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THE RESOLVENT FOR LAPLACE-TYPE OPERATORS ON ASYMPTOTICALLY CONIC SPACES

by A. HASSELL, A. VASY

1. Introduction.

Scattering metrics are a class of Riemannian metrics yielding manifolds which are geometrically complete, and asymptotically conic at infinity. We consider manifolds which have one end which is diffeomorphic to $Y \times [1,\infty)_r$, where Y is a closed manifold, and metrically asymptotic to $dr^2 + r^2 h$, where h is a Riemannian metric on Y, as $r \to \infty$. The precise definition is given in Definition 2.1 below. Examples include the standard metric and the Schwartzschild metric on Euclidean space.

In this paper we give a direct construction of the outgoing resolvent kernel, $R(\sigma + i0) = (H - (\sigma + i0))^{-1}$, for σ on the real axis, where H is a perturbation of the Laplacian with respect to a scattering metric. The incoming resolvent kernel, $R(\sigma - i0)$, may be obtained by taking the formal adjoint kernel.

The strategy of the construction is to compactify the space to a compact manifold X and use the scattering calculus of Melrose [10], as well as the calculus of Legendre distributions of Melrose-Zworski [12], extended by us in [4]. The oscillatory behaviour of the resolvent kernel is analyzed in terms of the 'scattering wavefront set' at the boundary. Using propagation

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of singularities theorems for the scattering wavefront set leads to an ansatz for the structure of the resolvent kernel as a sum of a pseudodifferential term and Legendre distributions of various types. The calculus of Legendre distributions allows us to construct a rather precise parametrix for the resolvent in this class, with a compact error term E. Using the parametrix, we show that one can make a finite rank correction to the parametrix which makes $\mathrm{Id} + E$ invertible, and thus can correct the parametrix to the exact resolvent.

As compared to the method of [4], where the authors previously constructed the resolvent, the construction is direct in two senses. First, we write down rather explicitly a parametrix for $R(\sigma + i0)$ and then solve away the error using Fredholm theory. In [4], by contrast, the resolvent was constructed via the spectral measure, which itself was constructed from the Poisson operator. Second, we make no use of the limiting absorption principle; that is, we work directly at the real axis in the spectral variable rather than taking a limit $R(\sigma + i\epsilon)$ as $\epsilon \to 0$. We then prove a posteriori that the operator constructed is equal to this limit.

Let us briefly describe the main result here. We consider an operator H of the form $H = \Delta + P$ acting on half densities, where P is, in the first place, a short range perturbation, that is, a first order self-adjoint differential operator with coefficients vanishing to second order at infinity. (Later, we show that there is a simple extension to metrics and perturbations of 'long range gravitational type', which includes the Newtonian or Coulomb potential and metrics of interest in general relativity.) We remark that the Riemannian half-density $|dg|^{1/2}$ trivializes the half-density bundle, and operators on half-densities can be regarded as operators on functions via this trivialization. Given $\lambda > 0$, we solve for a kernel $\tilde{R}(\lambda)$ on X^2 which satisfies

$$(H - \lambda^2)\tilde{R}(\lambda) = K_{\mathrm{Id}}, \tag{1.1}$$

where K_{Id} is the kernel of the identity operator on half densities. More precisely, we consider this equation on X^2_{b}, which is the space X^2 with the corner blown up. This allows us to use the scattering wavefront set at the 'front face' (the face resulting from blowing up the corner) to analyze singularities, which is an absolutely crucial part of the strategy. The kernel $\tilde{R}(\lambda)$ is also required to satisfy a wavefront set condition at the front face, which is the analogue of the outgoing Sommerfeld radiation condition.

We cannot find $\tilde{R}(\lambda)$ exactly in one step, so first we look for an approximation $G(\lambda)$ of it. The general strategy is to find $G(\lambda)$ which solves away the singularities of the right hand side, K_{Id}, of (1.1). Singularities should be understood both in the sense of interior singularities and oscillations, or growth, at the boundary, as measured by the scattering wavefront set.

The first step is to find a pseudodifferential approximation which solves away the interior singularities of K_{Id}, which is a conormal distribution supported on the diagonal. This can be done and removes singularities except at the boundary of the diagonal, where $H - \lambda^2$ is not elliptic (in the sense of the boundary wavefront set). In fact, the singularities which remain lie on a Legendrian submanifold $N^*\mathrm{diag}_{\mathrm{b}}$ at the boundary of $\mathrm{diag}_{\mathrm{b}}$ (see (4.5)). Singularities of $G(\lambda)$ can be expected to propagate in a Legendre submanifold $L_+(\lambda)$ which is the bicharacteristic flowout from the intersection of $N^*\mathrm{diag}_{\mathrm{b}}$ and the characteristic variety of $H - \lambda^2$. (The geometry here is precisely that of the fundamental solution of the wave operator in $\mathbb{R}^{n+1}$, which is captured by the intersecting Lagrangian calculus of Melrose-Uhlmann [11], but it takes place *at the boundary*.) This Legendre has conic singularities at another Legendrian, $L^{\#}(\lambda)$, which is 'outgoing'. Thus, in view of the calculus of Legendre distributions of Melrose-Zworski and the authors, the simplest one could hope for is that the resolvent on the real axis is the sum of a pseudodifferential term, an intersecting Legendre distribution associated to $(N^*\mathrm{diag}_{\mathrm{b}}, L_+(\lambda))$ and a Legendre conic pair associated to $(L_+(\lambda), L^{\#}(\lambda))$. This is the case:

