{n-1} \times [-y_M, y_M]$ . Moreover, we make requirements on the behaviour of  $c - c_0$  at infinity. We have, for  $y \neq 0$ ,

$$(4) \quad D_z^\alpha (c(z) - c_0(y)) = D_z^\alpha \sum_{j \geq J}^N \gamma_j \left( \frac{z}{|z|} \right) |z|^{-j} + \mathcal{O}(|z|^{-N-1-|\alpha|})$$

for any  $N$  and any multi-index  $\alpha$ , where  $\gamma_j \in C_b^\infty(\mathbb{S}^{n-1} \setminus \{(\bar{w}, 0)\})$ . Here we use the notation that  $C_b^\infty(X)$  is the space of smooth functions on  $X$  that have all derivatives bounded. We shall take  $J$  to be at least 2 everywhere, although sometimes we shall require it to be larger. Some of our results hold under less restrictive hypotheses.

Additionally, we shall often use one of the following hypotheses:

(H1)  $J = 2, c_+ = c_-, c$  and  $c_0$  are smooth.

(H2)  $J \geq 4$ .

We warn the reader that the choice of the total space dimension to be  $n$  rather than  $n+1$  is in disagreement with [5] and many other papers on the subject. We use the notation  $\langle w \rangle = (1 + |w|^2)^{1/2}$ . Throughout,  $\epsilon$  shall stand for a small positive quantity and  $C$  for a positive constant, either of which may change from line to line. For  $w \in \mathbb{R}^m, b(w, t) \sim \sum_{j \geq 0} |w|^{l-j} b_j(w/|w|, t)$  means

$$\left| D_w^\alpha D_t^\beta \left( b(w, t) - \sum_{j=0}^N |w|^{l-j} b_j(w/|w|, t) \right) \right| \leq C_{\alpha, \beta, K} \langle w \rangle^{l-|\alpha|-N-1}$$

for any  $N$  and  $t \in K, K$  a compact set in some manifold.

### 3. Spectral theory of $c_0^2\Delta$ .

In order to define the (absolute) scattering matrix for  $c^2\Delta$ , we will need some understanding of the generalized eigenfunctions of  $c_0^2\Delta$  and  $c^2\Delta$ , and in particular of the space that parameterizes them. Further details can be found in, for example, [3], [13], [32], [34].

The operators  $c_0^2\Delta$  and  $c^2\Delta$  are formally self-adjoint on  $L^2(\mathbb{R}^n, c_0^{-2}dz)$  and  $L^2(\mathbb{R}^n, c^{-2}dz)$ , respectively, and have unique self-adjoint extensions.

Roughly speaking, the spectral measure of  $c_0^2\Delta$  can be given in terms of two kinds of families of functions. At fixed energy  $\lambda$ , the first is parameterized by  $\mathbb{S}_c^{n-1}$ . Here

$$\mathbb{S}_c^{n-1} = \{\omega = (\bar{\omega}, \omega_n) \in \mathbb{S}^{n-1} : \omega_n \neq 0, \omega_n \neq \sqrt{1 - c_+^2/c_-^2}\}.$$

(Compare [32, Section 2.1]; we are modifying somewhat the notation of [32].) The generalized eigenfunctions are  $\Phi_0(z, \lambda, \omega)$ , where for  $\pm\omega_n > 0$ ,

$$(5) \quad \Phi_0(z, \lambda, \omega) = e^{i\lambda x \cdot \bar{\omega}/c_{\pm}} \phi_{\pm}(y, \lambda, \omega_n)$$

and  $\phi_{\pm}$  satisfies

$$(6) \quad c_0^2(D_y^2 + \lambda^2(1 - \omega_n^2)c_{\pm}^{-2})\phi_{\pm} = \lambda^2\phi_{\pm}.$$

Moreover, as  $y \rightarrow \infty$ ,

$$(7) \quad \phi_+(y) \sim e^{i\lambda y \omega_n/c_+} + R_+(\lambda, \omega_n)e^{-i\lambda \omega_n y/c_+}$$

and as  $y \rightarrow -\infty$ ,

$$(8) \quad \phi_+(y) \sim T_+(\lambda, \omega_n)e^{i\lambda y \sqrt{1/c_-^2 - 1/c_+^2 + \omega_n^2/c_+^2}}$$

where when  $1/c_-^2 - 1/c_+^2 + \omega_n^2/c_+^2 < 0$  we take the square root so that the right hand side of (8) is exponentially decreasing. The function  $\phi_-$  is similarly determined: as  $y \rightarrow -\infty$ ,

$$(9) \quad \phi_-(y) \sim e^{i\lambda y \omega_n/c_-} + R_-(\lambda, \omega_n)e^{-i\lambda \omega_n y/c_-}$$

and as  $y \rightarrow \infty$ ,

$$(10) \quad \phi_-(y) \sim T_-(\lambda, \omega_n)e^{-i\lambda y \sqrt{1/c_+^2 - 1/c_-^2 + \omega_n^2/c_-^2}}.$$

Properly normalized, these generalized eigenfunctions appear in the spectral decomposition of  $c_0^2\Delta$ .

A second type of generalized eigenfunction comes from eigenvalues of  $c_0^2(\kappa^2 + D_y^2)$  on  $L^2(\mathbb{R}, c_0^{-2}dy)$ , if there are any. If there are any eigenvalues, let  $\lambda_1^2(\kappa) < \lambda_2^2(\kappa) < \dots < \lambda_{k(\kappa)}^2(\kappa) < c_+^2\kappa^2$  denote the eigenvalues of  $c_0^2(\kappa^2 + D_y^2)$ . There are only finitely many (perhaps no) eigenvalues for fixed  $\kappa$  and the number is nondecreasing in  $\kappa^2$ . Additionally, if  $\kappa > 0$  and  $\lambda_j > 0$ , then  $\frac{d\lambda_j}{d\kappa} > 0$ ; this can be seen via an integration by parts argument (see, e.g., [5, Sect. 2.2]).

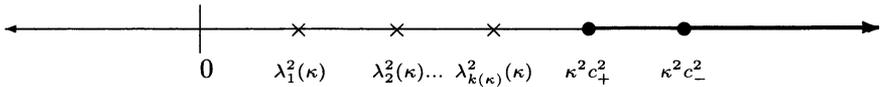


Figure 1. The spectrum of  $c_0^2(\kappa^2 + D_y^2)$ , for  $c_+ < c_-$ ,  $\min c_0 < c_+$ .

Let  $\kappa_j^0$  be the smallest positive number such that  $c_0^2(\kappa^2 + D_y^2)$  has  $j$  eigenvalues for all  $\kappa > \kappa_j^0$ . Let  $\kappa_j$  be the inverse function of  $\lambda_j$  (with the same sign), and let  $t_j = \lim_{\kappa \downarrow \kappa_j^0} \lambda_j^2(\kappa) = c_+^2(\kappa_j^0)^2$ . The  $\{t_j\}$  are called thresholds of  $c_0^2\Delta$ . Let  $T(\lambda)$  be the number of thresholds  $t_j$  less than  $\lambda^2$ . For  $0 < j \leq T(\lambda)$ ,  $\bar{\omega} \in \mathbb{S}^{n-1}$ , let

$$\Phi_j(z, \lambda, \omega) = e^{i\kappa_j(\lambda)x \cdot \bar{\omega}} f_j(y),$$

where  $f_j(y) \in L^2(\mathbb{R}, c_0^{-2}dy)$  satisfies

$$c_0^2(\kappa_j^2 + D_y^2)f_j = \lambda^2 f_j$$

and note that  $(c_0^2\Delta - \lambda^2)\Phi_j = 0$ . The functions  $f_j(y)$  are exponentially decreasing, as can be seen by noting that for  $\pm y > y_M$ , they are  $L^2$  solutions of  $D_y^2 f_j = (\lambda^2/c_\pm^2 - \kappa_j^2(\lambda))f_j$ .

At energy level  $\lambda^2$ , we can parameterize the generalized eigenfunctions of  $c_0^2\Delta$  by  $\mathbb{S}_c^{n-1}$  and  $T(\lambda)$  copies of  $\mathbb{S}^{n-2}$ . The continuous spectrum of  $c^2\Delta$  is parameterized by the same space as that of  $c_0^2\Delta$ . Therefore, the (absolute) scattering matrices of  $c_0^2\Delta$  and  $c^2\Delta$  are operators from  $L^2(\mathbb{S}_c^{n-1}) \oplus_{1 \leq j \leq T(\lambda)} L^2(\mathbb{S}^{n-2})$  into itself. In [5] a definition of the scattering matrix is given in terms of the generalized eigenfunctions. Here, however, it will be more useful to define the (absolute) scattering matrix using the Poisson operator, which we shall do in Section 5.

### 4. The Poisson operator.

The Poisson operator is defined as an operator

$$P(\lambda) : C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_{i=1}^{T(\lambda)} C^\infty(\mathbb{S}^{n-2}) \rightarrow \langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n).$$

(Again, we have  $C_c^\infty(\mathbb{S}_c^{n-1})$  because  $\mathbb{S}_c^{n-1}$  parameterizes part of the continuous spectrum.)

In order to define the Poisson operator, we introduce the notion of an “outgoing” function in this setting.

DEFINITION 4.1. — *A function  $u \in \langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$  will be called outgoing if it has a decomposition  $u = u_0 + \sum_1^{T(\lambda)} u_j + \tilde{u}$  with the following properties:*

$$\begin{aligned} \left( \frac{\partial}{\partial |z|} + i\lambda/c \right) u_0 &\in \langle z \rangle^\epsilon L^2(\mathbb{R}^n); \quad u_0 \in \langle y \rangle^{1/2+\epsilon} \langle z \rangle^\epsilon L^2(\mathbb{R}^n); \\ u_j &= \tilde{u}_j(x) f_j(y), \quad \tilde{u}_j \in \langle x \rangle^{1/2+\epsilon} L^2(\mathbb{R}^{n-1}), \\ \left( \frac{\partial}{\partial |x|} + i\kappa_j \right) \tilde{u}_j &\in \langle x \rangle^\epsilon L^2(\mathbb{R}^{n-1}), \quad 1 \leq j \leq T(\lambda); \\ \tilde{u}, \frac{\partial}{\partial |z|} \tilde{u} &\in \langle z \rangle^\epsilon L^2(\mathbb{R}^n), \end{aligned}$$

for any  $\epsilon > 0$ .

PROPOSITION 4.1. — *If  $g = (g_0, g_1, \dots, g_{T(\lambda)}) \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_{i=1}^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$ , then for  $\lambda \in \mathbb{R}, \lambda \neq 0$ , there is a unique  $u$  such that*

$$(c^2 \Delta - \lambda^2)u = 0$$

and, at infinity,

$$u \sim |z|^{-(n-1)/2} e^{i\lambda|z|/c} g_0 \left( \frac{z}{|z|} \right) + |x|^{-(n-2)/2} \sum_1^{T(\lambda)} e^{i\kappa_j(\lambda)|x|} g_j \left( \frac{x}{|x|} \right) f_j(y) + v$$

where  $v$  is outgoing as defined in Definition 4.1 and the  $f_j$  and  $\kappa_j$  are as defined in Section 3.

We will give the proof of the existence of such a  $u$  in Section 4.1. We postpone the proof of the uniqueness to Section 6. This proposition allows us to define the Poisson operator,  $P(\lambda)$ .

DEFINITION 4.2. — *If  $g = (g_0, g_1, \dots, g_{T(\lambda)}) \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_{i=1}^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$ ,  $\lambda \in \mathbb{R}, \lambda \neq 0$ , then  $P(\lambda)g = u$ , where  $u$  is the  $u$  of Proposition 4.1.*

Definition 5.1, using Proposition 5.1, defines the (absolute) scattering matrix via the Poisson operator.

### 4.1. Existence of the Poisson operator.

Here we prove the existence part of Proposition 4.1, using some results that we prove later in the paper. The first step is the construction of an “approximation” of the Poisson operator, which is carried out in Section 7. Other, simpler, proofs of the existence part of Proposition 4.1 are available, but this one facilitates the proofs of Theorem 1.1 and 1.2.

For  $\omega \in \mathbb{S}^{n-1}$ , let  $\delta_\omega(\theta)$  be the distribution such that

$$\int_{\theta \in \mathbb{S}^{n-1}} \delta_\omega(\theta) f(\theta) d\text{vol}_{\mathbb{S}^{n-1}} = f(\omega),$$

and similarly for  $\delta_{\bar{\omega}}(\bar{\theta})$ ,  $\bar{\omega} \in \mathbb{S}^{n-2}$ .

PROPOSITION 4.2. — *Let  $z \in \mathbb{R}^n$ ,  $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $\omega \in \mathbb{S}_c^{n-1}$ ,  $\bar{\omega} \in \mathbb{S}^{n-2}$ . There is a distribution  $\tilde{P}(\lambda) = (\tilde{P}_0(z, \lambda, \omega), \tilde{P}_1(z, \lambda, \bar{\omega}), \dots, \tilde{P}_{T(\lambda)}(z, \lambda, \bar{\omega}))$  such that, for  $j = 1, \dots, T(\lambda)$ ,*

$$\begin{aligned} D_\omega^\alpha (c^2 \Delta - \lambda^2) \tilde{P}_0(\cdot, \lambda, \omega), D_{\bar{\omega}}^\beta (c^2 \Delta - \lambda^2) \tilde{P}_j(\cdot, \lambda, \bar{\omega}) &\in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n), \\ D_\omega^\alpha (1 - \chi(y))(c^2 \Delta - \lambda^2) \tilde{P}_0(\cdot, \lambda, \omega), \\ D_{\bar{\omega}}^\beta (1 - \chi(y))(c^2 \Delta - \lambda^2) \tilde{P}_j(\cdot, \lambda, \bar{\omega}) &\in \mathcal{S}(\mathbb{R}^n), \end{aligned}$$

where  $\chi(y) \in C_c^\infty(\mathbb{R})$  is 1 for  $|y| \leq y_M + 1$  and  $\alpha, \beta$  are any multi-indices. Moreover, distributionally as  $|z| \rightarrow \infty$ ,

$$\begin{aligned} \tilde{P}_0(z, \lambda, \omega) &= |z|^{-(n-1)/2} (e^{i\lambda|z|/c_0(y)} \delta_\omega(z/|z|) + e^{-i\lambda|z|/c_0(y)} h_0(z/|z|, \omega)) \\ &\quad + \mathcal{O}(|z|^{-(n+1)/2}) \end{aligned}$$

where  $h_0(\omega, \theta) \in \mathcal{D}'(\mathbb{S}_c^{n-1} \times \mathbb{S}_c^{n-1})$ . If  $c_+ = c_-$ ,  $h_0$  is the Schwartz kernel of a Fourier integral operator associated to broken geodesic flow at time  $\pi$ . If  $c_- > c_+$ , then  $h_0$  is the Schwartz kernel of the sum of Fourier integral operators associated with the mapping

$$(\bar{\omega}, \omega_n) \mapsto (-\bar{\omega}, \omega_n)$$

and the mapping

$$(\bar{\omega}, \omega_n) \mapsto (-c_- \bar{\omega} / c_+, -\sqrt{1 - c_-^2 |\bar{\omega}|^2 / c_+^2}) \quad \text{if } \sqrt{1 - c_+^2 / c_-^2} < \omega_n$$

and

$$(\bar{\omega}, \omega_n) \mapsto (-c_+ \bar{\omega} / c_-, \sqrt{1 - c_+^2 |\bar{\omega}|^2 / c_-^2}) \quad \text{if } \omega_n < 0.$$

For  $j > 0$ , distributionally as  $|z| \rightarrow \infty$ ,

$$\tilde{P}_j(z, \lambda, \bar{\omega}) = |x|^{-(n-2)/2} (e^{i\kappa_j(\lambda)|x|} \delta_{\bar{\omega}}(x/|x|) + e^{-i\kappa_j(\lambda)|x|} h_j(x/|x|, \bar{\omega})) f_j(y) + \mathcal{O}(|x|^{-(n+1)/2} \langle y \rangle^{-\infty}).$$

Here  $h_j \in \mathcal{D}'(\mathbb{S}^{n-2} \times \mathbb{S}^{n-2})$  is the Schwartz kernel of a Fourier integral operator associated with the antipodal map on  $\mathbb{S}^{n-2}$ .

We will prove this proposition in Section 7.

We also use

PROPOSITION 4.3. — *If  $f \in \langle z \rangle^{-3/2+\epsilon} L^2(\mathbb{R}^n)$  for every  $\epsilon > 0$  and  $J \geq 2$ , then  $u = (\Delta - (\lambda - i0)^2 c^{-2})^{-1} f$  is outgoing in the sense of Definition 4.1.*

This proposition will be proved in Section 9.

*Proof of the Existence Part of Proposition 4.1.* — Let

$$Q(z, \lambda, \omega) = \tilde{P}(z, \lambda, \omega) - (\Delta - (\lambda - i0)^2 c^{-2})^{-1} (\Delta - \lambda^2 c^{-2}) \tilde{P}(z, \lambda, \omega),$$

and let  $Q = (Q_0, Q_1, \dots, Q_{T(\lambda)})$ . For  $g = (g_0, g_1, \dots, g_{T(\lambda)}) \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_{i=1}^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$ , let

$$u = \int_{\omega \in \mathbb{S}_c^{n-1}} Q_0(z, \lambda, \omega) g_0(\omega) + \sum_{j=1}^{T(\lambda)} \int_{\bar{\omega} \in \mathbb{S}^{n-2}} Q_j(z, \lambda, \bar{\omega}) g_j(\bar{\omega}).$$

Then

$$(c^2 \Delta - \lambda^2) u = 0$$

and  $u$  has an expansion at infinity as required in Proposition 4.1. Consequently,  $Q = P(\lambda)$ , the Poisson operator. □

### 5. The scattering matrix and the proof of Theorem 1.1.

We will define the (absolute) scattering matrix via the Poisson operator. In doing so, we assume that the Poisson operator is uniquely determined; we will prove this in Section 6.

We shall use the following lemma, which will be proved in Section 9.

LEMMA 5.1. — *If  $f \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ , then for  $y \in K$ ,  $K \subset \mathbb{R}$  compact,  $|x| > 1$ ,*

$$\begin{aligned}
 (\Delta - c^{-2}(\lambda - i0)^2)^{-1} f(z)|_{|x|>1, y \in K} \\
 = |x|^{-(n-2)/2} \sum_{j=1}^{T(\lambda)} e^{-i\kappa_j(\lambda)|x|} b_j(x/|x|) f_j(y) + v_1
 \end{aligned}$$

where  $b_j \in C^\infty(\mathbb{S}^{n-2})$  and  $v_1 \in \langle x \rangle^\epsilon L^2(\mathbb{R}^{n-1} \times K)$  for any  $\epsilon > 0$ . Moreover, for  $1 \leq j \leq T(\lambda)$  the  $u_j$  in the definition of outgoing functions can be taken to be  $u_j = |x|^{-(n-2)/2} e^{-i\kappa_j(\lambda)|x|} b_j(x/|x|) f_j(y)$ .

To define the scattering operator, we need

PROPOSITION 5.1. — *Let  $g = (g_0, g_1, \dots, g_{T(\lambda)}) \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$ . Let  $\theta = \frac{z}{|z|} \in K$ , for  $K$  a compact set in  $\mathbb{S}_c^{n-1}$ . Then, there exists  $g'_0 \in C^\infty(\mathbb{S}_c^{n-1})$  such that for  $\theta \in K$ , as  $|z| \rightarrow \infty$ ,*

$$P(\lambda)g = |z|^{-(n-1)/2} [e^{i\lambda|z|/c} g_0(\theta)|_K + e^{-i\lambda|z|/c} (g'_0(\theta))|_K] + \tilde{u}_K$$

where  $\tilde{u}_K \in \langle z \rangle^\epsilon L^2(\mathbb{R}^n \cap (K \times [1, \infty)))$  for any  $\epsilon > 0$ .

Let  $y \in K_y$ ,  $K_y \subset \mathbb{R}$  compact, and let  $\bar{\theta} = \frac{x}{|x|}$ . Then, there exists  $g'_j \in C^\infty(\mathbb{S}^{n-2})$  such that as  $|x| \rightarrow \infty$ ,

$$\begin{aligned}
 P(\lambda)g = |x|^{-(n-2)/2} \\
 \left[ \sum_{j=1}^{T(\lambda)} e^{i\kappa_j(\lambda)|x|} g_j(\bar{\theta}) f_j(y)|_{K_y} + \sum_{j=1}^{T(\lambda)} e^{-i\kappa_j(\lambda)|x|} g'_j(\bar{\theta}) f_j(y)|_{K_y} \right] + \tilde{u}_c
 \end{aligned}$$

where  $\tilde{u}_c \in \langle x \rangle^\epsilon L^2((\mathbb{R}^n \cap (\{|x| > 1\}) \times K_y))$  for any  $\epsilon > 0$ .

*Proof.* — We use the same notation to stand for an operator and its Schwartz kernel. Since

$$(11) \quad P(\lambda) = \tilde{P}(\lambda) - (\Delta - (\lambda - i0)^2 c^{-2})^{-1} (\Delta - \lambda^2 c^{-2}) \tilde{P},$$

the first part of this proposition follows from Propositions 4.2 and 4.3 and Theorem 1.3. The second part of the proposition follows again from the identity (11), Propositions 4.2 and 4.3, and Lemma 5.1. □

This information about the Poisson operator allows us to define the (absolute) scattering matrix  $A(\lambda)$ .

DEFINITION 5.1. — *The (absolute) scattering matrix  $A(\lambda)$  is given, for  $g \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$ , by  $A(\lambda)g = g' \in C^\infty(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$ , where for any compact set  $K \subset \mathbb{S}_c^{n-1}$ ,  $(g'_0)|_K$  is as in Proposition 5.1, and, for  $1 \leq j \leq T(\lambda)$ ,  $g'_j$  is as in Proposition 5.1.*

We remark that this definition differs slightly from the absolute scattering matrix discussed in [5]. However, as the two differ by a straightforward normalization, we shall use this definition here both to emphasize the similarities with the absolute scattering matrix as defined in [26] and because it is more convenient for the inverse results.

For completeness, in Appendix A we outline a proof that  $A(\lambda)$  can be extended to a bounded operator on  $L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$ .

For fixed  $\lambda$ ,  $A(\lambda)$  is a matrix  $(A_{ij}(\lambda))$ ,  $0 \leq i, j \leq T(\lambda)$ , with the  $A_{ij}$  operators. We shall call  $A_{00}(\lambda)$  the “main part” of the scattering matrix. If the operator  $c_0^2(D_y + \kappa^2)$  has no eigenvalues on  $L^2(\mathbb{R}, c_0^{-2}dy)$ , the main part of the scattering matrix is the entire scattering matrix.

We can now prove Theorem 1.1, on the structure of the (absolute) scattering matrix.

