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BESICOVITCH SUBSETS OF SELF-SIMILAR SETS

by J.-H. MA, Z.-Y. WEN and J. WU

1. Introduction.

Besicovitch [1] and Eggleston [2] considered subsets of points of the unit interval with given frequencies in the digits of their m-ary expansions; M. Moran and J. Rey in [5] extended the analysis to self-similar sets, and conjectured that "the Besicovitch subsets of self-similar sets have infinite Hausdorff measure in their dimension". Following the pioneer work of R. Kaufman [4] on the classical dyadic Besicovitch sets and that of Y. Peres [6] on the Mc Mullen sets, we give a complete classification of the Hausdorff and packing gauge functions of the Besicovitch subsets of self-similar sets, this extends the results of R. Kaufman, as a corollary, we prove the conjecture of Moran and Rey positively.

We recall first some result about self-similar sets, for more details, we refer to [3]. Given an integer $m \geqslant 2$, let $\{\phi_0, \phi_1, \cdots, \phi_{m-1}\}$ be similarity contracting with similarity ratio $\{r_0, r_1, \cdots, r_{m-1}\}$, and let E be the self-similar set of the family of the similarities. Suppose that the open set condition is satisfied (i.e., there exists a non-empty bounded open set V such that $V \supset \cup_{i=1}^{m} \phi_i V$ with the union disjoint), then $\dim_H E = \dim_P E = \dim_B E = s$ and moreover $0 < \mathcal{H}^s(E) \leqslant \mathcal{P}^s(E) < \infty$, where s is the unique positive solution of the equation $\sum_{j=0}^{m-1} r_j^s = 1$.

For the sake of simplicity, we shall work in the unit interval, and assume that $\{\phi_j([0,1])\}_{j=0}^{m-1}$ be a collection of disjoint sub-intervals of $[0,1]$.

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Let $S = \{0, 1, \cdots, m-1\}$ and $\Omega = \{0, 1, \cdots, m-1\}^{\mathbb{N}}$ to be the one-sided symbolic space. Define $I_{x_1,\cdots,x_n} = \phi_{x_1} \circ \phi_{x_2} \cdots \circ \phi_{x_n}(I)$ to be the n-level basic interval with $(x_1, \cdots, x_n) \in S^n$. In what follows, we adopt the following conventions:

1. The coding mapping $\pi\colon \Omega \to E$ is defined as
$$\pi((x_n)_{n=1}^{\infty}) = \bigcap_{n=1}^{\infty} I_{x_1,\cdots,x_n}.$$
2. If no confusion happens, $x = (x_n)_{n=1}^{\infty}$ denotes both an element of Ω and the point $\bigcap_{n=1}^{\infty} I_{x_1,\cdots,x_n}$ in E.
3. Let $x = x_1 \cdots x_n \cdots \in \Omega$, the n-level cylinder on Ω containing $x \in \Omega$, denoted by $I_n(x)$, is defined as
$$I_n(x) = \{y \in \Omega : \ y_1 = x_1, \cdots, y_n = x_n\}.$$
If no confusion happens, $I_n(x)$ also denotes the n-level basic interval $\pi(I_n(x))$ containing $\pi(x) \in E$.

Given a probability vector $\vec{p} = (p_0, p_1, \cdots, p_{m-1})$ $(p_j \geqslant 0, \sum_{j=0}^{m-1} p_j = 1)$, we can define a Besicovitch-type subset of E(see [5]) as follows:
$$E(\vec{p}) = \Big\{x \in E\colon \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_j(x_k) = p_j, \quad 0 \leqslant j \leqslant m-1\Big\},$$
where χ_j is the indicator function of the set $\{j\}$.

We first collect some known facts:

- $E(\vec{p})$ is a non-compact invariant set of the system $\{\phi_0, \phi_1, \cdots, \phi_{m-1}\}$;
- $E(\vec{p})$ is dense in E, hence, of box-dimension $s = \dim E$;
- $\dim_H(E(\vec{p})) = \dim_P(E(\vec{p})) = \dfrac{\sum_{j=0}^{m-1} p_j \log p_j}{\sum_{j=0}^{m-1} p_i \log r_j}$.