THEOREM 1.1.— *Let H be a short range perturbation of a short range scattering metric on X. Then, for $\lambda > 0$, the outgoing resolvent $R(\lambda^2 + i0)$ lies in the class (4.4), that is, it is the sum of a scattering pseudodifferential operator of order -2, an intersecting Legendre distribution of order $-1/2$ associated to $(N^*\mathrm{diag}_{\mathrm{b}}, L_+(\lambda))$ and a Legendrian conic pair associated to $(L_+(\lambda), L^{\#}(\lambda))$ of orders $-1/2$ at $L_+(\lambda)$, $(n-2)/2$ at $L^{\#}(\lambda)$ and $(n-1)/2$ at the left and right boundaries.*

If H is of long range gravitational type, then the same result holds except that the Legendre conic pair is multiplied by a complex power of the left and right boundary defining functions.

This theorem was already proved in our previous work [4], so it is the method that is of principal interest here. By comparison with [4], the proof is conceptually much shorter; it does not use any results from [10] or [12], although it makes substantial use of machinery from [12]. But the main point we wish to emphasize is that the proof works directly on the spectrum and nowhere uses the limiting absorption principle, a method of attack that we think will be useful elsewhere in scattering theory. It seems that things which are easy to prove with this method are difficult with the limiting absorption principle, and vice versa. For example, it is immediate from our results that if f is compactly supported in the interior of X, then $u = R(\lambda^2 + i0)f$ is such that $x^{-(n-1)/2}e^{-i\lambda/x}u \in \mathcal{C}^\infty(X)$, while it is not so easy to see that the resolvent is a bounded operator from $x^l L^2$ to $x^{-l}L^2$ for any $l > 1/2$. Using the limiting absorption principle, it is the second statement that is much easier to derive (following [1] for example). Thus, we hope that this type of approach will complement other standard methods in scattering theory.

In the next section, we describe the machinery required, including the scattering calculus on manifolds with boundary, the scattering-fibred calculus on manifolds with codimension two corners, and Legendre distributions in these contexts. The b-double space, X_{b}^2, which is a blown up version of the double space $X \times X$ which carries the resolvent kernel, is also described. The discussion here is rather concise, but there are more leisurely treatments in [12] and [4].

The third section gives a symbol calculus for Legendre distributions on manifolds with codimension two corners. This is a straightforward generalization from the codimension one case.

The fourth section is the heart of the paper, where we construct the parametrix $G(\lambda)$ for the resolvent kernel. Propagation of singularities theorems show that the simplest space of functions in which one could hope to find the resolvent kernel is given by (4.4). We can in fact construct a parametrix for the resolvent in this class. In the fifth section this is extended to long range metrics and perturbations.

In the final section we show that one can modify the parametrix so that the error term $E(\lambda)$ is such that $\mathrm{Id} + E(\lambda)$ is invertible. This is done by showing that the range of $H - \lambda^2$ on the sum of $\dot{\mathcal{C}}^\infty(X)$ and $G(\lambda)\dot{\mathcal{C}}^\infty(X)$ is dense on suitable weighted Sobolev spaces (see Lemma 6.1). Thus the parametrix may be corrected to an exact solution of (1.1). Such a result also shows the absence of positive eigenvalues for H. Finally, we show that the kernel so constructed has an analytic continuation to the upper half plane and agrees with the resolvent there.

Notation and conventions. — On a compact manifold with boundary, X, we use $\dot{\mathcal{C}}^\infty(X)$ to denote the class of smooth functions, all of whose derivatives vanish at the boundary, with the usual Fréchet topology, and $\mathcal{C}^{-\infty}(X)$ to denote its topological dual. On the radial compactification of $\mathbb{R}^n$ these correspond to the space of Schwartz functions and tempered distributions, respectively. The Laplacian Δ is taken to be positive. The space $L^2(X)$ is taken with respect to the Riemannian density induced by the scattering metric g. This density has the form $a\,dxdy/x^{n+1}$ near the boundary, where a is smooth.

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2. Preliminaries.

2.1. Scattering calculus.

Let X be a manifold with boundary $\partial X = Y$. Near the boundary we will write local coordinates in the form (x, y) where x is a boundary defining function and y are coordinates on Y extended to a collar neighbourhood of ∂X.