*Proof of Theorem 1.1.* — Again, we use the identity (11). Using the definition of the scattering matrix and Theorem 1.3, any singularities in the main part of the scattering matrix must come from  $\tilde{P}_0$ . Then Proposition 4.2 gives the structure of the singularities of the scattering matrix.  $\square$

Finally, we conclude this section with a proposition that describes the singularities of the other entries in the scattering matrix.

PROPOSITION 5.2. — *Let  $c, c_0$  satisfy the general conditions of Section 2 and either hypothesis (H1) or (H2). Let  $A(\lambda) = (A_{ij}(\lambda))$ ,  $0 \leq i, j \leq T(\lambda)$ . Then, for  $j > 0$ ,  $A_{jj}(\lambda)$  is a Fourier integral operator associated with the antipodal mapping on  $\mathbb{S}^{n-2}$ , and for  $i$  not equal to  $j$ ,  $A_{ij}(\lambda)$  is a smoothing operator.*

*Proof.* — Using the identity (11), Lemma 5.1, and Theorem 1.3, the singularities of the scattering matrix arise from  $\tilde{P}_j$ . Then Proposition 4.2 and the definition of the scattering matrix show that the singularities of the scattering matrix are as claimed.  $\square$

### 6. Proof of Proposition 4.1: uniqueness.

In proving Proposition 6.1, we shall use some results of Weder [31], [32] (See also [11]). We recall some of his results below. Let  $A = (-i/4)(z \cdot \nabla_z + \nabla_z \cdot z)$ . We define the commutator  $[\Delta - \lambda^2/c^2, A]$  as a quadratic form (See the proof of Theorem 5.4 of [32]). By [31, Lemma 3.1] for all  $\lambda > 0$ ,  $\mu > -\lambda^2/c^2_-$ , there is a compact operator  $K$ , a compact interval  $\Lambda$  containing  $\mu$ , and  $\beta > 0$  such that

$$(12) \quad iE_\Lambda[\Delta - \lambda^2c^{-2}, A]E_\Lambda \geq \beta E_\Lambda + K$$

where  $E_\Lambda = E_\Lambda(\Delta - \lambda^2c^{-2})$  is the spectral projector for  $\Delta - \lambda^2c^{-2}$ .

The following proposition and its proof, included for the convenience of the reader, are essentially adapted from [2, Lemma 4.17].

**PROPOSITION 6.1.** — *If  $u \in \langle z \rangle^\epsilon L^2(\mathbb{R}^n)$  for every  $\epsilon > 0$  and  $(\Delta - \lambda^2/c^{-2})u = 0$ , then  $u \equiv 0$ .*

*Proof.* — By the results of [31], [32], there is no nontrivial  $L^2$  null space of  $\Delta - \lambda^2/c^2$ , so it suffices to show that  $u \in L^2(\mathbb{R}^n)$ .

For  $\epsilon, \delta > 0$ , let  $u_{\epsilon\delta} = (1 + \delta\langle z \rangle)^{-\epsilon}u \in L^2(\mathbb{R}^n)$ . Let  $L = \Delta - \lambda^2/c^2$ , and let  $\Phi \in C_c^\infty(\mathbb{R})$  be 1 in a neighbourhood of 0. Note that

$$L\Phi(L)u_{\epsilon\delta} = \Phi(L) \left( \sum_j \frac{2\epsilon\delta z_j}{\langle z \rangle} (1 + \delta\langle z \rangle)^{-1} \frac{\partial}{\partial z_j} u_{\epsilon\delta} + \epsilon f_{\epsilon\delta}(z)u_{\epsilon\delta} \right)$$

where

$$(13) \quad f_{\epsilon\delta}(z) = \delta(1 + \delta\langle z \rangle)^{-1} \left( -\delta \frac{|z|^2}{\langle z \rangle^2} (1 - \epsilon)(1 + \delta\langle z \rangle)^{-1} + \frac{n}{\langle z \rangle} - \frac{|z|^2}{\langle z \rangle^3} \right).$$

Then

$$(14) \quad \begin{aligned} & ([L, A]\Phi(L)u_{\epsilon\delta}, \Phi(L)u_{\epsilon\delta}) \\ &= -2i \operatorname{Im} (AL\Phi(L)u_{\epsilon\delta}, \Phi(L)u_{\epsilon\delta}) \\ &= -2i \operatorname{Im} \left( A\Phi(L) \left( \sum_j \frac{2\epsilon\delta z_j}{\langle z \rangle} (1 + \delta\langle z \rangle)^{-1} \frac{\partial}{\partial z_j} u_{\epsilon\delta} + \epsilon f_{\epsilon\delta}(z)u_{\epsilon\delta} \right), \Phi(L)u_{\epsilon\delta} \right). \end{aligned}$$

Since

$$\Phi(L) : \langle z \rangle^{-\gamma} L^2(\mathbb{R}^n) \rightarrow \langle z \rangle^{-\gamma} L^2(\mathbb{R}^n),$$

we have

$$\left| \left( A\Phi(L) \sum_j \frac{2\epsilon\delta z_j}{\langle z \rangle} (1 + \delta\langle z \rangle)^{-1} \frac{\partial}{\partial z_j} u_{\epsilon\delta}, \Phi(L)u_{\epsilon\delta} \right) \right| \leq \epsilon C_\Phi \left\| \sum_j \delta z_j (1 + \delta\langle z \rangle)^{-1} \frac{\partial}{\partial z_j} u_{\epsilon\delta} \right\| \cdot \|\Phi(L)u_{\epsilon\delta}\|_{H^1} \leq \epsilon C_\Phi \|u_{\epsilon\delta}\| \cdot \|\Phi(L)u_{\epsilon\delta}\|.$$

Here and below  $C_\Phi$  is a constant that may change from line to line which depends on  $\Phi$ , and also on  $\lambda$  and  $c$ , but which is independent of  $\delta$  and  $\epsilon$ . Since, using (13), a similar bound holds for  $(A\Phi(L)\epsilon f_{\epsilon\delta}u_{\epsilon\delta}, \Phi(L)u_{\epsilon\delta})$ , we obtain from (14)

$$(15) \quad |([L, A]\Phi(L)u_{\epsilon\delta}, \Phi(L)u_{\epsilon\delta})| \leq \epsilon C_\Phi \|u_{\epsilon\delta}\| \cdot \|\Phi(L)u_{\epsilon\delta}\|.$$

Since  $(1 - \Phi(L))u = 0$ , we have

$$(16) \quad \|u_{\epsilon\delta}\| = \|\Phi(L)u_{\epsilon\delta} - [\Phi(L), (1 + \delta\langle z \rangle)^{-\epsilon}]u\| \leq \|\Phi(L)u_{\epsilon\delta}\| + C_\Phi \|\langle z \rangle^{-1/2}u\|.$$

The bound  $\|[\Phi(L), (1 + \delta\langle z \rangle)^{-\epsilon}]u\| \leq C_\Phi \|\langle z \rangle^{-1/2}u\|$  can be seen, for example, by using the Helffer-Sjöstrand representation of  $\Phi(L)$  (e.g. [17]).

However, using (12), the fact that 0 is not an eigenvalue of  $L$ , and [9, Lemma 4.2], we obtain

$$([L, A]\Phi(L)u_{\epsilon\delta}, \Phi(L)u_{\epsilon\delta}) \geq \beta_1 (\Phi(L)u_{\epsilon\delta}, \Phi(L)u_{\epsilon\delta})$$

for some  $\beta_1 > 0$ , if the support of  $\Phi$  is chosen sufficiently small. Thus we have, using (15) and (16), and choosing  $\epsilon$  sufficiently small,

$$\begin{aligned} \beta_1 \|\Phi(L)u_{\epsilon\delta}\|^2 &\leq \frac{\beta_1}{2} \|u_{\epsilon\delta}\| \cdot \|\Phi(L)u_{\epsilon\delta}\| \\ &\leq \frac{\beta_1}{2} (\|\Phi(L)u_{\epsilon\delta}\|^2 + C_\Phi \|\Phi(L)u_{\epsilon\delta}\| \cdot \|\langle z \rangle^{-1/2}u\|). \end{aligned}$$

Therefore

$$\|\Phi(L)u_{\epsilon\delta}\| \leq C_\Phi \|\langle z \rangle^{-1/2}u\|.$$

Using (16), this shows that for sufficiently small  $\epsilon > 0$ ,  $\|u_{\epsilon\delta}\|$  is bounded independent of  $\delta$ , and thus  $u \in L^2(\mathbb{R}^n)$ , and  $u \equiv 0$ .  $\square$

This allows us to show the uniqueness of “outgoing” solutions.

**PROPOSITION 6.2.** — *Given  $f \in L^2(\mathbb{R}^n)$ , there is at most one outgoing  $u \in \langle z \rangle^{1/2+\epsilon}L^2(\mathbb{R}^n)$  with  $(\Delta - \lambda^2/c^2)u = f$ .*

*Proof.* — Suppose there are two such  $u$ . Then by considering the difference we can reduce to the case where  $f \equiv 0$ . Then

$$\begin{aligned} 0 &= \int_{|z| < R} (\Delta - \lambda^2/c^2)u\bar{u} = - \int_{|z|=R} \left( \frac{\partial}{\partial|z|} u\bar{u} - u \frac{\partial}{\partial|z|} \bar{u} \right) \\ &= 2 \int_{|z|=R} \frac{i\lambda}{c} |u_0|^2 + \sum_{j,k=1}^{T(\lambda)} i\kappa_j(\lambda)u_j\bar{u}_k + i \operatorname{Re} \sum_{j=1}^{T(\lambda)} (\lambda/c + \kappa_j)u_0\bar{u}_j \\ &\qquad\qquad\qquad + i \operatorname{Im} \left( u_0e_0 + \sum_{j=1}^{T(\lambda)} u_j e_j \right) + i \operatorname{Im} \tilde{e}\tilde{f} \end{aligned}$$

where  $e_0, e_j, \tilde{e}, \tilde{f} \in \langle z \rangle^\epsilon L^2(\mathbb{R}^n)$  for all  $\epsilon > 0$ . Using the facts that  $\int f_j(y)\bar{f}_k(y)dy = 0$  if  $j \neq k$  and  $f_j$  is exponentially decreasing, this implies that as  $R \rightarrow \infty$

$$\begin{aligned} (17) \quad &\int_{|z|=R} \left| \frac{2i\lambda}{c} |u_0|^2 + \sum_{j=1}^{T(\lambda)} 2i\kappa_j(\lambda)|u_j|^2 \right| \\ &\leq C \int_{|z|=R} \left( \sum_{j=1}^{T(\lambda)} |u_0\bar{u}_j| + |u_0||e_0| + \sum_{j=1}^{T(\lambda)} |u_j e_j| + |\tilde{e}||\tilde{f}| \right) \\ &\qquad\qquad\qquad + \mathcal{O}(R^{-2-\epsilon}). \end{aligned}$$

Since  $u_j \in \langle y \rangle^{-\infty} \langle x \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$  and  $u_0 \in \langle y \rangle^{1/2+\epsilon} \langle z \rangle^\epsilon L^2(\mathbb{R}^n)$ , we have  $u_0\bar{u}_j \in \langle y \rangle^{-\infty} \langle z \rangle^{1/2+2\epsilon} L^1(\mathbb{R}^n)$ . Therefore, the right hand side of (17), considered as a function of  $R$  for large  $R$ , is in  $R^{1/2+2\epsilon} L^1(\mathbb{R}_+)$ , so that  $u_0, u_j \in \langle z \rangle^{1/4+\epsilon} L^2(\mathbb{R}^n)$ .

Now suppose we know that  $u_0, u_j \in \langle z \rangle^\beta L^2(\mathbb{R}^n)$  for some  $\beta > 0$ . Then, using (17) again, since  $u_0\bar{u}_j \in \langle y \rangle^{-\infty} \langle z \rangle^{\beta+\epsilon} L^1(\mathbb{R}^n)$ , and  $u_0e_0, u_j e_j, \tilde{e}\tilde{f} \in \langle z \rangle^{\beta+\epsilon} L^1(\mathbb{R}^n)$ , we obtain  $u_0, u_j \in \langle z \rangle^{\beta/2+\epsilon} L^2(\mathbb{R}^n)$  for any  $\epsilon > 0$ . By repeating this argument, we obtain that  $u_0, u_j \in \langle z \rangle^\delta L^2(\mathbb{R}^n)$  for any  $\delta > 0, 1 \leq j \leq T(\lambda)$ . Thus  $u \in \langle z \rangle^\delta L^2(\mathbb{R}^n)$  for any  $\delta > 0$ . By the previous proposition,  $u \equiv 0$ . □

Proposition 6.2 gives the uniqueness part of Proposition 4.1.

### 7. The approximate Poisson operator $\tilde{P}$ .

In this section we prove Proposition 4.2: we construct an approximation of the Poisson operator. We will use the following lemma.

LEMMA 7.1. — For  $f \in L^2([-y_M, y_M])$ , the boundary value problem

$$c_0^2(\lambda^2|\bar{\omega}|^2/c_+^2 + D_y^2)b - \lambda^2b = f$$

with boundary conditions

$$\begin{aligned} -i\lambda\omega_n/c_+b(y_M) - b'(y_M) &= \alpha_1 \\ i\lambda(1/c_-^2 - |\bar{\omega}|^2/c_+^2)^{1/2}b(-y_M) - b'(-y_M) &= \alpha_2 \end{aligned}$$

has a unique solution in  $L^2([-y_M, y_M], c_0^{-2}dy)$  if  $1/c_-^2 \geq |\bar{\omega}|^2/c_+^2$ .

*Proof.* — This boundary value problem can be reduced to the form

$$\begin{aligned} -i\lambda\omega_n/c_+b(y_M) - b'(y_M) &= 0 \\ i\lambda(1/c_-^2 - |\bar{\omega}|^2/c_+^2)^{1/2}b(-y_M) - b'(-y_M) &= 0 \\ c_0^2(\lambda^2|\bar{\omega}|^2/c_+^2 + D_y^2)b - \lambda^2b &= g. \end{aligned}$$

This has a solution if the null space of the adjoint operator is trivial, and the solution is unique if the only solution of the homogeneous equation is the trivial one.

The adjoint operator is the operator

$$c_0^2(\lambda^2|\bar{\omega}|^2/c_+^2 + D_y^2) - \lambda^2$$

with domain

$$(18) \quad \{h \in L^2([-y_M, y_M], c_0^{-2}dy) : -i\lambda\omega_n/c_+h(y_M) + h'(y_M) = 0 \\ \text{and } i\lambda(1/c_-^2 - |\bar{\omega}|^2/c_+^2)^{1/2}h(-y_M) + h'(-y_M) = 0\}.$$

Suppose  $g$  is a nontrivial element of the null space of the adjoint operator. Then

$$\begin{aligned} 0 &= \int_{-y_M}^{y_M} (c_0^2(\lambda^2|\bar{\omega}|^2/c_+^2 + D_y^2) - \lambda^2)g\bar{g}c_0^{-2}dy \\ &= -g'(y_M)\bar{g}(y_M) + g'(-y_M)\bar{g}(-y_M) \\ (19) \quad &+ g(y_M)\bar{g}'(y_M) - g(-y_M)\bar{g}'(-y_M). \end{aligned}$$

Using the boundary conditions, we find that this is

$$-2i\lambda\omega_n/c_+|g(y_M)|^2 - 2i\lambda(1/c_-^2 - |\bar{\omega}|^2/c_+^2)^{1/2}|g(-y_M)|^2.$$

Thus  $0 = g(y_M) = g'(-y_M)$ , or  $g(y) \equiv 0$ .

A similar calculation shows that the original operator has no nontrivial null space. □

In our proof of Proposition 4.2, the existence of an approximation to the Poisson operator, we shall concentrate on the construction of  $\tilde{P}_0$ . This is the most complicated component and also the one of primary interest, since our inverse results involve the main part of the scattering matrix,  $A_{00}(\lambda)$ . The construction owes a great deal to that of [22], and we refer the reader to it for full details.

We will show how to construct  $\tilde{P}_0(z, \lambda, \omega)$  when  $\omega_n > 0$ ; the case of  $\omega_n < 0$  is quite similar. The construction involves solving away errors at infinity. Since the model operator  $c_0^2 \Delta$  has different behaviour depending on the region “at infinity” ( $y > y_M$ ,  $|y| < y_M$ , or  $y < -y_M$ ) the techniques involved necessarily depend on the region in which  $z$  lies. The proof is additionally divided into two subcases:  $1/c_-^2 - |\bar{\omega}|^2/c_+^2 > 0$ , and  $1/c_-^2 - |\bar{\omega}|^2/c_+^2 < 0$ . Roughly speaking, the second subcase corresponds to total internal reflection and for these values of  $\omega$ ,  $\tilde{P}_0(z, \lambda, \omega)$  is exponentially decreasing in  $y$  as  $y \rightarrow -\infty$ . In this subcase we can handle the entire region  $y < y_M$  at once. The first subcase corresponds to angles of incidence in which some of the wave is transmitted, and for these values of  $\omega$ ,  $\tilde{P}_0$  is oscillatory at infinity in all directions.

Finally, there is a third division, involving the construction of  $\tilde{P}_0$  near the points where it is singular at infinity (and where the singularities of the scattering matrix arise). This we defer to Section 7.1.

In an attempt to make the proof more readable, we outline our construction of  $\tilde{P}_0(z, \lambda, \omega)$ ,  $\omega_n > 0$ ,  $\omega_n \neq \sqrt{1 - c_+^2/c_-^2}$ :

I.  $y > y_M$ , the “upper hemisphere”

a) “incident”

b) “reflected,” away from the singular point  $z/|z| = (-\bar{\omega}, \omega_n)$

II.  $1/c_-^2 - |\bar{\omega}|^2/c_+^2 > 0$ ,  $y < y_M$

a)  $y < -y_M$ , the “lower hemisphere,” away from the singular point (“transmitted”)

b)  $|y| < y_M$

III.  $1/c_-^2 - |\bar{\omega}|^2/c_+^2 < 0$ ,  $y < y_M$ .

Section 7.1. Construction near singular points.

The numbers correspond to numbering of the paragraphs.

**I.** Let  $\omega = (\bar{\omega}, \omega_n)$  with  $\omega_n > 0$ . In our construction of the approximation to the Poisson operator  $P_0$ , we begin with the function  $\Phi_0(z, \lambda, \omega)$  which is defined by (5)-(8). Note that, up to a constant multiple which depends only on  $n, \lambda$ , and  $c_{\pm}$ ,  $\Phi_0$  is the Schwartz kernel of the (partial) Poisson operator,  $P_{0,c_0}$ , for  $c_0^2\Delta$  when  $\omega$  is in the upper hemisphere of  $\mathbb{S}^{n-1}$ . We use this as our starting point, adding or subtracting terms to cancel the errors that result when we apply  $c^2\Delta - \lambda^2$  to  $\Phi_0$ .

When  $y > y_M$ , we use the techniques of [22] to construct  $\tilde{P}$ . Note that when we apply  $c^2\Delta - \lambda^2$  to  $\Phi_0$  we obtain an error which, for  $y > y_M$ , is of the form

$$(20) \quad (c^2\Delta - \lambda^2)\Phi_0 = (c^2 - c_+^2)\Delta\Phi_0 = a_1(z, \lambda, \omega)e^{i\lambda z \cdot \omega/c_+} + a_2(z, \lambda, \omega)e^{i\lambda(x \cdot \bar{\omega} - y\omega_n)/c_+}$$

where  $a_1(z, \lambda, \omega), a_2(z, \lambda, \omega) \in S_{phg}^{-2}$ . Here we say that  $b \in S_{phg}^{-j}(\mathbb{R}^n)$  if  $b \sim \sum_{k \geq j} |z|^{-k} b_k(\frac{z}{|z|}, \lambda, \omega)$ , with  $b_k$  smooth in  $z/|z|, \omega$ , and call it a polyhomogeneous symbol of order  $-j$ . We think of  $a_1(z, \lambda, \omega)e^{i\lambda z \cdot \omega/c_+}$  as an ‘‘incident’’ error and  $a_2(z, \lambda, \omega)e^{i\lambda(x \cdot \bar{\omega} - y\omega_n)/c_+}$  as the ‘‘reflected’’ error.

**Ia.** Note that if  $b(z, \lambda, \omega) \in S_{phg}^{-j}$ , then for  $y > y_M$ ,

$$(21) \quad (c^2\Delta - \lambda^2)b(z, \lambda, \omega)e^{i\lambda z \cdot \omega/c_+} = \left( -2i\lambda c_+ \omega \cdot \nabla_z b(z, \lambda, \omega) + c^2\Delta b(z, \lambda, \omega) + (c^2 - c_+^2) \frac{\lambda^2}{c_+^2} b(z, \lambda, \omega) - 2i\lambda \frac{(c^2 - c_+^2)}{c_+} \omega \cdot \nabla_z b(z, \lambda, \omega) \right) e^{i\lambda z \cdot \omega/c_+}.$$

Of the terms in parentheses on the right, the first is of the highest order, as  $\omega \cdot \nabla_z b \in S_{phg}^{-j-1}$ . The others are in  $S_{phg}^{-j-2}$ .

Suppose our construction to some point has resulted in an error of the form

$$(22) \quad \sum_{k \geq j} |z|^{-k} d_{I,-k} \left( \frac{z}{|z|}, \lambda, \omega \right) e^{i\lambda z \cdot \omega/c_+} + a_2(z, \lambda, \omega) e^{i\lambda(x \cdot \bar{\omega} - y\omega_n)/c_+},$$

similar to that of (20). To remove the term

$$|z|^{-j} d_{I,-j} \left( \frac{z}{|z|}, \lambda, \omega \right) e^{i\lambda z \cdot \omega/c_+}$$

from the error, we look for  $b_{I,-j+1}(\frac{z}{|z|}, \lambda, \omega)$  with

$$(23) \quad -2i\lambda c_+ \omega \cdot \nabla_z \left( |z|^{-j+1} b_{I,-j+1} \left( \frac{z}{|z|}, \lambda, \omega \right) \right) = |z|^{-j} d_{I,-j} \left( \frac{z}{|z|}, \lambda, \omega \right).$$

Using (21), if we then subtract

$$|z|^{-j+1} b_{I,-j+1} \left( \frac{z}{|z|}, \lambda, \omega \right) e^{i\lambda z \cdot \omega / c_+}$$

from our current approximation, the error term is again of the form (22), but the coefficient of  $e^{i\lambda z \cdot \omega / c_+}$  in the new error will vanish to one order faster at infinity. We choose  $b_{I,-j+1}(\frac{z}{|z|}, \lambda, \omega)$  so that it is smooth at  $z/|z| = \omega$  in order to keep the right coefficient of  $e^{i\lambda|z|/c_+}$  in the distributional asymptotic expansion. (Here the “I” in the subscript stands for “incident.” Later we shall see “R” for “reflected” and “T” for “transmitted.”)

