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## COMPUTING EXPLICITLY TOPOLOGICAL SEQUENCE ENTROPY: THE UNIMODAL CASE

by V. JIMÉNEZ LÓPEZ and J.S. CÁNOVAS PEÑA

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### 1. Introduction and statement of the results.

There are many tools to deal with the idea of “complex dynamical behaviour” for the family  $C(I)$  of continuous maps on a compact interval  $I$ . Among them topological entropy enjoys a steady popularity, one of the reasons being that it can be used as an indicator of the “size” of this dynamical complexity which, contrary to measure-theoretic approaches, is preserved under topological conjugacy.

DEFINITION 1.1 (see [1], [8]). — *Let  $(X, d)$  be a compact metric space, let  $f : X \rightarrow X$  be continuous and let  $A = (a_m)_{m=1}^{\infty}$  denote a (non necessarily strictly, but unbounded) increasing sequence of positive integers.*

*Let  $Y \subset X$  and set  $\epsilon > 0$ . We say that a set  $E \subset Y$  is  $(A, m, \epsilon, Y, f)$ -separated (by  $f$ ) if for any  $x, y \in E$ ,  $x \neq y$ , there exists  $i \in \{1, 2, \dots, m\}$  such that  $d(f^{a_i}(x), f^{a_i}(y)) > \epsilon$ . Denote by  $s_m(A, \epsilon, Y, f)$  the biggest cardinality of any  $(A, m, \epsilon, Y, f)$ -separated set in  $Y$  and write*

$$s(A, \epsilon, Y, f) = \limsup_{m \rightarrow \infty} \frac{1}{m} \log s_m(A, \epsilon, Y, f).$$

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The topological sequence entropy of  $f$  on the set  $Y$  (respect to the sequence  $A$ ) is defined by

$$h_A(f, Y) = \lim_{\epsilon \rightarrow 0} s(A, \epsilon, Y, f).$$

Finally we define the topological sequence entropy of  $f$  (respect to the sequence  $A$ ) as

$$h_A(f) = h_A(f, X).$$

If  $A = (m)_{m=1}^{\infty}$  then we get the standard topological entropy  $h(f)$ .

In this regard, the interest of having means to calculate (or at least to approximate) topological entropy in a useful way becomes obvious. In the setting of piecewise monotone maps such a mean is (sometimes) provided by symbolic dynamics, as follows from classical papers by Milnor and Thurston [19] and Misiurewicz and Szlenk [22] (cf. also the previous but little known work by Rothschild [23]).

It is interesting to recall the substance of the above results in the simpler case of *unimodal* maps, that is, maps from  $C(I)$ ,  $I = [a, b]$ , satisfying  $f(a) = f(b) \in \{a, b\}$  and for which there is a point  $c \in (a, b)$  such that the restrictions of  $f$  to  $[a, c]$  and  $[c, b]$  are strictly monotone.

With the notation above, associate to any point  $x \in I$  its *itinerary*  $\iota_f(x) = (\iota_n)_{n=0}^{\infty} \in \{-1, 0, 1\}^{\infty}$  defined by

$$\iota_n = \begin{cases} 1 & \text{if } f^n(x) < c, \\ 0 & \text{if } f^n(x) = c, \\ -1 & \text{if } f^n(x) > c, \end{cases}$$

and then the sequence  $\theta_f(x) = (\theta_n)_{n=0}^{\infty} \in \{-1, 0, 1\}^{\infty}$  given by

$$\theta_n = \iota_0 \iota_1 \cdots \iota_n \quad \text{for any } n.$$

It can be proved that the map  $x \mapsto \theta_f(x)$  is monotone (when  $\{-1, 0, 1\}^{\infty}$  is ordered with the lexicographic order) and then that the limit

$$\theta_f(x^-) := \lim_{y \uparrow x} \theta_f(y)$$

(when  $\{-1, 0, 1\}^{\infty}$  is endowed with the topology of pointwise convergence) exists for any  $x \in (a, b]$ . In particular let us consider

$$\theta_f(c^-) =: (\theta_n^c)_{n=0}^{\infty} \in \{-1, 1\}^{\infty}.$$

The formal power series

$$\nu_f := \sum_{n=0}^{\infty} \theta_n^c t^n$$

(or sometimes the sequence  $\theta_f(c^-)$  itself) is called the *kneading invariant* of  $f$  and, when  $t$  is seen as a real variable, it converges for all  $|t| < 1$ . Let  $r(f)$  be the smallest zero of  $\nu_f$  in  $(0, 1)$  or, if no such a number exists, let  $r(f) = 1$ .

THEOREM 1.2 (see [19], [22], [23]). — *Let  $f \in C(I)$  be unimodal. Then*

$$h(f) = \log \left( \frac{1}{r(f)} \right).$$

While dynamical complexity is certainly guaranteed for positive entropy maps, entropy zero maps need not be that “simple”: for instance some of them are chaotic (in the sense of Li and Yorke) [16], [14]. Recall that a map  $f \in C(I)$  is said to be *chaotic* in this sense if it possesses two points  $x, y$  satisfying simultaneously

$$\liminf_{n \rightarrow \infty} |f^n(x) - f^n(y)| = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| > 0.$$

A natural and more accurate lens to look at these maps is topological sequence entropy, which is emphasized by the following result by Franzová and Smítal:

THEOREM 1.3 (see [7]). — *Let  $f \in C(I)$ . Then it is chaotic if and only if its topological sequence entropy  $h_A(f)$  is positive respect to some appropriate increasing sequence  $A$ .*

In this context we must concentrate our attention on maps of type  $2^\infty$ , since they are the only entropy zero maps which are apt to enclose chaos [24], [15] (cf. also [25], pp. 73–74), [4], [20]. Recall here that  $p$  is a *periodic point* for  $f$  if there is some (minimal) number  $k \geq 1$  (which is called the *period* of  $p$ ) satisfying  $f^k(p) = p$ , and that  $f$  is said to be of *type  $2^\infty$*  if it has periodic points of periods exactly all powers of 2. Here the use of the sequence  $D = (2^{m-1})_{m=1}^\infty$  suggests itself. For instance the following was conjectured in [7] (and later disproved in [10]):

- (1) *Let  $f \in C(I)$ . Then it is chaotic if and only if  $h_D(f) > 0$ .*

The main aim of this paper is to calculate explicitly topological sequence entropy respect to the sequence  $D$  for unimodal maps of type  $2^\infty$ . Let us emphasize that if  $f \in C(I)$  has positive topological entropy then  $h_D(f) = \infty$  by [8], while if  $h(f) = 0$  but  $f$  is not of type  $2^\infty$  then it cannot be chaotic so  $h_D(f) = 0$  by Theorem 1.3.

In fact we will work in the slightly larger setting of weakly unimodal maps. Let  $I = [a, b]$ . A (non-constant) map  $f \in C(I)$  will be called *weakly unimodal* if  $f(a) = f(b) \in \{a, b\}$  and there is  $c \in (0, 1)$  such that the restrictions of  $f$  to  $[a, c]$  and  $[c, b]$  are (non necessarily strictly) monotone; hence, although  $f$  is not constant, it is allowed to have some constant pieces. We emphasize that the restriction  $f(a) = f(b) \in \{a, b\}$  is just cosmetic and could be easily removed. The (possibly degenerate) compact interval containing all absolute extrema of  $f$  will be called its *turning interval*. The family of weakly unimodal maps of type  $2^\infty$  will be denoted by  $W(I)$ . While in the multimodal case new and non-negligible technical difficulties arise, we do not expect essential differences to happen; we hope to deal with this general case in a future paper.

The problem here is that, contrary to the usual topological entropy case, standard symbolic dynamics seems now to be useless as they do not clearly emphasize the specific features of type  $2^\infty$  maps. Instead we will use an alternative approach introduced in [12], an extension of the well known idea of the “adding machine” [9] (although here it makes more sense to speak about a “subtracting” machine), the essential of which we presently recall (see also Section 2). Let us emphasize that the reader can find (sometimes implicit) proofs of all non-trivial results on maps from  $W(I)$  stated below (except of course those concerning sequence entropy) in [12].

As it is well known, weakly unimodal maps of type  $2^\infty$  are *infinitely renormalizable*, which means that there is a sequence  $I_1 \supset I_2 \supset I_3 \supset \dots$  of compact intervals, all containing the turning interval of  $f$ , such that, for any  $m \geq 1$ ,  $f^{2^m}(I_m) \subset I_m$  and its restriction  $f^{2^m}|_{I_m}$  belongs to  $W(I_m)$  (we will assume that the intervals  $I_m$  are minimal with this property). Moreover they possess *solenoidal structure*, that is, the intervals  $f^i(I_m)$  have pairwise disjoint interiors,  $i = 0, 1, \dots, 2^m - 1$ , and if we write

$$S_m(f) := \bigcup_{i=0}^{2^m-1} f^i(I_m), \quad m \geq 1,$$

then for any  $x \in I$  one of the following alternatives must occur: (a)  $x$  is *asymptotically periodic*, that is, there is a periodic point  $p$  such that

$\lim_{n \rightarrow \infty} |f^n(x) - f^n(p)| = 0$ ; (b) the orbit  $(f_n(x))_{n=0}^\infty$  of  $x$  eventually falls into  $S_m(f)$  for any  $m \geq 1$  and then its  $\omega$ -limit set (the set of limit points of its orbit) is infinite and included in the solenoid

$$S(f) := \bigcap_{m=1}^\infty S_m(f).$$

Next let us associate to any point  $x \in S(f)$  the sequence

$$\alpha_f(x) = (\alpha_i)_{i=1}^\infty \in \{0, 1\}^\infty$$

having for any  $m$  the property that  $\alpha_1 + 2\alpha_2 + \dots + 2^{m-1}\alpha_m$  is the first number  $k$  satisfying  $f^k(x) \in I_m$ . It turns out that this sequence is well defined, the map  $x \mapsto \alpha_f(x)$  is surjective and for any  $\alpha \in \{0, 1\}^\infty$  the set  $K_\alpha(f)$  of points  $x$  satisfying  $\alpha_f(x) = \alpha$  is a (possibly degenerate) compact interval. For instance, observe that  $K_0(f)$  includes (but not necessarily coincides with) the turning interval of  $f$ ,  $\mathbf{0} = (0, 0, \dots, 0, \dots)$ . Moreover, define a subinterval of  $I$  to be *essential (for  $f$ )* if it is a non-degenerate interval of the type  $K_\alpha(f)$  for which the sequence  $\alpha$  contains an infinite number of zeros. We have

**THEOREM 1.4** (see [12]). — *Let  $f \in W(I)$ . Then it is chaotic if and only if it possesses an essential interval.*

Since chaos and positive topological sequence entropy are so closely related, it is reasonable to expect the number  $h_D(f)$  to depend on the nature of the set  $\mathcal{N}(f) \subset \{0, 1\}^\infty$  of sequences  $\alpha$  for which  $K_\alpha(f)$  is non-degenerate. As we will immediately see, this is the case indeed.

*Remark 1.5.* — Essential intervals can be described in a somewhat simpler way. Recall that a *wandering interval* of  $f$  is an interval whose iterates are pairwise disjoint and which does not have any asymptotically periodic points. If the union set of all  $\omega$ -limit sets of  $f$  is denoted by  $\omega(f)$  then it can be proved that  $J \subset I$  is an essential interval of  $f \in W(I)$  if and only if it is a wandering interval whose both endpoints are in  $\omega(f)$ . Notice that the role of unimodality is not apparent here. In fact if a map  $f \in C(I)$  of type  $2^\infty$  is chaotic then it must possess a wandering interval (see e.g. [3]); we conjecture that a map  $f \in C(I)$  of type  $2^\infty$  is chaotic if and only if it possesses a wandering interval whose both endpoints are in  $\omega(f)$ .

To clarify the situation we need to introduce for any  $\alpha \in \{0, 1\}^\infty$  a kind of “symbolic entropy” which, essentially, inform us about the proportion of zeros in  $\alpha$  and how early they appear. Below,  $\text{Card } T$  denotes the cardinality of a set  $T$ .

DEFINITION 1.6. — Let  $\alpha = (\alpha_m)_{m=1}^\infty \in \{0, 1\}^\infty$ . For any  $m$  let  $\mathcal{S}(\alpha, m)$  be the family of subsets  $S$  of  $\{1, 2, \dots, m\}$  having for any  $1 \leq i \leq m$  the property that  $\text{Card}(S \cap \{1, 2, \dots, i\})$  is at most the number of zeros of the sequence  $(\alpha_1, \dots, \alpha_i)$ . Then the entropy  $h_\alpha$  of the sequence  $\alpha$  is defined by

$$h_\alpha = \limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{S}(\alpha, m).$$

For instance observe that if  $\alpha = (1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, \dots)$  then  $\mathcal{S}(\alpha, 11)$  includes all subsets from  $\{1, 2, \dots, 11\}$  simultaneously containing no points from  $\{1, 2, 3\}$ , at most one point from  $\{1, 2, \dots, 5\}$ , at most two points from  $\{1, 2, \dots, 9\}$  and at most three points from  $\{1, 2, \dots, 11\}$ .

We are now ready to state the first main result of the paper:

THEOREM A. — Let  $f \in W(I)$ . Then  $h_D(f) = \sup_{\alpha \in \mathcal{N}(f)} h_\alpha$ .

In the statement of Theorem A we implicitly assume  $\mathcal{N}(f) \neq \emptyset$ ; recall that if  $\mathcal{N}(f) = \emptyset$  then  $h_D(f) = 0$  by Theorems 1.3 and 1.4.

Notice that  $\text{Card } \mathcal{S}(\alpha, m) \leq 2^m$  for any  $\alpha$ . This implies that if  $f \in W(I)$  then  $h_D(f) \leq \log 2$  (more generally, in [6] has been recently proved that  $h_A(f) \leq \log 2$  for any sequence  $A$  and any piecewise monotone map  $f \in C(I)$ ), which is similar to the well known fact that  $h(f) \leq \log 2$  for any unimodal map. Observe also that  $\text{Card } \mathcal{S}(\mathbf{0}, m) = 2^m$ , so:

COROLLARY A.1. — If  $f \in W(I)$  has a non-degenerate turning interval then  $h_D(f) = \log 2$ .

Admittedly it may be uncomfortable to compute  $h_\alpha$  for an infinite number of sequences  $\alpha$ ; moreover  $\mathcal{N}(f)$ , though countable, is usually an infinite set. In a sense this cannot be helped, as for any countable subset  $\Lambda \subset \{0, 1\}^\infty$  there is a map  $f_\Lambda \in W(I)$  such that  $\Lambda \subset \mathcal{N}(f_\Lambda)$ . On the other hand there are fortunately many sequences having the same entropy. For instance, if we define in  $\{0, 1\}^\infty$  the pseudometric

$$(2) \quad \blacktriangle(\alpha, \beta) = \limsup_{m \rightarrow \infty} \left| \text{mean}(\alpha_1 - \beta_1, \alpha_2 - \beta_2, \dots, \alpha_m - \beta_m) \right|$$

(where  $\text{mean}(x_1, x_2, \dots, x_m)$  denotes the arithmetic mean of the numbers  $x_1, x_2, \dots, x_m$ ) then  $\blacktriangle(\alpha, \beta) = 0$  implies  $h_\alpha = h_\beta$ , see Corollary B.1 below (incidentally, the map  $f_\Lambda$  above may be constructed having no constant pieces and so that for any  $\alpha \in \mathcal{N}(f_\Lambda)$  there is  $\beta \in \Lambda$  such that  $\blacktriangle(\alpha, \beta) = 0$ ; hence  $h_D(f_\Lambda) = \sup_{\alpha \in \Lambda} h_\alpha$ ). In particular, if  $f \in W(I)$  and  $\blacktriangle(\alpha, \beta) = 0$  for any  $\alpha, \beta \in \mathcal{N}(f)$  then  $h_D(f) = h_\alpha$  for any  $\alpha \in \mathcal{N}(f)$ .