We begin by giving the definition of a scattering metric. The precise requirements for the metric (and many other things besides) are easiest to formulate in terms of a compactification of the space. Taking the function $x = r^{-1}$ as a boundary defining function and adding a copy of Y at $x = 0$ yields a compact manifold, X, with boundary $\partial X = Y$. Then the definition of scattering metric is given in terms of X in Definition 2.1 below. Regularity statements for the metric coefficients are in terms of the $\mathcal{C}^\infty$ structure on X; this is a strong requirement, being equivalent to the existence of a complete asymptotic expansion, together with all derivatives, in inverse powers of r as $r \to \infty$. The benefit of such a strong requirement is that we get complete asymptotic expansions for the resolvent kernel, and correspondingly, mapping properties of the resolvent on spaces of functions with complete asymptotic expansions.

DEFINITION 2.1. — *A (short range) scattering metric on X is a Riemannian metric g in the interior of X which takes the form*

$$g = \frac{dx^2}{x^4} + \frac{h'}{x^2}, \tag{2.1}$$

where h' is a smooth symmetric 2-cotensor on X which restricts to the boundary to be a metric h on Y [10]. A long range scattering metric is a metric in the interior of X which takes the form

$$g = a_{00}\frac{dx^2}{x^4} + \frac{h'}{x^2}, \tag{2.2}$$

where a_{00} is smooth on X, $a_{00} = 1 + O(x)$, and h' is as above [13]. If $a_{00} = 1 - cx + O(x^2)$ for some constant c we call g a gravitational long range scattering metric.

Examples. — Flat Euclidean space has a metric which in polar coordinates takes the form

$$dr^2 + r^2 d\omega^2,$$

where $d\omega^2$ is the standard metric on S^{n-1}. Compactifying Euclidean space as above, we obtain a ball with $x = r^{-1}$ as boundary defining function, and then the flat metric becomes

$$\frac{dx^2}{x^4} + \frac{d\omega^2}{x^2},$$

which is a short range scattering metric.

The Schwartzschild metric on $\mathbb{R}^n$ takes the form near infinity

$$\left(1 - \frac{2m}{r}\right) dr^2 + r^2 d\omega^2,$$

which under the same transformation leads to a gravitational long range scattering metric

$$(1 - 2mx)\frac{dx^2}{x^4} + \frac{d\omega^2}{x^2}$$

near $x = 0$. The constant $m = c/2$ is interpreted as the mass in general relativity. (Note that the boundary $r = 2m$ has a different structure.)

The natural Lie Algebra corresponding to the class of scattering metrics on X is the scattering Lie Algebra
$\mathcal{V}_{\mathrm{sc}}(X) = \{V | V = xW$, where W is a $\mathcal{C}^\infty$ vector field on X tangent to $\partial X\}$.

Clearly this Lie Algebra can be localized to any open set. In the interior of X, it consists of all smooth vector fields, while near the boundary it is equal to the $\mathcal{C}^\infty(X)$-span of the vector fields $x^2\partial_x$ and $x\partial_{y_i}$. Hence it is the space of smooth sections of a vector bundle, denoted ${}^{\mathrm{sc}}TX$, the scattering tangent bundle. Any scattering metric turns out to be a smooth fibre metric on ${}^{\mathrm{sc}}TX$. The dual bundle, denoted ${}^{\mathrm{sc}}T^*X$, is called the scattering cotangent bundle; near the boundary, smooth sections are generated over $\mathcal{C}^\infty(X)$ by dx/x^2 and dy_i/x. A general point in ${}^{\mathrm{sc}}T_p^*X$ can be thought of as the value of a differential $d(f/x)$ at p, where $f \in \mathcal{C}^\infty(X)$, and in terms of local coordinates (x, y) near ∂X can be written $\tau dx/x^2 + \mu_i dy_i/x$, yielding local coordinates (x, y, τ, μ) on ${}^{\mathrm{sc}}T^*X$ near ∂X.