To solve (23), introduce the “polar” coordinates  $(s, \tilde{\theta})$ , centered at  $\omega$ , in place of  $z/|z|$ . That is, let  $s$  be the geodesic distance on the sphere from  $\omega$  to  $z/|z|$  and let  $\tilde{\theta}$  be angular coordinates about  $\omega$ . Then equation (23) can be solved, just as in [22, Section 2], giving

$$(24) \quad b_{I,-j+1}(s, \tilde{\theta}, \lambda, \omega) = \frac{i}{2\lambda c_+ (\sin s)^{j-1}} \int_0^s (\sin s')^{j-2} d_{I,-j}(s', \tilde{\theta}; \omega) ds'.$$

We are abusing notation here, using the same notation for  $b_{I,-j+1}(z/|z|, \lambda, \omega)$  and  $b_{I,-j+1}(s, \tilde{\theta}, \lambda, \omega)$ , and similarly for  $d_{I,-j}$ . Note that as long as  $z/|z|$  is in the upper hemisphere we are away from  $-\omega$  so the transport equation has a smooth solution. Moreover, since  $\Phi_0(z, \lambda, \omega)$  is smooth in  $\omega \in \mathbb{S}_c^{n-1}$ , so is  $b_{I,-j+1}(z/|z|, \lambda, \omega)$ .

We find  $b_{I,-j}(z/|z|, \lambda, \omega)$  iteratively and then use Borel’s lemma to asymptotically sum  $|z|^{-j} b_{I,-j}$ , obtaining a  $b_I$  such that

$$(c^2 \Delta - \lambda^2)(\Phi - b_I(z, \lambda, \omega) e^{i\lambda z \cdot \omega / c_+}) = e^{i\lambda x \cdot \bar{\omega} / c_+} e^{-i\lambda y \omega_n / c_+} a_2 + \mathcal{O}(|z|^{-\infty})$$

when  $y > y_M$ . Note that the construction of  $b_I$  has not changed  $a_2$ .

**Ib.** We will apply almost the same technique to solve away the error  $e^{i\lambda x \cdot \bar{\omega} / c_+} e^{-i\lambda y \omega_n / c_+} a_2$ ,  $y > y_M$ , away from  $z/|z| = (-\bar{\omega}, \omega_n)$ . Here we will use solutions to the transport equation where we choose the initial condition at  $y/|z| = 0$ , and the solutions, in analogy to (24), are of the form

$$(25) \quad b_{R,-j+1}(s, \tilde{\theta}, \omega) = \frac{i}{2\lambda c_+ (\sin s)^{j-1}} \left[ \int_{s_{R_0}}^s (\sin s')^{j-2} d_{R,-j}(s', \tilde{\theta}; \omega) ds' + C_{R,j-1} \right].$$

Here  $s$  is the distance on  $\mathbb{S}^{n-1}$  from  $\theta = z/|z|$  to the point  $(\bar{\omega}, -\omega_n)$  and  $\tilde{\theta}$  is the angular coordinate about  $(\bar{\omega}, -\omega_n)$ . The value  $s = s_{R_0}$  corresponds to  $\theta_n = 0$ , and  $C_{R,j}$  depends only on  $\omega$  and  $\tilde{\theta}$  and will be determined below. We postpone to Section 7.1 discussion of the form of the parametrix near  $z/|z| = (-\bar{\omega}, \omega_n)$ , the singular point.

**IIa.** In the lower hemisphere, we use a similar technique if  $1/c_-^2 - |\bar{\omega}|^2/c_+^2 > 0$ . Here the error term is of the form

$$e^{i\lambda x \cdot \bar{\omega}/c_+} e^{i\lambda \sqrt{1/c_-^2 - |\bar{\omega}|^2/c_+^2} y} a_T(z, \lambda, \omega),$$

where  $a_T \in S_{phg}^{-2}$ . Again we have solutions like (24) to the transport equation, although this time  $s$  measures the distance on the sphere from the point  $(c_- \bar{\omega}/c_+, \sqrt{1 - c_-^2 |\bar{\omega}|^2/c_+^2})$ . We will have an additional term away from  $(-c_- \bar{\omega}/c_+, -\sqrt{1 - c_-^2 |\bar{\omega}|^2/c_+^2})$  of the form

$$e^{i\lambda x \cdot \bar{\omega}/c_+} e^{i\lambda \sqrt{1/c_-^2 - |\bar{\omega}|^2/c_+^2} y} \sum_{j \geq 1} b_{T,-j}(z/|z|, \lambda, \omega) |z|^{-j},$$

where  $b_{T,-j+1}$  is a solution to a transport equation like (23):

$$b_{T,-j+1}(s, \tilde{\theta}, \omega) = \frac{i}{2\lambda c_- (\sin s)^{j-1}} \left[ \int_{s_{T_0}}^s (\sin s')^{j-2} d_{-j,T}(s', \tilde{\theta}; \omega) ds' + C_{T,j-1} \right].$$

Again,  $s$  is the distance on  $\mathbb{S}^{n-1}$  from  $(c_- \bar{\omega}/c_+, \sqrt{1 - c_-^2 |\bar{\omega}|^2/c_+^2})$  and  $s_{T_0}$  corresponds to the distance at  $y/|z| = 0$ .

The constants (in  $s$ )  $C_{R,j}$  and  $C_{T,j}$  are to be determined. Of course their values affect subsequent errors and thus subsequent  $b_{R,-j}, b_{T,-j}$ .

**IIb.** We will use a different technique to construct the solutions when  $|y| < y_M$ . We point out that if  $c_0$  is not smooth, for example, if it is piecewise constant, we should expect it to be impossible to find a smooth Poisson operator on  $\mathbb{R}^n$ . We choose our approximations so that  $\tilde{P}_0$  is  $C^1$  on  $\mathbb{R}^n$ .

The values of  $C_{R,j}$ , and  $C_{T,j}$  are determined by solutions to boundary value problems that arise in constructing the parametrix when  $|y| < y_M$ , as described below.

When  $|y| < y_M$ , the errors are of the form

$$e^{i\lambda x \cdot \bar{\omega}/c_+} \sum_{j \geq 2} |z|^{-j} d_{M,-j}(x/|x|, y, \lambda, \omega).$$

We look for an approximate solution of the form

$$(26) \quad e^{i\lambda x \cdot \bar{\omega}/c_+} \sum_{j \geq 1} |z|^{-j} \left( b_{M,-j} \left( \frac{x}{|x|}, y, \lambda, \omega \right) + |z|^{-1} \tilde{b}_{M,-j} \left( \frac{x}{|x|}, y, \lambda, \omega \right) \right).$$

The term  $|z|^{-1} \tilde{b}_{M,-j}$  is of lower order and is included to improve the regularity at  $y = \pm y_M$ . We will suppress the dependence of  $b_{M,-j}$ ,  $b_{I,-j}$ ,  $b_{R,-j}$ , and  $\tilde{b}_{M,-j}$  on  $\lambda$  and  $\omega$  to simplify notation. Note that

$$\begin{aligned} & \left( c^2 \Delta - \lambda^2 \right) \left( |z|^{-j} b \left( \frac{x}{|x|}, y \right) e^{i\lambda x \cdot \bar{\omega}/c_+} \right) \\ &= (c_0^2 (D_y^2 + \lambda^2 |\bar{\omega}|^2 / c_+^2) b - \lambda^2 b) e^{i\lambda x \cdot \bar{\omega}/c_+} |z|^{-j} + \mathcal{O}(|z|^{-j-1}). \end{aligned}$$

Therefore, for  $|y| < y_M$ , to solve away an error of the form  $|z|^{-j} d_{M,-j}(\frac{x}{|x|}, y)$  we look for  $b_{M,-j}$  such that

$$c_0^2 (D_y^2 + \lambda^2 |\bar{\omega}|^2 / c_+^2) b_{M,-j}(x/|x|, y) - \lambda^2 b_{M,-j}(x/|x|, y) = d_{M,-j}(x/|x|, y).$$

The boundary conditions which  $b_{M,-j}$  must satisfy come from matching with the solutions in the top and bottom hemispheres in order to get a function which is  $C^1$  “at infinity”. They are

$$\begin{aligned} & b_{I,-j}(x/|x|, 0) e^{i\lambda \omega_n y_M / c_+} + e^{-i\lambda \omega_n y_M / c_+} \\ & \quad C_{Rj}(x/|x|) (\sin s_{R_0})^{-j} / 2i\lambda c_+ = b_{M,-j}(x/|x|, y_M) \\ & i\lambda \omega_n / c_+ [b_{I,-j}(x/|x|, 0) e^{i\lambda \omega_n y_M / c_+} - e^{-i\lambda \omega_n y_M / c_+} \\ & \quad C_{Rj}(x/|x|) (\sin s_{R_0})^{-j} / 2i\lambda c_+] = b'_{M,-j}(x/|x|, y_M) \\ & e^{-i\lambda \sqrt{1/c_-^2 - |\bar{\omega}|^2 / c_+^2} y_M} C_{Tj}(x/|x|) (\sin s_{T_0})^{-j} / 2i\lambda c_- = b_{M,-j}(x/|x|, -y_M) \\ & i\lambda (1/c_-^2 - 1/c_+^2 |\bar{\omega}|^2)^{1/2} e^{-i\lambda \sqrt{1/c_-^2 - |\bar{\omega}|^2 / c_+^2} y_M} \\ (27) \quad & \quad C_{Tj}(x/|x|) (\sin s_{T_0})^{-j} / 2i\lambda c_- = b'_{M,-j}(x/|x|, -y_M) \end{aligned}$$

where  $b_{I,-j+1}$  is known (it is determined by integrals over portions of geodesics of  $(\sin s)^{j-2} d_{I,-j}$ ) and  $C_{Tj}$ ,  $C_{Rj}$  are to be determined and are independent of  $y$ , and so can be treated as constants in solving the boundary value problem. They can be eliminated from this set of equations, resulting in a boundary value problem of the type considered in Lemma 7.1, which

guarantees us a unique solution to the problem when  $1/c_-^2 > |\bar{\omega}|^2/c_+^2$ . This then determines  $b_{R,-j}$  and  $b_{T,-j}$ , since  $C_{T,j}$  and  $C_{R,j}$  are determined by  $b_{M,-j}$ . We remark that if  $d_{M,-j} \equiv 0$ , then  $b_{M,-j}(x/|x|, y, \lambda, \omega) = b_{I,-j}(x/|x|, 0, \lambda, \omega)\phi_+(y, \lambda, \omega)$ ; this will be important when studying the inverse problem.

In order to ensure that our function will be  $C^1$  at  $y = y_M$  and at  $y = -y_M$  we will add an additional term  $\tilde{b}_{M,j}$  whose total contribution will be of order  $|z|^{-j-1}$ . Let  $\chi \in C_c^\infty(\mathbb{R})$ ,  $\chi(t) = 1$  for  $|t| < 1$  and  $\chi(t) = 0$  for  $|t| > 2$ . Let

$$\begin{aligned} \beta_{Uj} &= \lim_{y \downarrow y_M} \left( e^{i\lambda y \omega_n/c_+} b_{I,-j}(z/|z|) + e^{-i\lambda y \omega_n/c_+} b_{R,-j}(z/|z|) \right) \\ &\quad - b_{M,-j}(x/|x|, y_M) \\ \gamma_{Uj} &= \lim_{y \downarrow y_M} \left( \frac{\partial}{\partial y} (e^{i\lambda y \omega_n/c_+} (b_{I,-j}(z/|z|))) + \frac{\partial}{\partial y} (e^{-i\lambda y \omega_n/c_+} (b_{R,-j}(z/|z|))) \right) \\ &\quad - b'_{M,-j}(x/|x|, y_M) \\ \beta_{Lj} &= \lim_{y \uparrow -y_M} \left( e^{i\lambda y (1/c_-^2 - |\bar{\omega}|^2/c_+^2)^{1/2}} b_{T,-j}(z/|z|) \right) - b_{M,-j}(x/|x|, -y_M) \\ \gamma_{Lj} &= \lim_{y \uparrow -y_M} \frac{\partial}{\partial y} \left( e^{i\lambda y (1/c_-^2 - |\bar{\omega}|^2/c_+^2)^{1/2}} b_{T,-j}(z/|z|) \right) - b'_{M,-j}(x/|x|, -y_M). \end{aligned}$$

Note that by our choice of  $b_j, \beta_{Uj}, \gamma_{Uj}, \beta_{Lj}$  and  $\gamma_{Lj}$  all have leading order  $|z|^{-1}$ . Now, let

$$(28) \quad \tilde{b}_{M,-j} = - \left( \chi \left( \frac{3(y - y_M)}{y_M} \right) [\beta_{Uj} + (y - y_M)\gamma_{Uj}] + \chi \left( \frac{3(y + y_M)}{y_M} \right) [\beta_{Lj} + (y + y_M)\gamma_{Lj}] \right) |z|.$$

For  $|y| < y_M$ , this determines the approximate solution of the form (26).

**III.** If  $1/c_-^2 - |\bar{\omega}|^2/c_+^2 < 0$ , then we use a slightly different method for finding the approximate solution when  $y \leq y_M$ . Here, in a manner similar to that used for  $|y| < y_M$  above, we solve away the error term by using an approximation of the form

$$\sum e^{i\lambda x \bar{\omega}/c_+} \left( b_{L,-j} \left( \frac{x}{|x|}, y \right) + \tilde{b}_{L,-j} \left( \frac{x}{|x|}, y \right) |z|^{-1} \right) |z|^{-j}$$

where

$$c_0^2(\lambda^2 |\bar{\omega}|^2/c_+^2 + D_y^2) b_{L,-j} - \lambda^2 b_{L,-j} = d_{-j},$$

$d_{-j}$  is the coefficient of  $|z|^{-j} e^{i\lambda x \bar{\omega}/c_+}$  in the error term and is exponentially decreasing in  $y$  when  $y < y_M$ , and  $b_{L,-j}$  is square integrable on  $(-\infty, y_M]$ . We need in addition a boundary term at  $y = y_M$ , and this is provided by the first two equations of (27). An argument like that of Lemma 7.1 shows that there is a unique solution to this problem, and the solution is exponentially decreasing as  $y \rightarrow -\infty$ . As in the previous case,  $\tilde{b}_{L,-j}$  is chosen to improve the regularity at  $y = y_M$ .

*Remark 7.1.* — Note that this construction is smooth in  $\omega \in \mathbb{S}_c^{n-1}$ .

*Remark 7.2.* — We remark that this construction can be carried out, with some minor modifications, for sound speeds  $c_1 = c+d$ , where  $c-c_0$  has an asymptotic expansion of the type (4), and  $d$  is supported in  $|y| < y_M$ , with  $d \sim \sum_{j>0} |z|^{-j} d_j(x/|x|, y)$ .

**7.1. The approximate Poisson operator near its singularities.**

For  $\omega_n > 0$ , it remains to describe the approximation of the (partial) Poisson operator near  $z/|z| = \theta = (-\bar{\omega}, \omega_n)$  and, if  $1 - c_-^2 |\bar{\omega}|^2 / c_+^2 > 0$ , near  $z/|z| = \theta = (-c_- \bar{\omega} / c_+, -\sqrt{1 - c_-^2 |\bar{\omega}|^2 / c_+^2})$ . The approximation in these regions contributes to the scattering matrix. As these two are quite similar, we will concentrate on the first, using the techniques of [22, Section 3]. We refer to [22] for many of the details.

Let  $w = (w_1, \dots, w_n) = (w', w_n) \in \mathbb{R}^n$ , and rotate the coordinate system so that  $\omega$  is the north pole. Denote by  $I_\tau^{\gamma,\alpha}$  the class of operators with Schwartz kernel which can, near the south pole, be written as a Schwartz function plus a term of the form

$$(29) \quad \int_0^\infty \int \left( \frac{1}{S|w|} \right)^\gamma S^\alpha e^{i\tau(Sw' \cdot \mu - \sqrt{1+S^2}|w|)} a \left( \frac{1}{S|w|}, S, \mu \right) dS d\mu$$

with  $a \in C_c^\infty([0, \epsilon) \times [0, \epsilon) \times \mathbb{S}^{n-2})$ . From the results of [22], this class is asymptotically complete in  $\gamma$ . Moreover, a stationary phase computation which can be found in [22] shows that away from the south pole this is equivalent to the class of operators with Schwartz kernel of the form  $e^{-i\tau w_n} b$ , with  $b$  a polyhomogeneous symbol in  $w$  of order  $-\gamma + (n - 1)/2$ . This allows us to match it up with the part of  $\tilde{P}_0(z, \lambda, \omega)$  that we have constructed to this point. We recall below some additional facts about the operators  $I_\tau^{\gamma,\alpha}$  from [22].

We recall from [22]

PROPOSITION 7.1. — *If  $u(w, \omega) \in I_r^{\gamma, \alpha}$  and  $f \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ , then*

$$e^{i\tau|w|} \int u(|w|\theta, \omega) f(\theta, \omega) d\theta d\omega$$

*is a smooth symbolic function in  $|w|$  of order  $-1-\alpha$  and its lead coefficient is  $|w|^{-1-\alpha} \langle K, f \rangle$ . Here  $K$  is the pull-back of the Schwartz kernel of a pseudo-differential operator of order  $\alpha - \gamma - (n - 2)$  by the map  $\theta \mapsto -\theta$ . The principal symbol of  $K$  determines and is determined by the lead term of the symbol,  $a(t, S, \mu)$  of  $u$  as  $S \rightarrow 0 +$ .*

From [22, Propositions 3.1 and 3.2], we have

PROPOSITION 7.2. — *If  $u \in I_r^{\gamma, \alpha}$  and  $V \sim \sum_{j \geq 2} |w|^{-j} v_j(w/|w|)$ , then*

$$(\Delta - \tau^2)u \in I^{\gamma+1, \alpha+1}$$

and

$$Vu \in I^{\gamma+2, \alpha+2}.$$

From Lemma 3.2 of [22],

LEMMA 7.2. — *If  $u \in I_r^{\gamma, (n-3)/2}$ ,  $V \sim \sum_{j \geq 2} |w|^{-j} v_j(w/|w|)$ , and  $(\Delta + V - \tau^2)u \in I_r^{\gamma+1, \frac{n-1}{2}}$ , then  $(\Delta + V - \tau^2)u \in I_r^{\gamma+1, \frac{n+1}{2}}$ .*

Again from [22]

PROPOSITION 7.3. — *If  $u \in I_r^{\infty, \alpha} = \cap_\gamma I_r^{\gamma, \alpha}$ , then  $u = e^{-i\tau|w|} f(w)$ , with  $f$  a classical symbol of order  $-\alpha - 1$ .*

We proceed as described in [22, Section 3]. Using the first part of the construction, we have an approximation of the Poisson operator that blows up as  $\theta = z/|z|$  approaches  $(-\bar{\omega}, \omega_n)$  or  $(-c_-\bar{\omega}/c_+, -\sqrt{1 - c_-^2 |\bar{\omega}|^2 / c_+^2})$ . Recalling that  $\omega_n > 0$ , near  $\theta = (-\bar{\omega}, \omega_n)$ , we find an approximation of the “reflected” part of the Poisson operator of the form  $u\mathcal{R}_+$ , where  $u \in I_{\lambda/c_+}^{-(n-1)/2, (n-3)/2}$ , and  $\mathcal{R}_+ : (\bar{\omega}, \omega_n) \mapsto (\bar{\omega}, -\omega_n)$ . We will call the functions of the type of the previous section  $e^{i\lambda x \cdot \bar{\omega}/c_+} e^{-i\lambda y/c_+} b$  the “first ansatz” and functions of the form  $u\mathcal{R}_+$ , for  $u \in I_{\lambda/c_+}^{-(n-1)/2, (n-3)/2}$ , the “second ansatz”.

We outline briefly how the construction works, following [22, Section 3]. The “first” approximation of the “reflected” part of  $\hat{P}_0$  is  $R_+(\lambda, \omega_n) e^{i\lambda x \cdot \bar{\omega}/c_+} e^{-i\lambda y/c_+}$ . This corresponds to an element of  $I_{\lambda/c_+}^{-\frac{n-1}{2}, \frac{n-3}{2}} \mathcal{R}_+$ . Applying  $c^2 \Delta - \lambda^2$ , we obtain by Proposition 7.1, an error

$e_1$  with  $e_1 \mathcal{R}_+ \in I_{\lambda/c_+}^{-\frac{n-1}{2}+2, \frac{n-3}{2}+2}$ . Its symbol blows up to order  $n - 2$  as  $S \downarrow 0$  which corresponds to, in the first setup, blow up to order 0 as  $s \uparrow \pi$ .

The solution to the first transport equation for the first ansatz is a symbol which blows up to order  $-1$  as  $s \uparrow \pi$ . We can convert this solution to a symbol for the second ansatz, with this new symbol blowing up to order  $n - 3$  as  $S \downarrow 0$ . This gives us a symbol of  $u_1 \in I_{\lambda/c_+}^{-\frac{n-1}{2}+1, \frac{n-3}{2}}$   $\mathcal{R}_+$ . As the new error term for the first ansatz is  $e^{i\lambda(x\bar{\omega}-y\omega_n)/c_+} d_2$ , with  $d_2 \in S_{phg}^{-3}$ , we obtain

$$(c^2 \Delta - \lambda^2)(u_0 + u_1) \in I_{\lambda/c_+}^{-\frac{n-1}{2}+3, \frac{n+1}{2}}.$$

This technique can be iterated, and the terms  $(u_j)$  can then be asymptotically summed. The resulting error is in  $I_{\lambda/c_+}^{\infty, \frac{n+1}{2}}$ , and therefore, using Proposition 7.3, it is of the form  $e^{-i\lambda|z|/c_+} a$ , with  $a \in S_{phg}^{-(n+3)/2}$  (and smooth in  $\omega, z$ ). To remove this error modulo a Schwartz function, we use the fact that if  $b \in S_{phg}^{-j}$  has support in  $y > y_M$ , then

$$(c^2 \Delta - \lambda^2) b |z|^{-(n-1)/2} e^{-i\lambda|z|/c_+} + \frac{2ij\lambda c_+}{|z|} b |z|^{-(n-1)/2} e^{-i\lambda|z|/c_+} \in |z|^{-(n-1)/2} e^{-i\lambda|z|/c_+} S_{phg}^{-j-2}$$

to iteratively solve away the errors as in Proposition 12 of [26].