Generally speaking, maps  $f_\Lambda$  above need not be (even piecewise) differentiable. At this point one may be tempted to wonder whether computing sequence entropy via Theorem A makes any serious sense, specially taking into account that for many “natural” maps from  $W(I)$  (including all analytic ones) wandering intervals cannot exist [17] and then their sequence entropies are automatically zero. Moreover, as far as (at least) one significant class of maps is concerned (those consisting of a finite number of non-constant affine pieces), maps of type  $2^\infty$  *cannot* even exist, see [13], [18]. It is important then to stress that

$$f(x) = \min\{1 - |2x - 1|, \rho\}, \quad \rho = 0.8249\dots,$$

is still a very “natural” map belonging to  $W([0,1])$  [21] for which  $h_D(f) = \log 2$  by Corollary A.1. Let us also remark that combining some results from [12] and [17] it can be proved that if  $f \in W(I)$  consists of a finite number of “smooth” (possibly constant) pieces (the word “smooth” is used here in the same sense as in [17], p. 277) then for any  $\alpha \in \mathcal{N}(f)$  there is  $\beta \in \mathcal{N}(f)$  such that  $\blacktriangle(\alpha, \beta) = 0$  and  $f$  is constant on  $K_\beta(f)$ . Hence we can compute  $h_D(f)$  just evaluating a finite number of entropies  $h_\beta$ .

Let us return now to the general continuous case. According to Theorem A the computation of  $h_D$  depends on that of the corresponding entropies  $h_\alpha$ : our next result explains how to do it. Before stating it we must introduce some necessary notation.

For any  $m \geq 1$  let us define inductively the maps  $F_m: \mathbb{R}^m \rightarrow \mathbb{R}^m$  as follows. We begin by writing  $F_1(x) = x$  for any  $x$ . Next assume that the maps  $F_i$  have been already defined for any  $1 \leq i < m$  and define  $F_m(x_1, x_2, \dots, x_m) = (y_1, y_2, \dots, y_m)$  by

$$y_1 = y_2 = \dots = y_r = \text{mean}(x_1, x_2, \dots, x_r) \quad \text{and} \\ (y_{r+1}, y_{r+2}, \dots, y_m) = F_{m-r}(x_{r+1}, x_{r+2}, \dots, x_m),$$

where  $r$  is the (maximal) number  $1 \leq k \leq m$  at which  $\text{mean}(x_1, x_2, \dots, x_k)$

attains its minimum value. For instance

$$\begin{aligned} F_5(0.2, 0.4, 0.6, 0.2, 0.1) &= (0.2, 0.325, 0.325, 0.325, 0.325), \\ F_5(0.4, 0.2, 0.6, 0.2, 0.1) &= (0.3, 0.3, 0.3, 0.3, 0.3), \\ F_5(0.2, 0.35, 0.479, 0.5, 0.665) &= (0.2, 0.35, 0.479, 0.5, 0.665), \\ F_5(0.6, 0.45, 0.45, 0.3, 0.2) &= (0.4, 0.4, 0.4, 0.4, 0.4), \\ F_5(0.6, 0.45, 0.45, 0.2, 0.3) &= (0.4, 0.4, 0.4, 0.4, 0.4). \end{aligned}$$

Next let  $g : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$g(y) = \begin{cases} 0 & \text{if } y = 0, \\ -y \log y - (1 - y) \log(1 - y) & \text{if } 0 < y < \frac{1}{2}, \\ \log 2 & \text{if } \frac{1}{2} \leq y \leq 1 \end{cases}$$

(we emphasize that  $g$  is continuous, concave and increasing), and write

$$G_m(y_1, y_2, \dots, y_m) = \text{mean}(g(y_1), g(y_2), \dots, g(y_m)), \quad m \leq 1.$$

Now we have

**THEOREM B.** — *Let  $\alpha \in \{0, 1\}^\infty$ . Then*

$$(3) \quad h_\alpha = \limsup_{m \rightarrow \infty} G_m(F_m(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_m)).$$

Theorem B has a number of useful, relatively straightforward (but non-trivial) consequences which allows us to approximate or compute easily  $h_\alpha$  in many cases. We listed them below:

**COROLLARY B.1.** — *Let  $\epsilon > 0$  and let  $\delta_\epsilon$  be such that  $|x - y| \leq \delta_\epsilon$  implies  $|g(x) - g(y)| \leq \epsilon$  for any  $x, y \in [0, 1]$ . Then  $\blacktriangle(\alpha, \beta) \leq \delta_\epsilon$  implies  $|h_\alpha - h_\beta| \leq \epsilon$  for any  $\alpha, \beta \in \{0, 1\}^\infty$ . In particular, if  $\blacktriangle(\alpha, \beta) = 0$  then  $h_\alpha = h_\beta$ .*

**COROLLARY B.2.** — *Let  $\alpha \in \{0, 1\}^\infty$  and assume that*

$$\lambda = \lim_{m \rightarrow \infty} \text{mean}(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_m)$$

*exists. Then  $h_\alpha = g(\lambda)$ .*

COROLLARY B.3. — Let  $\alpha \in \{0, 1\}^\infty$  and

$$\lambda = \limsup_{m \rightarrow \infty} \text{mean}(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_m).$$

Let  $\kappa(\lambda)$  be the only positive solution of the equation

$$\left(\frac{\lambda + \kappa}{\lambda}\right)^\lambda = \frac{\lambda + \kappa}{\kappa}.$$

Then

$$(4) \quad \log\left(\frac{\lambda + \kappa(\lambda)}{\kappa(\lambda)}\right) \leq h_\alpha \leq g(\lambda);$$

moreover, both inequalities are sharp.

Regarding the last corollary, (4) means of course just  $h_\alpha = 0$  in the case  $\lambda = 0$ . It is easy to check that  $\kappa(\lambda) \geq \lambda$  for any  $\lambda$  (hence  $h_\alpha \geq \lambda \log 2$ ). In particular Corollary B.3 implies  $h_\alpha = \log 2$  when  $\limsup_{m \rightarrow \infty} \text{mean}(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_m) = 1$ , thus improving the corresponding result from Corollary B.2.

*Remark 1.7.* — As it is well known, the set of sequences converging to  $\frac{1}{2}$  has full measure in  $\{0, 1\}^\infty$  (when it is endowed with the usual product measure) so, in this sense,  $h_\alpha = \log 2$  for almost all sequences  $\alpha$ .

*Remark 1.8.* — Curiously enough, if  $(X, \mathcal{B}, \mu)$  is a probability space and  $\mathcal{P} = \{P_1, P_2\}$  is a partition of  $(X, \mathcal{B}, \mu)$  with corresponding measures  $\mu(P_1) = \lambda$ ,  $\mu(P_2) = 1 - \lambda$ ,  $\lambda \leq \frac{1}{2}$ , then the so-called *entropy*  $H(\mathcal{P})$  of the partition  $\mathcal{P}$  is defined to be precisely the number  $g(\lambda)$  (see e.g. [26], pp. 77–80).

Theorems 1.2 and A+B are worth comparing. Although the first one is of course “visually” much simpler, computing the zeros of the kneading invariant of  $f$  may be rather a difficult task (compare e.g. with Corollary B.2). It is true, however, that one can use well known continuity and monotonicity tools to compute approximately standard topological entropy in a very efficient way (cf. e.g. [5] and the references therein), tools which unfortunately are not available here. For instance we can modify slightly the construction in [11] to find a  $C^\infty$  map  $f \in W(I)$  having a non-degenerate turning interval and then satisfying  $h_D(f) = \log 2$  (resp. having no wandering intervals and then satisfying  $h_D(f) = 0$ ) and  $C^\infty$  maps  $g \in W(I)$  arbitrary close to  $f$  (in any fixed  $C^k$ -topology,  $0 \leq k < \infty$ ) having no wandering intervals (resp. having a non-degenerate turning interval). Hence  $h_D$  is not continuous in any reasonable sense. Regarding symbolic entropy things are much better, as Corollary B.1 emphasizes.

As we said before, if  $\alpha$  is fixed then there is a map  $f \in W(I)$  such that  $h_D(f) = h_\alpha$ . In particular, if  $\alpha$  is chosen to have an infinite number of zeros but satisfying  $\blacktriangle(\alpha, \mathbf{1}) = 0$ ,  $\mathbf{1} = (1, 1, \dots, 1, \dots)$ , then  $h_D(f) = 0$  by Corollary B.1 but it is chaotic because it has an essential interval. Hence Franzová and Smítal's conjecture (1) fails even in the setting of unimodal maps (Hric's counterexample [10] consists of infinitely many monotone pieces).

The contents of this paper can be briefly summarized as follows. In Section 2 the above-mentioned symbolic dynamics for maps from  $W(I)$  are described in more detail. Section 3 takes into account the specific properties of the sequence  $D$  and develops some necessary technical tools which are used to prove Theorem A. After some preparatory work in Section 4, Theorem B and its corollaries are proved in Section 5. We emphasize that  $D$  will denote the sequence  $(2^{m-1})_{m=1}^\infty$  throughout the paper. As usual we write  $\mathbb{N}$ ,  $\mathbb{Z}^-$ ,  $\mathbb{Z}$  and  $\mathbb{R}$  to describe the sets of positive integers, negative integers, integers and real numbers, respectively.  $\text{Cl } T$  will denote the closure of the set  $T$  and  $|J|$  will be the length of the interval  $J$ ;  $[x]$  will denote the integer part of the real number  $x$ .

## 2. Symbolic dynamics for maps from $W(I)$ .

If  $Z$  is a set then, as usual,  $Z^m$  (resp.  $Z^\infty$ ) will denote the set of finite sequences of length  $m$  (resp. infinite sequences) of elements from  $Z$ . If  $\theta \in \mathbb{Z}^m$  or  $\alpha \in \mathbb{Z}^\infty$  then we will often describe them through their components as  $(\theta_1, \dots, \theta_m)$  or  $(\alpha_i)_{i=1}^\infty$ , respectively. Also, if  $\alpha \in \mathbb{Z}^m$  with  $m \in \mathbb{N} \cup \{\infty\}$ , and  $n \in \mathbb{N}$  with  $n \leq m$  (of course we mean  $n \leq \infty$  for any  $n$ ) then the sequence  $\alpha|_n \in \mathbb{Z}^n$  is defined by

$$\alpha|_n = (\alpha_1, \dots, \alpha_n).$$

We remark that in the case  $m = 1$  we will indistinctly use the notations  $\theta = (\theta_1) = \theta_1 = \theta|_1$ . If  $\theta \in \mathbb{Z}^m$  and  $\alpha \in \mathbb{Z}^n$  with  $m \in \mathbb{N}$  and  $n \in \mathbb{N} \cup \{\infty\}$ , then  $\theta * \alpha \in \mathbb{Z}^{m+n}$  (where  $m + \infty$  means  $\infty$ ) will denote the sequence  $\beta$  defined by  $\beta|_m = \theta|_m$  and  $\beta_i = \alpha_{i-m}$  for any  $i > m$ . The *shift map*  $\sigma : \mathbb{Z}^\infty \rightarrow \mathbb{Z}^\infty$  is defined by  $\sigma((\alpha_i)_{i=1}^\infty) = (\alpha_{i+1})_{i=1}^\infty$ . For the sake of notation, if  $\beta$  is a sequence then we will use sometimes the "empty sequence"  $\beta|_0$  when, e.g.,  $\beta|_0 * \alpha$  is just the sequence  $\alpha$ .

Let  $f \in W(I)$ ,  $I = [a, b]$ , and let  $AP(f)$  be the set of asymptotically periodic points of  $f$ . In [12] was showed that it is possible to construct a

family  $\{K_\alpha(f)\}_{\alpha \in \mathbb{Z}^\infty}$  (or just  $\{K_\alpha\}_{\alpha \in \mathbb{Z}^\infty}$  if there is no ambiguity on  $f$ ) of pairwise disjoint (possibly degenerate) compact subintervals of  $[0, 1]$  such that  $\bigcup_{\alpha \in \mathbb{Z}^\infty} K_\alpha = I \setminus AP(I)$  and satisfying the key properties (P1)–(P4) listed below. Recall that we denote

$$\mathbf{0} = (0, 0, \dots, 0, \dots) \quad \text{and} \quad \mathbf{1} = (1, 1, \dots, 1, \dots).$$

- (P1) Interval  $K_{\mathbf{0}}$  contains all absolute maxima of  $f$  in the case  $f(a) = f(b) = a$ , and all absolute minima of  $f$  in the case  $f(a) = f(b) = b$ .
- (P2) Define in  $\mathbb{Z}^\infty$  the following total ordering: if  $\alpha, \beta \in \mathbb{Z}^\infty$ ,  $\alpha \neq \beta$  and  $k$  is the first integer such that  $\alpha_k \neq \beta_k$ , then  $\alpha < \beta$  if either  $\text{Card}\{1 \leq i < k : \alpha_i \leq 0\}$  is even and  $\alpha_k < \beta_k$  or  $\text{Card}\{1 \leq i < k : \alpha_i \leq 0\}$  is odd and  $\beta_k < \alpha_k$ . Then  $\alpha < \beta$  if and only if  $K_\alpha < K_\beta$  (that is,  $x < y$  for all  $x \in K_\alpha$ ,  $y \in K_\beta$ ) in the case  $f(a) = f(b) = a$ , and  $\alpha < \beta$  if and only if  $K_\beta < K_\alpha$  in the case  $f(a) = f(b) = b$ .
- (P3) Let  $\alpha \in \mathbb{Z}^\infty$ ,  $\alpha \neq \mathbf{0}$ , and let  $k$  be the first integer such that  $\alpha_k \neq 0$ . Define  $\beta \in \mathbb{Z}^\infty$  by  $\beta_i = 1$  for  $1 \leq i \leq k - 1$ ,  $\beta_k = 1 - |\alpha_k|$  and  $\beta_i = \alpha_i$  for  $i > k$ . Then  $f(K_\alpha) = K_\beta$ . Also, we have  $f(K_{\mathbf{0}}) \subset K_{\mathbf{1}}$ .
- (P4) For any  $m$  and any  $\alpha \in \mathbb{Z}^\infty$ , let  $K_{\alpha|_m}(f)$  (or just  $K_{\alpha|_m}$ ) be the smallest interval including all intervals  $K_\beta$ ,  $\beta \in \mathbb{Z}^\infty$ , such that  $\alpha|_m = \beta|_m$ . Then  $K_\alpha = \bigcap_{m=1}^\infty K_{\alpha|_m}$ .