Now we formulate our main theorem:

THEOREM 1.1. — *Let*
$$\alpha = \dim_H(E(\vec{p})) = \dim_P(E(\vec{p})) = \frac{\sum_{j=0}^{m-1} p_j \log p_j}{\sum_{j=0}^{m-1} p_i \log r_j}$$
and g a gauge function, then

(i) *If $\vec{p} = \vec{r} := (r_0^s, r_1^s, \cdots, r_{m-1}^s)$ (which we call the* **compatible case***), then we have*
$$\mathcal{H}^s(E(\vec{p})) = \mathcal{H}^s(E), \ \mathcal{P}^s(E(\vec{p})) = \mathcal{P}^s(E).$$

(ii) *If $\vec{p} = (p_0, p_1, \cdots, p_{m-1})$ is a positive probability vector other than $\vec{r}$, then the gauge functions can be partitioned as follows:*

$$\mathcal{H}^g(E(\vec{p})) = +\infty \iff \overline{\lim_{t\to 0}} \frac{\log g(t)}{\log t} \leqslant \alpha, \tag{1}$$

$$\mathcal{H}^g(E(\vec{p})) = 0 \iff \overline{\lim_{t\to 0}} \frac{\log g(t)}{\log t} > \alpha, \tag{2}$$

$$\mathcal{P}^g(E(\vec{p})) = +\infty \iff \underline{\lim_{t\to 0}} \frac{\log g(t)}{\log t} \leqslant \alpha, \tag{3}$$

$$\mathcal{P}^g(E(\vec{p})) = 0 \iff \underline{\lim_{t\to 0}} \frac{\log g(t)}{\log t} > \alpha. \tag{4}$$

Remark 1.2. — 1) Let $\phi_j(t) = \frac{t+j}{m}$, $0 \leqslant j \leqslant m-1$, then $E = [0,1]$ and $E(\vec{p})$ is the classical Besicovitch-Eggleston set.

2) Our argument can be passed through to the higher dimensional analogue, hence, by taking $g(t) = t^\alpha$ in (1) of Theorem 1.1, we give an affirmative answer to the conjecture of Moran and Rey in [5] that $E(\vec{p})$ has infinite Hausdorff measure in the dimension.

2. Preliminaries.

2.1. Gauge functions.

We call $g: [0, +\infty) \to [0, +\infty)$ a **gauge function** if it is non-decreasing, right-continuous such that

- $g(t) = 0 \iff t = 0$;
- $g(2t) \leqslant \zeta g(t)$, where $\zeta \geqslant 1$ is a constant.

Let $A \subset [0,1]$ and g a gauge function. For any $r > 0$, define

$$H_r^g(A) = \inf\Big\{\sum_{k=1}^{\infty} g(|U|) \colon \{U_k\}_{k=1}^{\infty} \text{ a } r\text{-covering of } A\Big\},$$

and

$$\mathcal{H}^g(A) = \lim_{r\to 0} H_r^g(A),$$

$\mathcal{H}^g(A)$ is called the g-**Hausdorff measure** of A. If $g(t) = t^s$, then $\mathcal{H}^g(A)$ is the classical s-dimensional Hausdorff measure of the set A.

By the same way, we can define g-**packing measure** $P^g(A)$ **of** A, for more details, see [3].

Let $A \subset [0,1]$, if there exists a gauge function g such that $0 < \mathcal{H}^g(A) < \infty$ (resp. $0 < P^g(A) < \infty$), then we say that g is the **Hausdorff gauge (resp. packing gauge)** of the set A, otherwise, we say that A has no Hausdorff (resp. packing) gauge.

2.2. Perturbation of the Bernoulli measures.

We denote by $\mathbf{M}(\Omega)$ the collection of Borel probability measures on Ω, for each positive probability vector $\vec{p} = (p_0, p_1, \cdots, p_{m-1})(p_j > 0, \sum p_j = 1, 0 \leqslant j \leqslant m-1)$, there is an associated Bernoulli measure $\mu_p \in \mathbf{M}(\Omega)$ which satisfies for each $x \in \Omega$:

$$\mu_p(I_n(x)) = \prod_{k=1}^{n} p_{x_k}.$$

This measure plays an important role in the study of classical Besicovich-Eggleston set, but we need a kind of more subtle measures in the present paper that will be obtained by following mainly the methods of [4] and [6].

