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MAXIMAL FUNCTIONS AND CAPACITIES

by Lennart CARLESON

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1. Let $f(x)$ be periodic with period 2π and assume $f(x) \in L^p(-\pi, \pi)$, some $p \geqslant 1$. The maximal function $f^*(x)$ associated with $f(x)$ was introduced by Hardy and Littlewood through the definition

$$(1.1) \qquad f^*(x) = \sup_t \frac{1}{t} \int_x^{x+t} f(u)\, du.$$

The inequalities

$$(1.2) \qquad \int_{-\pi}^{\pi} |f^*(x)|^p\, dx \leqslant A_p \int_{-\pi}^{\pi} |f(x)|^p\, dx, \qquad p > 1,$$

and

$$(1.3) \qquad m\{x | f^*(x) \geqslant \lambda\} \leqslant \frac{A}{\lambda} \int_{-\pi}^{\pi} |f(x)|\, dx$$

are basic in the theory of differentiation. (1.2) can alternatively be given as a theorem on harmonic functions. Assume $f > 0$ and let $u(z)$ be harmonic in $|z| < 1$ with boundary values $f(\theta)$. Then clearly

$$(1.4) \qquad \text{const.} f^*(\theta) \leqslant \sup_r u(re^{i\theta}) \leqslant \text{const.} f^*(\theta).$$

The inequality (1.2) follows if we can characterize those non-negative measures μ for which

$$(1.5) \qquad \iint_{|z|<1} u(z)^p\, d\mu(z) \leqslant A_p \int_{-\pi}^{\pi} f(x)^p\, dx.$$

It is sufficient to consider $p = 2$ and the complete solution was given in [3]: a necessary and sufficient condition on μ, is

$$\mu(S) \leqslant \text{const.}\, s$$

for every set $S : 1 - s < |z| < 1$, $|\arg(z) - \alpha| < s$.

The corresponding linear problem, i.e., to describe those μ for which

$$(1.6) \qquad \iint u(z)\, d\mu(z)$$

is bounded for $f \in L^p$ is clearly much simpler and the solution is that

$$\varphi(\theta) = \iint \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, d\mu(z) \tag{1.7}$$

belongs to L^q.

Although this result is in principle sufficient for differentiation purposes, it is of little help since no simple geometric characterization of μ seems to be available.

We shall now consider the corresponding problem for the class of functions $f(x)$,

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx},$$

such that

$$\|f\|_K^2 = \Sigma |c_n|^2 \lambda_{|n|} < \infty.$$

Here $\{\lambda_n\}$ is a positive sequence such that

$$K(x) \sim \sum_{0}^{\infty} \frac{\cos nx}{\lambda_n}$$

is a convex function $\in L^1$. The following theorem is quite easy to prove.

THEOREM 1.—*If* $\lambda_n = (n+1)^{1-\alpha}$, $0 \leqslant \alpha < 1$, (1.6) *is bounded if and only if*

$$E_\alpha(\mu) = \iint \frac{d\mu(a)\, d\mu(b)}{|1 - \bar{a}b|^\alpha} < \infty, \qquad 0 < \alpha < 1,$$

$$E_0(\mu) = \iint \log \left| \frac{1}{1 - \bar{a}b} \right| d\mu(a)\, d\mu(b) < \infty, \qquad \alpha = 0.$$

The bound of (1.6) *is* $\leqslant$ const. $\sqrt{E_\alpha}$.

If we specialize $d\mu$ to have the form $d\sigma(\theta)$ placed at a point on the radius from 0 to $e^{i\theta}$ we find using (1.4) and observing that $E_\alpha(\mu)$ essentially increases if we push the masses out to $|z| = 1$

$$\left(\int f^*(x)\, d\sigma(x) \right)^2 \leqslant A_\alpha \|f\|_K^2 I_\alpha(\sigma) \tag{1.8}$$

where I_α is the energy of σ with respect to the kernel $|x|^{-\alpha}$, resp. $\log \frac{1}{|x|}$. This inequality implies easily the existence of derivatives and boundary-values except on sets of capacities zero. This is a result by Beurling [1] and Broman [2].