The scattering differential operators of order k, denoted $\mathrm{Diff}^k_{\mathrm{sc}}(X)$, are those given by sums of products of at most k scattering vector fields. There are two symbol maps defined for $P \in \mathrm{Diff}^k_{\mathrm{sc}}(X)$. The first is the 'usual' symbol map, denoted here $\sigma^k_{\mathrm{int}}(P)$, which maps to $S^k({}^{\mathrm{sc}}T^*X)/S^{k-1}({}^{\mathrm{sc}}T^*X)$, where $S^k({}^{\mathrm{sc}}T^*X)$ denotes the classical symbols of order k on ${}^{\mathrm{sc}}T^*X$. The second is the boundary symbol, $\sigma_\partial(P) \in S^k({}^{\mathrm{sc}}T^*_{\partial X}X)$, which is the *full* symbol of P restricted to $x = 0$. This is well defined since the Lie Algebra $\mathcal{V}_{\mathrm{sc}}(X)$ has the property $[\mathcal{V}_{\mathrm{sc}}(X), \mathcal{V}_{\mathrm{sc}}(X)] \subset x\mathcal{V}_{\mathrm{sc}}(X)$, so commutators of scattering vector fields vanish to an additional order at the boundary. Dividing the interior symbol $\sigma^k_{\mathrm{int}}(P)$ by $|\xi|^k_g$, where $|\cdot|_g$ is the metric on ${}^{\mathrm{sc}}T^*X$ determined by the scattering metric, we get a function on the sphere bundle of ${}^{\mathrm{sc}}T^*X$. This may be combined with the boundary symbol into a joint symbol, $j^k_{\mathrm{sc}}(P)$, a function on $C_{\mathrm{sc}}(X)$ which is the topological space obtained by gluing together the sphere bundle of ${}^{\mathrm{sc}}T^*X$ with the the fibrewise radial compactification of ${}^{\mathrm{sc}}T^*_{\partial X}X$ along their common boundary.

The scattering pseudodifferential operators are defined in terms of the behaviour of their Schwartz kernels on the scattering double space X^2_{sc}, a blown up version of the double space X^2. This is defined by

$$X^2_{\mathrm{b}} = [X^2; (\partial X)^2] \qquad \text{and} \qquad X^2_{\mathrm{sc}} = [X^2_{\mathrm{b}}; \partial\mathrm{diag}_{\mathrm{b}}], \tag{2.3}$$

and $\mathrm{diag}_{\mathrm{b}}$ is the lift of the diagonal to X^2_{b}. The lift of $\mathrm{diag}_{\mathrm{b}}$ to X^2_{sc} is denoted $\mathrm{diag}_{\mathrm{sc}}$. The blowup notation $[;]$ is that of Melrose: see [8] or [9]. The boundary hypersurfaces are labelled lb, rb, bf and sf; see figure 1. The scattering pseudodifferential operators of order k, acting on half densities, $\Psi^k_{\mathrm{sc}}(X)$, are those given by $\mathrm{KD}^{\frac{1}{2}}_{\mathrm{sc}}$-valued distribution kernels which are classical conormal at $\mathrm{diag}_{\mathrm{sc}}$, of order k, uniformly to the boundary, and rapidly vanishing at lb, rb, bf. (Here $\mathrm{KD}^{\frac{1}{2}}_{\mathrm{sc}}$ is the pullback of the bundle $\pi_l^{*\mathrm{sc}}\Omega^{\frac{1}{2}}(X) \otimes \pi_r^{*\mathrm{sc}}\Omega^{\frac{1}{2}}(X)$ over X^2 to X^2_{b}.) The space $\Psi^{k,l}_{\mathrm{sc}}(X)$ is defined to be $x^l\Psi^k_{\mathrm{sc}}(X)$.

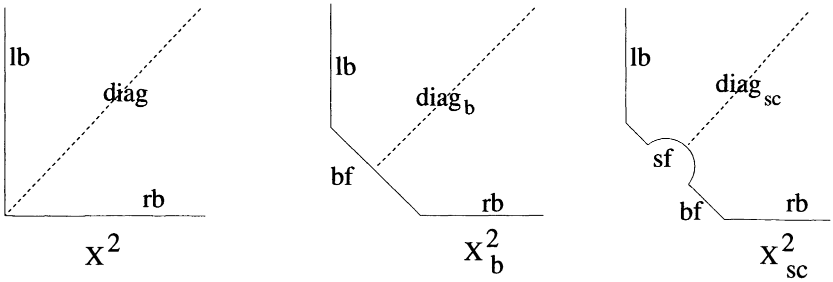

Figure 1. The b-double space and the scattering scattering space

The joint symbol map j^k_{sc} extends from $\mathrm{Diff}^k_{\mathrm{sc}}(X)$ to $\Psi^k_{\mathrm{sc}}(X)$ multiplicatively,

$$j^k_{\mathrm{sc}}(A) \cdot j^m_{\mathrm{sc}}(B) = j^{k+m}_{\mathrm{sc}}(AB),$$

and such that there is an exact sequence

$$0 \longrightarrow \Psi^{m-1,1}_{\mathrm{sc}}(X) \longrightarrow \Psi^m_{\mathrm{sc}}(X) \xrightarrow{j^m_{\mathrm{sc}}} \mathcal{C}^\infty(C_{\mathrm{sc}}(X)) \longrightarrow 0.$$

A scattering pseudodifferential operator $A \in \Psi^k_{\mathrm{sc}}(X)$ is said to be elliptic at a point $q \in C_{\mathrm{sc}}(X)$ if $j^k_{\mathrm{sc}}(A)(q) \neq 0$. It is said to have elliptic interior symbol (boundary symbol) if $j^k_{\mathrm{sc}}(A)$ does not vanish at fibre-infinity (spatial infinity), and is said to be totally elliptic if $j^k_{\mathrm{sc}}(A)$ vanishes nowhere. The characteristic variety of A, $\Sigma(A)$, is the zero set of $j^k_{\mathrm{sc}}(A)$.