Now, for any fixed  $m$  it can be easily checked that the intervals  $K_\theta$ ,  $\theta \in \mathbb{Z}^m$ , are open and pairwise disjoint, and (after replacing  $\infty$  by  $m$ ,  $\mathbf{0}$  by  $\mathbf{0}|_m$  and  $\mathbf{1}$  by  $\mathbf{1}|_m$ ) they also satisfy (P1)–(P3). For instance we have  $K_{(0,3,1,-1,7,0)} < K_{(0,3,1,-1,7,1)}$  and  $K_{(0,3,1,-1,0,1)} < K_{(0,3,1,-1,0,0)}$  in the case  $f(a) = f(b) = a$ , while  $f(K_{(7,0,0,1,2,0)}) = K_{(-6,0,0,1,2,0)}$  and  $f(K_{(0,0,0,1,6,0)}) = K_{(1,1,1,0,6,0)}$ . In general, notice that if  $f$  maps  $K_\theta$  into  $K_\vartheta$ ,  $\theta, \vartheta \in \mathbb{Z}^m$ , then

$$\sum_{i=1}^m |\vartheta_i| 2^{i-1} = \left( \sum_{i=1}^n |\theta_i| 2^{i-1} \right) - 1$$

(except in the case  $\theta = \mathbf{0}|_m$ ,  $\vartheta = \mathbf{1}|_m$ ). This is the reason why we used the expression “subtracting machine” in the Introduction. We emphasize that  $\text{Cl} K_{\mathbf{0}|_m}$  are the intervals  $I_m$  we mentioned there. In fact, it is easy to verify that

$$S(f) = \bigcap_{m=1}^\infty \bigcup_{i=0}^{2^m-1} f^i(I_m) = \bigcap_{m=1}^\infty \bigcup_{\theta \in \{0,1\}^m} K_\theta = \bigcup_{\alpha \in \{0,1\}^\infty} K_\alpha.$$

Thus, if  $x \in K_\alpha$ ,  $\alpha \in \{0, 1\}^\infty$ , we have that  $\sum_{i=1}^m |\alpha_i| 2^{i-1}$  is the first number  $k$  satisfying  $f^k(x) \in I_m$ , just as we stated then.

We finish this section by listing a number of additional properties of the above sets we will need later. They can be routinely obtained from (P2), (P3) and/or (P4).

- (P5)  $K_\alpha$  is a wandering interval for any  $\alpha \in \mathbb{Z}^\infty$ .
- (P6) Let  $\theta \in \{0, 1\}^m$ ,  $m \in \mathbb{N}$ . Then  $f^{2^m}(K_\theta) \subset K_\theta$ . Moreover, if  $K = \text{Cl } K_\theta$  and  $g = f^{2^m}|_K$  then  $g \in W(K)$ . Further,  $K_\alpha(g) = K_{\theta * \alpha}$  for any  $\alpha \in \mathbb{Z}^\infty$ .
- (P7) Let  $\theta \in \{0, 1\}^m$ ,  $m \in \mathbb{N}$ , define  $\vartheta \in \{0, 1\}^m$  by  $\vartheta|_{m-1} = \theta|_{m-1}$  and  $\vartheta_m = 1 - \theta_m$  and put  $K_\theta^\# = K_\theta^\#(f) = K_\vartheta$ . Then  $f^{2^{m-1}}(K_\theta) \subset K_\theta^\#$ .
- (P8) Let  $x \in K_\theta$  for some  $\theta \in \{0, 1\}^m$ . If  $i \geq m$ ,  $\ell \geq 0$  and  $f^{2^i}(x) \in K_{\theta * 1|_{\ell * p}}$  for some  $p \in \mathbb{Z}$ , then we have  $p \in \mathbb{Z}^- \cup \{0, 1\}$ .

### 3. Proof of Theorem A.

Before speaking about Theorem A and its proof let us recall two facts concerning sequence entropy respect to an arbitrary sequence  $A$ . The first one was proved in [2].

LEMMA 3.1. — *Let  $(X, d)$  be a compact metric space and let  $f : X \rightarrow X$  be a continuous map. Let  $Y \subset X$  and  $\epsilon > 0$ . Then*

$$s(A, 2\epsilon, Y, f) \leq s(\sigma^k(A), \epsilon, Y, f) \leq s(A, \epsilon, Y, f)$$

for any  $k \in \mathbb{N}$  and any sequence  $A$ . In particular  $h_A(f) = h_{\sigma^k(A)}(f)$  for any  $k$  and any sequence  $A$ .

LEMMA 3.2. — *Let  $f \in W(I)$  and  $\epsilon > 0$ . Then for any  $k \in \mathbb{N}$  and any sequence  $A$  there is  $\theta \in \{0, 1\}^k$  such that*

$$s(A, 2\epsilon, K_\theta(f), f) \leq s(A, 2\epsilon, I, f) \leq s(A, \epsilon, K_\theta(f), f).$$

*Proof.* — Say  $A = (a_m)_{m=1}^\infty$  and fix  $k$ . As the first inequality is trivial, and we can prove

$$s\left(A, \epsilon, \bigcup_{\theta \in \{0, 1\}^k} K_\theta(f), f\right) = \max_{\theta \in \{0, 1\}^k} s(A, \epsilon, K_\theta(f), f)$$

reasoning as in Theorem 7.5 from [26], it suffices to show that

$$s(A, 2\epsilon, I, f) \leq s\left(A, \epsilon, \bigcup_{\theta \in \{0,1\}^k} K_\theta(f), f\right).$$

Indeed, let  $J$  be the smallest interval including  $f^{2^k}(K_0(f)) \cup f^k(K_0(f))$  and  $T = \bigcup_{i=0}^{2^k-1} f^i(J)$ . By (P1), (P3) and (P6) we have  $f^k(J) = J$  (so  $f(T) = T$ ) and  $K_{0|_{k+1}}(f) \subset J \subset K_{0|_k}(f)$ . Then, in particular, we could just prove

$$(5) \quad s(A, 2\epsilon, I, f) \leq s(A, \epsilon, T, f).$$

To do this, let  $U_m = \{x \in I : f^{a_m}(x) \notin T\}$  for any  $m \geq 0$  (we mean  $a_0 = 0$ ) and write  $T_m = U_{m-1} \setminus U_m$  for any  $m \geq 1$ . Since  $f(T) = T$ , we can use [6], Lemma 2.3, where it is implicitly proved that

$$s_m(A, 2\epsilon, T_i, f) \leq s_{i-1}(A, \epsilon, T_i, f) s_m(A, \epsilon, T, f), \quad 1 \leq i \leq m,$$

with  $s_0(A, \epsilon, T_i, f) = 1$ . Then we take  $I = T \cup (\bigcup_{i=1}^m T_i) \cup U_m$  into account to get

$$\begin{aligned} (6) \quad s_m(A, 2\epsilon, I, f) &\leq s_m(A, 2\epsilon, T, f) + \sum_{i=1}^m s_m(A, 2\epsilon, T_i, f) + s_m(A, 2\epsilon, U_m, f) \\ &\leq s_m(A, \epsilon, T, f) + \sum_{i=1}^m s_{i-1}(A, \epsilon, U_{i-1}, f) s_m(A, \epsilon, T_i, f) \\ &\quad + s_m(A, \epsilon, U_m, f) \\ &\leq s_m(A, \epsilon, T, f) \left(2 + \sum_{i=1}^m s_i(A, \epsilon, U_i, f)\right). \end{aligned}$$

Now, because  $K_{0|_{k+1}}(f) \subset J$  and then  $\bigcup_{\theta \in \{0,1\}^{k+1}} K_\theta(f) \subset T$ , we get that if  $x \in \bigcap_{m=1}^\infty U_m$  then it is asymptotically periodic, and we can repeat the proof of Lemma 2.4 from [6] to prove

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log \left( \sum_{i=1}^m s_i(A, \epsilon, U_i, f) \right) = \lim_{m \rightarrow \infty} \frac{1}{m} \log s_m(A, \epsilon, U_m, f) = 0.$$

This and (6) imply (5). □

Let us concentrate now on the sequence  $D$ . Let  $f \in W(I)$ . On sight of Lemmas 3.1 and 3.2, in order to compute  $h_D(f)$  we must be able to calculate, for any given  $\epsilon > 0$  and an appropriate number  $k$ , the numbers  $s(\sigma^k(D), \epsilon, K_\theta(f), f) = s(D, \epsilon, K_\theta(f), f^{2^k})$  for every  $\theta \in \{0, 1\}^k$ . In fact we have (see [2])

LEMMA 3.3. — *Let  $f \in W(I)$  and  $\epsilon > 0$ . Let*

$$\mathcal{N}_\epsilon(f) = \{\alpha \in \mathbb{Z}^\infty : |K_\alpha(f)| \geq \epsilon\}.$$

*Then there is a number  $k_\epsilon$  such that, for any  $k \geq k_\epsilon$ , the following statements hold:*

(i) *if  $\alpha \in \mathcal{N}_\epsilon(f)$  and  $K_{\alpha|_k}^L(f)$  and  $K_{\alpha|_k}^R(f)$  denote the left and right-side components of  $K_{\alpha|_k}(f) \setminus K_\alpha(f)$  then  $|K_{\alpha|_k}^L(f)|, |K_{\alpha|_k}^R(f)| < \epsilon$ ;*

(ii) *if  $\theta \in \mathbb{Z}^k$  and  $\theta \neq \alpha|_k$  for any  $\alpha \in \mathcal{N}_\epsilon(f)$  then  $|K_\theta(f)| < \epsilon$ .*

This means that we can save some work if we take the number  $k = k_\epsilon$  from Lemma 3.3, since if  $\theta \neq \alpha|_k$  for any  $\alpha \in \mathcal{N}_\epsilon(f)$  then  $|K_\theta(f)| < \epsilon$  by (ii) and hence  $s(D, \epsilon, K_\theta(f), f^{2^k}) = 0$ .

These calculations will be done in the key Proposition 3.15, after which Theorem A follows almost immediately. Until the end of its proof,  $f \in W(I)$ ,  $\epsilon > 0$  and  $\alpha \in \mathcal{N}_\epsilon(f)$  will remain fixed, and we will write  $K = K_{\alpha|_{k_\epsilon}}(f)$ ,  $g = f^{2^{k_\epsilon}}$  and  $\beta = \sigma^{k_\epsilon}(\alpha)$ : with this notation, Proposition 3.15 will show that  $s(D, \epsilon, K, g) = h_\beta$ . Since  $g \in W(K)$  (we should say more precisely  $g|_{Cl K} \in W(Cl K)$ , cf. (P6), but hopefully this will not cause any confusion) we will keep the conventions in Section 2 and write  $K_\theta$  instead of  $K_\theta(g)$  whenever  $\theta \in \mathbb{Z}^m$ ,  $m \in \mathbb{N} \cup \{\infty\}$ . The left and right-side components of  $K \setminus K_\beta$  will be respectively denoted by  $K^L$  and  $K^R$ ; we will assume without loss of generality that

$$(7) \quad \alpha|_{k_\epsilon} \text{ has an even number of zeros,}$$

that is (cf. (P2)), that all intervals  $K_t$ ,  $t \leq -1$  (resp.  $t \geq 1$ ), are on the left (resp. right) of  $K_\beta$ . Recall that  $AP(g)$  is the set of asymptotically periodic points of  $g$ .

As the proof of Proposition 3.15 is complicated, we have divided it in several stages (Lemmas 3.4, 3.6, 3.10 and 3.14). To begin with, we return to more or less conventional symbolic dynamics and proceed as follows. Let

$$T = K \setminus \left( AP(g) \cup \bigcup_{m=1}^{\infty} g^{-2^{m-1}}(K_\beta) \right).$$

Thus,  $T$  consists of the non-asymptotically periodic points of  $g$  which never visit  $K_\beta$  under the action of the  $2^{m-1}$ -iterates of  $g$ ; for instance, with this definition,  $K_\beta \subset T$  because it is a wandering interval by (P5).

In what follows, and beginning with  $\nu(x)$  below, we are going to introduce some notations for any  $x \in T$ ; except when it could lead to confusion, we will not refer to  $x$  when using them.

Associate to each point  $x \in T$  a code

$$\nu(x) = (\nu_m)_{m=1}^\infty \in \{L, R\}^\infty$$

defined by  $\nu_m = L$  or  $\nu_m = R$  according to  $g^{2^{m-1}}(x) \in K^L$  or  $g^{2^{m-1}}(x) \in K^R$ ,  $m \in \mathbb{N}$ . Also, let

$$\mathcal{T}(m) = \{\nu(x)|_m : x \in T\}.$$

Then

LEMMA 3.4. — *With the above notation,*

$$s(D, \epsilon, K, g) = \limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{T}(m).$$

*Proof.* — As  $|K_\beta| \geq \epsilon$ , we obviously have  $s_m(D, \epsilon, K, g) \geq \text{Card } \mathcal{T}(m)$  so

$$s(D, \epsilon, K, g) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{T}(m)$$

is trivial. To prove the converse inequality fix  $m$ , let

$$T_i = \{x \in K : g^{2^{i-1}}(x) \in K_\beta\}, \quad 1 \leq i \leq m,$$

and put  $T_0 = K \setminus (\bigcup_{i=1}^m T_i)$ . Since  $K = \bigcup_{i=0}^m T_i$  we have

$$(8) \quad s_m(D, \epsilon, K, g) \leq \sum_{i=0}^m s_m(D, \epsilon, T_i, g).$$

For any  $0 \leq i \leq m$ , let  $E_i \subset T_i$  be a maximal  $(D, \epsilon, T_i, g)$ -separated set. If  $x \in K$  is given then there is a point  $y = y(x) \in T$  such that if  $g^{2^{i-1}}(x) \in K^L$  (resp.  $g^{2^{i-1}}(x) \in K^R$ ) then  $g^{2^{i-1}}(y) \in K^L$  (resp.  $g^{2^{i-1}}(y) \in K^R$ ) as well,  $1 \leq i \leq m$ . (For instance, if  $x \in AP(g)$  and  $J$  is the connected component of  $AP(g)$  containing it, then  $J$  is closed by (P2) and (P4). Moreover, notice that neither of the iterates of  $J$  meets  $K|_\beta$ . Now the existence of the point  $y$  follows from the uniform continuity of  $g$  and the fact that, by (P2) and (P3), there are points of  $T$  as close to  $J$  as required.) In particular,

this means that if  $x \neq x'$  are in  $E_0$  then  $\nu(y(x))|_m \neq \nu(y(x'))|_m$  (use also Lemma 3.3 (i)); hence,

$$(9) \quad s_m(D, \epsilon, T_0, g) \leq \text{Card } \mathcal{T}(m).$$

Fix now  $1 \leq i \leq m$  and let  $\xi \in \mathcal{T}(m)$ . We claim that the set of points  $x \in E_i$  such that  $\nu(y(x)) = \xi$  has cardinality at most  $\lceil |K_\beta|/\epsilon \rceil + 1$ . In fact, if  $x, x'$  are two such points then  $g^{2^j-1}(x)$  and  $g^{2^j-1}(x')$  are on the same side of  $K_\beta$  for any  $j = 1, 2, \dots, m, j \neq i$  (notice that, since  $K_\beta$  is a wandering interval,  $g^{2^j-1}(x), g^{2^j-1}(x') \notin K_\beta$  for any  $j \neq i$ ). Use now that  $E_i$  is a separated set and Lemma 3.3 (i) to conclude  $|g^{2^i-1}(x) - g^{2^i-1}(x')| > \epsilon$ . From this, the claim easily follows. Thus,

$$s_m(D, \epsilon, T_i, g) \leq (\lceil |K_\beta|/\epsilon \rceil + 1) \text{Card } \mathcal{T}(m),$$

which together with (9) and (8) imply

$$s(D, \epsilon, K, g) \leq \limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{T}(m).$$

The lemma follows. □

Next we define the sets

$$U_m = \{x \in T : g^{2^m}(x) \notin K_{\beta|_i}^\#, 1 \leq i \leq m\}, \quad m \geq 1.$$

We emphasize that, because of (P6),

$$(10) \quad \text{if } x \in U_m \text{ then } g^{2^j}(x) \notin K_{\beta|_i}^\#, 1 \leq i \leq j \leq m.$$

We will denote

$$\mathcal{U}(m) = \{\nu(x)|_m : x \in U_m\}.$$

Also, for any  $x \in T$  and any  $m$ , let  $d_m = d_m(x) \geq 0$  be such that  $g^{2^{m-1}}(x) \in K_{\beta|_{d_m}} \setminus K_{\beta|_{d_m+1}}$  (we mean  $K_{\beta|_0} = K$ ). The following simple fact will be very useful later.