A sequence $\delta = \{\delta_n\}_{n=1}^{\infty}$ is called a **perturbation factor** provided

$$\lim_{n\to\infty} \delta_n = 0, \tag{5}$$

$$\frac{1}{\log(n+2)} \leqslant \delta_n < 1. \tag{6}$$

Take another Bernoulli measure μ_q, called **perturbation source**, defined by a probability vector $\vec{q} = (q_0, q_1, \cdots, q_{m-1})$. We now define a sequence of positive probability vectors as follows:

$$\vec{p}^{(k)} = (1-\delta_k)\vec{p} + \delta_k \vec{q}, \tag{7}$$

which introduces a measure $\mu_p^{(\delta,q)} \in \mathbf{M}(\Omega)$ such that

$$\mu_p^{(\delta,q)}(I_n(x)) = \prod_{k=1}^{n} p_{x_k}^{(k)}. \tag{8}$$

The measure $\mu_p^{(\delta,q)}$ is called the **perturbation measure** of μ_p with **perturbation factor** δ and **perturbation source** μ_q(or $\vec{q} = (q_0, q_1, \cdots, q_{m-1})$).

The following two propositions (the first is classical, see [7]) will be used in the proof of Theorem 1.1.

PROPOSITION 2.1. — *Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent random variables with finite second moments. If there are positive numbers a_n such that $a_n \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} \frac{\mathbb{V}[X_n]}{a_n^2} < \infty,$$

then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k - \sum_{k=1}^n \mathbb{E}[X_k]}{a_n} = 0 \quad \text{a.e.},$$

where $\mathbb{V}$ and $\mathbb{E}$ stand for mathematical expectation and variance respectively.

PROPOSITION 2.2. — *Given a perturbation factor $\delta = \{\delta_n\}_{n=1}^{\infty}$ and a perturbation source $\vec{q} = (q_0, q_1, \cdots, q_{m-1}) \neq \vec{p} = (p_0, p_1, \cdots, p_{m-1})$, then for $\mu_p^{(\delta,q)}$ –* a.e. *$x \in \Omega$:*

(i)

$$\log[\mu_p^{(\delta,q)}(I_n(x))] = n \sum_{j=0}^{m-1} p_j \log p_j - (\Delta + o(1)) \sum_{k=1}^{n} \delta_k$$

where $\Delta = \sum_{j=0}^{m-1} (p_j - q_j) \log p_j$;

(ii)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_j(x_k) = p_j,\ 0 \leqslant j \leqslant m-1.$$

Proof. — With respect to the probability measure $\mu_p^{(\delta,q)}$, $\{x_k\}_{k=1}^{\infty}$ can be regarded as a sequence of independent random variables.

(i) Let $X_k = \log p_{x_k}^{(k)}$, a simple calculation yields

$$\mathbb{E}[X_k] = \sum_{j=0}^{m-1} p_j^{(k)} \log p_j^{(k)}$$

and

$$\mathbb{V}[X_k] = \sum_{j=0}^{m-1} p_j^{(k)} (\log p_j^{(k)})^2 - (\mathbb{E}[X_k])^2.$$

Since $\sum_{k=2}^{\infty} \frac{1}{k(\log k)^2} < \infty$, by Proposition 2.1 we get

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} \mathbb{E}\,[X_k]}{\sqrt{n}\log n} = 0, \quad \mu_p^{(\delta,q)} - \text{a.e.} \tag{9}$$

From (7), $p_j^{(k)} = (1-\delta_k)p_j + \delta_k q_j$, therefore by expanding the function $z\log z$ at $z_0 = p_j$, the Taylor's formula gives

$$p_j^{(k)} \log p_j^{(k)} = p_j \log p_j - \delta_k(p_j - q_j)(1 + \log p_j) + O(\delta_k^2),$$

thus

$$\mathbb{E}\,[X_k] = \sum_{j=0}^{m-1} p_j \log p_j - \Delta\delta_k + O(\delta_k^2)$$

where $\Delta = \sum_{j=0}^{m-1}(p_j - q_j)\log p_j$. On the other hand, by (5) and (6), we get $\sum_{k=1}^{n} \delta_k^2 = o(1)\sum_{k=1}^{n}\delta_k$, $n \to \infty$, we get thus

$$\sum_{k=1}^{n} \mathbb{E}\,[X_k] = n\sum_{j=0}^{m-1} p_j \log p_j - (\Delta + o(1))\sum_{k=1}^{n}\delta_k.$$

By (6), $\sum_{k=1}^{n}\delta_k \geqslant \frac{n}{\log(n+2)} \geqslant \sqrt{n}\log n$ for n large enough, then by (9), we conclude that for $\mu_p^{(\delta,q)}-$ a.e. $\omega \in \Omega$:

$$\sum_{K=1}^{n} X_k = \sum_{k=1}^{n} \mathbb{E}\,[X_k] + o(1)\sqrt{n}\log n = n\sum_{j=0}^{m-1} p_j \log p_j - (\Delta + o(1))\sum_{k=1}^{n}\delta_k$$

which yields the conclusion of (i).