The scattering wavefront set of a tempered distribution $u \in \mathcal{C}^{-\infty}(X)$ (the dual space of $\dot{\mathcal{C}}^\infty(X)$, the space of smooth functions on X vanishing with all derivatives at the boundary) is the closed subset of $C_{\mathrm{sc}}(X)$ whose complement is

(2.4) $$^{\mathrm{sc}}\mathrm{WF}(u)^{\complement} = \{q \in C_{\mathrm{sc}}(X) \mid \exists A \in \Psi^0_{\mathrm{sc}}(X) \text{ elliptic at } q \text{ such that } Au \in \dot{\mathcal{C}}^\infty(X)\}.$$

The interior part of the wavefront set (at fibre-infinity) is a familiar object, the standard wavefront set introduced by Hörmander (except that each ray of the standard wavefront set is thought of here as a point in the cosphere bundle). In this paper we are mostly interested in the part of the scattering wavefront set at spatial infinity. In fact, the operators H we shall study will have elliptic interior symbol, uniformly to the boundary, so in view of the next theorem, solutions of $(H - \sigma)u = 0$ must have wavefront set contained in the part of $C_{\mathrm{sc}}(X)$ at spatial infinity, which we denote K (that is, $K = {}^{\mathrm{sc}}T^*_{\partial X}X$).

There is a natural contact structure on K induced by the symplectic form ω on T^*X. Writing ω in terms of the rescaled cotangent variables τ, μ and contracting with the vector $x^2\partial_x$ yields the 1-form

$$\chi = \iota_{x^2\partial_x}\omega = d\tau + \mu \cdot dy,$$

which is nondegenerate, and therefore a contact form. A change of boundary defining function $x' = ax$ changes χ by a factor a^{-1}, so the contact *structure* is totally well-defined. Given a Hamiltonian h on K, the Hamiltonian vector field on K determined by the contact form is

$$\frac{\partial h}{\partial \mu_i}\frac{\partial}{\partial y_i} + \Big(-\frac{\partial h}{\partial y_i} + \mu_i\frac{\partial h}{\partial \tau}\Big)\frac{\partial}{\partial \mu_i} + \Big(h - \mu_i\frac{\partial h}{\partial \mu_i}\Big)\frac{\partial}{\partial \tau}.$$

This is the same as $x^{-1}V_h$ restricted to $x = 0$, where V_h is the Hamilton vector field on ${}^{\mathrm{sc}}T^*X$ induced by h. Integral curves of this vector field are called bicharacteristics of h (or of A, if h is the boundary symbol of A).

Under a coordinate change, the variables τ and μ change according to

$$\tau' = a\tau, \quad \mu' = a\mu\frac{\partial y}{\partial y'} - \tau\frac{\partial a}{\partial y'} \qquad x' = ax.$$

Since $a > 0$, this shows that the subset

$$K_- = \{(y, \tau, \mu) \in K \mid \tau \leqslant 0\} \tag{2.5}$$

is invariantly defined. This is important in the definition of the outgoing resolvent, see (4.2).

The boundary part of the scattering wavefront set behaves very much as the interior wavefront set part behaves, and in particular we have a propagation of singularities result for operators of real principal type:

THEOREM 2.2. — *Suppose $A \in \Psi^k_{\mathrm{sc}}(X)$ has elliptic interior symbol, and real boundary symbol. Then for $u \in \mathcal{C}^\infty(X^o) \cap \mathcal{C}^{-\infty}(X)$, we have*

$$\mathrm{WF}_{\mathrm{sc}}(Au) \subset \mathrm{WF}_{\mathrm{sc}}(u), \tag{2.6}$$

$$\mathrm{WF}_{\mathrm{sc}}(u) \setminus \mathrm{WF}_{\mathrm{sc}}(Au) \subset \Sigma(A), \tag{2.7}$$

and

(2.8) $\mathrm{WF}_{\mathrm{sc}}(u) \setminus \mathrm{WF}_{\mathrm{sc}}(Au)$ *is a union of maximally extended bicharacteristics of A inside $\Sigma(A) \setminus \mathrm{WF}_{\mathrm{sc}}(Au)$.*

Thus, if $Au = 0$, then ${}^{\mathrm{sc}}\mathrm{WF}(u) \subset K$ and consists of a union of maximally extended bicharacteristics of A inside $\Sigma(A)$.