LEMMA 3.5. — *If  $x \in U_m$  then  $d_i < i$  for any  $1 \leq i \leq m$ .*

*Proof.* — Assume  $d_j \geq j$  for some  $1 \leq j \leq m$ . Then

$$g^{2^j}(x) = g^{2^j-1}(g^{2^j-1}(x)) \in g^{2^j-1}(K_{\beta|_{d_j}}) \subset g^{2^j-1}(K_{\beta|_j}) \subset K_{\beta|_j}^\#,$$

by (P7) and we arrive to a contradiction with (10). □

Moreover

LEMMA 3.6. — *With the above notation,*

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{T}(m) = \limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{U}(m).$$

*Proof.* — The inequality

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{T}(m) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{U}(m)$$

is trivial. Conversely, fix  $m$  and let  $x \in T$ . If  $x \notin U_1$  then  $g^2(x) \in K_{\beta_1}^\#$  and in general  $g^{2^{i-1}}(x) \in K_{\beta_1}^\#$  for any  $i \geq 2$  (cf. (P6)). Hence either  $\nu(x)|_m = (C, L, \dots, L)$  or  $\nu(x)|_m = (C, R, \dots, R)$ , where “ $C$ ” denotes indistinctly “ $L$ ” or “ $R$ ”. More precisely, if  $\beta_1 = 1$  then  $C = L$  (when  $\mathcal{U}(1) = \{L\}$ ) while if  $\beta_1 = 0$  then both  $C = L$  and  $C = R$  are possible (when  $\mathcal{U}(1) = \{L, R\}$ ): use (P2) and (P3); cf. also (7). Thus  $\nu(x)|_m = \xi * (L, \dots, L)$  or  $\nu(x)|_m = \xi * (R, \dots, R)$  for some  $\xi \in \mathcal{U}(1)$ . Also, if  $x \in U_i \setminus U_{i+1}$  for some  $1 \leq i < m$  then we have  $\nu(x)|_m = \xi * (L, \dots, L)$  or  $\nu(x)|_m = \xi * (R, \dots, R)$  for some  $\xi \in \mathcal{U}(i)$ . Hence

$$\begin{aligned} \text{Card } \mathcal{T}(m) &\leq 2(\text{Card } \mathcal{U}(1) + \dots + \text{Card } \mathcal{U}(m-1)) + \text{Card } \mathcal{U}(m) \\ &< 2m \sum_{i=1}^m \text{Card } \mathcal{U}(i), \end{aligned}$$

and

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{T}(m) &\leq \limsup_{m \rightarrow \infty} \frac{1}{m} \log \left( \sum_{i=1}^m \text{Card } \mathcal{U}(i) \right) \\ &= \limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{Card } \mathcal{U}(m) \end{aligned}$$

as required. □

Now we face the hardest part of the section: we are going to show (Lemmas 3.10 and 3.14) that, for any  $m$ ,

$$(11) \quad \text{Card } \mathcal{U}(m) = \text{Card } \mathcal{S}(\beta, m),$$

where recall that  $\mathcal{S}(\alpha, m)$  is the family of subsets  $S$  of  $\{1, 2, \dots, m\}$  having for any  $1 \leq i \leq m$  the property that  $\text{Card}(S \cap \{1, 2, \dots, i\}) \leq z_i$ , where  $z_i$  is the number of zeros of the sequence  $\beta|_i$ .

We begin by introducing the useful notion of *irregularity*. Let  $x \in T$ . We define the sequence  $\phi(x) = (\phi_m)_{m=1}^\infty \in \{N, Y\}^\infty$  by  $\phi_m = N$  or  $\phi_m = Y$  according to, respectively,  $z_{d_{m-1}} = z_{d_m}$  or not (we mean  $z_0 = d_0 = 0$ ). Notice that if  $x \in U_m$  for some  $m$  then Lemma 3.5 implies that  $d_{i-1} \leq d_i$

for any  $1 \leq i \leq m + 1$ , so  $\phi_i = Y$  means  $z_{d_i-1} < z_{d_i}$  for these indexes. Also, observe that  $\phi_1 = N$  regardless  $x$ . Now

DEFINITION 3.7. — *Let  $x \in U_m$ . We say that  $1 \leq i \leq m$  is irregular (for  $x$ ) if the following properties hold:*

- (i)  $z_{d_i}$  is even and  $\nu_i = R$  (resp.  $z_{d_i}$  is odd and  $\nu_i = L$ );
- (ii)  $\beta_i = 0$ ;
- (iii)  $d_i = i - 1$ ;
- (iv)  $\phi_{i+1} = Y$ .

LEMMA 3.8. — *Let  $x \in U_m$ . With the above notation, suppose that  $1 \leq r \leq s \leq m$  are such that that  $\phi_i = N$  for any  $r \leq i \leq s$ . Also, suppose that  $z_{d_r}$  is even (resp. odd). Then*

- (i)  $\nu_i = L$  (resp.  $\nu_i = R$ ) for any  $r \leq i < s$ ;
- (ii) if  $s \leq t \leq m$  and  $(\nu_s, \dots, \nu_t) = (R, L, R, L, \dots)$  (resp.  $(\nu_s, \dots, \nu_t) = (L, R, L, R, \dots)$ ) then all numbers  $i$  with  $s \leq i \leq t$  are irregular.

*Proof.* — We can for example assume that  $z_{d_r}$  is even.

Let  $r \leq i \leq s$ . We claim that

$$(12) \quad g^{2^{i-1}}(x) \in K_{\beta|_{d_i} * p} \quad \text{for some } p \in \mathbb{Z}^- \cup \{0, 1\}.$$

The statement is obvious if  $i = r = 1$ . If  $i > r$  or  $r > 1$  then we have  $g^{2^{i-2}}(x) \in K_{\beta|_{d_{i-1}}}$  and

$$g^{2^{i-1}}(x) \in K_{\beta|_{d_i} * p} = K_{\beta|_{d_{i-1}} * \mathbf{1}|_{d_i-d_{i-1}} * p}$$

for some  $p \in \mathbb{Z}$ ; by Lemma 3.5 and (P8) we get  $p \in \mathbb{Z}^- \cup \{0, 1\}$  and (12) follows.

Now consider the following possibilities:

(a) If  $\beta_{d_i+1} = 1$  then  $p < 1$  by the definition of  $d_i$ , and  $\nu_i = L$  (recall that  $z_{d_i}$  is even and use (P2) and (7)).

(b) If  $d_i < i - 1$  then  $p \in \{0, 1\}$  is impossible, because it would contradict either the definition of  $d_i$  (in the case  $p = \beta_{d_i+1}$ ) or (10) (in the case  $p = 1 - \beta_{d_i+1}$ ). Then  $p \in \mathbb{Z}^-$  and we get again  $\nu_i = L$ .

(c) If  $d_{i+1} = d_i$  then  $p \notin \{0, 1\}$ , as otherwise we use (P7) or (P6) and get  $g^{2^i}(x) \in K_{\beta|_{d_{i+1}*p'}}$  with  $p' \in \{0, 1\}$ , arriving to a contradiction as in (b). Hence  $p \in \mathbb{Z}^-$  and  $\nu_i = L$ .

Thus, if  $\nu_i = R$ , we must have  $d_i = i - 1, \beta_{d_{i+1}} = \beta_i = 0$  and  $d_i < d_{i+1}$ . In particular we get  $i = s$ , since if  $i < s$  then we can use  $d_i < d_{i+1}$  to get  $\beta_{d_{i+1}} = 1$ , a contradiction. Then  $\nu_i = L$  for any  $r \leq i < s$  and we have proved (i).

Concerning (ii) assume  $\nu_s = R$ , when

$$g^{2^{s-1}}(x) \in K_{\beta|_{s-1*(p_1,p_2,p_3,\dots)}}$$

with  $p_1 \in \mathbb{Z}^- \cup \{0, 1\}$  (recall (12)),  $\beta_s = 0$  and  $s - 1 = d_s < d_{s+1}$ . As  $z_{d_s} = z_{d_r}$  is even,  $s$  is irregular. Moreover,  $\nu_s = R$  gives  $p_1 = 1$ , which means that  $p_2 \in \mathbb{Z}^- \cup \{0, 1\}$  by (P8). Observe also that

$$g^{2^s}(x) \in K_{\beta|_{s-1*(0,p_2,p_3,\dots)}} = K_{\beta|_s*(p_2,p_3,\dots)}.$$

Since we are done in the case  $s = m$ , we can assume  $s < m$ , when  $d_{s+1} = s$  by Lemma 3.5. Now, if  $\nu_{s+1} = L$ , this forces  $\beta_{s+1} = 0$  and  $p_2 = 1$  (and then  $d_{s+1} < d_{s+2}$ ), which together with the fact that  $z_{d_{s+1}} = z_s$  is odd (because  $\beta_s = 0$ ) implies the irregularity of  $s + 1$ . Also,  $p_2 = 1$  implies  $p_3 \in \mathbb{Z}^- \cup \{0, 1\}$  by (P8). Then, if  $s + 2 \leq m$  and  $\nu_{s+2} = R$ , we reason similarly and prove the irregularity of  $s + 2$  and so on. Hence (ii) follows.  $\square$

Next, if  $x \in T$  let

$$\Sigma(x) = (\Sigma_m(x))_{m=1}^\infty = (\Sigma_m)_{m=1}^\infty$$

be the sequence of sets  $\Sigma_m \subset \{1, \dots, m\}$  containing the indexes  $1 \leq i \leq m$  such that  $\nu_{i-1} \neq \nu_i$  (here we mean  $\nu_0 = L$ ). Notice that if  $i < m$  then  $\Sigma_i = \Sigma_m \cap \{1, \dots, i\}$ . We also mean  $\Sigma_0 = \emptyset$ .

LEMMA 3.9. — *Let  $x \in U_m$  and  $1 \leq i \leq m$ . Then  $\text{Card } \Sigma_i \leq z_{d_i} + 1$ . Moreover, if  $\text{Card } \Sigma_i = z_{d_i} + 1$  then  $i$  is irregular.*

*Proof.* — We will prove the lemma inductively. In fact, it follows easily from Lemma 3.8 in the case  $i = 1$ .

Now, fixed  $1 < r \leq m$ , assume it to be true for all  $1 \leq i < r$ . We distinguish between several possibilities:

(a)  $\phi_r = Y$ . Let  $\ell$  be the last integer  $i < r$  satisfying  $\phi_i = N$  (this makes sense because, recall,  $\phi_1 = N$ ). As  $\text{Card } \Sigma_{\ell-1} \leq z_{d_{\ell-1}}$  because either  $\ell = 1$  or  $\ell - 1$  is regular (use the induction hypothesis), and  $z_{d_{i+1}} > z_{d_i}$  for any  $\ell \leq i < r$ , we get that  $\text{Card } \Sigma_r \leq z_{d_r} + 1$  and, moreover, that if  $\text{Card } \Sigma_r = z_{d_r} + 1$  then  $\ell$  is irregular and  $\text{Card } \Sigma_{i+1} = \text{Card } \Sigma_i + 1$  for any  $\ell \leq i < r$ . Assume for instance  $(\nu_\ell, \dots, \nu_r) = (R, L, R, L, \dots)$ . Since  $\ell$  is irregular we know that  $z_{d_\ell}$  is even, so we can apply Lemma 3.8 (ii) to deduce that all numbers  $i$  with  $\ell \leq i \leq r$  (in particular  $i = r$ ) are irregular.

(b)  $(\phi_{r-1}, \phi_r) = (N, N)$ . Then  $r - 1$  is regular and  $\text{Card } \Sigma_{r-1} \leq z_{d_{r-1}} = z_{d_r}$ , so we automatically get  $\text{Card } \Sigma_r \leq z_{d_r} + 1$ . If we exactly have  $\text{Card } \Sigma_r = z_{d_r} + 1$ , that is  $(\nu_{r-1}, \nu_r) = (L, R)$  (or  $(\nu_{r-1}, \nu_r) = (R, L)$ ) then Lemma 3.8 (namely,  $r - 1$  and  $r$  plays respectively the role of  $r$  and  $s = t$  there) implies that  $r$  is irregular and we are done.

(c)  $(\phi_{r-1}, \phi_r) = (Y, N)$ . This is the most complicated case. As in case (b),  $\text{Card } \Sigma_r \leq z_{d_r} + 1$  is guaranteed by the induction hypothesis. Now suppose that  $\text{Card } \Sigma_r = z_{d_r} + 1$  and for example assume  $\nu_r = R$ , when  $(\nu_{r-1}, \nu_r) = (L, R)$  as before. Let  $\ell$  be the last integer  $i < r - 1$  satisfying  $\phi_i = N$ , assume for instance that  $r - \ell$  is odd (the other case is analogous) and consider the following possibilities:

(c1)  $\ell$  is irregular and  $\nu_\ell = L$ . Then we have  $(\nu_\ell, \dots, \nu_{r-1}, \nu_r) = (L, \dots, L, R)$ . Since the combination  $(\nu_\ell, \dots, \nu_{r-1}) = (L, R, \dots, L)$  is impossible because  $r - 1$  is regular (cf. Lemma 3.8 (ii)), and any other combination implies the contradiction  $\text{Card } \Sigma_r \leq z_{d_r}$  as it is very simple to check (take  $\text{Card } \Sigma_{\ell-1} \leq z_{d_{\ell-1}}$  into account), we conclude that this case is in fact impossible.

(c2)  $\ell$  is irregular and  $\nu_\ell = R$ . Now the only feasible combination is one such that  $\text{Card } \Sigma_r - \text{Card } \Sigma_\ell = r - \ell - 1$ , say e.g.  $(\nu_\ell, \dots, \nu_{r-1}, \nu_r) = (R, R, L, R, \dots, L, R)$ ; moreover we must have  $z_{d_r} - z_{d_\ell} = r - \ell - 1$ . But  $\ell$  is irregular, so  $z_{d_\ell}$  (and then  $z_{d_r}$ ) are even. Hence  $r$  is irregular by Lemma 3.8 (ii).

(c3)  $\ell$  is regular and  $\nu_\ell = L$ . Here  $(\nu_\ell, \dots, \nu_{r-1}, \nu_r) = (L, R, \dots, L, R)$ , with  $z_{d_r} - z_{d_\ell} = r - \ell - 1$ , is the only possibility. Then  $z_{d_\ell}$  and  $z_{d_r}$  are again even and  $r$  is irregular.

(c4)  $\ell$  is regular and  $\nu_\ell = R$ . This case is again impossible, because  $\Sigma_\ell \leq z_{d_\ell}$  implies  $\Sigma_r \leq z_{d_r}$  for any possible structure for  $(\nu_\ell, \dots, \nu_{r-1}, \nu_r)$ .