(ii) By (5), $\lim_{k\to\infty}\delta_k = 0$, so $\sum_{k=1}^{n}\delta_k = o(n)$, on the other hand, notice that

$$\mathbb{E}(\chi_j(x_k)) = p_j^{(k)} = p_j + \delta_k(q_j - p_j),$$

we have therefore

$$\mathbb{E}\Big(\sum_{k=1}^{n}\chi_j(x_k)\Big) = np_j + (q_j - p_j)\sum_{k=1}^{n}\delta_k = n(p_j + o(1)).$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$ and $\mathbb{V}(\chi_j(x_k)) = p_j^{(k)} - (\mathbb{E}(\chi_j(x_k)))^2$, we have by Proposition 2.1,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_j(x_k) = p_j,\ 0 \leqslant j \leqslant m-1, \quad \mu_p^{(\delta,q)} - \text{a.e.}$$

□

3. Proof of the main theorem.

The following variant of the density theorem is essentially due to Taylor, see p. 684 of [8],

PROPOSITION 3.1. — *Let A be a subset of the self-similar set E, μ a probability measure supported by E with $\mu(A) > 0$. Let g be a gauge function and $a > 0$ a constant then*

1. *If for each $x \in A$, $\overline{\lim}_{n\to\infty} \frac{\mu(I_n(x))}{g(|I_n(x)|)} \leqslant a$, then $\mathcal{H}^g(A) \geqslant c_1 a^{-1}$.*
2. *If for each $x \in A$, $\underline{\lim}_{n\to\infty} \frac{\mu(I_n(x))}{g(|I_n(x)|)} \leqslant a$, then $\mathcal{P}^g(A) \geqslant c_2 a^{-1}$,*

where c_1 and c_2 are positive constants.

PROPOSITION 3.2. — *Except for the compatible case that $p_j = r_j^s (0 \leqslant j \leqslant m-1)$, there must exist $0 \leqslant l \leqslant m-1$, such that $p_l < r_l^\alpha$.*

Proof. — By the concavity of the function "log", we have
$$\sum_{j=0}^{m-1} p_j \log \frac{r_j^s}{p_j} \leqslant \log\left(\sum_{j=0}^{m-1} p_j \frac{r_j^s}{p_j}\right) = 0$$
where the equality holds if and only if $p_j = r_j^s (0 \leqslant j \leqslant m-1)$.

Thus $\alpha = s$ if $p_j = r_j^s$, and $\alpha < s$ otherwise.

So in the non-compatible cases, we have
$$\sum_{j=0}^{m-1} r_j^\alpha > \sum_{j=0}^{m-1} r_j^s = 1 = \sum_{j=0}^{m-1} p_j$$
which gives the desired result. □

Let λ denote the image measure of $\mu_{\vec{p}}^{\delta,\vec{q}}$ under the coding mapping π.

LEMMA 3.3. — *Suppose that there is a sequence $\{t_n \downarrow 0\}_{n\geqslant 1} \subset (0, \min_j\{r_j\})$ with $\lim_{n\to\infty} \frac{\log g(t_n)}{\log t_n} = \alpha$. Then there exists a Borel set $E_1 \subset E$ with $\lambda(E_1) > 0$, a sequence of non-negative numbers $\{\epsilon_n\}_{n=1}^\infty$, and a perturbation factor $\delta = \{\delta_k\}_{k=1}^\infty$ such that*

(1) $\lim_{n\to\infty} \frac{n\epsilon_n}{\sum_{k=1}^n \delta_k} = 0$,

(2) *for any $x \in E_1$ and $n \in \mathbb{N}$,*
$$\frac{\log g(|I_{k(x,n)}(x)|)}{\log |I_{k(x,n)}(x)|} \leqslant \alpha + \epsilon_n,$$

where $k(x,n)$ is determined by the inequality

$$\prod_{j=1}^{1+k(x,n)} r_{x_j} < t_n \leqslant \prod_{j=1}^{k(x,n)} r_{x_j}. \tag{10}$$

Proof. — For any $x \in E$, set

$$\epsilon_n(x) = \begin{cases} 0, & \text{if } \frac{\log g(|I_{k(x,n)}(x)|)}{\log |I_{k(x,n)}(x)|} \leqslant \alpha, \\ \frac{\log g(|I_{k(x,n)}(x)|)}{\log |I_{k(x,n)}(x)|} - \alpha, & \text{if } \frac{\log g(|I_{k(x,n)}(x)|)}{\log |I_{k(x,n)}(x)|} > \alpha. \end{cases}$$