In the lower hemisphere, a similar argument shows that for  $\theta = z/|z|$  near  $(-c_- \bar{\omega}/c_+, -\sqrt{1 - c_-^2 |\bar{\omega}|^2 / c_+^2})$ , the approximation of the Poisson operator is of the form  $u \mathcal{R}_-$ , where  $u \in I_{\lambda/c_-}^{(n-1)/2, (n-3)/2}$  and  $\mathcal{R}_- : (\bar{\omega}, \omega_n) \mapsto (c_- \bar{\omega}/c_+, \sqrt{1 - c_-^2 |\bar{\omega}|^2 / c_+^2})$ . Putting all of this together (and multiplying by an appropriate constant depending on  $c, \lambda$ , and  $n$ ), we get an approximation  $\tilde{P}_0$  to the (partial) Poisson operator with a remainder term  $(c^2 \Delta - \lambda^2) \tilde{P}_0$  that is in  $\langle z \rangle^{-\infty} L^2$ , and is Schwartz after multiplication by a function  $\phi \in C_b^\infty(\mathbb{R})$  which vanishes for  $|y| < y_M + 1$ . This completes the proof of the first part of Proposition 4.2.

### 7.2. Construction of $\tilde{P}_j(\lambda), 1 \leq j \leq T(\lambda)$ .

For completeness, we briefly outline how to construct an approximation  $\tilde{P}_j(\lambda)$  to  $P_j(\lambda), 1 \leq j \leq T(\lambda)$ . The approximation will have the properties

$$(c^2 \Delta - \lambda^2) \tilde{P}_j \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n), (1 - \phi(y))(c^2 \Delta - \lambda^2) \tilde{P}_j \in \mathcal{S}(\mathbb{R}^n)$$

if  $\phi \in C_c^\infty(\mathbb{R})$ ,  $\phi(y) = 1$  when  $|y| < y_M + 1$ , and, distributionally as  $|z| \rightarrow \infty$ ,

$$\begin{aligned} \tilde{P}_j(z, \lambda, \bar{\omega}) &= |x|^{-(n-2)/2} \left( e^{i\kappa_j(\lambda)|x|} \delta_{\bar{\omega}} \left( \frac{x}{|x|} \right) f_j(y) \right. \\ &\quad \left. + e^{-i\kappa_j(\lambda)|x|} h \left( \frac{x}{|x|}, \bar{\omega} \right) f_j(y) \right) + \mathcal{O}(|x|^{-n/2} \langle y \rangle^{-\infty}). \end{aligned}$$

Here  $\bar{\omega} \in \mathbb{S}^{n-2}$ .

For the construction of  $\tilde{P}_j$ , we begin with  $\Phi_j = e^{i\kappa_j(\lambda)x \cdot \bar{\omega}} f_j(y)$ , which is, up to a constant multiple depending on  $n$  and  $\kappa_j$ , the ( $j$ th partial) Poisson operator for  $c_0^2 \Delta$ . We have

$$(c^2 \Delta - \lambda^2) \Phi_j \sim \sum_{k \geq 2} |x|^{-k} e^{i\kappa_j(\lambda)x \cdot \bar{\omega}} d_{-k}(x/|x|, y, \bar{\omega}),$$

where  $d_{-k}$  is smooth in  $x/|x|$ , exponentially decreasing in  $y$ , and smooth in  $y$  when  $|y| > y_M$ .

To solve away the error with  $k = 2$ , we write

$$d_{-2}(x/|x|, y, \bar{\omega}) = d_{-2,1}(x/|x|, y, \bar{\omega}) + d_{-2,2}(x/|x|, \bar{\omega}) f_j(y)$$

where  $\int d_{-2,1}(x/|x|, y, \bar{\omega}) f_j(y) c_0^{-2}(y) dy = 0$ . Since  $d_{-2,1}$  is orthogonal to  $f_j$ , we can find  $g_{-2}$  such that

$$(c_0^2(D_y^2 + \kappa_j^2) - \lambda^2) g_{-2}(x/|x|, y, \bar{\omega}) = d_{-2,1}(x/|x|, y, \bar{\omega})$$

and use  $e^{i\kappa_j(\lambda)x \cdot \bar{\omega}} |x|^{-2} g_{-2}(x/|x|, y, \bar{\omega})$  to solve away the

$$e^{i\kappa_j(\lambda)x \cdot \bar{\omega}} |x|^{-2} d_{-2,1}(x/|x|, y, \bar{\omega})$$

error, up to a term vanishing one order faster at infinity. We note that  $g_{-2}$  is exponentially decreasing in  $y$ , since  $d_{-2,1}$  is, and that this term does not contribute anything to the scattering matrix, since distributionally it is  $\mathcal{O}(|x|^{-(n+2)/2})$ .

To solve away the error  $|x|^{-2} e^{i\kappa_j(\lambda)x \cdot \bar{\omega}} d_{-2,2}(x/|x|, \bar{\omega}) f_j(y)$ , we use the techniques of [22] in the  $x$  variables only. That is, essentially as in our construction of  $\tilde{P}_0$ , we solve transport equations along geodesics on  $\mathbb{S}^{n-2}$  beginning at  $x/|x| = \bar{\omega}$ . Near  $x/|x| = -\bar{\omega}$ , we must use the second ansatz, as in Section 7.1 or [22]. Our new approximation of  $P_j$  then has the same properties as before, but with (new) error of order  $|x|^{-3}$ .

The subsequent errors are solved away in exactly the same manner, resulting in an approximation as claimed. This finishes the proof of Proposition 4.2.

### 8. The inverse problem.

In this section we prove our central inverse result, Theorem 1.2. In proving the results for the inverse problem, we use the techniques of [22], [23] and much of their language. We recall the arguments from these papers, noting the adjustments that must be made for the stratified case.

Theorem 1.2 follows from the following theorem, which is somewhat stronger.

**THEOREM 8.1.** — *Suppose  $n \geq 3$ ,  $c_1$  and  $c_2$  satisfy the general requirements of Section 2, and either (H1) or (H2), for the same  $c_0$ . Let  $S_1(\lambda), S_2(\lambda)$  be the corresponding scattering matrices for some  $\lambda \in \mathbb{R} \setminus \{0\}$ . If, for  $c_+ = c_-$ , the transmitted part of the main part of  $S_1(\lambda) - S_2(\lambda)$  is of order  $-l$ , then  $c_1(z) - c_2(z) = \mathcal{O}(|z|^{-l-1})$ . If for  $c_+ < c_-$  the reflected part of the main part of  $S_1(\lambda) - S_2(\lambda)$  is of order  $-l$ , then  $c_1(z) - c_2(z) = \mathcal{O}(|z|^{-l-1})$ .*

*Proof.* — Let  $a \in S_{cl,s}^k(\mathbb{R}^n)$  if  $a(z) \sim \sum_{j \geq 0} |z|^{k-j} a_j(z/|z|)$ , where  $a_j(z/|z|) \in C_b^\infty(\mathbb{S}^{n-1} \setminus \{(\bar{\omega}, 0)\})$ .

Suppose that  $c_1 - c_2 = W \in S_{cl,s}^{-k}(\mathbb{R}^n)$  and that the scattering matrices associated to  $c_1^2 \Delta$  and  $c_2^2 \Delta$  have the same transmitted (if  $c_+ = c_-$ ) or reflected (if  $c_+ \neq c_-$ ) singular parts, to order  $l \geq k$ . Then we shall show that actually  $c_1 - c_2 \in S_{cl,s}^{-k-1}(\mathbb{R}^n)$ , and thus by induction  $c_1 - c_2 \in S_{cl,s}^{-l-1}(\mathbb{R}^n)$ .

If  $c_1 - c_2 \in S_{cl,s}^{-k}(\mathbb{R}^n)$ , then  $\lambda^2(c_1^{-2} - c_2^{-2}) = \lambda^2 c_1^{-2} c_2^{-2} (c_1 + c_2)(c_2 - c_1) = U$ , with

$$U_{|y > y_M} \sim \sum_{j \geq k} |z|^{-j} U_{-j,+}(z/|z|), \quad U_{|y < -y_M} \sim \sum_{j \geq k} |z|^{-j} U_{-j,-}(z/|z|),$$

and  $U = \mathcal{O}(|z|^{-k})$ , where  $U_{-j,\pm} \in C_b^\infty(\mathbb{S}_\pm^{n-1})$ . Let

$$W_{-j}(\bar{\omega}, \omega_n) = \begin{cases} U_{-j,+}(\bar{\omega}, \omega_n) & \text{if } \omega_n > 0 \\ U_{-j,-}(\bar{\omega}, \omega_n) & \text{if } \omega_n < 0. \end{cases}$$

Note then that the first  $k - 2$  terms in the construction of the Poisson operator carried out in Section 7 are the same for  $c_1$  and  $c_2$ , and the difference comes in the  $k - 1$ st term.

Although many of the underlying techniques are the same, we shall treat the cases  $c_+ = c_-$  and  $c_+ \neq c_-$  serially. We begin with the case  $c_+ = c_-$ , and take  $\omega = (\bar{\omega}, \omega_n)$  with  $\omega_n > 0$ . In the construction of the

approximation of the Poisson operator, the transmitted parts (that is,  $z/|z|$  in the lower hemisphere) of the  $k - 1$ st terms differ by

$$(30) \quad \frac{i|z|^{1-k}}{2\lambda c_+ (\sin s)^{k-1}} T_+(\omega_n, \lambda) \int_0^s W_{-k}(s', \theta; \omega) (\sin s')^{k-2} ds',$$

almost as in (2.3) of [22]. Here  $T_+(\lambda, \omega_n)$  is the transmission coefficient determined by equations (6)-(8). We remark that in case  $c_+ = c_-$ ,  $T_+ = T_-$ . The transmission coefficient is nonzero for  $\lambda \in \mathbb{R}$ ,  $\lambda \neq 0$ ,  $\omega_n \neq 0$ ,  $0 < \omega_n < 1$ . Therefore, as described in [22, Section 4], we can recover from the difference of the transmitted parts of the scattering matrices

$$\int_0^\pi W_{-k}(s, \theta; \omega) (\sin s)^{k-2} ds$$

as long as  $\omega = (\bar{\omega}, \omega_n)$  with  $\omega_n \neq 0$ .

If the transmitted parts of the two scattering matrices are the same to order  $k - 1$ , then

$$(31) \quad I_k = \int_0^\pi W_{-k}(s, \theta; \omega) (\sin s)^{k-2} ds = 0.$$

Since this is true for all  $\omega$  with  $\omega_n \neq 0$ , we can differentiate with respect to the starting point twice, use  $\sin^2 s + \cos^2 s = 1$ , and find that  $I_{k-2} = \int_0^\pi W_{-k}(s, \theta; \omega) (\sin s)^{k-4} ds = 0$  as well (see [23, Section 5]). Therefore, if  $k$  is even, we reduce eventually to the case with  $k - 2 = 0$  and if  $k$  is odd, to  $k - 2 = 1$ . When  $k$  is even, differentiating one more time with respect to the starting point shows that  $W_{-k}$  is even; for odd  $k$  two more differentiations show that  $W_{-k}$  is odd.

When  $k$  is even, we obtain

$$\int_\gamma W_{-k} = 0$$

for each closed geodesic  $\gamma$  by joining together two half-geodesics. However, by [16, Theorem 4.7], for  $n \geq 3$  the  $x$ -ray transform on  $\mathbb{S}^{n-1}$  with domain restricted to smooth even functions is 1-1. Although  $W_{-k}$  may have a jump discontinuity at  $\omega_n = 0$ , it is smooth elsewhere. As in the proof of [16, Corollary 4.19], by first taking a convolution of  $W_{-k}$  with a test function (and then letting the test function approach the identity), we may assume that  $W_{-k}$  is smooth and, applying the theorem, it is thus 0.

If  $k$  is odd, we consider  $\frac{z_n}{|z|} W_{-k}$ , which is even. Since for each geodesic beginning at  $z_n = 0$ ,  $\sin s$  is a constant multiple of  $z_n/|z|$ , we obtain  $\int_\gamma \frac{z_n}{|z|} W_{-k} = 0$  for each geodesic  $\gamma$ , and again  $W_{-k} = 0$ .

When  $c_+ \neq c_-$ , we use the reflected singularities in the inverse problem. Recalling that  $W \in S_{cl,s}^{-k}$ , we can recover from the reflected singularities, when  $\omega_n > 0$ ,

$$(32) \quad R_+(\omega_n, \lambda) \left( \int_0^{s_0} W_{-k}(s, \theta; \omega) (\sin s)^{k-2} ds + \int_{s_0}^\pi W_{-k}(s', \theta; \omega) (\sin s')^{k-2} ds' \right).$$

The first integral is along a geodesic originating at  $\omega$  and continuing to  $\{(\bar{\phi}, 0)\} \subset \mathbb{S}^{n-1}$ ; the second integral is along the reflection of the first geodesic when it meets  $\phi_n = 0$  and the path of integration ends at the point  $(-\bar{\omega}, \omega_n)$ . The variable  $s'$  is the distance from the point  $(\bar{\omega}, -\omega_n)$ .

It is, however, more convenient to think of the sum (32) as the single integral

$$(33) \quad R_+(\omega_n, \lambda) \int_0^\pi \tilde{W}_{-k,+}(s, \theta; \omega) (\sin s)^{k-2} ds$$

where

$$(34) \quad \tilde{W}_{-k,+}(\phi) = \begin{cases} W_{-k}(\phi) & \text{if } \phi_n \geq 0 \\ W_{-k}(\bar{\phi}, -\phi_n) & \text{if } \phi_n < 0 \end{cases}$$

and  $s$  is the distance from  $\omega$ . It is fairly straightforward to see by symmetry that (32) and (33) are the same.

If we can show that (33) is sufficient for recovering

$$\int_0^\pi \tilde{W}_{-k,+}(s, \theta; \omega) (\sin s)^{k-2} ds$$

for all  $\omega$  with  $\omega_n > 0$ , then the analysis used in the case  $c_+ = c_-$  will show that  $\tilde{W}_{-k,+}$  is 0 if the reflection coefficients agree to order  $-k$ .

It suffices that  $R_+(\omega_n, \lambda)$  is 0 for at most an isolated set of  $\omega_n$  with  $0 < \omega_n \leq 1$ , for we can obtain the integrals for these isolated values of  $\omega_n$  by continuity. We recall from [34, Chapter 3] that for  $0 < \omega_n < \sqrt{1 - c_+^2/c_-^2}$ ,  $|R_+(\omega_n, \lambda)| = 1$ . Moreover, because  $c_0(y) - c_\pm$  is compactly supported for  $\pm y > 0$ , for fixed  $\lambda \in \mathbb{R}$ ,  $R_+(\omega_n, \lambda)$  can be extended to a meromorphic function of  $\omega_n$  in a neighborhood of  $0 < \omega_n < 1$ , except near  $\omega_n = \sqrt{1 - c_+^2/c_-^2}$ , where it is a meromorphic function of  $(1 - c_-^2/c_+^2 + c_-^2\omega_n^2/c_+^2)^{1/2}$ . Therefore, its zeros are isolated, and we have shown that it is possible to recover  $W_{-k}(\omega)$  for  $\omega_n > 0$ .

A very similar analysis works for the lower hemisphere, proving the theorem. □

We remark that this proof shows that if  $c_1 - c_2 \in S_{\text{cl}}^{-k}(\mathbb{S}^{n-1})$ , then the main part of  $S_1(\lambda) - S_2(\lambda)$  is of order  $-k + 1$ .

**COROLLARY 8.1.** — *Let  $c, c_0$  satisfy the general conditions of Section 2, and either (H1) or (H2), and let  $n \geq 3$ . Then  $c_+, c_-, \lambda$ , and the main part of the scattering matrix at nonzero fixed energy determine  $c$  modulo terms vanishing faster than the reciprocal of any polynomial at infinity.*

*Proof.* — We need only show that  $c_0$  can be recovered from  $c_+, c_-, \lambda$ , and knowledge of the scattering matrix at fixed energy. The leading order singularity of the scattering matrix  $A(\lambda)$  determines and is determined by  $R_{\pm}(\lambda, \omega_n), T_{\pm}(\lambda, \omega_n), \lambda$  and  $c_{\pm}$ , where  $R_{\pm}, T_{\pm}$  are defined by equations (6-10).

Fix  $\lambda \in \mathbb{R} \setminus \{0\}$ . We can think of equation (6) in the slightly more general form

$$(35) \quad (D_y^2 - \lambda^2(1/c_0^2 - 1/c_+^2) - k^2)\phi = 0,$$

a Schrödinger operator with potential  $-\lambda^2(1/c_0^2 - 1/c_+^2)$  which is either compactly supported (if  $c_+ = c_-$ ) or “steplike” (if  $c_+ < c_-$ ). We can define the reflection and transmission coefficients  $\tilde{R}_{\pm}(k), \tilde{T}_{\pm}(k)$ , for (35) as usual, as in (6-10), and  $\tilde{R}_{\pm}(k) = R_{\pm}(\lambda, \omega_n), \tilde{T}_{\pm}(k) = T_{\pm}(\lambda, \omega_n)$ , when  $k = \lambda\omega_n/c_+$ . Moreover,  $\tilde{R}_{\pm}, \tilde{T}_{\pm}$  are meromorphic functions of  $k \in \mathbb{C}$  if  $c_+ = c_-$ , and if  $c_+ < c_-$ , they are meromorphic functions on  $\hat{Z}$ , the Riemann surface on which  $k$  and  $(k^2 - \lambda^2/c_+^2 + \lambda^2/c_-^2)^{1/2}$  are single-valued holomorphic functions. Therefore, knowing  $R_+(\lambda, \omega_n), T_+(\lambda, \omega_n)$  for  $0 < \omega_n < \sqrt{1/c_+^2 - 1/c_-^2}$  determines  $\tilde{R}_+(k), \tilde{T}_+(k)$  on the whole plane (if  $c_+ = c_-$ ) or  $\hat{Z}$  (if  $c_+ < c_-$ ). This in turn determines the eigenvalues of  $D_y^2 - \lambda^2(1/c_0^2 - 1/c_+^2)$  and the norming constants. These, together with  $c_{\pm}$  and  $\tilde{R}_+$ , determine the potential  $-\lambda^2(1/c_0^2 - 1/c_+^2)$  (e.g. [8], [12]).  $\square$

## 9. “Outgoing” solutions.

This section and the next one are devoted to some proofs which have been postponed. These include Proposition 4.3, used in the proof of the existence of the Poisson operator. The next section culminates in the proof of Theorem 1.3. These two sections use similar techniques.

For  $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $\epsilon > 0$ , the limit

$$\begin{aligned} \lim_{\delta \downarrow 0} \langle z \rangle^{-1/2-\epsilon} (c^2 \Delta - (\lambda - i\delta)^2)^{-1} c^2 \langle z \rangle^{-1/2-\epsilon} \\ = \lim_{\delta \downarrow 0} \langle z \rangle^{-1/2-\epsilon} (\Delta - c^{-2}(\lambda - i\delta)^2)^{-1} \langle z \rangle^{-1/2-\epsilon} \\ = \langle z \rangle^{-1/2-\epsilon} (\Delta - c^{-2}(\lambda - i0)^2)^{-1} \langle z \rangle^{-1/2-\epsilon} \end{aligned}$$

as an operator on  $L^2(\mathbb{R}^n)$  exists in the norm topology [4], [10]. In this section and the next, we study further properties of  $(\Delta - (\lambda - i0)^2 c^{-2})^{-1}$  when it is applied to a function  $f \in \langle z \rangle^{-3/2+\epsilon} L^2(\mathbb{R}^n)$  or to a function  $f \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ ,  $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$  for some  $\phi \in C_c^\infty(\mathbb{R})$ . In particular, we are interested in the asymptotics at infinity of the resulting function.

For simplicity of exposition, we shall assume  $\lambda > 0$  throughout this section. The results for  $\lambda < 0$  can be proved in a similar way.

For the remainder of the paper we shall repeatedly use the fact that if

$$u = \lim_{\epsilon \downarrow 0} (\Delta - c^{-2}(\lambda - i\epsilon)^2)^{-1} f = (\Delta - c^{-2}(\lambda - i0)^2)^{-1} f,$$

then

$$(36) \quad u = (\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} (Vu + f), \text{ where } V = \lambda^2 (c^{-2} - c_0^{-2}).$$

Additionally,

$$(37) \quad \begin{aligned} & (\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} g(z) \\ &= (2\pi)^{1-n} \int e^{ix \cdot \xi} ((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} \hat{g}(\xi, \cdot))(y) d\xi \end{aligned}$$

where  $\hat{g}(\xi, y) = \int e^{-ix \cdot \xi} g(x, y) dx$  is the Fourier transform in the  $x$  variables only. We shall repeatedly use this notation for the Fourier transform in the  $x$  variables only, and  $V$  is given by (36).

We make several remarks about the operator  $(D_y^2 + t^2 - (\lambda - i0)^2 c_0^{-2})^{-1}$  as an operator from  $L^2_{\text{comp}}(\mathbb{R})$  to  $H^2_{\text{loc}}(\mathbb{R})$  for any  $\epsilon > 0$ . It is smooth for  $|t| < \lambda/c_+$ ,  $t$  away from  $\lambda/c_\pm$ . Near  $t = \lambda/c_+$ ,  $(\lambda^2/c_+^2 - t^2)^{1/2} (D_y^2 + t^2 - (\lambda - i0)^2 c_0^{-2})^{-1}$  is a smooth function of  $(\lambda^2/c_+^2 - t^2)^{1/2}$ , and, if  $c_- > c_+$ , near  $t = \lambda/c_-$  it is a smooth function of  $(\lambda^2/c_-^2 - t^2)^{1/2}$ . When  $t$  is sufficiently large,  $(D_y^2 + t^2 - (\lambda - i0)^2 c_0^{-2})^{-1}$  is a smooth function of  $t$ , bounded on  $L^2(\mathbb{R})$  by  $C(|\xi|^2 - C)^{-1}$ . These properties follow essentially as for one-dimensional Schrödinger operators.

We shall use the following lemma in the proof of Proposition 4.3.

LEMMA 9.1. — If  $f \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ , let

$$u = \lim_{\epsilon \downarrow 0} (\Delta - (\lambda - i\epsilon)^2 c^{-2})^{-1} f.$$

Then

$$\widehat{Vu}(\xi, y) \in H^{J-1/2-\epsilon-\beta}(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\beta} L^2(\mathbb{R}_y))$$

for  $0 \leq \beta < J - 1/2 - \epsilon$ ,  $\epsilon > 0$ . Moreover, for any  $Q = Q_1 Q_2 \cdots Q_k$ , where  $Q_j = \xi_{\mu_j} \frac{\partial}{\partial \xi_{\nu_j}} - \xi_{\nu_j} \frac{\partial}{\partial \xi_{\mu_j}}$ ,  $1 \leq \nu_j, \mu_j \leq n$ ,

$$\chi(\xi) Q \widehat{Vu}(\xi, y) \in H^{J-1/2-\epsilon-\beta}(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\beta} L^2(\mathbb{R}_y))$$

for  $0 \leq \beta < J - 1/2 - \epsilon$ ,  $\epsilon > 0$ , for any  $\chi \in C_c^\infty(\mathbb{R}^{n-1})$ .

*Proof.* — Throughout the proof  $\epsilon > 0$  is small, and may change from line to line.