Thus, in all cases, we have shown that  $\text{Card } \Sigma_r \leq z_{d_r} + 1$  and that  $r$

is irregular if  $\text{Card } \Sigma_r = z_{d_r} + 1$ . The lemma is proved.  $\square$

We are ready to prove the first part of (11):

LEMMA 3.10. — *Let  $m \in \mathbb{N}$ . Then  $\text{Card } \mathcal{U}(m) \leq \text{Card } \mathcal{S}(\beta, m)$ .*

*Proof.* — Notice that if  $x, y \in U_m$  and  $\nu(x)|_m \neq \nu(y)|_m$ , then  $\Sigma_m(x) \neq \Sigma_m(y)$ . Thus, in order to prove the lemma, we must just show that if  $x \in U_m$  then  $\Sigma_i$  has cardinality at most  $z_i$  for any  $1 \leq i \leq m$ .

Two possibilities arise. If  $\text{Card } \Sigma_i \leq z_{d_i}$  then we just apply Lemma 3.5 to get  $d_i < i$  and  $\text{Card } \Sigma_i \leq z_i$ . If  $\text{Card } \Sigma_i = z_{d_i} + 1$  then we use the irregularity of  $i$  (cf. Lemma 3.9) to get  $z_{i-1} = z_i - 1$  and  $z_{d_i} = z_{i-1}$ . Then  $\text{Card } \Sigma_i \leq z_{i-1} + 1 = z_i$  as desired.  $\square$

To prove the other inequality in (11) we state and prove below three technical, more or less similar lemmas. In what follows, if  $r \geq 0$  and  $z_r$  is even (resp. odd) then we say that a point  $x \in K$  is  $r$ -central if  $K_{\beta|_r * 0} < x < K_{\beta|_r * 1}$  (resp.  $K_{\beta|_r * 1} < x < K_{\beta|_r * 0}$ ).

LEMMA 3.11. — *Let  $1 \leq r < s$  and suppose  $\beta_r = \beta_s = 0$  and  $\beta_t = 1$  for any  $r < t < s$ . Let  $i < j$  with  $i \geq r, j \geq s$ .*

(i) *If  $i > r, j > s$  and  $x \in K_{\beta|_s}$  then there is  $y \in K_{\beta|_r}$  such that  $g^{2^{j-1}-2^{i-1}}(y) = x$ .*

(ii) *If  $i = r, j > s$  and  $x \in K_{\beta|_s}$  then there is an  $(i - 1)$ -central point  $y \in K_{\beta|_{i-1}}$  such that  $g^{2^{j-1}-2^{i-1}}(y) = x$ .*

(iii) *If  $i > r, j = s$  and  $x \in K_{\beta|_{j-1}}$  is  $(j - 1)$ -central then there is  $y \in K_{\beta|_r}$  such that  $g^{2^{j-1}-2^{i-1}}(y) = x$ .*

(iv) *If  $i = r, j = s$  and  $x \in K_{\beta|_{j-1}}$  is  $(j - 1)$ -central then there is an  $(i - 1)$ -central point  $y \in K_{\beta|_{i-1}}$  such that  $g^{2^{j-1}-2^{i-1}}(y) = x$ .*

Moreover, if  $z_r$  is even (resp. odd) then  $y$  can be got so that  $g^{2^{i-1}-2^{i-1}}(y) < K_\beta$  (resp.  $g^{2^{i-1}-2^{i-1}}(y) > K_\beta$ ) for any  $i \leq t < j$  in cases (i) and (iii), and for any  $i < t < j$  is cases (ii) and (iv).

*Proof.* — We will assume in what follows that, e.g.,  $z_r$  is even. As far as the final statement of the lemma is concerned, we will prove indeed the more general facts

$$(13) \quad g^{u2^r}(y) < K_\beta, \quad 0 \leq u < 2^{j-1-r} - 2^{i-1-r},$$

in cases (i) and (iii), and

$$(14) \quad g^{u2^t} (g^{2^{t-1}}(y)) < K_\beta, \quad 0 \leq u < 2^{j-i-1} - 1,$$

in cases (ii) and (iv). Concerning this last equation, notice that if  $i < t$  then

$$2^{t-1} - 2^{i-1} = 2^{i-1}(2^{t-i} - 1) = 2^{i-1}(1 + 2(2^{t-i-1} - 1)) = 2^{i-1} + 2^i(2^{t-i-1} - 1).$$

(i) Let

$$\theta = \begin{cases} \beta|_r * (2^{i-1-r} - 2^{j-1-r}) & \text{if } s - r = 1, \\ \beta|_r * (1 + 2^{i-1-r} - 2^{j-1-r}) * \mathbf{0}|_{s-r-2} * 1 & \text{if } s - r > 1. \end{cases}$$

Using (P6) it is easy to check that if we are in the first case then

$$g^{u2^r} (K_\theta) = K_{\beta|_{r*(u+2^{i-1-r}-2^{j-1-r})}}, \quad 0 \leq u \leq 2^{j-1-r} - 2^{i-1-r},$$

while in the second case we have

$$g^{u2^r} (K_\theta) = K_{\beta|_{r*(u+1+2^{i-1-r}-2^{j-1-r})*\mathbf{0}|_{s-r-2}*1}}, \quad 0 \leq u < 2^{j-1-r} - 2^{i-1-r},$$

and then

$$g^{2^{j-1}-2^{i-1}} (K_\theta) = g^{(2^{j-1-r}-2^{i-1-r})2^r} (K_\theta) = K_{\beta|_{r*1}|_{s-r-1}*0} = K_{\beta|_s}.$$

This proves (i) and (13).

(ii) Put

$$\theta = \begin{cases} \beta|_{i-1} * (1, 1 - 2^{j-i-1}) & \text{if } s - i = 1, \\ \beta|_{i-1} * (1, 2 - 2^{j-i-1}) * \mathbf{0}|_{s-i-2} * 1 & \text{if } s - i > 1. \end{cases}$$

Observe that in the first case  $j - i \geq 2$ , while in the second case  $j - i \geq 3$ . Thus, in any case,  $\theta_{i+1} < 0$  and all points from  $K_\theta$  are  $(i - 1)$ -central. Moreover, we respectively have

$$g^{2^{i-1}} (K_\theta) = K_{\beta|_{i-1}*(0,1-2^{j-i-1})}$$

and

$$g^{2^{i-1}} (K_\theta) = K_{\beta|_{i-1}*(0,2-2^{j-i-1})*\mathbf{0}|_{s-i-2}*1}$$

and we can see similarly as above that (14) holds and

$$g^{2^{j-1}-2^{i-1}} (K_\theta) = K_{\beta|_s},$$

which implies (ii).

(iii) Notice that in this case we must have  $s - r = j - r > 1$ . Let  $x \in K_{\beta|_{j-1}}$  be  $(j - 1)$ -central, when there must exist some  $\ell \geq 1$  and  $p \leq 0$  such that  $x \in K_{\beta|_{j-1} * \mathbf{1}|_{\ell} * p}$ . Now, if we take

$$\theta = \beta|_r * (1 + 2^{i-1-r} - 2^{j-1-r}) * \mathbf{0}|_{j-r-2+\ell} * (p - 1),$$

we have

$$g^{u2^r}(K_\theta) = K_{\beta|_r * (u+1+2^{i-1-r}-2^{j-1-r}) * \mathbf{0}|_{j-r-2+\ell} * (p-1)}$$

for  $0 \leq u < 2^{j-1-r} - 2^{i-1-r}$  and

$$g^{2^{j-1}-2^{i-1}}(K_\theta) = K_{\beta|_{j-1} * \mathbf{1}|_{\ell} * p},$$

which proves (iii) and (13).

(iv) Let  $x \in K_{\beta|_{j-1}}$  be  $(j - 1)$ -central, with  $\ell \geq 1$  and  $p \leq 0$  such that  $x \in K_{\beta|_{j-1} * \mathbf{1}|_{\ell} * p}$ . We have again two cases. Let

$$\theta = \begin{cases} \beta|_{i-1} * \mathbf{1}|_{1+\ell} * p & \text{if } j - i = 1, \\ \beta|_{i-1} * (1, 2 - 2^{j-i-1}) * \mathbf{0}|_{j-i-2+\ell} * (p - 1) & \text{if } j - i > 1. \end{cases}$$

Then, in any case, all points from  $K_\theta$  are  $(i - 1)$ -central. Notice that if  $j - i = 1$  then we directly get

$$g^{2^{j-1}-2^{i-1}}(K_\theta) = g^{2^{i-1}}(K_\theta) = K_{\beta|_i * \mathbf{1}|_{\ell} * p} = K_{\beta|_{j-1} * \mathbf{1}|_{\ell} * p}$$

and (iv) follows (here (14) means nothing). In the case  $j - i > 1$  we have

$$g^{2^{i-1}}(K_\theta) = K_{\beta|_i * (2 - 2^{j-i-1}) * \mathbf{0}|_{j-i-2+\ell} * (p-1)}$$

and now it is easy to realize that (14) holds and

$$g^{2^{j-1}-2^{i-1}}(K_\theta) = K_{\beta|_{j-1} * \mathbf{1}|_{\ell} * p};$$

hence (iv) follows.

The lemma is proved. □

LEMMA 3.12. — *Let  $1 \leq s \leq j$  and suppose  $\beta_s = 0$  and  $\beta_t = 1$  for any  $1 \leq t < s$ . If  $j > s$  (resp.  $j = s$ ) and  $x \in K_{\beta|_s}$  (resp.  $x \in K_{\beta|_{j-1}}$  is  $(j - 1)$ -central) then there is  $y \in K$  such that  $g^{2^{j-1}}(y) = x$  and  $g^{2^{t-1}}(y) < K_\beta$  for any  $1 \leq t < j$ .*

*Proof.* — If  $j > s > 1$  (resp.  $j > s = 1$ ) let  $\theta = (1 - 2^{j-1}) * \mathbf{0}_{|s-2} * 1$  (resp.  $\theta = (-2^{j-1})$ ); if  $j = s$ ,  $x \in K_{\beta_{|j-1*1|_\ell * p}}$  is  $(j - 1)$ -central with  $\ell \geq 1$  and  $p \leq 0$ , and  $j > 1$  (resp.  $j = 1$ ), let  $\theta = (1 - 2^{j-1}) * \mathbf{0}_{|j-2+\ell} * (p - 1)$  (resp.  $\theta = \mathbf{0}_{|\ell} * (p - 1)$ ). In the first cases we get  $g^{2^{j-1}}(K_\theta) = K_{\beta_{|s}}$ , and in the second ones we get  $g^{2^{j-1}}(K_\theta) = K_{\beta_{|j-1*1|_\ell * p}}$ ; it is easy to check that the lemma holds.  $\square$

LEMMA 3.13. — *Let  $r < i \leq m$  (resp.  $r = i \leq m$ ) and suppose  $\beta_r = 0$ . Then there is a point  $x \in K_{\beta_{|r}}$  (resp. an  $(i - 1)$ -central point  $x \in K_{\beta_{|i-1}}$ ) such that, for any  $i \leq t \leq m + 2$ ,  $g^{2^{t-1}-2^{i-1}}(x) < K_{\beta_{|r+1}}$  if  $z_r$  is even and  $g^{2^{t-1}-2^{i-1}}(x) > K_{\beta_{|r+1}}$  if  $z_r$  is odd; moreover,  $g^{2^{m+2}-2^{i-1}}(x) \in K_{\beta_{|r}}^\#$ .*

*Proof.* — We will assume that  $z_r$  is even. If we are in the case  $i > r$  then it suffices to take any point  $x \in K_{\beta_{|r*(2^{i-1}-r-2^{m+2-r})}}$  when  $\beta_{r+1} = 1$ , and any point  $x \in K_{\beta_{|r*(1+2^{i-1}-r-2^{m+2-r})}}$  when  $\beta_{r+1} = 0$ . If we are in the case  $i = r$  then we can take any point  $x \in K_{\beta_{|r-1*(1,1-2^{m+2-r})}}$  if  $\beta_{r+1} = 1$ , and any point  $x \in K_{\beta_{|r-1*(1,2-2^{m+2-r})}}$  if  $\beta_{r+1} = 0$ .  $\square$

Now we can complete the proof of (11):

LEMMA 3.14. — *Let  $m \in \mathbb{N}$ . Then  $\text{Card } \mathcal{U}(m) \geq \text{Card } \mathcal{S}(\beta, m)$ .*

*Proof.* — Let  $S \subset \{1, \dots, m\}$  be such that  $\text{Card}(S \cap \{1, \dots, i\}) \leq z_i$  for any  $1 \leq i \leq m$ . We must find a point  $x \in U_m$  such that  $\Sigma_m(x) = S$ .

If  $S = \emptyset$  then it is easy to find a point  $x \in K$  such that  $f^{2^{t-1}}(x) < K_{\beta_1}$  for any  $1 \leq t \leq m + 1$  and  $f^{2^{m+1}}(x) \in K_{\beta_1}^\#$  (and thus  $f^{2^{t-1}} \in K_{\beta_1}^\#$  for any  $t \geq m + 2$ ). Then  $x \in T$  and, in fact,  $x \in U_m$ . Obviously  $\Sigma_m(x) = \emptyset = S$  and we are done.

Assume now  $S \neq \emptyset = \{i_1, i_2, \dots, i_q\}$  with  $q \geq 1$  and  $1 \leq i_1 < \dots < i_q \leq m$ . By the definition of  $S$ , if  $1 \leq c_1 < \dots < c_q$  are the  $q$  first indexes  $\ell$  satisfying  $\beta_\ell = 0$  then we must have  $i_\ell \geq c_\ell$  for any  $\ell$ . Now, using Lemmas 3.13, 3.11 and 3.12 it is simple to find a point  $x \in K$  such that

(a) if  $t = i_\ell$  for some  $1 \leq \ell \leq q$  then  $d_t = c_\ell$  or  $d_t = c_\ell - 1$  according to whether  $i_\ell > c_\ell$  or  $i_\ell = c_\ell$ ; moreover, in the second case the point  $f^{2^{t-1}}(x)$  is  $(t - 1)$ -central;

(b) if  $i_{\ell-1} < t < i_\ell$  for some  $1 \leq \ell \leq q + 1$  then  $d_t = c_{\ell-1}$  (here we mean  $i_0 = c_0 = 0$  and  $i_{q+1} = \infty$ );

(c) if  $1 \leq \ell \leq q + 1$  is odd (resp. even) and  $i_{\ell-1} \leq t < i_\ell$  then  $f^{2^{t-1}}(x) < K_\beta$  (resp.  $f^{2^{t-1}}(x) > K_\beta$ ); also,  $f^{2^{m+2}}(x) \in K_{\beta|c_q}^\#$  (here we mean  $i_0 = 1$  and  $i_{q+1} = m + 2$ ).

In particular, (c) implies that  $x \in U_{m+1} \setminus U_{m+2}$  (and then  $x \in U_m$ ), and  $\Sigma_m(x) = S$ . We are done. □

Finally we are ready to prove the result we formulated at the beginning of the section:

PROPOSITION 3.15. — *With the notation above,  $s(D, \epsilon, K, g) = h_\beta$ .*

*Proof.* — It follows from Lemmas 3.4, 3.6, 3.10 and 3.14. □

And now a final easy lemma before proving Theorem A:

LEMMA 3.16. — *Let  $\alpha \in \{0, 1\}^\infty$ . Then  $h_{\sigma^k(\alpha)} = h_\alpha$  for any  $k \in \mathbb{N}$ .*

*Proof.* — Of course it suffices to show  $h_{\sigma(\alpha)} = h_\alpha$ .