The proof of Theorem 1 in the case $\alpha = 0$ is particularly simple. Consider first the case when μ has its support strictly inside $|z| < 1$. Consider the harmonic function

$$u_0(z) = \iint \log|1 - z\bar{\zeta}|\, d\mu(\zeta)$$

and let (u, v) denote scalar product in the space of harmonic functions with finite Dirichletintegral and with $u(0) = 0$. Then by Poisson's formula

$$(u, u_0) = \int_{|z|=1} u \frac{\partial u_0}{\partial n}\, ds = 2\pi \iint u(z)\, d\mu(z).$$

Hence

$$2\pi \left| \iint u\, d\mu \right| \leqq \| u_0 \| \cdot \| u \|$$

with equality if $u = u_0$, and the linear functional (1.6) has norm $(2\pi)^{-\frac{1}{2}}\sqrt{E_0(\mu)}$. The case of a general μ follows immediately.

The restriction $u(0) = 0$, i.e., $\int f\, dx = 0$, is clearly inessential. Let us also observe that we here (as well as in Section 2) also may restrict ourselves to $f > 0$ since $|f(x)|$ has a smaller norm than f (see (2.1)).

In the case $0 < \alpha < 1$ we write

$$\int u(a)\, d\mu(a) = \int f(\theta)\, d\theta \frac{1}{2\pi} \int \frac{1 - |a|^2}{|e^{i\theta} - a|^2}\, d\mu(a) = \int f(\theta) g(\theta)\, d\theta.$$

The function $v(r, \theta)$ harmonic in $|z| < 1$ with boundary values $g(\theta)$ is

$$(1.9) \qquad v(r, \theta) = \frac{1}{2\pi} \int \frac{1 - |a|^2 r^2}{|e^{i\theta} - ar|^2}\, d\mu(a) = \Sigma\, b_n r^{|n|} e^{in\theta}.$$

We wish to prove

$$(1.10) \qquad \iint v(r, \theta)^2 (1 - r)^{-\alpha}\, dr\, d\theta < \infty,$$

since this inequality is equivalent to $\Sigma\, |b_n|^2 (|n| + 1)^{\alpha - 1} < \infty$. Inserting (1.9) in (1.10) we see that (1.10) holds if $E_\alpha(\mu) < \infty$.

2. It is clearly possible to use the same method for general kernels $K(x)$ and corresponding weights λ_n. However the formulas become so involved that they cannot be used to deduce inequalities of the form (1.8). Of particular interest is the case

$$\lambda_n = (\log(n + 2))^\alpha, \qquad 0 < \alpha < \infty.$$

For functions f with corresponding $\|f\|_K$ finite and $0 < \alpha < 1$, nothing is known on convergence of Fourier series and no better result on existence of derivatives than Lebesgue's theorem. The kernel K_α that is associated with this sequence is

$$K_\alpha(x) \sim \frac{1}{|x|(\log 1/|x|)^{1+\alpha}}, \qquad x \to 0.$$

The following theorem holds

THEOREM 2. — *There is a constant* B_α *such that*

$$C_{K_\alpha}\Big[\{x | f^*(x) \geqslant \lambda\}\Big] \leqslant \frac{B_\alpha}{\lambda^2}\|f\|^2_{K_\alpha}, \qquad 0 < \alpha < \infty.$$

By standard methods this implies that the primitive function of f has a derivative except on a set of K_α-capacity zero. It is interesting to compare this result with what is known on convergence of Fourier series. It has been proved by Temko [4], that if $\|f\|_{K_{\alpha+1}} < \infty$ then the Fourier series converges except on a set of K_α-capacity zero, while we here get a stronger result on existence of boundary values.