As well as a boundary principal symbol defined on K, scattering pseudodifferential operators also have a boundary subprincipal symbol. This is the $O(x)$ term of the full symbol at the boundary when the operator is written in Weyl form. It is important to keep in mind that it depends on a choice of product structure at the boundary; it does not enjoy quite the same invariance properties as does the standard (interior) subprincipal symbol. A practical formula to use for differential operators with symbol in left-reduced form, ie, such that

$$\sigma\left(a(x,y)(x^2D_x)^j(xD_y)^\alpha\right) = a(x,y)\tau^j\mu^\alpha, \;\; D = -i\partial, \;\; \alpha = (\alpha_1, \dots, \alpha_{n-1}),$$

is that for $\sigma(P) = p(y,\tau,\mu) + xq(y,\tau,\mu) + O(x^2)$, the boundary subprincipal symbol of P is given by

$$\sigma_{\mathrm{sub}}(P) = q + \frac{i}{2}\left(\frac{\partial^2 p}{\partial y_i \partial \mu_i} - (n-1)\frac{\partial p}{\partial \tau} + \mu_i \frac{\partial^2 p}{\partial \mu_i \partial \tau}\right). \tag{2.9}$$

LEMMA 2.3. — *Let g be a short range scattering metric, let x be a boundary defining function with respect to which $g = dx^2/x^4 + h'/x^2$, and let H be a short range perturbation of the Laplacian with respect to g. Then in local coordinates (x, y), the sub-principal symbol of H vanishes at $\mu = 0$.*

Proof. — The operator H may be written

$$H = (x^2D_x)^2 + (n-1)ix^3D_x + x^2\Delta_\partial + a_{ij}(x,y)x^3D_{y_i}D_{y_j} + Q, \quad Q \in x^2\,\mathrm{Diff}_{\mathrm{sc}}(X).$$

Thus the left-reduced symbol as above is

$$\sigma(H) = \tau^2 + h_{ij}\mu_i\mu_j + x\left((n-1)i\tau + a_{ij}\mu_i\mu_j\right) + O(x^2).$$

Hence the sub-principal symbol is

$$\sigma_{\mathrm{sub}}(H) = i\frac{\partial h_{ij}}{\partial y_i}\mu_j + a_{ij}\mu_i\mu_j, \tag{2.10}$$

which vanishes when $\mu = 0$. □

We now define the gravitational condition for the perturbation P.

DEFINITION 2.4. — *A first order scattering differential operator P on X is said to be short range if it lies in $x^2\mathrm{Diff}^1_{\mathrm{sc}}(X)$, and long range if it* *lies in* $x\mathrm{Diff}^1_{\mathrm{sc}}(X)$. *Let* g *be a scattering metric and* x *a boundary defining function with respect to which* g *takes the form (2.1) or (2.2).* P *is said to be of long range gravitational type with respect to* g *if it has the form*

$$P = x\left(\sum_i a_i(x\partial_{y_i}) + bx^2\partial_x + c\right),$$

near $x = 0$, *where* a_i, b *and* c *are in* $C^\infty(X)$, *and for some constants* b_0 *and* c_0, $b = b_0 + O(x)$ *and* $c = c_0 + O(x)$.

The point of the short range condition is that then the subprincipal symbol of both $H = \Delta + P$ vanishes at the radial sets $\mu = 0, \tau = \pm\lambda$ of $H - \lambda^2$, whilst in the long range gravitational case, the subprincipal symbol is constant. In the general long range case, the subprincipal symbol is an arbitrary function on the radial set, which causes some inconvenience (but not insuperable difficulties) in constructing the parametrix for $(H - \lambda^2 - i0)^{-1}$.

2.2. Legendre distributions.

An important special case that occurs often is that ${}^{\mathrm{sc}}\mathrm{WF}(u)$ is a Legendre submanifold, or union of Legendre submanifolds, of K; moreover, in many cases, u is a Legendre distribution, which means that it has a WKB-type expansion, the product of a oscillatory and smooth term, as discussed below, which makes it particularly amenable to analysis.

We let $\dim X = n$, so that $\dim K = 2n - 1$. A Legendre submanifold of K is a submanifold G of dimension $n - 1$ such that $\chi \restriction G = 0$. Such submanifolds have several nice properties. One is that if a Hamiltonian, h, is constant on G then its Hamilton vector field is tangent to G. Another is that Legendre submanifolds may be generated in the following way: If F is a submanifold of dimension $n - 2$, such that χ vanishes on F, and if the Hamilton vector field of h is nowhere tangent to F, then the union of bicharacteristics of h passing through F is (locally) a Legendre submanifold.