If  $R \in \mathcal{S}(m, \sigma(\alpha))$  for some  $m$  then we obviously have

$$S = \{r + 1 : r \in R\} \in \mathcal{S}(m + 1, \alpha).$$

Therefore  $\text{Card } \mathcal{S}(m, \sigma(\alpha)) \leq \text{Card } \mathcal{S}(m + 1, \alpha)$  and  $h_{\sigma(\alpha)} \leq h_\alpha$  follows.

Conversely, let  $S \in \mathcal{S}(m + 1, \alpha)$  be nonempty, let  $S' = S \setminus \{\min S\}$  and  $R = R(S) = \{s - 1 : s \in S'\}$ . It is easy to check that  $R \in \mathcal{S}(m, \sigma(\alpha))$ . Moreover, if  $R \in \mathcal{S}(m, \sigma(\alpha))$  is given then there are at most  $m + 1$  sets  $S$  in  $\mathcal{S}(m + 1, \alpha)$  satisfying  $R(S) = R$ . Hence  $\text{Card } \mathcal{S}(m + 1, \alpha) \leq 1 + (m + 1) \text{Card } \mathcal{S}(m, \sigma(\alpha))$  and  $h_\alpha \leq h_{\sigma(\alpha)}$  follows. □

*Proof of Theorem A.* — Let  $\epsilon > 0$  and let  $k = k_\epsilon$  be the number from Lemma 3.3. Combining Lemmas 3.1 and 3.2 we can find a sequence  $\alpha \in \mathcal{N}_\epsilon(f)$  such that, if  $g = f^{2^k}$  and  $K = K_{\alpha|k}(f)$ , then

$$s(D, 4\epsilon, I, f) \leq s(\sigma^k(D), 2\epsilon, I, f) \leq s(D, \epsilon, K, g).$$

Since Proposition 3.15 and Lemma 3.16 give  $s(D, \epsilon, K, g) = h_\alpha$ , we conclude  $s(D, 4\epsilon, I, f) \leq \sup_{\alpha \in \mathcal{N}(f)} h_\alpha$ . Then  $h_D(f) \leq \sup_{\alpha \in \mathcal{N}(f)} h_\alpha$ .

Conversely, take  $\alpha \in \mathcal{N}(f)$ , put  $\epsilon = |K_\alpha(f)|$  (thus  $\alpha \in \mathcal{N}_\epsilon(f)$ ), take  $k = k_\epsilon$ ,  $g = f^{2^k}$  and  $K = K_{\alpha|k}(f)$  as before, and use Proposition 3.15 and Lemmas 3.16 and 3.1 to get

$$h_\alpha = s(D, \epsilon, K, g) \leq s(D, \epsilon, I, g) = s(\sigma^k(D), \epsilon, I, f) \leq s(D, \epsilon, I, f) \leq h_D(f).$$

Hence  $\sup_{\alpha \in \mathcal{N}(f)} h_\alpha \leq h_D(f)$ , which ends the proof. □

### 4. An ordering preserved by increasing concave maps.

The main ideas behind the proof of Theorem B can be roughly described as follows. According to Definition 1.6, computing  $h_\alpha$  implies basically to evaluate the (logarithmic) means of large amounts of combinatorial numbers constrained by certain conditions. This leads naturally to the consideration of the map  $g$  we defined in the Introduction, thus allowing us to dispose of combinatorics and simplifying the calculations a lot. When the above-mentioned constraints are translated into this new language one arrives immediately to the ordering “ $\preceq$ ” described below Lemma 4.2 and discovers that if  $(x_1, \dots, x_n)$  is given then  $F_n(x_1, \dots, x_n)$  turns out to be a point at which  $G_n$  attains the constrained maximum by the restriction  $(y_1, \dots, y_n) \preceq (x_1, \dots, x_n)$  (here it is essential that  $g$  is concave and increasing). From this point on, the pieces of the puzzle are relatively easy to adjust (cf. Proposition 4.9).

Thus we begin by recalling some simple properties of the maps  $F_n$ . We omit the easy proofs. In what follows, if  $x \in \mathbb{R}$  then  $x-s-x$  has exactly length  $s$ ; thus, e.g.,  $x-2-x = x, x$ .

LEMMA 4.1. — *Let  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $(c_1, \dots, c_n) = F_n(x_1, \dots, x_n)$ ,  $\delta \in \mathbb{R}$ , and  $s, i \in \mathbb{N}$ . Then we have*

- (i)  $F_n(x_1 + \delta, \dots, x_n + \delta) = (c_1 + \delta, \dots, c_n + \delta)$ ;
- (ii)  $F_n(\delta x_1, \dots, \delta x_n) = (\delta c_1, \dots, \delta c_n)$  (if  $\delta \geq 0$ );
- (iii)  $F_{n+i}(0, \dots, 0, x_1, \dots, x_n) = (0, \dots, 0, c_1, \dots, c_n)$  (if  $0 \leq x_j \leq 1$  for any  $j$ );
- (iv)  $F_{n+i}(x_1, \dots, x_n, 1, \dots, 1) = (c_1, \dots, c_n, 1, \dots, 1)$  (if  $0 \leq x_j \leq 1$  for any  $j$ );
- (v)  $F_{sn}(x_1-s-x_1, x_2-s-x_2, \dots, x_n-s-x_n)$   
 $= (c_1-s-c_1, c_2-s-c_2, \dots, c_n-s-c_n)$ .

LEMMA 4.2. — Let  $(x_1, \dots, x_n) \in \mathbb{R}^n$  and  $(c_1, \dots, c_n) = F_n(x_1, \dots, x_n)$ . Then there are numbers  $0 = \ell_0 < \ell_1 < \dots < \ell_r = n$  and  $d_1 < d_2 < \dots < d_r$  (uniquely determined by  $(x_1, \dots, x_n)$ ) such that

(i)  $d_s = c_{\ell_{s-1}+1} = c_{\ell_{s-1}+2} = \dots = c_{\ell_s} = \text{mean}(x_{\ell_{s-1}+1}, x_{\ell_{s-1}+2}, \dots, x_{\ell_s})$ ,  $1 \leq s \leq r$ ;

(ii)  $d_s \leq \text{mean}(x_{\ell_{s-1}+1}, x_{\ell_{s-1}+2}, \dots, x_i)$ ,  $\ell_{s-1} < i \leq \ell_s$ ,  $1 \leq s \leq r$ .

In what follows we will use the notation  $(y_1, \dots, y_n) \preceq (x_1, \dots, x_n)$  whenever

$$\text{mean}(y_1, \dots, y_i) \leq \text{mean}(x_1, \dots, x_i) \quad \text{for any } 1 \leq i \leq n$$

(or, equivalently,  $\sum_{j=1}^i y_j \leq \sum_{j=1}^i x_j$  for any  $i$ ). Regarding this the following lemma will be needed later:

LEMMA 4.3. — Let  $(x_1, \dots, x_n), (z_1, \dots, z_n) \in \mathbb{R}^n$ . With the notation of Lemma 4.2 for  $(x_1, \dots, x_n)$ , assume that there are numbers  $e_1, e_2, \dots, e_r$  such that

$$e_s = z_{\ell_{s-1}+1} = z_{\ell_{s-1}+2} = \dots = z_{\ell_s}, \quad 1 \leq s \leq r.$$

Then  $(z_1, \dots, z_n) \preceq (x_1, \dots, x_n)$  if and only if  $\text{mean}(z_1, \dots, z_{\ell_s}) \leq \text{mean}(x_1, \dots, x_{\ell_s})$  for any  $1 \leq s \leq r$ .

*Proof.* — The “only if” part of the lemma is trivial. Conversely, assume  $\text{mean}(z_1, \dots, z_{\ell_s}) \leq \text{mean}(x_1, \dots, x_{\ell_s})$  for any  $1 \leq s \leq r$  and fix a number  $1 \leq i \leq n$ , say  $\ell_{s-1} < i \leq \ell_s$ . Then

$$\begin{aligned} & \text{mean}(z_1, \dots, z_i) \\ &= \frac{1}{i} (z_1 + \dots + z_{\ell_{s-1}}) + \frac{i - \ell_{s-1}}{i(\ell_s - \ell_{s-1})} (z_{\ell_{s-1}+1} + \dots + z_{\ell_s}) \\ &= \frac{\ell_s - i}{i(\ell_s - \ell_{s-1})} (z_1 + \dots + z_{\ell_{s-1}}) + \frac{i - \ell_{s-1}}{i(\ell_s - \ell_{s-1})} (z_1 + \dots + z_{\ell_s}) \\ &\leq \frac{\ell_s - i}{i(\ell_s - \ell_{s-1})} (x_1 + \dots + x_{\ell_{s-1}}) + \frac{i - \ell_{s-1}}{i(\ell_s - \ell_{s-1})} (x_1 + \dots + x_{\ell_s}) \\ &= \frac{1}{i} (x_1 + \dots + x_{\ell_{s-1}}) + \frac{i - \ell_{s-1}}{i(\ell_s - \ell_{s-1})} (x_{\ell_{s-1}+1} + \dots + x_{\ell_s}) \\ &\leq \frac{1}{i} (x_1 + \dots + x_{\ell_{s-1}}) + \frac{1}{i} (x_{\ell_{s-1}+1} + \dots + x_i) \\ &= \text{mean}(x_1, \dots, x_i). \quad \square \end{aligned}$$

Observe that Lemma 4.3 implies in particular  $F_n(x_1, \dots, x_n) \preceq (x_1, \dots, x_n)$ . Hence the statement of the next proposition, the first key result of the section, makes sense.

PROPOSITION 4.4. — Let  $v : [a, b] \rightarrow \mathbb{R}$  be an increasing concave map and let  $V_n : [a, b]^n \rightarrow \mathbb{R}$  be defined by

$$V_n(y_1, \dots, y_n) = \text{mean}(v(y_1), \dots, v(y_n)).$$

Let  $(x_1, \dots, x_n) \in [a, b]^n$ . Then  $V_n$  has a constrained maximum at  $F_n(x_1, \dots, x_n)$  by the restriction  $(y_1, \dots, y_n) \preceq (x_1, \dots, x_n)$ .

*Proof.* — We will make induction on  $r$  (we are using the notation from Lemma 4.2).

Assume first  $r = 1$  and  $(y_1, \dots, y_n) \preceq (x_1, \dots, x_n)$ . Then

$$\begin{aligned} V_n(y_1, \dots, y_n) &\leq v(\text{mean}(y_1, \dots, y_n)) \\ &\leq v(\text{mean}(x_1, \dots, x_n)) \\ &= V_n(F_n(x_1, \dots, x_n)); \end{aligned}$$

we have used that  $v$  is concave and increasing and that

$$F_n(x_1, \dots, x_n) = (\text{mean}(x_1, \dots, x_n), \text{mean}(x_1, \dots, x_n), \dots, \text{mean}(x_1, \dots, x_n))$$

because  $r = 1$ .

Let us prove now the statement in the general case. We must show that if  $(y_1, \dots, y_n) \preceq (x_1, \dots, x_n)$  is given then

$$V_n(y_1, \dots, y_n) \leq V_n(c_1, \dots, c_n) = V_n(F_n(x_1, \dots, x_n)).$$

For  $1 \leq s \leq r$  write

$$e_s = z_{\ell_{s-1}+1} = z_{\ell_{s-1}+2} = \dots = z_{\ell_s} = \text{mean}(y_{\ell_{s-1}+1}, y_{\ell_{s-1}+2}, \dots, y_{\ell_s}).$$

Then  $(z_1, \dots, z_n) \preceq (x_1, \dots, x_n)$  by Lemma 4.3 and (because  $v$  is concave)  $V_n(y_1, \dots, y_n) \leq V_n(z_1, \dots, z_n)$ . Hence it suffices to prove  $V_n(z_1, \dots, z_n) \leq V_n(c_1, \dots, c_n)$  or, equivalently,

$$(15) \quad \sum_{j=1}^r (\ell_j - \ell_{j-1}) v(e_j) \leq \sum_{j=1}^r (\ell_j - \ell_{j-1}) v(d_j).$$

Since  $v$  is continuous, we can trivially also assume that  $V_n$  has a constrained maximum at  $(y_1, \dots, y_n)$  and then at  $(z_1, \dots, z_n)$ , that is,

$$(16) \quad \text{if } (w_1, \dots, w_n) \preceq (x_1, \dots, x_n) \text{ then } V_n(w_1, \dots, w_n) \leq V_n(z_1, \dots, z_n).$$

If  $e_j \leq d_j$  for any  $j$  then there is nothing to prove. As  $(z_1, \dots, z_n) \preceq (x_1, \dots, x_n)$  implies  $e_1 \leq d_1$ , we can assume  $e_s \leq d_s < d_{s+1} < e_{s+1}$  for some  $1 \leq s < r$  (recall that the numbers  $d_j$  form a strictly increasing sequence, cf. Lemma 4.2).

We claim that  $v$  is affine on  $[e_s, e_{s+1}]$ . Fix otherwise a number  $\epsilon > 0$  small enough, write  $e'_s = e_s + \epsilon/(\ell_s - \ell_{s-1})$  and  $e'_{s+1} = e_{s+1} - \epsilon/(\ell_{s+1} - \ell_s)$ , and observe that

$$\frac{v(e'_s) - v(e_s)}{e'_s - e_s} > \frac{v(e_{s+1}) - v(e'_{s+1})}{e_{s+1} - e'_{s+1}}$$

and hence

$$(17) \quad (\ell_s - \ell_{s-1})v(e_s) + (\ell_{s+1} - \ell_s)v(e_{s+1}) < (\ell_s - \ell_{s-1})v(e'_s) + (\ell_{s+1} - \ell_s)v(e'_{s+1}).$$

Define  $(w_1, \dots, w_n)$  by

$$w_i = \begin{cases} e'_s & \text{if } \ell_{s-1} < i \leq \ell_s, \\ e'_{s+1} & \text{if } \ell_s < i \leq \ell_{s+1}, \\ z_i & \text{otherwise} \end{cases}$$

and apply

$$(\ell_s - \ell_{s-1})e_s + (\ell_{s+1} - \ell_s)e_{s+1} = (\ell_s - \ell_{s-1})e'_s + (\ell_{s+1} - \ell_s)e'_{s+1}$$

to deduce  $\text{mean}(w_1, \dots, w_{\ell_j}) = \text{mean}(z_1, \dots, z_{\ell_j}) \leq \text{mean}(x_1, \dots, x_{\ell_j})$  if  $j \neq s$ . Moreover,  $d_{s+1} < e_{s+1}$  implies  $\text{mean}(z_1, \dots, z_{\ell_s}) < \text{mean}(x_1, \dots, x_{\ell_s})$ , so it is not restrictive to assume  $\text{mean}(w_1, \dots, w_{\ell_s}) \leq \text{mean}(x_1, \dots, x_{\ell_s})$  as well. Hence  $(w_1, \dots, w_n) \preceq (x_1, \dots, x_n)$  by Lemma 4.3, which contradicts (17) and (16).