Since  $u \in \langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$ , the decay properties of  $V$  ensure that  $\widehat{Vu} \in H^{J-1/2-\epsilon-\beta}(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\beta} L^2(\mathbb{R}_y))$ . Of course,  $\widehat{f}(\xi, y) \in H^\infty(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\infty} L^2(\mathbb{R}_y))$ .

We have  $(\Delta - \lambda^2/c_0^2)u = Vu + f$ . Then

$$(38) \quad \widehat{Vu}(\xi, y) = C \int e^{-ix \cdot \xi} V(x, y) \int e^{ix \cdot \eta} ((D_y^2 + |\eta|^2 - c_0^2(\lambda - i0)^2)^{-1} [\widehat{f}(\eta, \cdot) + \widehat{Vu}(\eta, \cdot)])(y) d\eta dx.$$

Note that if  $\psi \in C_b^\infty(\mathbb{R}^{n-1})$ ,  $\text{supp}(\chi) \cap \text{supp}(\psi) = \emptyset$ , then

$$(39) \quad \begin{aligned} \chi(\xi) \int e^{-ix \cdot \xi} V(x, y) \int e^{ix \cdot \eta} ((D_y^2 + |\eta|^2 - c_0^2(\lambda - i0)^2)^{-1} \\ \Psi(\eta)(\widehat{Vu} + \widehat{f})(\eta, \cdot))(y) d\eta dx \\ = \chi(\xi) \int e^{-ix \cdot \xi} |\xi - \eta|^{-2m} \Delta_x^m V(x, y) \int e^{ix \cdot \eta} ((D_y^2 + |\eta|^2 - c_0^2(\lambda - i0)^2)^{-1} \\ \Psi(\eta)(\widehat{Vu} + \widehat{f})(\eta, \cdot))(y) d\eta dx. \end{aligned}$$

Since, for large  $|x|$ ,

$$(40) \quad |D_x^\alpha V(x, y)| \leq C_\alpha (1 + |z|)^{-J-|\alpha|},$$

we have, using (39) and (40),

$$(41) \quad \begin{aligned} D_\xi^\gamma \chi(\xi) \int e^{-ix \cdot \xi} V(x, y) \int e^{ix \cdot \eta} ((D_y^2 + |\eta|^2 - c_0^2(\lambda - i0)^2)^{-1} \\ \Psi(\eta)(\widehat{Vu} + \widehat{f})(\eta, \cdot))(y) d\eta dx \\ \in H^{J-1/2-\epsilon-\beta}(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\beta} L^2(\mathbb{R}_y)) \end{aligned}$$

for any multi-index  $\gamma$ .

We note that, for any  $\chi_1 \in C_c^\infty(\mathbb{R}^{n-1})$ ,  $j \neq k$ ,

$$\begin{aligned}
 (42) \quad & \left( \xi_k \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_k} \right) \int e^{-ix \cdot \xi} V(x, y) \\
 & \int e^{ix \cdot \eta} ((D_y^2 + |\eta|^2 - c_0^2(\lambda - i0)^2)^{-1} \chi_1(\eta) \widehat{Vu + f(\eta, \cdot)})(y) d\eta dx \\
 = & \int e^{-ix \cdot \xi} V(x, y) \int e^{ix \cdot \eta} \left( (D_y^2 + |\eta|^2 - c_0^2(\lambda - i0)^2)^{-1} \right. \\
 & \left. \chi_1(\eta) \left( \eta_k \frac{\partial}{\partial \eta_j} - \eta_j \frac{\partial}{\partial \eta_k} \right) \widehat{Vu + f(\eta, \cdot)} \right)(y) d\eta dx \\
 + & \int e^{-ix \cdot \xi} \left( \left( x_k \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_k} \right) V(x, y) \right) \\
 & \int e^{ix \cdot \eta} ((D_y^2 + |\eta|^2 - c_0^2(\lambda - i0)^2)^{-1} \chi_1(\eta) \widehat{Vu + f(\eta, \cdot)})(y) d\eta dx.
 \end{aligned}$$

Since

$$\left( \eta_k \frac{\partial}{\partial \eta_j} - \eta_j \frac{\partial}{\partial \eta_k} \right) \chi_1(\eta) \widehat{Vu}(\eta, y) \in H^{J-3/2-\epsilon-\beta}(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\beta} L^2(\mathbb{R}_y))$$

for  $0 \leq \beta < J - 1/2 - \epsilon$ , and using (40), we obtain from (38), (41), and (42), that

$$\begin{aligned}
 \left( \xi_k \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_k} \right) \chi(\xi) \widehat{Vu}(\xi, y) & \in H^{J-1/2-\epsilon-\beta}(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\beta} L^2(\mathbb{R}^n)), \\
 & 0 \leq \beta \leq J - 1/2 - \epsilon; \epsilon > 0.
 \end{aligned}$$

This argument can then be iterated, proving the lemma. □

The vector fields  $\xi_j \frac{\partial}{\partial \xi_k} - \xi_k \frac{\partial}{\partial \xi_j}$ ,  $j \neq k$ ,  $1 \leq j, k \leq n$  span the vector fields tangent to  $|\xi| = \text{const} \neq 0$ . Therefore, we have the following corollary.

**COROLLARY 9.1.** — *For fixed  $t \in \mathbb{R} \setminus \{0\}$ ,  $\bar{\theta} \in \mathbb{S}^{n-2}$ ,  $\widehat{Vu}(t\bar{\theta}, y) \in C^\infty(\mathbb{S}^{n-2}; \langle y \rangle^{-\beta} L^2(\mathbb{R}_y))$ ,  $\beta < J - 1$ .*

**LEMMA 9.2.** — *If  $w, \eta \in \mathbb{R}^m$ , and*

$$|D_w^\alpha D_\eta^\beta a(w, \eta)| \leq C_{\alpha\beta} (1 + |w|)^{-|\alpha|}$$

*for all multi-indices  $\alpha, \beta$ , then the operator  $A$  defined by*

$$(Ag)(\eta) = \int e^{iw \cdot \eta} a(w, \eta) g(w) dw$$

is a continuous map

$$A : \langle w \rangle^\gamma L^2(\mathbb{R}^m) \rightarrow H^{-\gamma}(\mathbb{R}^m)$$

for any  $\gamma \in \mathbb{R}$ .

*Proof.* — We can write

$$(Ag)(\eta) = \int e^{iw \cdot (\eta - \eta')} a(w, \eta) \mathcal{F}^{-1} g(\eta') d\eta' dw$$

with  $\mathcal{F}^{-1} g \in H^{-\gamma}(\mathbb{R}^m)$  the inverse Fourier transform of  $g$  on  $\mathbb{R}^m$ . Since  $a$  is a symbol of order 0 in  $w$ , this is a pseudodifferential operator acting on  $\mathcal{F}^{-1} g$  and the result follows from standard pseudodifferential operator theory.  $\square$

LEMMA 9.3. — Let  $a(w, \eta)$  be a symbol of order 0 in  $w = (\bar{w}, w_{m+1}) \in \mathbb{R}^{m+1}$ , and let  $h(\eta) \in H^{-\gamma}(\mathbb{R}^m)$  be supported in  $|\eta|^2 \leq t^2$ , where  $t$  is a nonzero constant. Then

$$(Ah)(w) = \int e^{i\bar{w} \cdot \eta + iw_{m+1} \sqrt{t^2 - |\eta|^2}} a(w, \eta) h(\eta) d\eta \in \langle w \rangle^{\gamma+1/2} L^2(\mathbb{R}^{m+1})$$

provided  $\gamma > 0$ .

*Proof.* — Let  $g \in \langle w \rangle^{-\gamma-1/2} L^2(\mathbb{R}^{m+1})$ . Then

$$(43) \quad (g, Ah) = \left( \int \bar{a}(w, \eta) e^{-i\bar{w} \cdot \eta - iw_{m+1} \tau} g(w) dw \Big|_{\tau = \sqrt{t^2 - |\eta|^2}}, h(\eta) \right).$$

By the previous lemma and the restriction properties of elements of Sobolev spaces,

$$\int \bar{a}(w, \eta) e^{-i\bar{w} \cdot \eta - iw_{m+1} \tau} g(w) dw \Big|_{\tau = \sqrt{t^2 - |\eta|^2}} \in H^\gamma$$

if  $\gamma > 0$ . The pairing (43) is then well-defined for all such  $g$ , and  $Ah \in \langle w \rangle^{\gamma+1/2} L^2(\mathbb{R}^{m+1})$ .  $\square$

Although the following fact may be well-known, we are unaware of a reference and so include it for the convenience of the reader.

LEMMA 9.4. — Suppose  $w \in \mathbb{R}^m$ ,  $f \in \langle w \rangle^{-1-\delta} L^2(\mathbb{R}^m)$  for some  $\delta > 0$ , and  $\mu \in \mathbb{R} \setminus \{0\}$ . Then

$$\begin{aligned} \left( \frac{\partial}{\partial |w|} + i\mu \right) \lim_{\gamma \downarrow 0} (\Delta - (\mu - i\gamma)^2)^{-1} f \\ = \left( \frac{\partial}{\partial |w|} + i\mu \right) (\Delta - (\mu - i0)^2)^{-1} f \in \langle w \rangle^\epsilon L^2(\mathbb{R}^m) \end{aligned}$$

for any  $\epsilon > 0$ .

*Proof.* — The proof is an adaptation of [19, Lemma 14.2.1]. We will assume for simplicity that  $\mu > 0$ . We have

$$(\Delta - (\mu - i0)^2)^{-1} f(w) = \lim_{\gamma \downarrow 0} (2\pi)^{-m} \int e^{iw \cdot \eta} \frac{1}{|\eta|^2 - (\mu - i\gamma)^2} \mathcal{F}(f)(\eta) d\eta$$

where  $\mathcal{F}(f)(\eta) = \int e^{-iw' \cdot \eta} f(w') dw'$ . Clearly, the only issue is the integration near  $|\eta| = \mu$ . Using a partition of unity, we can reduce the problem to considering integrals of the type

$$(2\pi)^{-m} \int e^{iw \cdot \eta} \frac{1}{|\eta|^2 - (\mu - i0)^2} g_{j\pm}(\eta) d\eta$$

where  $g_{j\pm}(\eta)$  is supported near  $|\eta| = \mu$  and  $\pm \eta_j > 0$  on the support of  $g_{j\pm}$ .

On the support of  $g_{1+}$ , we write  $\eta = (\eta_1, \eta')$  and

$$\frac{1}{|\eta| - (\mu - i\gamma)} \frac{g_{1+}}{|\eta| + \mu} = (\eta_1 - \sqrt{\mu^2 - |\eta'|^2} + i\gamma q(\eta))^{-1} h(\eta)$$

where

$$q(\eta) = \frac{\eta_1 - \sqrt{\mu^2 - |\eta'|^2}}{|\eta| - \mu} > 0$$

and  $h(\eta) = g_{1+}(\eta)(|\eta| + \mu)^{-1} q(\eta) \in H^{1+\delta}(\mathbb{R}^m)$ . We write

$$\begin{aligned} h(\eta) &= h(\sqrt{\mu^2 - |\eta'|^2}, \eta') + h(\eta) - h(\sqrt{\mu^2 - |\eta'|^2}, \eta') \\ &= h(\sqrt{\mu^2 - |\eta'|^2}, \eta') + h_1(\eta). \end{aligned}$$

Clearly,

$$\int e^{iw \cdot \eta} \frac{1}{\eta_1 - \sqrt{\mu^2 - |\eta'|^2} + i0} h_1(\eta) d\eta \in \langle w \rangle^{-\delta} L^2(\mathbb{R}^m).$$

Let  $w = (w_1, w')$ . For the other term, we use that

$$\begin{aligned} \lim_{\gamma \downarrow 0} (2\pi)^{-1} \int e^{iw_1 \eta_1} \frac{1}{\eta_1 - \sqrt{\mu^2 - |\eta'|^2} + i\gamma q(\eta)} h(\sqrt{\mu^2 - |\eta'|^2}, \eta') d\eta_1 \\ = -ih(\sqrt{\mu^2 - |\eta'|^2}, \eta') e^{iw_1 \sqrt{\mu^2 - |\eta'|^2}} H(-w_1) \end{aligned}$$

with  $H$  the Heaviside function. Since

$$\begin{aligned} H(-w_1) \left( \frac{\partial}{\partial |w|} + i\mu \right) \int e^{iw' \cdot \eta'} e^{iw_1 \sqrt{\mu^2 - |\eta'|^2}} h(\sqrt{\mu^2 - |\eta'|^2}, \eta') d\eta' \\ = H(-w_1) \int \left( \frac{w'}{|w|} \cdot \eta' + \frac{w_1}{|w|} \sqrt{\mu^2 - |\eta'|^2} + i\mu \right) e^{iw' \cdot \eta'} e^{iw_1 \sqrt{\mu^2 - |\eta'|^2}} \\ h(\sqrt{\mu^2 - |\eta'|^2}, \eta') d\eta' \end{aligned}$$

and the integrand vanishes on the stationary points of the phase, by integrating by parts and applying Lemma 9.3, we obtain that this is an element of  $\langle w \rangle^\epsilon L^2(\mathbb{R}^m)$  for any  $\epsilon > 0$ .

A similar analysis works for the remaining  $g_{j\pm}$ . □

We recall Proposition 4.3, the main result of this section.

**PROPOSITION 4.3.** — *If  $f \in \langle z \rangle^{-3/2+\epsilon} L^2(\mathbb{R}^n)$  for every  $\epsilon > 0$  and  $J \geq 2$ , then  $u = (\Delta - (\lambda - i0)^2 c^{-2})^{-1} f$  is outgoing in the sense of Definition 4.1.*

The proof of this proposition includes the proof of Lemma 5.1.

*Proofs of Proposition 4.3 and Lemma 5.1.* — Throughout the proof,  $\epsilon$  stands for any  $\epsilon > 0$  but may change from line to line.

We use (36) and (37). Note that  $V\widehat{u} + f(\xi, y) \in H^{3/2-\epsilon-\beta}(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\beta} L^2(\mathbb{R}_y))$  for any  $0 \leq \beta \leq 3/2 - \epsilon$ . Choose  $\Psi \in C_c^\infty(\mathbb{R})$  to be 1 for  $|\xi| \leq \lambda^2/c_+^2$  and supported in a slightly larger neighborhood. In particular, choose  $\Psi$  so that  $\Psi(\kappa_j(\lambda)) = 0$  for  $j = 1, 2, \dots, T(\lambda)$ . We use the fact that we can write  $(1 - \Psi(\xi))((D_y^2 + |\xi|^2 - (\lambda - i\delta)^2 c_0^{-2})^{-1})$  as a sum of an operator bounded on  $L^2(\mathbb{R})$  and an operator involving projection onto  $\{f_j\}$ ,  $1 \leq j \leq T(\lambda)$ . The part corresponding to an operator bounded on  $L^2(\mathbb{R})$  gives, upon integration, an element of  $L^2(\mathbb{R}^n)$ . Choose  $\Psi_j \in C_c^\infty$  so that  $\Psi_j(\kappa_j(\lambda)) = 1$  and so that  $\Psi_j$  is supported very near  $\kappa_j(\lambda)$ . Then

$$\begin{aligned} (2\pi)^{-n+1} \int e^{ix \cdot \xi} \Psi_j(|\xi|) ((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} V\widehat{u} + f(\xi, \cdot))(y) dy d\xi \\ = u_j(z) + w_j(z). \end{aligned}$$

Here  $w_j \in H^2(\mathbb{R}^n)$ ,

$$u_j(z) = ((\Delta_{\mathbb{R}^{n-1}} - (\kappa_j(\lambda) - i0)^2)^{-1} g_j)(x) f_j(y),$$

and

$$\begin{aligned} g_j(x) = (2\pi)^{-n+1} \|f_j\|^{-1} \int e^{ix \cdot \xi} \Psi_j(|\xi|) \int V\widehat{u} + f(\xi, y) \bar{f}_j(y) dy d\xi \\ \in \langle x \rangle^{-3/2+\epsilon}(\mathbb{R}^{n-1}). \end{aligned}$$

Then Lemma 9.4 shows that  $u_j$  has the desired properties.

Suppose for the moment that  $f \in \langle z \rangle^{-\infty}(\mathbb{R}^n)$ . Then we may write

$$u_j(z) = ((\Delta_{\mathbb{R}^{n-1}} - (\kappa_j(\lambda) - i0)^2)^{-1} \tilde{g}_j)(x) f_j(y) + w_{1j}(z)$$

where  $w_{1j}(z) \in \langle z \rangle^\epsilon H^2(\mathbb{R}^n)$  and, using Corollary 9.1,

$$\mathcal{F}(\tilde{g}_j)(\xi) = \Psi_j(|\xi|) \|f_j\|^{-1} \int V \widehat{u} + f(\kappa_j(\lambda)\xi/|\xi|, y) \bar{f}_j(y) dy \in C_c^\infty(\mathbb{R}^n).$$

Then the standard results for the resolvent applied to Schwartz functions then give us that

$$u_j = e^{i\kappa_j(\lambda)|x|} |x|^{-(n-2)/2} b_j(x/|x|) f_j(y) + w_{2j}(z)$$

where  $b_j \in C^\infty(\mathbb{S}^{n-2})$  and  $w_{2j} \in \langle z \rangle^\epsilon H^2(\mathbb{R}^n)$ . This  $b_j$  is the  $b_j$  appearing in Lemma 5.1.

Now we return to the assumption that  $f \in \langle z \rangle^{-3/2+\epsilon} L^2(\mathbb{R}^n)$  and consider the integral over the support of  $\Psi(\xi)$  ( $|\xi| \leq \lambda/c_+ + \delta$ ).

We may write

$$\begin{aligned} (44) \quad & ((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} g)(y) \\ &= \varphi_-(y, |\xi|) \int_{y' < y} \frac{\varphi_+(y', |\xi|) g(y')}{[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]} dy' \\ & \quad + \varphi_+(y, |\xi|) \int_{y' > y} \frac{\varphi_-(y', |\xi|) g(y')}{[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]} dy'. \end{aligned}$$

Here

$$(D_y^2 + |\xi|^2 - c_0^{-2} \lambda^2) \varphi_\pm = 0$$

and, for  $\pm y > y_M$ ,

$$\varphi_\pm(y, |\xi|) = e^{\mp iy \sqrt{\lambda^2/c_\pm^2 - |\xi|^2}}$$

with  $\text{Im} \sqrt{\lambda^2/c_\pm^2 - |\xi|^2} \leq 0$ . These functions are smooth functions of  $|\xi|$  except near  $|\xi| = \lambda/c_\pm$ , where they are smooth functions of  $(|\xi|^2 - \lambda^2/c_\pm^2)^{1/2}$ . The Wronskian  $[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]$  is nonzero except, perhaps, at  $|\xi| = \lambda/c_+$ , where it behaves like  $\alpha_1 + \alpha_2 \sqrt{|\xi|^2 - \lambda^2/c_+^2}$  with at least one of  $\alpha_1, \alpha_2$  nonzero. We suppress the dependence of  $\varphi_\pm$  on  $\lambda$  for notational simplicity. Notice that

$$\begin{aligned} (45) \quad & \frac{\partial}{\partial y} ((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} g)(y) \\ &= \left( \frac{\partial}{\partial y} \varphi_-(y, |\xi|) \right) \int_{y' < y} \frac{\varphi_+(y', |\xi|) g(y')}{[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]} dy' \\ & \quad + \left( \frac{\partial}{\partial y} \varphi_+(y, |\xi|) \right) \int_{y' > y} \frac{\varphi_-(y', |\xi|) g(y')}{[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]} dy'. \end{aligned}$$

When  $|y| < C$  for some constant  $C$ , we have that away from  $|\xi| = \lambda/c_+$ ,

$$w(\xi, y) = \Psi(|\xi|)((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1}(\widehat{V}u + \widehat{f})(\xi, \cdot))(y) \in L^2(\mathbb{R}_\xi^{n-1})$$

with a norm bounded independent of  $y$ ,  $|y| < C$ . Near  $|\xi| = \lambda/c_+$ ,  $w(\xi, y) \in L^p(\mathbb{R}_\xi^{n-1})$  for any  $p < 2$ , again with norm bounded independent of  $y$ ,  $|y| < C$ , so using the mapping properties of the Fourier transform

$$\int e^{ix \cdot \xi} w(\xi, y) d\xi_{\{|y| < C\}} \in \langle x \rangle^\epsilon H^2(\mathbb{R}_x^{n-1} \times [-C, C]_y).$$

Now we consider what happens for  $y > y_M$ . When  $y > y_M$ , using (44), for  $|\xi| \leq \lambda/c_+$  we have

$$(46) \quad ((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1}(\widehat{V}u + \widehat{f})(\xi, \cdot))(y) = h_1(\xi) e^{-iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}} + h_2(\xi, y),$$

with

$$h_1(\xi) = \int_{-\infty}^{\infty} \frac{\varphi_+(y', |\xi|)}{[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]} (\widehat{V}u + \widehat{f})(\xi, y') dy'.$$

For  $|\xi| \leq \lambda/c_+$ ,  $h_2(\xi, y) \in L^2(\mathbb{R}_\xi^{n-1}; L^2(\mathbb{R}_y))$ . Then taking its inverse Fourier transform in  $\xi$  gives an element of  $L^2(\mathbb{R}^n \cap \{y > y_M\})$ , and similarly if we take a radial derivative.

If we stay away from  $|\xi| = \lambda/c_+$ ,  $h_1(\xi) \in H^{1-\epsilon}(\mathbb{R}^{n-1})$ . Notice that

$$(47) \quad \left( \frac{\partial}{\partial |z|} + i\lambda/c_+ \right) e^{ix \cdot \xi - iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}} = \left( \frac{x}{|z|} \cdot \xi - i \frac{y}{|z|} \sqrt{\lambda^2/c_+^2 - |\xi|^2} + i\lambda/c_+ \right) e^{ix \cdot \xi - iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}},$$

that is, it vanishes to first order on the critical points of the phase. Therefore, if we smoothly cut-off to stay in  $|\xi| < \lambda/c_+ - \delta$ , we may integrate by parts and, applying Lemma 9.3 (with  $m = n - 1$ ), we obtain an element of  $\langle z \rangle^\epsilon L^2(\mathbb{R}^n)$ .