Let G be a Legendre submanifold, and let $q \in G$. A local (nondegenerate) parametrization of G near q is a function $\phi(y, v)$ defined in a neighbourhood of $y_0 \in Y$ and $v_0 \in \mathbb{R}^k$, such that $d_v\phi = 0$ at $q' = (y_0, v_0)$,

$q = (y, d_{(x,y)}(\phi/x))$ at q', ϕ satisfies the nondegeneracy hypothesis

$$d\Big(\frac{\partial\phi}{\partial v_i}\Big) \text{ are linearly independent at } C = \{(y,v) \mid d_v\phi = 0\}, 1 \leqslant i \leqslant k,$$

and near q,

$$G = \left\{ \left(y, d_{(x,y)}\left(\frac{\phi}{x}\right)\right) \mid (y,v) \in C \right\}. \tag{2.11}$$

A Legendre distribution of order m associated to G is a half-density of the form $u = (u_0 + \sum_{j=1}^N u_j)\nu$, where ν is a smooth section of the scattering half density bundle, $u_0 \in \dot{\mathcal{C}}^\infty(X)$, and u_j is supported in a coordinate patch (x,y) near the boundary, with an expression

$$u_j = x^{m+n/4-k/2} \int_{\mathbb{R}^k} e^{i\phi_j(y,v)/x} a_j(x,y,v)\, dv,$$

where ϕ_j locally parametrizes G and $a_j \in \mathcal{C}^\infty(X \times \mathbb{R}^k)$, with compact support in v. Melrose and Zworski showed that u_j can be written with respect to any local parametrization, up to an error in $\dot{\mathcal{C}}^\infty(X)$. The set of such half-densities is denoted $I^m(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$. The scattering wavefront set of $u \in I^m(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$ is contained in G.

An intersecting Legendre distribution is associated to a pair of Legendre submanifolds, $\tilde{L} = (L_0, L_1)$, where L_1 is a manifold with boundary such that L_0 and L_1 intersect cleanly at ∂L_1. A local parametrization of (L_0, L_1) near $q \in L_0 \cap L_1$ is a function $\phi(y,v,s)$ defined in a neighbourhood of $q' = (y_0, v_0, 0)$ in $Y \times \mathbb{R}^k \times [0,\infty)$ such that $d_v\phi = 0$ at q', $q = (y, d_{(x,y)}(\phi/x))(q')$, ϕ satisfies the nondegeneracy hypothesis

$$ds,\ d\phi, \text{ and } d\Big(\frac{\partial\phi}{\partial v_i}\Big) \text{ are linearly independent at } q',\ 1 \leqslant i \leqslant k,$$

and near q,

$$L_0 = \left\{ \left(y, d_{(x,y)}\left(\frac{\phi}{x}\right)\right) \mid s = 0, d_v\phi = 0 \right\},$$
$$L_1 = \left\{ \left(y, d_{(x,y)}\left(\frac{\phi}{x}\right)\right) \mid s \geqslant 0, d_s\phi = 0,\ d_v\phi = 0 \right\}.$$

A Legendre distribution of order m associated to $\tilde{L}$ is a half-density of the form $u = u_0 + (\sum_{j=1}^N u_j)\nu$, where ν is a smooth scattering half-density, $u_0 \in I_c^m(X, L_1; {}^{\mathrm{sc}}\Omega^{\frac12}) + I^{m+1/2}(X, L_0; {}^{\mathrm{sc}}\Omega^{\frac12})$ (the subscript c indicates that the microlocal support does not meet the boundary of L_1), and u_j

is supported in a coordinate patch (x, y) near the boundary, with an expression

$$u_j = x^{m+n/4-(k+1)/2} \int_0^\infty ds \int e^{i\phi_j(y,v,s)/x} a_j(x, y, v, s)\, dv,$$

where ϕ_j locally parametrizes (L_0, L_1) and $a_j \in \mathcal{C}^\infty(X \times \mathbb{R}^k \times [0, \infty))$, with compact support in v and s. Again, u_j can be written with respect to any local parametrization, up to an error in $\dot{\mathcal{C}}^\infty(X)$. The set of such half-densities is denoted $I^m(X, \tilde{L}; {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$. The scattering wavefront set of $u \in I^m(X, \tilde{L}; {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$ is contained in $L_0 \cup L_1$.

A Legendre distribution associated to a conic Legendrian pair is associated to a pair of Legendre submanifolds $\tilde{G} = (G, G^\sharp)$ where $G^\sharp$ is a projectable Legendrian (that is, the projection from ${}^{\mathrm{sc}}T^*X$ to Y is a diffeomorphism restricted to $G^\sharp$) and G is an open Legendrian submanifold such that $\overline{G} \setminus G$ is contained in $G^\sharp$ and $\overline{G}$ has at most a conic singularity at $G^\sharp$. We further assume that $\tau \neq 0$ on $G^\sharp$, so that we may change coordinates to a new boundary defining function such that $G^\sharp$ is parametrized by the phase function 1. In these coordinates, the condition that $\overline{G}$ has a conic singularity at $G^\sharp$ means that $\overline{G}$ lifts to a smooth submanifold with boundary, $\hat{G}$, on the blown-up space

$$[{}^{\mathrm{sc}}T^*X;\ \{x = 0, \mu = 0\}], \tag{2.12}$$

intersecting the front face of (2.12) transversally. In local coordinates (x, y, τ, μ), coordinates near the front face are

$$x/|\mu|,\ y,\ \tau,\ |\mu| \text{ and } \hat{\mu}, \tag{2.13}$$

and we require that $\hat{G}$ is given by the vanishing of n smooth functions of these variables with linearly independent differentials, and that $d|\mu| \neq 0$ at $\partial\hat{G}$.