Denote

$$\bar{d} = d_s \frac{\ell_s - \ell_{s-1}}{\ell_{s+1} - \ell_{s-1}} + d_{s+1} \frac{\ell_{s+1} - \ell_s}{\ell_{s+1} - \ell_{s-1}},$$

$$\bar{e} = e_s \frac{\ell_s - \ell_{s-1}}{\ell_{s+1} - \ell_{s-1}} + e_{s+1} \frac{\ell_{s+1} - \ell_s}{\ell_{s+1} - \ell_{s-1}}$$

and define  $(x'_1, \dots, x'_n), (z'_1, \dots, z'_n)$  by

$$x'_i = \begin{cases} \bar{d} & \text{if } \ell_{s-1} < i \leq \ell_{s+1}, \\ x_i & \text{otherwise} \end{cases}$$

and

$$z'_i = \begin{cases} \bar{e} & \text{if } \ell_{s-1} < i \leq \ell_{s+1}, \\ z_i & \text{otherwise.} \end{cases}$$

Clearly (adding apostrophes to describe the corresponding parameters for the sequence  $(x'_1, \dots, x'_n)$  in Lemma 4.2) we have  $r' = r - 1$ ,

$$\ell'_j = \begin{cases} \ell_j & \text{if } j < s, \\ \ell_{j+1} & \text{if } j \geq s, \end{cases} \quad \text{and} \quad d'_j = \begin{cases} d_j & \text{if } j < s, \\ \bar{d} & \text{if } j = s, \\ d_{j+2} & \text{if } j > s. \end{cases}$$

Apply now Lemma 4.3 to get  $(z'_1, \dots, z'_n) \preceq (x'_1, \dots, x'_n)$  and use the induction hypothesis to deduce  $V_n(z'_1, \dots, z'_n) \leq V_n(c'_1, \dots, c'_n)$ ,  $(c'_1, \dots, c'_n) = F_n(x'_1, \dots, x'_n)$ . Finally notice that

$$c'_i = \begin{cases} \bar{d} & \text{if } \ell_{s-1} < i \leq \ell_{s+1}, \\ c_i & \text{otherwise} \end{cases}$$

and take into account that  $v$  is affine in  $[e_s, e_{s+1}]$  to get

$$\begin{aligned} \sum_{j=1}^r (\ell_j - \ell_{j-1})v(e_j) &= \sum_{j=1}^{s-1} (\ell_j - \ell_{j-1})v(e_j) + (\ell_{s+1} - \ell_{s-1})v(\bar{e}) \\ &\quad + \sum_{j=s+2}^r (\ell_j - \ell_{j-1})v(e_j) \\ &= nV_n(z'_1, \dots, z'_n) \\ &\leq nV_n(c'_1, \dots, c'_n) \\ &= \sum_{j=1}^{s-1} (\ell_j - \ell_{j-1})v(d_j) + (\ell_{s+1} - \ell_{s-1})v(\bar{d}) \\ &\quad + \sum_{j=s+2}^r (\ell_j - \ell_{j-1})v(d_j) \\ &= \sum_{j=1}^r (\ell_j - \ell_{j-1})v(d_j); \end{aligned}$$

inequality (15) is proved. □

Proposition 4.4 can be reformulated as follows:

**COROLLARY 4.5.** — *Let  $V_n$  be defined as in Proposition 4.4. Then  $V_n \circ F_n : [a, b]^n \rightarrow \mathbb{R}$  is increasing, that is,*

*if  $(y_1, \dots, y_n) \preceq (x_1, \dots, x_n)$  then  $V_n(F_n(y_1, \dots, y_n)) \leq V_n(F_n(x_1, \dots, x_n))$ .*

*Proof.* — Just notice that  $F_n(y_1, \dots, y_n) \preceq (y_1, \dots, y_n) \preceq (x_1, \dots, x_n)$  by Lemma 4.3 and use Proposition 4.4. □

As we explained at the beginning of the section, the above results will be used mainly for the maps  $g$  and  $G_n$  from Theorem B. The other key property of  $g$  is the following one; recall that  $[x]$  denotes the integer part of  $x$ .

PROPOSITION 4.6. — For any  $n \in \mathbb{N}$  let  $g_n : [0, \frac{1}{2}] \rightarrow \mathbb{R}$  be defined by

$$g_n(x) = \frac{1}{n} \log \left( \frac{n}{[xn]} \right).$$

Then  $(g_n)_n$  converges uniformly to  $g$  on  $[0, \frac{1}{2}]$ .

Proof. — Let  $x_0 \in (0, \frac{1}{2}]$  be arbitrarily fixed. First we will prove that  $(g_n)_n$  converges uniformly to  $g$  on  $[x_0, \frac{1}{2}]$ .

Consider the maps  $t_n, u_n, v_n : [x_0, \frac{1}{2}] \rightarrow \mathbb{R}$  defined respectively by

$$t_n(x) = \frac{[xn]}{[n]}, \quad u_n(x) = \frac{1}{n} \log \left( \frac{\sqrt{2\pi n}}{\sqrt{2\pi[xn]} \sqrt{2\pi(n - [xn])}} \right)$$

and

$$\begin{aligned} v_n(x) &= g(t_n(x)) + u_n(x) \\ &= \frac{1}{n} \log \left( \frac{n^n \sqrt{2\pi n}}{[xn][xn](n - [xn])^{n - [xn]} \sqrt{2\pi[xn]} \sqrt{2\pi(n - [xn])}} \right). \end{aligned}$$

Clearly these maps are well defined if  $n$  is large enough. Further it is easy to check that  $(u_n)_n$  converges uniformly to 0 and  $(t_n)_n$  converges uniformly to the identity map. Since  $g$  is uniformly continuous,  $(g \circ t_n)_n$  and then  $(v_n)_n$  converge uniformly to  $g$  on  $[x_0, \frac{1}{2}]$ .

Next use Stirling’s formula and realize that if  $m$  is large enough then

$$(18) \quad \frac{1}{2} m^m e^{-m} \sqrt{2\pi m} \leq m! \leq 2m^m e^{-m} \sqrt{2\pi m}$$

or, equivalently,

$$(19) \quad \frac{1}{2} \frac{1}{m^m e^{-m} \sqrt{2\pi m}} \leq \frac{1}{m!} \leq 2 \frac{1}{m^m e^{-m} \sqrt{2\pi m}}.$$

Replacing  $m$  by  $n$  in (18) and by  $[xn]$  and  $n - [xn]$  in (19), multiplying the resultant chains of inequalities, taking logarithms and dividing by  $n$  we get

$$\frac{\log \frac{1}{8}}{n} + v_n(x) \leq g_n(x) \leq \frac{\log 8}{n} + v_n(x)$$

if  $x \in [x_0, \frac{1}{2}]$  and  $n$  is large enough. Thus  $(g_n)_n$  converges uniformly to  $g$  on  $[x_0, \frac{1}{2}]$ .

Fix  $\epsilon > 0$  and find a number  $x_0$  small enough so that  $g(x_0) \leq \frac{1}{3}\epsilon$ , and then a number  $n_\epsilon$  such that  $|g_n(x) - g(x)| \leq \frac{1}{3}\epsilon$  for any  $x \in [x_0, \frac{1}{2}]$  and any  $n \geq n_\epsilon$ . Let  $x \in [0, x_0]$ . Then

$$|g_n(x) - g(x)| \leq g_n(x) + g(x) \leq g_n(x_0) + g(x_0) \leq |g_n(x_0) - g(x_0)| + 2g(x_0) \leq \epsilon$$

for any  $n \geq n_\epsilon$ . This guarantees the uniform convergence of  $(g_n)_n$  to  $g$  on the whole interval  $[0, \frac{1}{2}]$ . □

Next lemmas show how to combine Propositions 4.4 and 4.6 with Definition 1.6; they culminate in Proposition 4.9. Until the end of the section the sequence  $\alpha \in \{0, 1\}^\infty$  will remain fixed. If  $s < r$  then the number of zeros of the sequence  $\alpha_{s+1}, \alpha_{s+2}, \dots, \alpha_r$  will be denoted by  $z_{s,r}$ . For any  $n \in \mathbb{N}$  the map  $k_n : \{0, 1, \dots, [\frac{1}{2}n]\}^n \rightarrow \mathbb{N}$  is defined by

$$k_n(p_1, p_2, \dots, p_n) = \binom{n}{p_1} \binom{n}{p_2} \cdots \binom{n}{p_n};$$

say that it has a constrained maximum at  $(q_{1,n}, q_{2,n}, \dots, q_{n,n})$  by the restriction

$$(p_1, p_2, \dots, p_n) \preceq (z_{0,n}, z_{n,2n}, \dots, z_{(n-1)n,nn}).$$

LEMMA 4.7. — *Let  $m \in \mathbb{N}$  and  $n = \lfloor \sqrt{m} \rfloor$ . Then we have*

$$\begin{aligned} & \frac{-n \log 2}{(n+1)^2} + \frac{1}{(n+1)^2} \log k_n(q_{1,n}, \dots, q_{n,n}) \\ & < \frac{1}{m} \log \text{Card } \mathcal{S}(\alpha, m) \\ & < \frac{(2n+1) \log 2}{n^2} + \frac{\log(n+1)}{n} + \frac{1}{n^2} \log k_n(q_{1,n}, \dots, q_{n,n}). \end{aligned}$$

*Proof.* — Clearly the statement of the lemma follows once we prove

$$(20) \quad \begin{aligned} 2^{-n} k_n(q_{1,n}, \dots, q_{n,n}) & < \text{Card } \mathcal{S}(\alpha, m) \\ & < 2^{2n+1} (n+1)^n k_n(q_{1,n}, \dots, q_{n,n}). \end{aligned}$$

First of all observe that if  $S \in \mathcal{S}(\alpha, m)$  and we write  $p_r = \text{Card}(S \cap \{(r-1)n+1, \dots, rn\})$ ,  $1 \leq r \leq n$ , then Definition 1.6 implies  $(p_1, \dots, p_n) \preceq (z_{0,n}, \dots, z_{(n-1)n,nn})$ .

Consider now any sequence  $(p_1, \dots, p_n) \in \{0, 1, \dots, n\}^n$  satisfying

$$(p_1, \dots, p_n) \preceq (z_{0,n}, \dots, z_{(n-1)n,nn}).$$

As  $m - n^2 < 2n + 1$ , the family of subsets of  $\{1, \dots, m\}$  having exactly  $p_r$  elements from  $\{(r - 1)n + 1, \dots, rn\}$  for any  $r$  has cardinality less than

$$2^{2n+1} k_n \left( \min\{p_1, \lfloor \frac{1}{2}n \rfloor\}, \dots, \min\{p_n, \lfloor \frac{1}{2}n \rfloor\} \right)$$

and then less than  $2^{2n+1} k_n(q_{1,n}, \dots, q_{n,n})$ . Since  $\{0, 1, \dots, n\}^n$  has exactly  $(n + 1)^n$  elements, we get the second inequality in (20).

On the other hand suppose that  $S \subset \{1, \dots, m\}$  does not intersect  $\{1, \dots, n\}$  nor  $\{n^2 + 1, \dots, m\}$  and has exactly  $q_{r-1,n}$  elements from  $\{(r - 1)n + 1, \dots, rn\}$ ,  $2 \leq r \leq n$ . Notice that if  $1 \leq i \leq m$  with  $(r - 1)n < i \leq rn$ ,  $2 \leq r \leq n$ , then  $S \cap \{1, \dots, i\}$  has at most  $q_{1,n} + \dots + q_{r-1,n}$  elements. Since  $q_{1,n} + \dots + q_{r-1,n} \leq z_{0,(r-1)n} \leq z_{0,i}$  we conclude that  $S$  belongs to  $\mathcal{S}(\alpha, m)$ . Thus

$$\binom{n}{q_{1,n}} \binom{n}{q_{2,n}} \dots \binom{n}{q_{n-1,n}} \leq \text{Card } \mathcal{S}(\alpha, m)$$

and since  $\binom{n}{q_{n,n}} < 2^n$ , the first inequality in (20) is proved (when writing  $2 \leq r \leq n$  we have assumed  $n \geq 2$ ; if  $n = 1$  then the statement is trivial).  $\square$

LEMMA 4.8. — *With the above notation,*

$$G_n \left( F_n \left( \frac{z_{0,n}}{n}, \frac{z_{n,2n}}{n}, \dots, \frac{z_{(n-1)n,nn}}{n} \right) \right) - \frac{1}{n^2} \log k_n(q_{1,n}, q_{2,n}, \dots, q_{n,n}) \rightarrow 0$$

as  $n \rightarrow \infty$ .

*Proof.* — Fix  $\epsilon$  and suppose  $n$  be large enough so that  $|x - y| \leq 1/n$  implies  $|g(x) - g(y)| \leq \frac{1}{2}\epsilon$  and  $|g_n(x) - g(x)| \leq \frac{1}{2}\epsilon$  for any  $x \in [0, \frac{1}{2}]$  (cf. Proposition 4.6). Denote  $(c_1, \dots, c_n) = F_n(z_{0,n}, \dots, z_{(n-1)n,nn})$  and write  $c'_i = \min\{c_i, \frac{1}{2}n\}$ ,  $1 \leq i \leq n$ . Then

$$\begin{aligned} G_n(F_n(z_{0,n}/n, \dots, z_{(n-1)n,nn}/n)) &= G_n(c_1/n, \dots, c_n/n) \\ &= G_n(c'_1/n, \dots, c'_n/n) \\ &= \text{mean}(g(c'_1/n), \dots, g(c'_n/n)) \\ &\leq \frac{\epsilon}{2} + \text{mean}(g([c'_1]/n), \dots, g([c'_n]/n)) \\ &\leq \epsilon + \text{mean}(g_n([c'_1]/n), \dots, g_n([c'_n]/n)) \\ &= \epsilon + \frac{1}{n^2} \log k_n([c'_1], \dots, [c'_n]) \\ &\leq \epsilon + \frac{1}{n^2} \log k_n(q_{1,n}, \dots, q_{n,n}); \end{aligned}$$

we have used Lemma 4.1 (ii).