By (10), we have

$$g(t_n) \leqslant g\left(\prod_{j=1}^{k(x,n)} r_{x_j}\right)$$

and

$$\prod_{j=1}^{k(x,n)} r_{x_j} = \frac{\prod_{j=1}^{1+k(x,n)} r_{x_j}}{r_{1+k(x,n)}} < \frac{t_n}{\min_{0\leqslant j\leqslant m-1}\{r_j\}},$$

from this inequality and noting that $|I_{k(x,n)}(x)| = \prod_{j=1}^{k(x,n)} r_{x_j}$, we have

$$\frac{\log g(|I_{k(x,n)}(x)|)}{\log |I_{k(x,n)}(x)|} < \frac{\log g(t_n)}{\log t_n - \log(\min_{0\leqslant j\leqslant m-1}\{r_j\})}$$

which implies

$$\varlimsup_{n\to\infty} \frac{\log g(|I_{k(x,n)}(x)|)}{\log |I_{k(x,n)}(x)|} \leqslant \lim_{n\to\infty} \frac{\log g(t_n)}{\log t_n} = \alpha,$$

hence for any $x \in E$, by the definition of $\epsilon_n(x)$, we have $\lim_{n\to\infty} \epsilon_n(x) = 0$. Thus Egorov's theorem asserts that there exists a Borel set $E_1 \subset E$ with $\lambda(E_1) > 0$ such that $\epsilon_n(x)$ converges to 0 uniformly on E_1. Now set

$$\epsilon_n = \sup_{x\in E_1} \{\epsilon_n(x)\},$$

then $\lim_{n\to\infty} \epsilon_n = 0$.

Finally, put

$$\delta_k = \max\left\{ \sup_{j\geqslant k}\{\sqrt{\epsilon_j}\},\ \frac{1}{\log(k+2)} \right\},$$

then

$$\frac{n\epsilon_n}{\sum_{k=1}^{n} \delta_k} \leqslant \frac{n\epsilon_n}{n\sqrt{\epsilon_n}} = \sqrt{\epsilon_n} \to 0(n \to \infty),$$

which is the conclusion (1) of the lemma. The conclusion (2) follows from directly the definition of $\epsilon_n(x)$ and ϵ_n. □

Now we are going to prove the main theorem.

(i) In the compatible case, take $\mu(A) = \frac{\mathcal{H}^s(A\cap E)}{\mathcal{H}^s(E)}$ for any Borel set A, then by the scaling property and translation invariance of $\mathcal{H}^s$ (see [3])

$$\mu(I_n(x)) = \prod_{k=1}^{n} r_{x_k}^s,$$

thus $\mu(E(\vec{p})) = 1$ by the strong law of large number with respect to the random variables $\{x_n\}_{n\geqslant 1}$ (see [7]), hence $\mathcal{H}^s(E(\vec{p})) = \mathcal{H}^s(E)$. The same argument yields $\mathcal{P}^s(E(\vec{p})) = \mathcal{P}^s(E)$.

(ii) Now consider the non-compatible case, in this case, take a perturbation measure $\mu_{\vec{p}}^{\delta,\vec{q}} \in \mathbf{M}(\Omega)$, where $\vec{p} = (p_0, p_1, \cdots, p_{m-1})$, the perturbation factor $\delta = \{\delta_n\}_{n=1}^{\infty}$ is taken as in Lemma 3.3 and the perturbation source $\vec{q} = (q_0, q_1, \cdots, q_{m-1})$ will be determined later.

First, by (i) of Proposition 2.2, for λ – a.e. $x \in E$:

$$\log[\lambda(I_n(x))] = n\sum_{j=0}^{m-1} p_j \log p_j - (\Delta + o(1))\sum_{k=1}^{n}\delta_k. \tag{11}$$

Next, we sketch the estimate of $\log|I_n(x)|$ by an analogous discussion as in Proposition 2.2.

Since

$$\mathbb{E}\left[\log r_{x_k}\right] = \sum_{j=0}^{m-1} p_j^{(k)} \log r_j,$$

we get

$$\sum_{k=1}^{n}\mathbb{E}(\log r_{x_k}) = n\sum_{j=0}^{m-1} p_j \log r_j + \Delta^*\sum_{k=1}^{n}\delta_k,$$

where $\Delta^* = \sum_{j=0}^{m-1}(q_j - p_j)\log r_j$.