It remains to consider the integral near  $|\xi| = \lambda/c_+$ . Using a partition of unity, we can work on sets where  $\xi_i > 0$  for some  $i$  or  $\xi_i < 0$  for some  $i$ , and  $|\xi| \sim \lambda/c_+$ . We will consider the case  $\xi_1 > 0$  as the other cases follow similarly, and let  $\Psi_{1+} \in C^\infty(\mathbb{R}^{n-1})$  be a cut-off function with such support properties. For  $|\xi| < \lambda/c_+$ , introduce the coordinates  $t = \sqrt{\lambda^2/c_+^2 - |\xi|^2}$ ,

$\tilde{\xi} = (\xi_2, \dots, \xi_{n-1})$ . We will use  $\xi(t, \tilde{\xi})$  or sometimes just  $\xi$  to denote  $\xi$  as a function of  $(t, \tilde{\xi})$ . We must consider

$$\left( \frac{\partial}{\partial |z|} + i\lambda/c_+ \right) \int_{t \geq 0} \int e^{i(x_1 \sqrt{\lambda^2/c_+^2 - t^2 - |\tilde{\xi}|^2 + \tilde{x} \cdot \tilde{\xi})} e^{-iyt} h_{12}(t, \tilde{\xi}) d\tilde{\xi} dt$$

using the notation  $\tilde{x} = (x_2, \dots, x_{n-1})$ . Here

$$\begin{aligned} h_{12}(t, \tilde{\xi}) &= \Psi_{1+}(\xi(t, \tilde{\xi})) \frac{t}{\xi_1(t, \tilde{\xi})} h_1(\xi(t, \tilde{\xi})) \\ &= \Psi_{1+}(\xi(t, \tilde{\xi})) \frac{t}{\xi_1(t, \tilde{\xi})} \int_{-\infty}^{\infty} \frac{\varphi_+(y', |\xi|(t))}{[\varphi_+(y', |\xi|(t)), \varphi_-(y', |\xi|(t))]} \\ &\quad \widehat{Vu + f(\xi(t, \tilde{\xi}), y')} dy' \in H^{1-\epsilon}(\mathbb{R}_\xi^{n-1}). \end{aligned}$$

Then, using the fact that the application of  $\frac{\partial}{\partial |z|} + i\lambda/c_+$  produces a function that vanishes on the stationary points of the phase, we may integrate by parts and use Lemma 9.3 (with  $m = n - 1$ ) and a similar result for the boundary term to obtain an element of  $\langle z \rangle^\epsilon L^2(\mathbb{R}^n)$  for the contribution of  $h_{12}$ . We have already considered the contribution of  $h_2(\xi, y)$  when  $|\xi| \leq \lambda/c_+$ .

What now remains is an integral over  $|\xi| \geq \lambda/c_+$ , with  $|\xi|$  near  $\lambda/c_+$ . First consider the term corresponding to the integral over  $y' > y$  in (44). Since

$$\begin{aligned} \left| \int_{|\xi|=s} \varphi_+(y, |\xi|) \int_{y' > y > y_M} \frac{\varphi_-(y', |\xi|)}{[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]} \widehat{Vu + f(\xi, y')} dy' \right| \\ \leq \frac{C}{\sqrt{s^2 - \lambda^2/c_+^2}} \langle y \rangle^{-1/2+\epsilon} \end{aligned}$$

for  $\lambda/c_+ < s < \lambda/c_+ + \delta$ , we have

$$\begin{aligned} &\left\| \Psi(|\xi|) \varphi_+(y, |\xi|) \int_{y' > y > y_M} \frac{\varphi_-(y', |\xi|)}{[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]} \widehat{Vu + f(\xi, y')} dy' \right\|_{L^p(\{\xi \in \mathbb{R}^{n-1} : \lambda/c_+ \leq |\xi| \})} \\ (48) &\leq C \langle y \rangle^{-1/2+\epsilon} \end{aligned}$$

for any  $p < 2$ . Then taking the inverse Fourier transform in  $\xi$  and using the mapping properties of the Fourier transform we obtain an element of  $\langle z \rangle^\epsilon H^2(\mathbb{R}^n)$ .

For  $y > y_M$ , the last remaining term to consider is an integral over  $|\xi| \geq \lambda/c_+$ ,  $|\xi|$  near  $\lambda/c_+$ , and the term of (44) which involves integrating over  $y' < y$ . Here, we again work on a coordinate patch with  $\xi_1 > 0$ , and use the coordinates  $t = \sqrt{|\xi|^2 - \lambda^2/c_+^2}$ ,  $\tilde{\xi} = (\xi_2, \dots, \xi_{n-1})$ . We remark that

$$\left\| \Psi_{1,+}(\xi(t, \tilde{\xi})) \frac{\partial}{\partial \xi_j} t\varphi_-(y, |\xi|(t)) \int_{y' < y} \frac{\varphi_+(y', |\xi|(t))}{[\varphi_+(y', |\xi|(t)), \varphi_-(y', |\xi|(t))]} \widehat{Vu + f(\xi(t, \tilde{\xi}), y')} dy' \right\|_{H^{-\epsilon}(\mathbb{R}_{\tilde{\xi}}^{n-2})} \leq Ct^{-1/2}$$

and

$$\left\| \frac{\partial}{\partial t} t\varphi_-(y, |\xi|(t)) \int_{y' < y} \frac{\varphi_+(y', |\xi|(t))}{[\varphi_+(y', |\xi|(t)), \varphi_-(y', |\xi|(t))]} \widehat{Vu + f(\xi(t, \tilde{\xi}), y')} dy' \right\|_{L^2(\mathbb{R}_{\tilde{\xi}}^{n-2})} \leq C\langle y \rangle^\epsilon t^{-1/2}.$$

Therefore, we may as before integrate by parts and use Lemma 9.3 to obtain that

$$\begin{aligned} & \int_{|\xi| > \lambda/c_+} \left( \left( \frac{\partial}{\partial |z|} + i\lambda/c_+ \right) e^{i(x_1 \sqrt{\lambda^2/c_+^2 + t^2 - |\tilde{\xi}|^2 + \tilde{x} \cdot \tilde{\xi}})} \varphi_-(y, |\xi|(t)) \right) \\ & \quad \times \int_{y' < y} \frac{t\varphi_+(y', |\xi|(t))}{\xi_1(t, \tilde{\xi})[\varphi_+(y', |\xi|(t)), \varphi_-(y', |\xi|(t))]} \Psi_{1,+}(\xi(t, \tilde{\xi})) \\ & \quad \widehat{Vu + f(\xi(t, \tilde{\xi}), y')} dy' dt d\tilde{\xi}_{|y > y_M} \\ & \in \langle z \rangle^{-1} \langle y \rangle^\epsilon \langle x \rangle^{1/2 + \epsilon} L^2(\mathbb{R}_x^{n-1}; L^\infty(\mathbb{R}_{y > y_M})) \subset \langle z \rangle^\epsilon L^2(\mathbb{R}^n \cap \{y > y_M\}). \end{aligned}$$

A similar analysis works for  $y < -y_M$ , proving the proposition.  $\square$

### 10. Proof of Theorem 1.3.

It remains to prove Theorem 1.3. We divide the proof into two cases, depending on which of the hypotheses (H1) or (H2) holds. The first is the simpler case.

**10.1. Proof of Theorem 1.3 in case hypothesis (H1) holds.**

In case hypothesis (H1) holds we can use some results from the study of the  $n$ -body problem to prove Theorem 1.3. Our proof uses the notion of the scattering wave front set,  $WF_{sc}$ , introduced by Melrose in [26]. Roughly speaking, the scattering wave front set provides a microlocal description of the lack of decay of a function as well as capturing information about its singularities. We refer the reader to [26, Section 7] for the full definition of  $WF_{sc}$  but give some indications here for the convenience of the reader.

Let  $A$  be the operator with Schwartz kernel

$$A(z, z') = (2\pi)^{-n} \int e^{i(z-z') \cdot \xi} a_L(z, \xi) d\xi,$$

where

$$|D_z^\alpha D_\xi^\beta a_L(z, \xi)| \leq c_{\alpha\beta} (1 + |z|)^{-l - |\alpha|} (1 + |\xi|)^{m - |\beta|}.$$

We will say  $A \in \Psi_{sc}^{m,l}(\mathbb{R}^n)$  (compare [18] and [26, Section 4]). For a smooth  $v \in \mathcal{S}'(\mathbb{R}^n)$ ,  $\tau \in \mathbb{R}$  we say

$$WF_{sc}(v) \subset R^+(\tau) \Leftrightarrow$$

$$Av \in \mathcal{S}(\mathbb{R}^n) \text{ for all } R > 0, m, l, \text{ and for all } A \in \Psi_{sc}^{m,l}(\mathbb{R}^n) \text{ with } a_L(z, \xi) \text{ vanishing to infinite order on the set } \left\{ \frac{z}{|z|} \cdot \xi = -\tau, |z| > R > 0 \right\}.$$

For the definition of  $R^+(\tau)$  see [26, Section 8]. (To put this in the setting of [26] we are assuming a radial compactification of  $\mathbb{R}^n$ .)

The following proposition implies Theorem 1.3 in one case.

**PROPOSITION 10.1.** — *Suppose  $c$  and  $c_0$  satisfy the hypothesis (H1). Let  $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$  and let  $f \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ ,  $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$  for some  $\phi \in C_c^\infty(\mathbb{R})$ . Then*

$$\begin{aligned} \chi(z/|z|)(\Delta - c^{-2}(\lambda - i0)^2)^{-1} f \\ = |z|^{-(n-1)/2} e^{-i\lambda|z|/c_0(y)} \sum_{j=0}^N |z|^{-j} a_j(z/|z|) + \mathcal{O}(|z|^{-(n-1)/2-N-1}) \end{aligned}$$

where  $a_j \in C^\infty(\mathbb{S}_c^{n-1})$ .

*Proof.* — As  $c_+ = c_-$ , we can view this as a perturbation of a simple  $n$ -body problem and use the fact that much is known about asymptotic

expansions of such resolvents applied to Schwartz functions. First, we note that

$$(\Delta - c^{-2}(\lambda - i0)^2)^{-1} = (\Delta - \lambda^2(c^{-2} - c_+^{-2}) - c_+^{-2}(\lambda - i0)^2)^{-1}.$$

The operator  $\Delta - \lambda^2(c_0^{-2} - c_+^{-2})$  is a particularly simple example of a class of  $n$ -body operators widely studied. The operator  $\Delta - \lambda^2(c^{-2} - c_+^{-2})$ , while not quite an  $n$ -body operator since the potential depends on all variables, is a perturbation that has many of the same properties we desire.

The paper [28], which builds on results of [14], [26], shows that

$$\begin{aligned} \chi(z/|z|)(\Delta - \lambda^2(c_0^{-2} - c_+^{-2}) - c_+^{-2}(\lambda - i0)^2)^{-1} f \\ = |z|^{-(n-1)/2} e^{-i\lambda|z|/c_0(y)} \sum_0^N |z|^{-j} b_j(z/|z|) + \mathcal{O}(|z|^{-(n-1)/2-N-1}) \end{aligned}$$

with  $b_j$  smooth. The proof is such that the results of [28] hold with  $c_0$  replaced by a sound speed  $c$  of the type considered here. Roughly speaking, this is because [14] requires that the operator  $\Delta + V_1$  (where for us  $V_1 = -\lambda^2(c_0^{-2} - c_+^{-2})$  or  $V_1 = -\lambda^2(c^{-2} - c_+^{-2})$ ) satisfy a Mourre estimate and some regularity and decay properties, both of which are satisfied for either  $V_1$ . Vasy remarks already that the results of [28, Section 2], which are local versions of results of [26], will hold in our case. Then the results of [28, Section 3] hold for our case, since just as in that paper we can argue that from the results of [14] and [26] that

$$\text{WF}_{\text{sc}}(\chi(z/|z|)(\Delta - c^{-2}(\lambda - i0)^2)^{-1} f) \subset R^+(\lambda/c_+),$$

and the asymptotic expansion follows from [28, Proposition 2.8] and the remarks made there. □

Before giving the proof in the case  $c_+ \neq c_-$ , we give some explanation as to why the proof of [28] does not apply in this case. The argument of [28] uses very strongly the fact that, for  $f \in \mathcal{S}$ ,

$$\text{WF}_{\text{sc}}(\chi(z/|z|)(\Delta - \lambda^2(c_0^{-2} - c_+^{-2}) - c_+^{-2}(\lambda - i0)^2)^{-1} f) \subset R^+(\lambda/c_+)$$

where  $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$ .

This is not, however, true in general when  $c_+ \neq c_-$ . The results of [5, Theorem 4.1] show that in general for  $f \in \mathcal{S}(\mathbb{R}^n)$ ,

$$\begin{aligned} ((\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} f)(z) &= e^{-i\lambda|z|/c_+} |z|^{-(n-1)/2} a_0(z/|z|) \\ &+ |z|^{-(n+1)/2} \left( e^{-i\lambda|z|/c_+} a_1(z/|z|) + e^{-i\lambda|x|/c_- - iy\lambda\sqrt{1/c_+^2 - 1/c_-^2}} b_1(z/|z|) \right) \\ &+ \mathcal{O}(|z|^{-(n+3)/2}) \end{aligned}$$

when  $0 < \epsilon < y/|z| < (1 - c_+^2/c_-^2)^{1/2} - \epsilon$  and  $|z| \rightarrow \infty$ . If  $b_1 \neq 0$ , then the scattering wave front set is not contained in  $R^+(\lambda/c_+)$ . This means that the scattering wave front set of  $\chi(\Delta - (\lambda - i0)^2/c^2)^{-1}f$  is in general more complicated than in the case where  $c_+ = c_-$  (and unknown, to the best of our knowledge), and the techniques of [26], [28] cannot be immediately applied. Similar differences can be seen in the resolvent estimates of [14, Theorem 1.1] for the  $n$ -body problem, and [20, Theorem 3.1], for a particular stratified medium.

Instead, we will take a different approach.

**10.2. Preliminaries for the proof of Theorem 1.3  
in case hypothesis (H2) holds.**

In order to prove Theorem 1.3 when hypothesis (H2) holds, we will make heavy use of equations (36) and (37). We use the fact that the more rapidly  $g$  decays at infinity, the more we can say about  $(\Delta - (\lambda - i0)^2 c_0^{-2})^{-1}g$ , using equation (37). To take advantage of this, roughly speaking, we find approximations  $w$  of  $u = (\Delta - (\lambda - i0)^2 c^{-2})^{-1}f$  so that  $(\Delta - \lambda^2/c_0^2)w = Vu + e$ , with the error  $e$  decaying faster than  $Vu$  does, and  $w$  decaying faster than  $u$ . Then  $u = w - (\Delta - (\lambda - i0)^2 c_0^{-2})^{-2}(f + e)$  (compare (36)). The better rate of decay of  $e$  improves our knowledge of  $u$ .

In actuality, the proof is somewhat more complicated. We study the behaviour at infinity of  $\chi(z/|z|)u$ ,  $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$ , and we introduce a “microlocal” cut-off  $\Psi(D_x)$  so that  $\chi\Psi(D_x)u$  has the same leading behaviour as  $\chi u$  at infinity (Lemma 10.1), but is easier to understand.

If  $\Psi(\xi) \in C_b^\infty(\mathbb{R}^{n-1})$ , we use the notation

$$(\Psi(D_x)f)(z) = (2\pi)^{1-n} \int e^{i(x-x')\cdot\xi} \Psi(\xi) f(x', y) dx' d\xi.$$

Note that  $[\Psi(D_x), \Delta] = 0$  and  $[\Psi(D_x), c_0(y)] = 0$ .

Lemma 10.1 shows that, for suitable  $\Psi$ ,  $\chi(1 - \Psi(D_x))u \in \langle z \rangle^{\epsilon-1/2} L^2(\mathbb{R}^n)$ , so that  $\chi\Psi(D_x)u$  captures the leading behaviour of  $\chi u$ . Lemmas 10.2-10.7 are preliminaries. Lemma 10.8 constructs a first approximation of  $\chi\Psi(D_x)u$ . Then in Proposition 10.2, we use Lemma 10.8 as well as results from Lemmas 10.2-10.7 to make successive approximations of  $\chi\Psi(D_x)u$ , showing that it has an asymptotic expansion with smooth coefficient in the leading order term.

The next lemma shows us that, for suitable  $\Psi$  and  $\chi$ ,  $\chi\Psi(D_x)u$  is the leading order term of  $\chi u$ .

LEMMA 10.1. — *Let  $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$ ,  $\Psi \in C_b^\infty(\mathbb{R}^{n-1})$ , and suppose that if  $(x/|z|, y/|z|) \in \text{supp}\chi$  and  $y > 0$ , then  $\pm\lambda x/(c_+|z|) \notin \text{supp}(1 - \Psi)$ , and if  $(x/|z|, y/|z|) \in \text{supp}\chi$  and  $y < 0$ , then  $\pm\lambda x/(c_-|z|) \notin \text{supp}(1 - \Psi)$ . If  $J \geq 4$ ,  $u = (\Delta - (\lambda - i0)^2 c^{-2})^{-1}f$ , and  $f \in \langle z \rangle^{-\infty}L^2(\mathbb{R}^n)$ , then*

$$\chi(z/|z|)(1 - \Psi(D_x))u, \frac{\partial}{\partial|z|}\chi(z/|z|)(1 - \Psi(D_x))u \in \langle z \rangle^{-1/2+\epsilon}L^2(\mathbb{R}^n)$$

for any  $\epsilon > 0$ .

*Proof.* — We give the proof for  $\chi$  supported in  $y > 0$ , as the proof for  $\chi$  with support in  $y < 0$  is quite similar. Moreover, for simplicity we assume that  $\lambda > 0$ .

The proof closely resembles that of Proposition 4.3. Let  $\Psi_1 \in C_b^\infty(\mathbb{R})$  be such that  $\text{supp}\Psi_1(t) \subset \{|t| > \lambda/c_+\}$ ,  $\text{supp}(1 - \Psi_1) \subset \{|t| \leq \lambda/c_+ + \delta\}$ , some small  $\delta > 0$ . In particular,  $\text{supp}(1 - \Psi_1)$  should not include  $\kappa_j(\lambda)$ ,  $j = 1, \dots, T(\lambda)$ . Then, by the same type of arguments as in the proof of Proposition 4.3, since the eigenfunctions of  $D_y^2 + c_0^{-2}\lambda^2$  are exponentially decreasing in  $y$ , we have

$$\chi(z/|z|) \int e^{ix \cdot \xi} (1 - \Psi(\xi)) \Psi_1(|\xi|) (D_y^2 + |\xi|^2 - c_0^{-2}(\lambda - i0)^2)^{-1} (\widehat{V}u(\xi, \cdot) + \widehat{f}(\xi, \cdot))(y) d\xi \in \langle z \rangle^{-1/2+\epsilon}L^2(\mathbb{R}^n).$$

When  $|\xi| \leq \lambda/c_+$  and  $y > y_M$ , we use (46):

$$((D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} \widehat{V}u + \widehat{f}(\xi, \cdot))(y) = h_1(\xi) e^{-iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}} + h_2(\xi, y).$$

Here we have  $\|h_2(\xi, y)\|_{L^2(\mathbb{R}_\xi^{n-1} \cap \{|\xi| \leq \lambda/c_+\})} \leq C\langle y \rangle^{-J+1+\epsilon}$ , so that

$$\chi(z/|z|) \int_{|\xi| \leq \lambda/c_+} (1 - \Psi(\xi)) e^{ix \cdot \xi} h_2(y, \xi) d\xi \in \langle z \rangle^{-J+3/2+\epsilon}L^2(\mathbb{R}^n).$$

Now consider the term corresponding to  $e^{-iy\sqrt{\lambda^2 - |\xi|^2}} h_1(\xi)$ . Away from  $|\xi| = \lambda/c_\pm$ ,  $h_1(\xi) \in H^{J-1-\epsilon}(\mathbb{R}^{n-1})$ . There are no stationary points of the phase  $x \cdot \xi - y\sqrt{\lambda^2/c_+^2 - |\xi|^2}$  when  $z$  is restricted to the support of  $\chi(z/|z|)$  and  $\xi$  to the support of  $(1 - \Psi)$ . Therefore, we may integrate by parts after smoothly cutting off away from  $|\xi| = \lambda/c_\pm$ . This gives an element of

$\langle z \rangle^{-1/2+\epsilon} L^2(\mathbb{R}^n)$ . We need only examine the integration near  $|\xi| = \lambda/c_{\pm}$  more closely.

If  $c_+ < c_-$ , near  $|\xi| = \lambda/c_-$ ,  $y > y_M$ , we have

$$(49) \quad h_1(\xi) = a_1(\xi) + (\lambda^2/c_-^2 - |\xi|^2)^{1/2} a_2(\xi)$$

where  $a_1, a_2 \in H^l(\mathbb{R}^{n-1})$ , and  $l$  is an integer with  $l < J - 1$ . This comes from using (44) and the subsequent comments about smoothness. Since again there are no stationary points of the phase  $x \cdot \xi - \sqrt{\lambda^2/c_+^2 - |\xi|^2}$  with  $z/|z|$  in the support of  $\chi$  and  $\xi$  in the support of  $1 - \Psi$ , we integrate by parts. Since, for any  $\Psi_3 \in C_c^\infty(\mathbb{R})$ , the Fourier transform in  $x$  of  $\Psi_3(|\xi| - \lambda/c_+) (\lambda^2/c_-^2 - |\xi|^2)^{-1/2}$  is in  $L^p(\mathbb{R}^{n-1}) \cap C^\infty(\mathbb{R}^{n-1})$  for any  $p > 2$ , we have that the contribution of the integration of the  $a_i$  terms over this region is in  $\langle z \rangle^{-1/2+\epsilon} L^2(\mathbb{R}^n)$  for any  $\epsilon > 0$ .

Near  $|\xi| = \lambda/c_+$ , as in the proof of Proposition 4.3, we divide the integration into two pieces. When  $|\xi| < \lambda/c_+$ , to handle the  $h_1$  term we introduce the coordinate  $t = \sqrt{\lambda^2/c_+^2 - |\xi|^2}$  as in the proof of Proposition 4.3 and we can integrate by parts, which results in a boundary term plus an element of  $\langle z \rangle^{-1/2+\epsilon} L^2(\mathbb{R}^n)$ . The boundary term, when added to the corresponding boundary term for the integral over  $|\xi| \geq \lambda/c_+$ , results in an element of  $\langle z \rangle^{-1/2+\epsilon} L^2(\mathbb{R}^n)$ .

Now consider  $\lambda/c_+ + \delta \geq |\xi| \geq \lambda/c_+$ . There are two terms from (44) to consider. We first consider the integral over  $y' > y$ . Note that, as in (48), for any  $p < 2$ :

$$\begin{aligned} & \left\| \varphi_+(y, |\xi|) \int_{y' > y > y_M} \frac{\varphi_-(y', |\xi|)}{[\varphi_+(y', |\xi|), \varphi_-(y', |\xi|)]} \widehat{Vu + f(\xi, y')} dy' \right\|_{L^p(\{|\xi| \in \mathbb{R}^{n-1} : \lambda/c_+ \leq |\xi| \leq \lambda/c_+ + \delta\})} \\ & \leq C \langle y \rangle^{-J+1+\epsilon}. \end{aligned}$$

By the mapping properties of the Fourier transform, this contributes an element of  $\langle z \rangle^{-J+3/2+\epsilon} L^2(\mathbb{R}^n)$ .