In the proof we use the equivalent norm

$$(2.1) \qquad \iint_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{\varphi(x-y)}\, dx\, dy, \qquad \varphi(t) = |t|\left(\log \frac{8}{|t|}\right)^{1-\alpha}$$

and the following potential theoretic lemma:

LEMMA. — *If σ is an interval of length d on $(-\pi, \pi)$, denote by $T\sigma$ an interval of length $3d$ and having the same midpoint as σ. We assume that $\{\sigma_\nu\}$ are disjoint and denote by $E = \cup\sigma_\nu$ and $E' = \cup T\sigma_\nu$. Then there is a constant Q only depending on K such that*

$$C_K(E') \leqslant QC_K(E)$$

provided $K(x) = O(K(2x)), x \to 0$.

In an outline, the proof of theorem 2 proceeds as follows. Let $\sigma_{\nu n}$ denote the 2^n disjoint intervals of length $2\pi . 2^{-n}$ on $(-\pi, \pi)$. Let λ be given and denote by $M_\alpha(f)$ the mean value of f over the interval α. We choose intervals $\sigma_1, \sigma_2, \ldots,$ such that

$$(2.2) \qquad M\sigma_\nu(f) \geqq \lambda$$

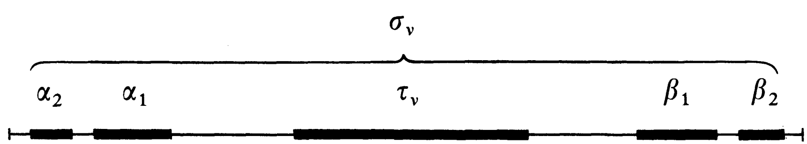
by first choosing those $\sigma_{\nu 1}$ that satisfy (2.2), then $\sigma_{\mu 2}$ disjoint from those chosen before, etc. It follows easily from the lemma that it is sufficient to prove $C\{\cup\sigma_\nu\} \leqslant$ const. $\|f\|^2 . \lambda^{-2}$.

Let τ_ν be intervals such that $T\tau_\nu = \sigma_\nu$. We want to construct $f_1(x)$ such that $\|f_1\| \leqslant$ const. $\|f\|$ and $f_1(x) \equiv M\sigma_\nu(f)$, $x \in \tau_\nu$. We first modify f on each σ_ν according to the following rule where we have normalized σ_ν to $(-1, 1)$:

$$f_2(x) = \begin{cases} f(2x), & -\frac{1}{2} < x < \frac{1}{2} \\ f(-x - \frac{3}{2}), & -\frac{3}{4} < x < -\frac{1}{2} \\ f(x), & -1 < x < -\frac{3}{4} \\ \text{analogously on } (\frac{1}{2}, 1). & \end{cases}$$

Outside $\cup\sigma_\nu$ we define $f_2(x) = f(x)$. From (2.1) it follows that $\|f_2\|_K \leqslant$ const. $\|f\|_K$.

Let 4δ be the length of the shortest of the intervals σ_ν. We have the following picture:



where we construct α_i and β_i until their length $< \delta$. α_i and β_i have lengths $= 3^{-i-1}$ (length σ_ν). We define

$$f_1(x) = \begin{cases} M_{\tau_\nu}(f_2) = M_{\sigma_\nu}(f), & x \in \tau_\nu\,; \\ M_{\alpha_i}(f_2), & x \in \alpha_i\,; \\ M_{\beta_i}(f_2), & x \in \beta_i\,; \\ \text{linear between the intervals.} & \end{cases}$$

We do the same construction on each σ_ν and each complementary interval. A computation in (2.1) shows that $\|f_1\| <$ const. $\|f_2\|$.

To complete the proof, let μ be a distribution of unit mass on $E'' = \cup\tau_\nu$. Then

$$\lambda \leqslant \int_{E''} f_2(x)\, d\mu(x) \leqslant \|f_2\|_K \,.\, I_K(\mu)^{\frac{1}{2}} \leqslant \text{const. } \|f\|_K \,.\, I_K(\mu)^{\frac{1}{2}}.$$

The lemma now yields theorem 2.

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