On the other hand, by the boundedness of $\mathbb{V}\left[\log r_{x_k}\right]$ and the fact $\sum_{k=2}^{\infty}\frac{1}{k(\log k)^2} < \infty$, we get by Proposition 2.1

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\log r_{x_k} - \sum_{k=1}^{n}\mathbb{E}\left[\log r_{x_k}\right]}{\sqrt{n}\log n} = 0, \quad \lambda - \text{a.e.}$$

From (6), $\sum_{k=1}^{n}\delta_k \geqslant \frac{n}{\log(n+2)}$, we have for λ – a.e. $x \in E$:

$$\log|I_n(x)| = \sum_{k=1}^{n}\log r_{x_k} = n\sum_{j=0}^{m-1} p_j \log r_j + (\Delta^* + o(1))\sum_{k=1}^{n}\delta_k. \tag{12}$$

Now we prove the conclusions (1) and (2) of the theorem.

It is ready to see that the assertions (1) and (2) are equivalent to the following three implications:

$$\overline{\lim_{t\to 0}}\frac{\log g(t)}{\log t} < \alpha \Longrightarrow \mathcal{H}^g(E(\vec{p})) = \infty;$$

$$\underline{\lim_{t\to 0}}\frac{\log g(t)}{\log t} > \alpha \Longrightarrow \mathcal{H}^g(E(\vec{p})) = 0;$$

and

$$\overline{\lim_{t\to 0}}\frac{\log g(t)}{\log t} = \alpha \Longrightarrow \mathcal{H}^g(E(\vec{p})) = +\infty.$$

1) Suppose first that $\overline{\lim}_{t\to 0}\frac{\log g(t)}{\log t} < \alpha$, then there exist $\epsilon > 0$ and $t_0 > 0$ such that for $t \leqslant t_0$, we have $\log g(t)/\log t < \alpha - \epsilon$, so $g(t) > t^{\alpha-\epsilon}$. Therefore from the definitions of g-Hausdorff measure, Hausdorff measure and Hausdorff dimension that,

$$\mathcal{H}^g(E(\vec{p})) \geqslant \mathcal{H}^{\alpha-\epsilon}(E(\vec{p})) = \infty,$$

we thus obtain the first implication.

2) Suppose that $\underline{\lim}_{t\to 0}\frac{\log g(t)}{\log t} > \alpha$, then there exist $\epsilon > 0$ and a sequence $\{t_n\}$ decreasing to zero as $n \to \infty$ such that $\log g(t_n)/\log t_n > \alpha + \epsilon$, so $g(t_n) < t_n^{\alpha+\epsilon}$.

Let $A(\vec{p}) = \left\{x \in \Omega\colon \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_j(x_k) = p_j,\quad 0 \leqslant j \leqslant m-1\right\}$ and fix $\epsilon' > 0$. Then for any $x \in A(\vec{p})$, there exists $N(x)$ such that for any $n \geqslant N(x)$, $0 \leqslant j \leqslant m-1$, we have

$$n(p_j - \epsilon') \leqslant \sum_{k=1}^{n}\chi_j(x_k) \leqslant n(p_j + \epsilon'). \tag{13}$$

For $n \geqslant 1$, set

$$A(n) := \Big\{x = (x_1, x_2, \cdots, x_n) \in S^n\ n(p_j - \epsilon') \leqslant \sum_{k=1}^{n}\chi_j(x_k) \leqslant n(p_j + \epsilon'), 0 \leqslant j \leqslant m-1\Big\}.$$

Then from (13) and the definition of $I_n(x)$, we have for $x \in A(n)$,

$$|I_n(x)| \leqslant \prod_{j=0}^{m-1} r_j^{n(p_j-\epsilon')}. \tag{14}$$

From the definition of $A(n)$, for any $n \geqslant 1$,

$$\sharp A(n) = \sum \binom{a_1}{n} \binom{a_2}{n-a_1} \cdots \binom{a_n}{n-a_1-a_2\cdots-a_{n-1}},$$

where the sum runs over the set $\{(a_1,\cdots,a_n) : n(p_j-\epsilon') \leqslant a_j \leqslant n(p_j+\epsilon'), 0 \leqslant j \leqslant m-1, \sum_{j=0}^{m-1} a_j = n\}$.