For the one remaining term, we change coordinates as in the proof of Proposition 4.3, using, in the region with  $\xi_1 > \delta > 0$ , the coordinates  $t = \sqrt{|\xi|^2 - \lambda^2/c_+^2}$  and  $\tilde{\xi} = (\xi_2, \dots, \xi_{n-1})$ . (Other regions can be handled similarly.)

We have

$$\chi(z/|z|) \int \int_{t>0} (1 - \Psi(\xi))(1 - \Psi_1(|\xi|))\Psi_{1+}(\xi)e^{i(x_1\sqrt{t^2+\lambda^2-|\tilde{\xi}|^2+\tilde{x}\cdot\tilde{\xi}}-yt)} \\ \times \int_{y'<y} \frac{t}{\xi_1} \frac{\varphi_+(y',|\xi|)}{[\varphi_+(y',|\xi|),\varphi_-(y',|\xi|)]} V\widehat{u} + f(\xi,y)dy'dt d\tilde{\xi}.$$

Here we are using  $\xi, \xi_1$  to stand for  $\xi(t, \tilde{\xi}), \xi_1(t, \tilde{\xi})$  for notational simplicity. There are no stationary points of  $i(x_1\sqrt{t^2 + \lambda^2 - |\tilde{\xi}|^2 + \tilde{x} \cdot \tilde{\xi}} - yt)$  on the support of  $\chi(z/|z|)(1 - \Psi(\xi))$  so we may integrate by parts twice. The first boundary term adds to the boundary term from the integral over  $|\xi| \leq \lambda/c_+$  to give an element of  $\langle z \rangle^{-1/2+\epsilon} L^2(\mathbb{R}^n)$ , using Lemma 9.3. The remaining boundary term and the remaining integral both give elements of  $\langle z \rangle^{-1/2+\epsilon} L^2(\mathbb{R}^n)$ .

A similar argument gives the same result for  $\frac{\partial}{\partial|z|} \chi(1 - \Psi(D_x))u$ .  $\square$

We use the notation

$$\mathbb{S}_{\pm}^{n-1} = \{\omega = (\bar{\omega}, \omega_n) \in \mathbb{S}^{n-1} : \pm\omega_n > 0\}$$

and will use differential operators of the following type.

DEFINITION 10.1. — We shall say that a differential operator  $P$  is in  $\text{Diff}_r^l(\mathbb{R}^n)$  if  $P$  is a differential operator of the form

$$\sum_{\pm} \sum_{|\alpha|+j \leq l} b_j \left(\frac{1}{|z|}\right) a_{j,\alpha,\pm} \left(\frac{z}{|z|}\right) |z|^j \left(\frac{\partial}{\partial|z|} + i\lambda/c_{\pm}\right)^j D_{z/|z|}^{\alpha}$$

where  $b_j \in C_b^{\infty}([0, \infty))$ ,  $a_{j,\alpha,\pm} \in C_c^{\infty}(\mathbb{S}_{\pm}^{n-1})$  and  $D_{z/|z|}^{\alpha}$  is a differential operator of order  $|\alpha|$  in the  $z/|z|$  variables.

We shall make use of the following lemma. Its proof follows by a straightforward computation.

LEMMA 10.2. — If  $P \in \text{Diff}_r^l(\mathbb{R}^n)$  and  $b \in C^{\infty}(\mathbb{S}^{n-1})$ , then  $[\frac{\partial}{\partial|z|}, P] \in |z|^{-1}\text{Diff}_r^l(\mathbb{R}^n)$ ,  $[\frac{\partial}{\partial z/|z|}, P] \in \text{Diff}_r^l(\mathbb{R}^n)$ ,  $[|z|^{-m}b(z/|z|), P] \in |z|^{-m}\text{Diff}_r^{l-1}(\mathbb{R}^n)$ , and  $[\Delta, P] \in |z|^{-2}\text{Diff}_r^{l+1}(\mathbb{R}^n)$ . Here  $\frac{\partial}{\partial z/|z|}$  stands for a derivative in the  $z/|z|$  variables.

In order to construct the desired approximation, we shall use the following lemma. In practice, when we apply this lemma, the first term will be used in solving away the error, and the subsequent terms will be of lower

order. In particular, in applications  $h$  will vanish faster than  $g$  at infinity and so will  $\chi_{\pm}(\frac{\partial}{\partial|z|} + \frac{i\lambda}{c_{\pm}})g$ .

LEMMA 10.3. — Suppose  $(\Delta - \lambda^2/c_0^2)g = h$ ,  $P \in \text{Diff}_r^l(\mathbb{R}^n)$ , and  $\chi_{\pm}, v \in C^\infty(\mathbb{S}^{n-1})$  with the support of  $\chi_{\pm}$  contained in  $\mathbb{S}_{\pm}^{n-1}$ . Then

$$\begin{aligned} & (\Delta - \lambda^2/c_0^2) (|z|^{-j+1}v(z/|z|)\chi_{\pm}(z/|z|)Pg) \\ &= (2j - 2)|z|^{-j}v\chi_{\pm} \frac{-i\lambda}{c_{\pm}} Pg + (2j - 2)|z|^{-j}v\chi_{\pm}P(\frac{\partial}{\partial|z|} + \frac{i\lambda}{c_{\pm}})g \\ & \quad + |z|^{-j-1}v\chi_{\pm}(P_{l+1}g) + |z|^{-j-1}\nabla_0(v\chi_{\pm}) \cdot \nabla_0Pg \\ & \quad + |z|^{-j+1}v\chi_{\pm}Ph + \Delta(|z|^{-j+1}v\chi_{\pm})Pg \\ &= (2j - 2)|z|^{-j}v\chi_{\pm} \frac{-i\lambda}{c_{\pm}} Pg + |z|^{-j-1}P'_{l+1}g + |z|^{-j+1}v\chi_{\pm}Ph \end{aligned}$$

with  $P_{l+1}, P'_{l+1} \in \text{Diff}_r^{l+1}(\mathbb{R}^n)$ . Here  $\nabla_0$  stands for the gradient on  $\mathbb{S}^{n-1}$ .

Proof. — The proof follows from a straightforward computation, using Lemma 10.2. □

LEMMA 10.4. — If  $\chi_1, \chi_2 \in C_c^\infty(\mathbb{S}^{n-1})$  with  $\text{supp}\chi_1 \cap \text{supp}\chi_2 = \emptyset$ ,  $\Psi \in \mathcal{S}(\mathbb{R}^{n-1})$ , and

$$Au = \chi_1(z/|z|)\Psi(D_x)\chi_2(z/|z|)u,$$

then  $A : \langle z \rangle^\alpha L^2(\mathbb{R}^n) \rightarrow \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$  for any  $\alpha \in \mathbb{R}$ .

Proof. — The crucial observation is that if  $(x, y)/|(x, y)| \in \text{supp}\chi_1$  and  $(x', y)/|(x', y)| \in \text{supp}\chi_2$ , then  $|x - x'| \geq \beta|(x, y)|$ ,  $|x - x'| \geq \beta|(x', y)|$  for some  $\beta > 0$ .

The Schwartz kernel of  $\Psi(D_x)$  is given by  $(2\pi)^{1-n}\hat{\Psi}(x' - x)$ , where  $\hat{\Psi} \in \mathcal{S}(\mathbb{R}^{n-1})$ . Thus, for  $f \in L^2(\mathbb{R}^n)$ ,

$$\begin{aligned} |Af|(z) &= (2\pi)^{1-n}|\chi_1(z/|z|) \int \hat{\Psi}(x' - x)\chi_2((x', y)/|(x', y)|)f(x', y)dx'| \\ &\leq C|\chi_1(z/|z|) \int (1 + |x - x'|)^{-m-\alpha}(1 + |x - x'|)^{m+\alpha} \\ & \quad \hat{\Psi}(x' - x)\chi_2((x', y)/|(x', y)|)f(x', y)dx'| \\ &\leq C\langle z \rangle^{-m}(\int (1 + |x - x'|)^{2m+2\alpha}|\hat{\Psi}(x' - x)|^2dx')^{1/2} \\ & \quad (\int \langle (x', y) \rangle^{-2\alpha}|f(x', y)|^2dx')^{1/2} \\ &\leq C\langle z \rangle^{-m}(\int \langle (x', y) \rangle^{-2\alpha}|f(x', y)|^2dx')^{1/2} \end{aligned}$$

for any  $m$  (where the constant depends on  $m, \Psi$ , and  $\alpha$ ) and thus it follows that  $Af \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ .  $\square$

LEMMA 10.5. — *If  $\Psi \in C_c^\infty(\mathbb{R}^{n-1})$  and  $g \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$ , then  $\Psi(D_x)g \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$ . Suppose that  $D_z^\alpha g, D_z^\alpha P g \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$  for all  $P \in \text{Diff}_r^k(\mathbb{R}^n)$  and for all multi-indices  $\alpha$ . Then  $D_z^\alpha P \Psi(D_x)g \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$  for all  $P \in \text{Diff}_r^k(\mathbb{R}^n)$  and all multi-indices  $\alpha$ .*

*Proof.* — To show that if  $g$  is in  $\langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$ , then so is  $\Psi(D_x)g$ , we take the Fourier transform of  $\Psi(D_x)g$ :

$$\mathcal{F}(\Psi(D_x)g)(\eta, \tau) = \Psi(\eta)\mathcal{F}(g)(\eta, \tau)$$

where  $\mathcal{F}(h)(\eta, \tau)$  denotes the Fourier transform of  $h$  in all variables. Then, since  $\mathcal{F}(g) \in H^\beta(\mathbb{R}^n)$ , and  $\mathcal{F}(\Psi(D_x)g) \in H^\beta(\mathbb{R}^n)$ , we have  $\Psi(D_x)g \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$ .

We give an indication of the proof of the remainder of the lemma. Let  $k \geq 1$  and suppose  $D_z^\alpha g, D_z^\alpha P g \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$  for all  $P \in \text{Diff}_r^k(\mathbb{R}^n)$  and for all multi-indices  $\alpha$ . For  $\chi_+ \in C_c^\infty(\mathbb{S}_+^{n-1})$ , consider

$$\begin{aligned} & (2\pi)^{n-1} \chi_+(z/|z|) \left( |z| \frac{\partial}{\partial |z|} + \frac{i\lambda|z|}{c_+} \right) \Psi(D_x)g \\ &= \chi_+(z/|z|) \int \hat{\Psi}(x' - x) (x' \cdot \nabla_{x'} + y \frac{\partial}{\partial y} + i\lambda|(x', y)|/c_+) f(x', y) dx' \\ & \quad + i\lambda/c_+ \chi_+(z/|z|) \int \hat{\Psi}(x' - x) (|(x, y)| - |(x', y)|) f(x', y) dx' \\ & \quad + \chi_+(z/|z|) \int \sum \widehat{D_{\xi_j}} \Psi(x' - x) \frac{\partial}{\partial x'_j} f(x', y) dx'. \end{aligned}$$

Let  $\tilde{\chi} \in C_c^\infty(\mathbb{S}_+^{n-1})$  be 1 on the support of  $\chi$ . Then, using the first part of the lemma,

$$\begin{aligned} & \chi_+(z/|z|) \int \hat{\Psi}(x' - x) \tilde{\chi}(|(x', y)|/|(x, y)|) (x' \cdot \nabla_{x'} + y \frac{\partial}{\partial y} + i\lambda|(x', y)|/c_+) \\ & \quad f(x', y) dx' \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n) \end{aligned}$$

since  $\tilde{\chi}(x' \cdot \nabla_{x'} + y \frac{\partial}{\partial y} + i\lambda|(x', y)|/c_+) f(x', y) \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$ . By the same reasoning,

$$\chi_+(z/|z|) \int \sum \widehat{D_{\xi_j}} \Psi(x' - x) \frac{\partial}{\partial x'_j} f(x', y) dx' \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n).$$

Moreover, by Lemma 10.4,

$$\chi_+(z/|z|) \int \hat{\Psi}(x' - x)(1 - \tilde{\chi}((x', y)/|(x', y)|))(x' \cdot \nabla_{x'} + y \frac{\partial}{\partial y} + i\lambda|(x', y)|/c_+) f(x', y) dx' \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n).$$

To finish, note that  $\langle x - x' \rangle^m \hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|)$  is a bounded function of  $x, y$ , and  $x'$  for any  $m$ . Then, if  $\beta \geq 0$ ,

$$\begin{aligned} & \langle z \rangle^{2\beta} \int \hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|) f(x', y) dx'{}^2 \\ & \leq C \left| \int \langle (x', y) \rangle^\beta \langle x - x' \rangle^\beta \hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|) f(x', y) dx' \right|^2 \\ & \leq C \left( \int \langle (x', y) \rangle^{2\beta} |\hat{\Psi}(x - x')(|(x, y)| - |(x', y)|)| f(x', y)|^2 dx' \right) \\ & \quad \times \left( \int \langle x - x' \rangle^{2\beta} |\hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|)| dx' \right). \end{aligned}$$

Since  $\int \langle x - x' \rangle^{2\beta} |\hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|)| dx' < C$ , where we allow the constant  $C$  to change from line to line, we have

$$\begin{aligned} & \int \langle z \rangle^{2\beta} \int \hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|) f(x', y) dx'{}^2 dz \\ & \leq C \iint \langle (x', y) \rangle^{2\beta} |\hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|)| |f(x', y)|^2 dx' dz \\ & \leq C \iint \langle (x', y) \rangle^{2\beta} \left( \int (|\hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|)|) dx \right) |f(x', y)|^2 dx' dy \\ & \leq C \iint \langle (x', y) \rangle^{2\beta} |f(x', y)|^2 dx' dy \end{aligned}$$

where for the last inequality we used that  $|\int (|\hat{\Psi}(x' - x)(|(x, y)| - |(x', y)|)|) dx| < C$ . A similar argument can be used when  $\beta < 0$ , using instead in the first step that for  $\beta < 0$ ,  $\langle z \rangle^\beta \leq C \langle x - x' \rangle^{-\beta} \langle (x', y) \rangle^\beta$ .

A similar argument works for a derivative in the  $z/|z|$  direction, and the argument can be iterated to get the lemma. □

LEMMA 10.6. — *Let  $\phi, \phi_1 \in C_c^\infty(\mathbb{R})$ , with  $(1 - \phi)(1 - \phi_1) = 1 - \phi_1$  and  $\phi_1(y) = 1$  if  $|y| \leq y_M + 1$ . If  $D_z^\alpha(1 - \phi(y))g \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$  for  $|\alpha| \leq l < \beta - 1/2$ ,  $\phi(y)g \in \langle z \rangle^{-\beta} L^2(\mathbb{R}^n)$ ,  $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$ , and  $\Psi \in C_b^\infty(\mathbb{R}^{n-1})$  with  $\text{supp } \Psi \cap \{\xi : |\xi| = |\lambda|/c_\pm\} = \emptyset$ , then*

$$P\chi(z/|z|)(1 - \phi_1(y))\Psi(D_x)(\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} g \in \langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$$

for every  $\epsilon > 0$  and  $P \in \text{Diff}_r^l$ , with  $l < \beta - 1/2$ .

*Proof.* — We give the proof for  $\chi$  supported in  $y/|z| > 0$ ; the proof for  $\chi$  with support in  $y/|z| < 0$  is similar.

We use a cut-off function,  $\Psi_1 \in C_c^\infty(\mathbb{R})$  with  $\Psi_1(|\xi|) \equiv 1$  when  $|\xi| \leq \lambda/c_+$  and supported in a small neighborhood of that region, so that  $\text{supp}(\Psi_1(|\xi|)\Psi(\xi)) \subset \{|\xi| < \lambda/c_+\}$ . We write

$$(50) \quad (1 - \phi_1(y))\chi(z/|z|)\Psi(D_x)(\Delta - (\lambda - i0)^2c_0^{-2})^{-1}g \\ = (2\pi)^{1-n}(1 - \phi_1(y))\chi(z/|z|) \int e^{ix \cdot \xi} \Psi(\xi)(\Psi_1(|\xi|) + 1 - \Psi_1(|\xi|)) \\ (D_y^2 + |\xi|^2 - (\lambda - i0)^2c_0^{-2})^{-1} \hat{g}(\xi, \cdot)(y) d\xi.$$

The main contribution is

$$(51) \quad (2\pi)^{1-n}(1 - \phi_1(y))\chi(z/|z|) \\ \int e^{ix \cdot \xi} \Psi(\xi)\Psi_1(|\xi|)e^{ix \cdot \xi}(D_y^2 + |\xi|^2 - (\lambda - i0)^2c_0^{-2})^{-1} \hat{g}(\xi, \cdot)(y) d\xi.$$

Here we may write, for  $y > y_M$  and  $|\xi| < \lambda/c_+$ ,

$$(52) \quad (D_y^2 + |\xi|^2 - c_0^{-2}(\lambda - i0)^2)^{-1} \hat{g}(\xi, \cdot)(y) = e^{-iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}} \tilde{g}(\xi) + g_1(\xi, y)$$

where  $D_y^k(1 - \phi_1(y))g_1 \in L^2(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\beta+1}L^2(\mathbb{R}_y))$  for all  $k$ . Putting the first term of (52) into (51), we obtain

$$(2\pi)^{1-n}(1 - \phi_1(y))\chi(z/|z|) \int \Psi(\xi)\Psi_1(|\xi|)e^{ix \cdot \xi} e^{-iy\sqrt{\lambda^2/c_+^2 - |\xi|^2}} \tilde{g}(\xi) d\xi.$$

Note that if we apply  $\frac{\partial}{\partial|z|} + i\lambda/c_+$ , then the integrand vanishes on the critical set of the phase function. Since on the support of  $\Psi\Psi_1$ ,  $\tilde{g} \in H^l$ ,  $l$  an integer with  $l < \beta - 1/2$ , we can integrate by parts to see that we have an element of  $\langle z \rangle^{-1/2+\epsilon}L^2$ .

For the tangential derivatives (in the  $z/|z|$  directions), notice that if we have a derivative in a direction orthogonal to  $y$ , it commutes with  $((D_y^2 + |\xi|^2) - c_0^{-2}(\lambda - i0)^2)^{-1}$ . That is, for example,

$$\left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}\right) \\ \int e^{ix \cdot \xi} \Psi_1(|\xi|)\Psi(\xi) (D_y^2 + |\xi|^2 - (\lambda - i0)^2c_0^{-2})^{-1} \hat{g}(\xi, \cdot)(y) d\xi \\ = \int e^{ix \cdot \xi} \Psi_1(|\xi|) \\ \left(D_y^2 + |\xi|^2 - (\lambda - i0)^2c_0^{-2}\right)^{-1} \left(\xi_1 \frac{\partial}{\partial \xi_2} - \xi_2 \frac{\partial}{\partial \xi_1}\right) \Psi(\xi) \hat{g}(\xi, \cdot)(y) d\xi.$$

By the decay properties of  $g$  and the regularity and decay properties of  $(1 - \phi)g$ , this yields an element of  $\langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$  after multiplication by  $\chi(1 - \phi_1)$ , if  $\beta$  is greater than  $3/2$ .

After applying a derivative of the form  $y \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y}$ , as in the radial case the integrand vanishes on the critical set of the phase function, and so we can integrate by parts. This argument can be iterated up to  $l < \beta - 1/2$ .

The second term of (52) contributes to (51) an element of  $\langle z \rangle^{-\beta+1} H^\infty(\mathbb{R}^n)$ .

On the support of  $(1 - \Psi_1(|\xi|))$ ,  $((D_y^2 + |\xi|^2) - (\lambda - i0)^2 c_0^{-2})^{-1}$  is a smooth function of  $|\xi|$ , except near a finite number of points for which  $\lambda^2$  is an eigenvalue of  $c_0^2(D_y^2 + |\xi|^2)$ . Since the eigenfunctions of this operator are exponentially decreasing in  $y$  and  $\chi$  is supported in  $y/|z| > \delta > 0$  for some  $\delta > 0$ , these eigenfunctions do not contribute to the asymptotics here. Projecting off the eigenfunctions, we have

$$(1 - \Psi_1)(D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} \Pi_e : \langle y \rangle^{-\beta} L^2(\mathbb{R}_y) \rightarrow \langle y \rangle^{-\beta} L^2(\mathbb{R}_y)$$

with bound  $C(|\xi|^2 - C)^{-2}$ . Therefore,

$$P(1 - \phi_1(y))\chi(z/|z|) \int \Psi(\xi)(1 - \Psi_1(|\xi|))(D_y^2 + |\xi|^2 - (\lambda - i0)^2 c_0^{-2})^{-1} \hat{g}(\xi, \cdot)(y) d\xi \in \langle z \rangle^{-\beta+l} H^\infty(\mathbb{R}^n)$$

where we used the fact that the inverse Fourier transform is an isomorphism on  $L^2$ , and the regularity properties of  $D_z^\alpha(1 - \phi(y))g$ . □

We shall also need the following lemma.

LEMMA 10.7. — *If  $w(x, y) \in \langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$ ,  $\Psi \in C_b^\infty(\mathbb{R}^{n-1})$ , and  $\text{supp } \hat{w}(\xi, y) \cap \text{supp } \Psi = \emptyset$ , then*

$$\Psi(\xi) \widehat{Vw}(\xi, y) \in H^\infty(\mathbb{R}_\xi^{n-1}; \langle y \rangle^{-\infty} L^2(\mathbb{R}_y)).$$

*Proof.* — Observe that

$$(53) \quad \Psi(\xi) \widehat{Vw}(\xi, y) = \Psi(\xi) \int \int e^{-ix \cdot \xi} V(x, y) e^{ix \cdot \eta} \hat{w}(\eta, y) d\eta dx$$

$$(54) \quad = \sum_j \Psi(\xi) \int \int \Psi_j(\xi, \eta) (\xi_j - \eta_j)^{-1} e^{-ix \cdot (\xi - \eta)} D_{x_j} V(x, y) \hat{w}(\eta, y) d\eta dx$$

where  $\Psi_j$  is a partition of unity with  $\xi_j \neq \eta_j$  on  $\text{supp}\Psi_j$ . We may repeat this integration by parts as many times as desired. Since  $|D_x^\alpha V(x, y)| \leq C_\alpha \langle z \rangle^{-J-|\alpha|}$ , the lemma follows.  $\square$

**10.3. Proof of Theorem 1.3 in case hypothesis (H2) holds.**

The following lemma will provide the first step in the successive approximations that will allow us to show that the leading order coefficient of  $\chi(z/|z|)\Psi(D_x)(\Delta - (\lambda - i0)^2 c^{-2})^{-1} f$  is smooth.