Conversely,

$$\begin{aligned} \frac{1}{n^2} \log k_n(q_{1,n}, \dots, q_{n,n}) &= \text{mean}(g_n(q_{1,n}/n), \dots, g_n(q_{n,n}/n)) \\ &\leq \frac{\epsilon}{2} + G_n(q_{1,n}/n, \dots, q_{n,n}/n) \\ &\leq \frac{\epsilon}{2} + G_n(F_n(z_{0,n}/n, \dots, z_{(n-1)n,nn}/n)); \end{aligned}$$

here we have used Proposition 4.4. The lemma is proved. □

Combining Lemmas 4.7 and 4.8 we finally get:

PROPOSITION 4.9. — *With the above notation,*

$$h_\alpha = \limsup_{n \rightarrow \infty} G_n \left( F_n \left( \frac{z_{0,n}}{n}, \frac{z_{n,2n}}{n}, \dots, \frac{z_{(n-1)n,nn}}{n} \right) \right).$$

### 5. Proof of Theorem B and its corollaries.

*Proof of Theorem B.* — Write  $\sigma = 1 - \alpha$ , that is,  $\sigma_i = 1 - \alpha_i$  for any  $i$ . Define the numbers  $z_{r,s}$  as in the previous section, fix for the moment  $n \in \mathbb{N}$  and write

$$\begin{aligned} \gamma_{(r-1)n+1} &= \gamma_{(r-1)n+1} = \dots = \gamma_{rn} \\ &= z_{(r-1)n,rn}/n = \text{mean}(\sigma_{(r-1)n+1}, \sigma_{(r-1)n+2}, \dots, \sigma_{rn}) \end{aligned}$$

for any  $1 \leq r \leq n$ . First of all we emphasize that

$$(21) \quad G_n \left( F_n \left( \frac{z_{0,n}}{n}, \frac{z_{n,2n}}{n}, \dots, \frac{z_{(n-1)n,nn}}{n} \right) \right) = G_{n^2} (F_{n^2}(\gamma_1, \dots, \gamma_{n^2}))$$

by Lemma 4.1 (v). Further, we claim that

$$(22) \quad \left| G_{n^2} (F_{n^2}(\sigma_1, \dots, \sigma_{n^2})) - G_{n^2} (F_{n^2}(\gamma_1, \dots, \gamma_{n^2})) \right| \leq \frac{\log 2}{n}.$$

In fact observe that

$$\begin{aligned} G_{(n+1)n} (F_{(n+1)n}(0, \dots, 0, \sigma_1, \dots, \sigma_{n^2})) &= \frac{n}{n+1} G_{n^2} (F_{n^2}(\sigma_1, \dots, \sigma_{n^2})), \\ G_{(n+1)n} (F_{(n+1)n}(\gamma_1, \dots, \gamma_{n^2}, 1, \dots, 1)) &= \frac{n}{n+1} G_{n^2} (F_{n^2}(\sigma_1, \dots, \sigma_{n^2})) \\ &\quad + \frac{\log 2}{n+1} \end{aligned}$$

by Lemma 4.1 (iii) and (iv); since

$$(0, \dots, 0, \sigma_1, \dots, \sigma_{n^2}) \preceq (\gamma_1, \gamma_2, \dots, \gamma_{n^2}, 1, \dots, 1)$$

and then

$$\begin{aligned} G_{(n+1)n}(F_{(n+1)n}(0, \dots, 0, \sigma_1, \dots, \sigma_{n^2})) \\ \leq G_{(n+1)n}(F_{(n+1)n}(\gamma_1, \dots, \gamma_{n^2}, 1, \dots, 1)) \end{aligned}$$

by Corollary 4.5, we deduce

$$G_{n^2}(F_{n^2}(\sigma_1, \dots, \sigma_{n^2})) \leq G_{n^2}(F_{n^2}(\gamma_1, \dots, \gamma_{n^2})) + \frac{\log 2}{n}.$$

Analogously,  $(0, \dots, 0, \gamma_1, \dots, \gamma_{n^2}) \preceq (\sigma_1, \dots, \sigma_{n^2}, 1, \dots, 1)$  implies

$$G_{n^2}(F_{n^2}(\gamma_1, \dots, \gamma_{n^2})) \leq G_{n^2}(F_{n^2}(\sigma_1, \dots, \sigma_{n^2})) + \frac{\log 2}{n}$$

and (22) follows. Finally, observe that if  $n^2 < m \leq (n + 1)^2$  then

$$\begin{aligned} (23) \quad G_m(F_m(\sigma_1, \dots, \sigma_m)) &\leq G_m(F_m(\sigma_1, \dots, \sigma_{n^2}, 1, \dots, 1)) \\ &= \frac{n^2}{m} G_{n^2}(F_{n^2}(\sigma_1, \dots, \sigma_{n^2})) + \frac{m - n^2}{m} \log 2 \\ &< G_{n^2}(F_{n^2}(\sigma_1, \dots, \sigma_{n^2})) + \frac{2n + 1}{n^2} \log 2. \end{aligned}$$

Merging Proposition 4.9 and (21), (22) and (23), the theorem follows.  $\square$

*Proof of Corollary B.1.* — Fix  $n \in \mathbb{N}$  and assume  $\rho > 0$  be small enough so that  $|x - y| \leq \delta_\epsilon + \rho$  implies  $|g(x) - g(y)| \leq \epsilon + 1/n$  for any  $x, y \in [0, 1]$ . Write  $\sigma = \mathbf{1} - \alpha$ ,  $\tau = \mathbf{1} - \beta$  and use  $\blacktriangle(\alpha, \beta) \leq \delta_\epsilon$  to find an integer  $r$  large enough so that if

$$\gamma = (0-r-0, \sigma_1, \sigma_2, \dots)$$

then

$$(\gamma_1, \dots, \gamma_m) \preceq (\tau_1 + \delta_\epsilon + \rho, \dots, \tau_m + \delta_\epsilon + \rho) \quad \text{for any } m.$$

Finally define  $v : [0, 1 + \delta_\epsilon + \rho] \rightarrow \mathbb{R}$  by  $v(y) = g(y)$  if  $y \in [0, 1]$  and  $v(y) = \log 2$  otherwise. Since  $v$  is concave and increasing we can apply

Corollary 4.5 to the corresponding maps  $V_m$ . Let  $m > r$ . Then

$$\begin{aligned} G_m(F_m(\sigma_1, \dots, \sigma_m)) &\leq \frac{m-r}{m} G_{m-r}(F_{m-r}(\sigma_1, \dots, \sigma_{m-r})) + \frac{r}{m} \log 2 \\ &= G_m(F_m(\gamma_1, \dots, \gamma_m)) + \frac{r}{m} \log 2 \\ &= V_m(F_m(\gamma_1, \dots, \gamma_m)) + \frac{r}{m} \log 2 \\ &\leq V_m(F_m(\tau_1 + \delta_\epsilon + \rho, \dots, \tau_m + \delta_\epsilon + \rho)) + \frac{r}{m} \log 2 \\ &\leq V_m(F_m(\tau_1, \dots, \tau_m)) + \epsilon + \frac{1}{n} + \frac{r}{m} \log 2 \\ &= G_m(F_m(\tau_1, \dots, \tau_m)) + \epsilon + \frac{1}{n} + \frac{r}{m} \log 2 \end{aligned}$$

(we have used Lemma 4.1 (i)) and hence  $h_\alpha \leq h_\beta + \epsilon + 1/n$ . Since we can prove analogously  $h_\beta \leq h_\alpha + \epsilon + 1/n$  and  $n$  is arbitrary, the proof is finished.  $\square$

*Proof of Corollary B.2.* — Notice that if we use Theorem B to define  $h_\gamma$  for any sequence  $\gamma \in [0, 1]^\infty$  then Corollary B.1 makes sense and holds true for all sequences from  $[0, 1]^\infty$ . In particular if  $\gamma$  is the sequence given by  $\gamma_i = 1 - \lambda$  for any  $i$  then we have  $\blacktriangle(\alpha, \gamma) = 0$  and so  $h_\alpha = h_\gamma = g(\lambda)$ .  $\square$

*Proof of Corollary B.3.* — For any  $m$  let  $z_m$  be the number of zeros of the sequence  $\alpha_1, \dots, \alpha_m$ , when  $\lambda = \limsup_{m \rightarrow \infty} z_m/m$ . We will simply write  $\kappa = \kappa(\lambda)$ .

We begin by proving the second inequality in (4). Trivially we can assume  $\lambda < 1$ . Fix  $0 < \epsilon < 1 - \lambda$ . Then  $z_m/m < \lambda + \epsilon$  for any  $m$  large enough. Since  $(1 - \alpha_1, \dots, 1 - \alpha_m) \preceq (1 - z_m - 1, 0, \dots, 0)$  we get

$$G_m(F_m(1 - \alpha_1, \dots, 1 - \alpha_m)) \leq g\left(\frac{z_m}{m}\right) \leq g(\lambda + \epsilon)$$

for all large  $m$ . Thus  $h_\alpha \leq g(\lambda + \epsilon)$  for any  $\epsilon > 0$  and then  $h_\alpha \leq g(\lambda)$ .

Proving the other inequality requires a bit more of effort. Fix again  $\epsilon > 0$  and take a small  $0 < \delta < \kappa$  with the property that

$$(24) \quad \frac{\lambda + \kappa - 2\delta}{1 + \kappa + \delta} g\left(\frac{\lambda - \delta}{\lambda + \kappa + 2\delta}\right) > \log\left(\frac{\lambda + \kappa}{\kappa}\right) - \epsilon;$$

this makes senses since

$$(25) \quad \frac{\lambda + \kappa}{1 + \kappa} g\left(\frac{\lambda}{\lambda + \kappa}\right) = \frac{\lambda + \kappa}{1 + \kappa} \log \left[ \left(\frac{\lambda}{\lambda + \kappa}\right)^{-\frac{\lambda}{\lambda + \kappa}} \left(\frac{\kappa}{\lambda + \kappa}\right)^{-\frac{\kappa}{\lambda + \kappa}} \right]$$

$$\begin{aligned} &= \frac{1}{1 + \kappa} \log \left[ \left( \frac{\lambda}{\lambda + \kappa} \right)^{-\lambda} \left( \frac{\kappa}{\lambda + \kappa} \right)^{-\kappa} \right] \\ &= \frac{1}{1 + \kappa} \log \left[ \frac{\lambda + \kappa}{\kappa} \left( \frac{\kappa}{\lambda + \kappa} \right)^{-\kappa} \right] \\ &= \log \left( \frac{\lambda + \kappa}{\kappa} \right). \end{aligned}$$

Let  $m_0$  be fixed. Then there is  $m > \max\{m_0, 1/\delta\}$  such that

$$(26) \quad \lambda - \delta < \frac{z_m}{m} < \lambda + \delta;$$

further,  $1/m < \delta$  implies the existence of an integer  $r_m$  satisfying

$$(27) \quad \kappa - \delta < \frac{r_m}{m} < \kappa + \delta.$$

Now  $(0 - m - z_m - 0, 1 - z_m - 1, 0 - r_m - 0) \preceq (1 - \alpha_1, \dots, 1 - \alpha_{m+r_m})$  and (24), (26) and (27) guarantee

$$\begin{aligned} G_m(F_m(1 - \alpha_1, \dots, 1 - \alpha_{m+r_m})) &\geq \frac{(z_m + r_m)}{m + r_m} g\left(\frac{z_m}{z_m + r_m}\right) \\ &= \frac{(z_m/m + r_m/m)}{1 + r_m/m} g\left(\frac{z_m/m}{z_m/m + r_m/m}\right) \\ &> \log\left(\frac{\lambda + \kappa}{\kappa}\right) - \epsilon. \end{aligned}$$

Thus  $h_\alpha \geq \log((\lambda + \kappa)/\kappa)$  as we desired to prove.

It only rests to show that the first equality in (4) is possible (for the second one just use Corollary B.2). To this end fix  $0 < \lambda < 1$ , put  $\kappa = \kappa(\lambda)$  and construct sequences of positive integers  $(o_n)_n$  and  $(p_n)_n$  with the properties

$$(28) \quad \frac{o_1}{o_1 + p_1} > \frac{o_1 + o_2}{o_1 + p_1 + o_2 + p_2} > \dots > \frac{o_1 + \dots + o_n}{o_1 + p_1 + \dots + o_n + p_n} \rightarrow 0,$$

$$(29) \quad \frac{p_1 + \dots + p_n}{p_{n+1}} \rightarrow 0,$$

$$(30) \quad \frac{o_{n+1}}{o_{n+1} + p_n} \rightarrow \lambda.$$

Finally define

$$\alpha = (0 - o_1 - 0, 1 - p_1 - 1, 0 - o_2 - 0, 1 - p_2 - 1, \dots, 0 - o_n - 0, 1 - p_n - 1, \dots).$$

Clearly, the above conditions imply

$$\limsup_{m \rightarrow \infty} \text{mean}(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_m) = \lambda;$$

let us prove that  $h_\alpha = \log((\lambda + \kappa)/\kappa)$ . Denote first of all

$$u(x) = \frac{\lambda + x}{1 + x} g\left(\frac{\lambda}{\lambda + x}\right), \quad x \in [0, \infty);$$

it is rutinary to verify that

$$u'(x) = \begin{cases} \frac{(1 - \lambda) \log 2}{(1 + x)^2} & \text{if } 0 \leq x \leq \lambda, \\ \frac{\log((\lambda + x)/x) - \lambda \log((\lambda + x)/\lambda)}{1 + x^2} & \text{if } x > \lambda, \end{cases}$$

and then that  $u(x)$  has a strict absolute maximum at  $\kappa$ . Fix  $\epsilon > 0$  and find  $\delta > 0$  satisfying

$$(31) \quad \frac{\lambda + \delta + x}{1 + x} g\left(\frac{\lambda + \delta}{\lambda - \delta + x}\right) < u(x) + \frac{\epsilon}{2}$$

for any  $x$ ; clearly, such a number  $\delta$  exists. Next apply (28), (29) and (30) to find a number  $n_0$  large enough so that

$$(32) \quad g\left(\frac{o_1 + \dots + o_n}{o_1 + p_1 \dots + o_n + p_n}\right) < \frac{\epsilon}{2},$$

$$(33) \quad \lambda - \delta < \frac{o_{n+1}}{o_1 + p_1 \dots + o_n + p_n + o_{n+1}} < \lambda + \delta$$

for any  $n \geq n_0$ . We claim that

$$(34) \quad \text{if } m \geq \sum_{i=1}^{n_0} (o_i + p_i) \text{ then} \\ G_m(F_m(1 - \alpha_1, \dots, 1 - \alpha_m)) < \log\left(\frac{\lambda + \kappa}{\kappa}\right) + \epsilon;$$

this will finish the proof.

Say  $\sum_{i=1}^n (o_i + p_i) \leq m \leq \sum_{i=1}^{n+1} (o_i + p_i)$  and put  $o = \sum_{i=1}^n o_i$ ,  $r = \sum_{i=1}^n (o_i + p_i)$ ,  $s = m - r$ . Also, let  $\ell < p_{n+1}$  be the real number satisfying (cf. (28))

$$\frac{o}{r} = \frac{o + o_{n+1}}{r + o_{n+1} + \ell}.$$

Observe that if  $s \leq o_{n+1}$  then

$$\begin{aligned} G_m(F_m(1 - \alpha_1, \dots, 1 - \alpha_m)) &= \frac{r}{m} g\left(\frac{o}{r}\right) + \frac{s}{m} \log 2 \\ &\leq \frac{r}{r + o_{n+1}} g\left(\frac{o}{r}\right) + \frac{o_{n+1}}{r + o_{n+1}} \log 2 \\ &= G_{r+o_{n+1}}(F_{r+o_{n+1}}(1 - \alpha_1, \dots, 1 - \alpha_{r+o_{n+1}})) \end{aligned}$$

by (28), while if  $s \geq o_{n+1} + \ell$  then

$$\begin{aligned} G_m(F_m(1 - \alpha_1, \dots, 1 - \alpha_m)) &= g\left(\frac{o + o_{n+1}}{m}\right) \leq g\left(\frac{o + o_{n+1}}{r + o_{n+1} + \ell}\right) = g\left(\frac{o}{r}\right) \\ &= G_r(F_r(1 - \alpha_1, \dots, 1 - \alpha_r)) \\ &< G_{r+o_{n+1}}(F_{r+o_{n+1}}(1 - \alpha_1, \dots, 1 - \alpha_{r+o_{n+1}})) \end{aligned}$$

again by (28); hence, to prove (34), it is not restrictive to assume  $o_{n+1} \leq s < o_{n+1} + \ell$ , when  $x = (s - o_{n+1})/(r + o_{n+1}) \geq 0$ . Now use (32), (33), (31) and (25) to get

$$\begin{aligned} G_m(F_m(1 - \alpha_1, \dots, 1 - \alpha_m)) &= \frac{r}{m} g\left(\frac{o}{r}\right) + \frac{s}{m} g\left(\frac{o_{n+1}}{s}\right) \\ &< \frac{\epsilon}{2} + \frac{x + o_{n+1}/(r + o_{n+1})}{1 + x + o_{n+1}/(r + o_{n+1})} g\left(\frac{o_{n+1}/(r + o_{n+1})}{x + o_{n+1}/(r + o_{n+1})}\right) \\ &< \epsilon + u(x) \leq \epsilon + u(\kappa) = \epsilon + \log((\lambda + \kappa)/\kappa). \quad \square \end{aligned}$$

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