Set $a_j = nq_j (0 \leqslant j \leqslant m-1)$, then $p_j - \epsilon' \leqslant q_j \leqslant p_j + \epsilon'$ and $\sum_{j=0}^{m-1} q_j = 1$, we get thus by Stirling formula,

$$\sharp A(n) \leqslant (2n\epsilon')^m c^m \max \left\{ \left(\prod_{j=0}^{m-1} q_j^{nq_j} \right) \right\}^{-1}, \tag{15}$$

where max is taken over the set $\{a_j - \epsilon' \leqslant q_j \leqslant a_j + \epsilon', 0 \leqslant j \leqslant m-1, \sum_{j=0}^{m-1} q_j = 1\}$ and c is a positive constant independent of n.

Now for any $n \geqslant 1$, take N_n such that

$$\prod_{j=0}^{m-1} r_j^{(N_n+1)(p_j-\epsilon')} < t_n \leqslant \prod_{j=0}^{m-1} r_j^{N_n(p_j-\epsilon')}.$$

Notice that by the definition of $E(\vec{p})$ and above discussions, we have

$$E(\vec{p}) \subset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(\bigcup_{x\in A(N_k)} I_n(x) \right).$$

Now by the condition $g(2t) \leqslant \xi g(t)(t>0)$, (14) and (15), we have

$$\begin{aligned}
\mathcal{H}^g(E(\vec{p})) &\leqslant \varliminf_{n\to\infty} \sum_{k=n}^{\infty} \sum_{x\in A(N_k)} g(|I_x|) \\
&\leqslant c_1 \varliminf_{n\to\infty} \sum_{k=n}^{\infty} \sum_{x\in A(N_k)} g(t_k) \\
&\leqslant c_1 \varliminf_{n\to\infty} \sum_{k=n}^{\infty} \sum_{x\in A(N_k)} t_k^{\alpha+\epsilon} \\
&\leqslant c_1 \varliminf_{n\to\infty} \sum_{k=n}^{\infty} \sum_{x\in A(N_k)} \left(\prod_{j=0}^{m-1} r_j^{N_k(p_j-\epsilon')} \right)^{\alpha+\epsilon} \\
&\leqslant c_1 \varliminf_{n\to\infty} \sum_{k=n}^{\infty} (2N_k\epsilon)^m c^{N_k} \max \left(\prod_{j=0}^{m-1} q_j^{N_k q_j} \right)^{-1} \left(\prod_{j=0}^{m-1} r_j^{N_k(p_j-\epsilon')} \right)^{\alpha+\epsilon},
\end{aligned}$$

the last term is zero if ϵ' is small enough, where c_1 is an independent constant. We get thus $\mathcal{H}^g(\vec{E}) = 0$ and we prove the second implication.

3) Now we prove the third implication.

Suppose $\overline{\lim}_{t\to 0}\log g(t)/\log t = \alpha$. Take $t_n = \prod_{j=1}^{n} r_{x_j} (n \geqslant 1)$, then $t_n \to 0$ as $n \to \infty$ and $\overline{\lim}_{n\to\infty}\log g(t_n)/\log t_n \leqslant \alpha$, in this case, $k(x,n) = n$, where $k(x,n)$ is defined as in (10). It is easy to check that if we replace the condition $\lim_{n\to\infty}\log g(t_n)/\log t_n = \alpha$ by $\overline{\lim}_{n\to\infty}\log g(t_n)/\log t_n \leqslant \alpha$, the conclusion of Lemma 3.3 still hold. Thus from this variant of Lemma 3.3, there is a Borel set $E_1 \subset E$ with $\lambda(E_1) > 0$ such that for any $x \in E_1$ we have

$$\log g(|I_n(x)|) \geqslant (\alpha + \epsilon_n)\log |I_n(x)|,$$

where the sequence $\{\epsilon_n\}_{n\geqslant 1}$ is defined as in Lemma 3.3. Thus by (12) and the definition of α, we have for $\lambda-$a.e. $x \in E_1$:

$$\log g(|I_n(x)|) \geqslant n\sum_{j=0}^{m-1} p_j \log p_j + n\epsilon_n \sum_{j=0}^{m-1} p_j \log r_j + (\alpha\Delta^* + o(1))\sum_{k=1}^{n}\delta_k,$$

where $\Delta^* = \sum_{j=0}^{m-1}(q_j - p_j)\log r_j$.