LEMMA 10.8. — *Let  $J \geq 4$ ,  $f \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ ,  $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$  for some  $\phi \in C_c^\infty(\mathbb{R})$ , and let  $u = (\Delta - (\lambda - i0)^2 c^{-2})f$ . If  $\Psi_0 \in C_b^\infty(\mathbb{R})$  is 0 in a neighborhood of  $|\xi| = \lambda/c_\pm$  and  $\chi_0 \in C_c^\infty(\mathbb{S}_c^{n-1})$ , then there is a  $w_0 = \sum_\pm \tilde{\chi}_{n\pm} \sum_{j=0}^{J-3} |z|^{-j-J+1} P_j \Psi_0(D_x)u$ , with  $P_j \in \text{Diff}_r^j(\mathbb{R}^n)$ , such that*

$$(\Delta - \lambda^2/c_0^2)w_0 = \chi_0(z/|z|)V\Psi_0(D_x)u + e_0$$

where  $e_0 \in \langle z \rangle^{-2J+5/2+\epsilon} L^2(\mathbb{R}^n)$ . Moreover,  $\text{supp } w_0(z) \subset \text{supp}\chi_0(z/|z|)$ .

*Proof.* — Let  $\chi_0 = \chi_+ + \chi_-$ , with  $\chi_\pm$  supported in  $\pm y > 0$ . We will outline the proof for  $\chi_0 = \chi_+$ , as the proof for  $\chi_0 = \chi_-$  is similar, and the functions can be added to get the general case.

Recall that on the support of  $\chi_+$ ,

$$V = \lambda^2(c^{-2} - c_0^{-2}) \sim \sum_{j \geq J} |z|^{-j} v_j \left( \frac{z}{|z|} \right).$$

We find  $w = \sum_{j=0}^{J-3} w_{0j}$ . Let

$$w_{00} = (1 - \phi_1(y))|z|^{-J+1} v_J \frac{ic_+}{\lambda(2J - 2)} \chi_+ \Psi_0(D_x)u$$

with  $\phi_1 \in C_c^\infty(\mathbb{R})$ ,  $\phi_1 \phi = \phi$ , and  $\phi_1(y) = 1$  for  $|y| \leq y_M$ . Then, by Lemma 10.3

$$(\Delta - \lambda^2/c_0^2)w_{00} = |z|^{-J} v_J \left( \frac{z}{|z|} \right) \chi_+ \Psi_0(D_x)u + |z|^{-J-1} P'_1 \Psi_0(D_x)u + |z|^{-J+1} \Psi_0(D_x)(Vu + f) + e_t$$

where  $P'_1 \in \text{Diff}_r^1(\mathbb{R}^n)$  and  $e_t \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ , so that  $(\Delta - \lambda^2/c_0^2)w_{00} = \chi_+ V \Psi(D_x)u + e_{00}$ ,  $e_{00} \in \langle z \rangle^{-J-1/2+\epsilon} L^2(\mathbb{R}^n)$ .

In the same manner, we choose  $w_{01}$  so that

$$(\Delta - \lambda^2/c_0^2)w_{01} = |z|^{-J-1} v_{J+1} \left( \frac{z}{|z|} \right) \chi_+ \Psi_0(D_x)u - |z|^{-J-1} P'_1 \Psi_0(D_x)u + |z|^{-J-2} P'_2 \Psi_0(D_x)u + e_{01r}$$

with  $P'_2 \in \text{Diff}_r^2(\mathbb{R}^n)$  and  $e_{01r} \in \langle z \rangle^{-2J+1/2+\epsilon} L^2(\mathbb{R}^n)$ . Then  $(\Delta - \lambda^2/c_0^2)(w_{00} + w_{01}) = V\chi + \Psi(D_x) + e_{01}$ ,  $e_{01} \in \langle z \rangle^{-J-3/2+\epsilon} L^2(\mathbb{R}^n)$ . This can be continued, with  $w_{0j}$  removing the terms in  $\langle z \rangle^{-J+1/2+\epsilon-j} L^2(\mathbb{R}^n)$ , modulo terms in  $\langle z \rangle^{-J-1/2+\epsilon-j} L^2(\mathbb{R}^n)$ , up to  $j = J - 3$ .  $\square$

PROPOSITION 10.2. — Suppose  $J \geq 4$ ,  $\chi \in C_c^\infty(\mathbb{S}_c^{n-1})$ ,  $\Psi \in C_c^\infty(\mathbb{R}^{n-1})$  has  $\text{supp}\Psi \cap \{\xi : |\xi| = |\lambda|/c_\pm\} = \emptyset$ ,  $\text{supp}\Psi \cap \{\xi : |\xi| = \kappa_j(\lambda)\} = \emptyset$ ,  $j = 1, 2, \dots, T(\lambda)$ , and if  $(x, y)/|(x, y)| \in \text{supp}(1 - \chi)$ ,  $y > 0$ , then  $\pm\lambda x/(c_+|z|) \notin \text{supp}\Psi$ . Moreover, suppose if  $(x, y)/|(x, y)| \in \text{supp}(1 - \chi)$ ,  $y < 0$ , then  $\pm\lambda x/(c_-|z|) \notin \text{supp}\Psi$ , and  $f \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^{n-1})$ ,  $(1 - \phi(y))f \in \mathcal{S}(\mathbb{R}^n)$  for some  $\phi \in C_c^\infty(\mathbb{R})$ . Then

$$\begin{aligned} \chi(z/|z|)\Psi(D_x)(\Delta - (\lambda - i0)^2/c^2)^{-1}f \\ = e^{-i\lambda|z|/c_0}\chi(z/|z|)|z|^{-(n-1)/2}(a_0(z/|z|) + \mathcal{O}(|z|^{-1})) \end{aligned}$$

with  $\chi a_0 \in C_c^\infty(\mathbb{S}_c^{n-1})$ .

Proof. — Recall that if  $u = (\Delta - (\lambda - i0)^2/c^2)^{-1}f$ , then  $u = (\Delta - (\lambda - i0)^2/c_0^2)^{-1}(Vu + f)$ . If  $\Psi(\xi)\widehat{Vu} + f(\xi, y)$  were in  $C^\infty(\mathbb{R}^{n-1}; \langle y \rangle^{-\infty} L^2(\mathbb{R}))$ , then using

$$(55) \quad \begin{aligned} (\Delta - (\lambda - i0)^2/c_0^2)^{-1}g \\ = (2\pi)^{1-n} \int e^{ix \cdot \xi} ((|\xi|^2 + D_y^2 - (\lambda - i0)^2/c_0^2)^{-1}\hat{g}(\xi, \cdot)) (y)d\xi \end{aligned}$$

and stationary phase, we would be done. However, it is not clear that  $\widehat{Vu}$  is in  $C^\infty(\mathbb{R}^{n-1}; \langle y \rangle^{-\infty} L^2(\mathbb{R}))$ . We shall show that  $\chi\Psi(D_x)u$  can be written as a sum of two terms: one vanishing faster than  $u$  at infinity, and another of the form  $(\Delta - (\lambda - i0)^2/c_0^2)^{-1}g_k$ , where  $\Psi(\xi)\hat{g}_k(\xi, y) \in C^k(\mathbb{R}^{n-1}; \langle y \rangle^{-k} L^2(\mathbb{R}))$ , where we can make  $k$  as large as desired. Then (55) and stationary phase will finish the proof.

To do this, we follow an iterative procedure. The first step has been done in Lemma 10.8. We will iteratively construct functions  $w_l$  which have the property that  $(\Delta - \lambda^2/c_0^2)(u - w_l)$  improves with increasing  $l$  in an appropriate sense.

Let  $\Psi_0, \Psi_1, \Psi_2, \dots \in C_c^\infty(\mathbb{R}^{n-1})$  be such that, for all  $i$ ,  $\Psi_i \equiv 1$  on the support of  $\Psi$ ,  $\Psi_{i+1}\Psi_i = \Psi_{i+1}$ , and  $\Psi_i$  satisfies the support requirements placed on  $\Psi$  in the statement of the Proposition. Let  $\chi_0, \chi_1, \chi_2, \dots \in C_c^\infty(\mathbb{S}_c^{n-1})$  be such that  $\chi_0\chi = \chi$  and  $\chi_{i+1}\chi_i = \chi_i$ ,  $i = 0, 1, 2, \dots$ . Let  $w_0$  be the function constructed in Lemma 10.8 for this  $\Psi_0$  and  $\chi_0$ . Using the notation of that lemma, let

$$t_0 = (1 - \chi_0)V\Psi_0(D_x)u + f - e_0 + V(1 - \Psi_0(D_x))u.$$

Note that the first three terms are in  $\langle z \rangle^{-2J+5/2+\epsilon} L^2(\mathbb{R}^n)$ , where we use the support properties of  $\chi_0$  and  $\Psi_0$  along with an integration by parts argument as in the proof of Lemma 10.1 to obtain the result for the first term. Additionally, the support of the Fourier transform in the  $x$  variables of  $(1 - \Psi_0(D_x))u$  is disjoint from the support of  $\Psi_1$ . Then, using Lemmas 10.6 and 10.7, we have

$$P\chi_1\Psi_1(D_x)(\Delta - (\lambda - i0)^2/c_0^2)^{-1}t_0 \in \langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n),$$

$$\text{for all } P \in \text{Diff}_r^l(\mathbb{R}^n), l \leq 2J - 4.$$

Since  $u = w_0 + (\Delta - (\lambda - i0)^2/c_0^2)^{-1}t_0$ , which can be seen by Proposition 4.3 and the uniqueness result (Proposition 6.2), this in turn means that  $P\Psi_1(D_x)w_0 \in \langle z \rangle^{-J+3/2+\epsilon} L^2(\mathbb{R}^n)$  for all  $P \in \text{Diff}_r^l$ ,  $l \leq 2J - 4$ , using Lemma 10.5.

We now iteratively construct  $w_l$  for  $l \geq 1$  such that

$$\begin{aligned} (\Delta - \lambda^2/c_0^2)w_l &= V\chi_l\Psi_l(D_x)u + e_l \\ &= V\chi_l\Psi_l(D_x)w_{l-1} + V\chi_l\Psi_l(D_x)(\Delta - (\lambda - i0)^2/c_0^2)^{-1}t_{l-1} + e_l, \end{aligned}$$

where  $e_l \in \langle z \rangle^{-J+1/2+\epsilon-l(J-2)} L^2(\mathbb{R}^n)$  and  $t_l$  is defined by

$$\begin{aligned} t_l &= V((1 - \chi_l)\Psi_l(D_x) + (1 - \Psi_l(D_x)))w_{l-1} \\ &\quad + V((1 - \Psi_l(D_x)) + (1 - \chi_l)\Psi_l(D_x))(\Delta - (\lambda - i0)^2/c_0^2)^{-1}t_{l-1} - e_l + f. \end{aligned}$$

Then

$$u = w_l + (\Delta - (\lambda - i0)^2/c_0^2)^{-1}t_l.$$

Moreover,  $t_l = t'_l + t''_l$ , where  $\Psi_{l+1}(\xi)\hat{t}'_l(\xi, y) \in H^\infty(\mathbb{R}_\xi^{n-1}, \langle y \rangle^{-\infty} L^2(\mathbb{R}))$ ,  $t''_l \in \langle z \rangle^{-(l+1)(J-2)-J+1/2+\epsilon} L^2(\mathbb{R}^n)$ , and  $P\Psi_{l+1}(D_x)w_l \in \langle z \rangle^{-J+3/2+\epsilon} L^2(\mathbb{R}^n)$  for all  $P \in \text{Diff}_r^m(\mathbb{R}^n)$ ,  $m \leq (l + 2)(J - 2)$ . Additionally,  $\text{supp } w_l(z) \subset \text{supp } \chi_l(z/|z|)$ .

Supposing that  $w_{l-1}, t_{l-1}$  are as above, we show how to construct  $w_l$ . Since

$$P\chi_l\Psi_l(D_x)w_{l-1} \in \langle z \rangle^{-J+3/2+\epsilon} L^2(\mathbb{R}^n)$$

$$\text{and } P\chi_l\Psi_l(D_x)(\Delta - (\lambda - i0)^2/c_0^2)^{-1}t_{l-1} \in \langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$$

for all  $P \in \text{Diff}_r^{(l+1)(J-2)}$ , we can, just as in Lemma 10.8, find  $w_l = \sum_{j=0}^{(l+1)(J-2)-1} w_{lj}$  so that

$$(\Delta - \lambda^2/c_0^2)w_l = V\chi_l\Psi_l(D_x)w_{l-1} + V\chi_l\Psi_l(D_x)(\Delta - (\lambda - i0)^2/c_0^2)^{-1}t_{l-1} + e_l$$

with  $e_l \in \langle z \rangle^{-J+1/2+\epsilon-(l+1)(J-2)} L^2(\mathbb{R}^n)$ . Let

$$t'_l = V(1 - \Psi_l(D_x))(w_l + (\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} t_{l-1});$$

then  $\Psi_{l+1}(\xi) \hat{t}'_l(\xi, y) \in H^\infty(\mathbb{R}_\xi^{n-1}, \langle y \rangle^{-\infty} L^2(\mathbb{R}))$  by Lemma 10.7. Since  $\text{supp } w_{l-1}(z) \subset \text{supp } \chi_{l-1}(z/|z|)$ , by Lemma 10.4,  $V(1 - \chi_l) \Psi_l(D_x) w_{l-1} \in \langle z \rangle^{-\infty} L^2(\mathbb{R}^n)$ , and

$$V(1 - \chi_l) \Psi_l(D_x) (\Delta - (\lambda - i0)^2 c_0^{-2})^{-1} t_{l-1} \in \langle z \rangle^{-(l+1)(J-2)-1/2-J+\epsilon} L^2(\mathbb{R}^n)$$

using the support properties of  $\Psi_l$  and  $\chi_l$  and an integration by parts argument as in Lemma 10.6. Thus  $t''_l = t_l - t'_l \in \langle z \rangle^{-(l+1)(J-2)-J+1/2+\epsilon} L^2(\mathbb{R}^n)$ .

Note that

$$\begin{aligned} w_{lj} &= |z|^{-j-J+1} P_{lj}(1 - \phi(y)) \Psi_l(D_x) [w_{l-1} + (\Delta - (\lambda - i0)^2/c_0^2)^{-1} t_{l-1}] \\ (56) \qquad \qquad \qquad &= |z|^{-j-J+1} P_{lj}(1 - \phi(y)) \Psi_l(D_x) u, \end{aligned}$$

with  $P_{lj} \in \text{Diff}_r^j(\mathbb{R}^n)$ ,  $\text{supp } P_{lj} \subset \text{supp } \chi_l(z/|z|)$ ,  $\phi \in C_c^\infty(\mathbb{R})$ . Since  $(\Delta - \lambda^2/c_0^2)(u - w_l) = t_l$ , we have

$$(57) \qquad \qquad \qquad u = w_l + (\Delta - (\lambda - i0)^2/c_0^2)^{-1} t_l.$$

Using the properties of  $t'_l$  and  $t''_l$ , we obtain that  $P\Psi_{l+1}(D_x)(\Delta - (\lambda - i0)^2/c_0^2)^{-1} t_l \in \langle z \rangle^{1/2+\epsilon} L^2(\mathbb{R}^n)$  for  $P \in \text{Diff}_r^{(l+2)(J-2)}(\mathbb{R}^n)$ . Using (56), (57), and Lemma 10.5, this in turn means that  $P\Psi_{l+1}(D_x) w_l \in \langle z \rangle^{3/2+\epsilon-J} L^2(\mathbb{R}^n)$  for all  $P \in \text{Diff}_r^{(l+2)(J-2)}(\mathbb{R}^n)$ . Thus, for any  $l \geq 1$ ,  $w_l$  and  $t_l$  can be constructed to have the desired properties.

To prove the proposition, we use (57). Since  $t_l = t'_l + t''_l$  with  $\Psi_{l+1}(\xi) \hat{t}'_l(\xi, y) \in C^\infty(\mathbb{R}_\xi^{n-1}, \langle y \rangle^{-\infty} L^2(\mathbb{R}_y))$ ,  $t''_l \in \langle z \rangle^{-(l+1)(J-2)-J+1/2+\epsilon} L^2(\mathbb{R}^n)$ , we have  $\Psi(\xi) \hat{t}_l(\xi, y) \in H^s(\mathbb{R}_\xi^{n-1}, \langle y \rangle^{-(l+1)(J-2)+1/2+\epsilon-J+s} L^2(\mathbb{R}_y))$  for  $s < (l+1)(J-2) + J - 1/2 - \epsilon$ . Then, using equation (55) and stationary phase, we see that

$$\chi \Psi(D_x) (\Delta - (\lambda - i0)^2/c_0^2)^{-1} t_l = \chi e^{-i\lambda|z|/c} |z|^{-(n-1)/2} (a_0(\frac{z}{|z|}) + \mathcal{O}(|z|^{-1}))$$

with  $a_0 \in C^{(l+1)(J-2)+J-n-4}(\mathbb{S}_c^{n-1})$  when  $l$  is sufficiently large.

To finish, then, we only need show that  $\Psi(D_x) w_l$  is of order  $|z|^{-(n+1)/2}$ . But we recall that

$$w_l, D_{z/|z|}^\alpha w_l, |z|(\frac{\partial}{\partial |z|} + i\lambda/c) D_{z/|z|}^\alpha w_l \in \langle z \rangle^{-J+3/2+\epsilon} L^2(\mathbb{R}^n).$$

Therefore  $\Psi(D_x) w_l = \mathcal{O}(|z|^{-(n-1)/2-J+1+\epsilon})$ . □

*Proof of Theorem 1.3 in case hypothesis (H1) holds.* — Since

$$\begin{aligned} \chi(\Delta - c^{-2}(\lambda - i0)^2)^{-1}f &= \chi\Psi(D_x)(\Delta - c^{-2}(\lambda - i0)^2)^{-1}f \\ &\quad + \chi(1 - \Psi(D_x))(\Delta - c^{-2}(\lambda - i0)^2)^{-1}f, \end{aligned}$$

the theorem follows by choosing  $\Psi$  as in Lemma 10.1, and then applying Proposition 10.2 and Lemma 10.1.  $\square$

**Appendix A. The absolute scattering operator  $A(\lambda)$  is bounded on  $L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$ .**

In this appendix we prove that  $A(\lambda)$  can be extended to a map  $L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2}) \rightarrow L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$ .

For  $g, h \in L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$ , let

$$\begin{aligned} \langle g, h \rangle_{\mathcal{H}(\lambda)} &= \int_{\omega \in \mathbb{S}_c^{n-1}} g_0(\omega) \overline{h_0(\omega)} + \sum_1^{T(\lambda)} \int_{\omega' \in \mathbb{S}^{n-2}} f_j(\omega') \overline{g_j(\omega')}, \\ \langle g, h \rangle_{\lambda, \mathcal{H}(\lambda)} &= \sum_{\pm} \int_{\omega \in \mathbb{S}_c^{n-1}; \pm\omega_n > 0} \frac{\lambda}{c_{\pm}} g_0(\omega) \overline{h_0(\omega)} \\ &\quad + \sum_1^{T(\lambda)} \kappa_j(\lambda) \int_{\omega' \in \mathbb{S}^{n-2}} f_j(\omega') \overline{g_j(\omega')}. \end{aligned}$$

First, we consider what happens for  $A_{c_0}(\lambda)$ , the absolute scattering matrix associated to the operator  $c_0^2\Delta$ , defined as for  $A(\lambda)$ .

LEMMA A.1. — For  $g, h \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$ ,  $\lambda \in \mathbb{R} \setminus \{0\}$ ,

$$\langle g, h \rangle_{\lambda, \mathcal{H}(\lambda)} = \langle A_{c_0}(\lambda)g, A_{c_0}(\lambda)h \rangle_{\lambda, \mathcal{H}(\lambda)}.$$

*Proof.* — We use  $P_{c_0}(\lambda)$  to stand for the Poisson operator associated to  $c_0^2\Delta$ . Then

$$\begin{aligned} 0 &= \lim_{R \rightarrow \infty} \int_{|z| < R} (c_0^2\Delta - \lambda^2)P_{c_0}g \overline{P_{c_0}h} dz \\ &= - \lim_{R \rightarrow \infty} \int_{|z|=R} \left( \frac{\partial}{\partial|z|} P_{c_0}g \overline{P_{c_0}h} - P_{c_0}g \frac{\partial}{\partial|z|} \overline{P_{c_0}h} \right) \\ &= -2i\langle g, h \rangle_{\lambda, \mathcal{H}(\lambda)} + 2i\langle A_{c_0}(\lambda)g, A_{c_0}(\lambda)h \rangle_{\lambda, \mathcal{H}(\lambda)}. \end{aligned}$$

For the last equality we have used the definition of  $P_{c_0}(\lambda)$  and  $A_{c_0}(\lambda)$ , and the uniform asymptotic expansion of  $P_{c_0}g, P_{c_0}h$  that follows from the results of this paper and [5].  $\square$

As a consequence of this lemma, we have that  $A_{c_0}(\lambda)$  can be extended to a map from  $L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$  to itself, with norm bounded by  $\sqrt{\lambda/(c_+ \kappa_1(\lambda))}$  if  $T(\lambda) \geq 1$  and  $\sqrt{|\lambda|/c_+}$  otherwise.

Now, a method similar to the proof of Lemma A.1 shows that

LEMMA A.2. — For  $g, h \in C_c^\infty(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} C^\infty(\mathbb{S}^{n-2})$ ,  $\lambda \in \mathbb{R} \setminus \{0\}$ , we have

$$\int P(\lambda)g(c^2\Delta - \lambda^2)\overline{P_{c_0}(-\lambda)h}dz = 2i\langle g, A_{c_0}(-\lambda)h \rangle_{\lambda, \mathcal{H}(\lambda)} - 2i\langle A(\lambda)g, h \rangle_{\lambda, \mathcal{H}(\lambda)}.$$

Now, because  $P_{c_0}(\lambda), P(\lambda): L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2}) \rightarrow \langle z \rangle^{1/2+\epsilon} H^2(\mathbb{R}^n)$  are bounded, we obtain

$$|\langle g, A_{c_0}(-\lambda)h \rangle_{\lambda, \mathcal{H}(\lambda)} - \langle A(\lambda)g, h \rangle_{\lambda, \mathcal{H}(\lambda)}| \leq C (\langle g, g \rangle_{\mathcal{H}(\lambda)} \langle h, h \rangle_{\mathcal{H}(\lambda)})^{1/2}.$$

This must hold for any  $h \in L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$ , using the fact that  $A_{c_0}(\lambda)$  is bounded and invertible ( $(A_{c_0}(\lambda))^{-1} = A_{c_0}(-\lambda)$ ). Therefore,  $A(\lambda)g \in L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$ , and we obtain

$$\langle A(\lambda)g, A(\lambda)g \rangle_{\mathcal{H}(\lambda)} \leq C \langle g, g \rangle_{\mathcal{H}(\lambda)}$$

so that  $A(\lambda)$  can be extended to a bounded operator on all of  $L^2(\mathbb{S}_c^{n-1}) \oplus_1^{T(\lambda)} L^2(\mathbb{S}^{n-2})$ .

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