Combine with (11) and Lemma 3.3 (i), we can claim that there exists a Borel set $E_2 \subset E$ with $\lambda(E_2) = \lambda(E_1) > 0$ such that for any $x \in E_2$:

$$\begin{aligned}
&\log[\lambda(I_n(x))] - \log[g(|I_n(x)|)]\\
&\leqslant -n\epsilon_n \sum_{j=0}^{m-1} p_j \log r_j - (\Delta + \alpha\Delta^* + o(1))\sum_{k=1}^{n}\delta_k \qquad (16)\\
&= -(\Delta + \alpha\Delta^* + o(1))\sum_{k=1}^{n}\delta_k.
\end{aligned}$$

By Proposition 3.2, there is an integer $0 \leqslant l \leqslant m-1$ such that $p_l < r_l^{\alpha}$, we specify the perturbation resource $\vec{q} = (q_0, q_1, \cdots, q_{m-1})$ by letting

$$q_j = \begin{cases} 1, & \text{if } j = l; \\ 0, & \text{otherwise.} \end{cases}.$$

Thus by the definitions of Δ, Δ^* and α, we have

$$\Delta + \alpha\Delta^* = -\log p_l + \log r_l^{\alpha} > 0,$$

which, together with (17), yields that for each $x \in E_2$,

$$\log[\lambda(I_n(x))] - \log[g(|I_n(x)|)] \leqslant -[\Delta + \alpha\Delta^* + o(1)]\sum_{k=1}^{n}\delta_k \to -\infty$$

as n tends to infinity. This asserts for each $x \in E_2$, we have

$$\lim_{n\to\infty}\frac{\lambda(I_n(x))}{g(|I_n(x)|)} = 0,$$

therefore by Proposition 3.1, we get $\mathcal{H}^g(E_2) = +\infty$, so $\mathcal{H}^g(E(\vec{p})) = +\infty$. We thus prove the third implication, so complete the proof of the conclusions (1) and (2).

By the same discussion, to prove the assertions (3) and (4), we need only to prove the implications

$$\varliminf_{t\to 0} \frac{\log g(t)}{\log t} < \alpha \Longrightarrow \mathcal{P}^g(E(\vec{p})) = +\infty;$$
$$\varliminf_{t\to 0} \frac{\log g(t)}{\log t} > \alpha \Longrightarrow \mathcal{P}^g(E(\vec{p})) = 0;$$
$$\varliminf_{t\to 0} \frac{\log g(t)}{\log t} = \alpha \Longrightarrow \mathcal{P}^g(E(\vec{p})) = +\infty.$$

a) The second implication can be proved by the same way as the proof of the case the Hausdorff gauge.

b) Suppose that $\varliminf_{t\to 0} \frac{\log g(t)}{\log t} = \alpha$, then there is a sequence $\{t_n\}_{n\geqslant 1}$ decreasing to zero as $n \to \infty$ such that $\lim_{n\to 0} \frac{\log g(t_n)}{\log t_n} = \alpha$. By using Lemma 3.3 and the same discussions as above, we can prove that $\lim_{n\to\infty} \frac{\lambda(I_{k(x,n)}(x))}{g(|I_{k(x,n)}(x)|)} = 0$ so $\varliminf_{n\to\infty} \frac{\lambda(I_n(x))}{g(|I_n(x)|)} = 0$ holds on a subset of $E(\vec{p})$ with positive λ-measure, hence the third implication follows from Proposition 3.1.

c) Suppose now $\varliminf_{t\to 0} \frac{\log g(t)}{\log t} := \beta < \alpha$. Let $G(t) := g^{\alpha/\beta}(t)$, then

$$\varliminf_{t\to 0} \frac{\log G(t)}{\log t} = \frac{\alpha}{\beta} \varliminf_{t\to 0} \frac{\log g(t)}{\log t} = \alpha.$$

Therefore from the proof of b), we have $\mathcal{P}^G(E(\vec{p})) = +\infty$. On the other hand, since $\alpha/\beta > 1$, we have $\lim_{t\to 0} G(t)/g(t) = 0$, so $\mathcal{P}^g(E(\vec{p})) = +\infty$. We prove thus the first implication. $\square$

Remark 3.4. — In the cases $p_j = 0$ for some j, the conclusions of the main theorem remain valid. To see this, we only need to modify the perturbation measure by letting

$$p_j^{(k)} = \begin{cases} \frac{\delta_k}{m^*}, & \text{if } p_j = 0; \\ p_j - \frac{\delta_k}{m-m^*}, & \text{otherwise.} \end{cases}$$

where $m^* = \#\{j : p_j = 0\}$.

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