A local parametrization of $\tilde{G}$ near $q \in \overline{G} \cap G^\sharp$ is a function $\phi(y, v, s) = 1 + s\psi(y, v, s)$ defined in a neighbourhood of $q' = (y_0, v_0, 0)$ in $Y \times \mathbb{R}^k \times [0, \infty)$ such that ϕ_0 parametrizes $G^\sharp$ near q, $d_v\phi = 0$ at q', $q = (y, d_{(x,y)}(\phi/x))(q')$, ϕ satisfies the nondegeneracy hypothesis

$$ds,\ d\psi,\ \text{and } d\Big(\frac{\partial\psi}{\partial v_i}\Big) \text{ are linearly independent at } q',\ 1 \leqslant i \leqslant k,$$

and near q,

$$\hat{G} = \{(0, y, -\phi, s d_y\psi, \widehat{d_y\psi}) \mid d_s\phi = 0,\ d_v\psi = 0, s \geqslant 0\},$$

in the coordinates (2.13). A Legendre distribution of order (m,p) associated to $(G,G^\sharp)$ is a half-density of the form $u = u_0 + (\sum_{i=1}^N u_i)\nu$, where ν is as above, $u_0 \in I_c^m(X,G;{}^{\mathrm{sc}}\Omega^{\frac12}) + I^p(X,G^\sharp;{}^{\mathrm{sc}}\Omega^{\frac12})$ (the subscript c indicates that the microlocal support does not meet $G^\sharp$), and u_j is supported in a coordinate patch (x,y) near the boundary, with an expression

$$u_j = \int_0^\infty ds \int e^{i\phi_j(y,v,s)/x} a_j(y,v,x/s,s)\Big(\frac{x}{s}\Big)^{m+n/4-(k+1)/2} s^{p+n/4-1}\, dv,$$

where ϕ_j locally parametrizes $(G,G^\sharp)$ and $a_j \in \mathcal{C}^\infty(X \times \mathbb{R}^k \times [0,\infty) \times [0,\infty))$, with compact support in v, x/s and s. Here u_j can be written with respect to any local parametrization, up to an error in $I^p(X,G^\sharp;{}^{\mathrm{sc}}\Omega^{\frac12})$. The set of such half-densities is denoted $I^{m,p}(X,\tilde{G};{}^{\mathrm{sc}}\Omega^{\frac12})$. The wavefront set of $u \in I^m(X,G;{}^{\mathrm{sc}}\Omega^{\frac12})$ is contained in $G \cup G^\sharp$.

2.3. Codimension 2 corners.

In this subsection we briefly review the extension of the theory of Legendre distributions to manifolds with codimension 2 corners and fibred boundaries given in [4].

Let M be a compact manifold with codimension 2 corners. The boundary hypersurfaces will be labelled $\mathrm{mf}, H_1,\dots,H_d$, where the H_i are endowed with fibrations $\pi_i : H_i \to Z_i$ to certain closed manifolds Z_i and mf (the 'main face') is given the trivial fibration $\mathrm{id} : \mathrm{mf} \to \mathrm{mf}$. The collection of fibrations is denoted Φ. It is assumed that $H_i \cap H_j = \varnothing$ if $i \neq j$. It is also assumed that the fibres of π_i intersect $H_i \cap \mathrm{mf}$ transversally and therefore induce a fibration from $H_i \cap \mathrm{mf} \to Z_i$. Further, it is assumed that a total boundary defining function x is given, which is distinguished up to multiplication by positive functions which are constant on the fibres of ∂M.

Near $H \cap \mathrm{mf}$, where $H = H_i$ for some i, there are coordinates x_1, x_2, y_1, y_2 such that x_1 is a boundary defining function for H, x_2 is a boundary defining function for mf, $x_1 x_2 = x$, and the fibration on H takes the form

$$(y_1, x_2, y_2) \mapsto y_1.$$

Associated with this structure is a Lie Algebra of vector fields

$$\mathcal{V}_{\mathrm{s}\Phi}(M) = \{V \mid V \in \mathcal{C}^\infty, V \text{ is tangent to } \Phi \text{ at } \partial M, V(x) = O(x^2)\}.$$

This is the space of smooth sections of a vector bundle, denoted ${}^{\mathrm{s}\Phi}TM$. The dual space is denoted ${}^{\mathrm{s}\Phi}T^*M$. A point in ${}^{\mathrm{s}\Phi}T^*_pM$ may be thought of as a differential $d(f/x)$ at p, where f is a smooth function on M constant on the fibres at ∂M. A basis for ${}^{\mathrm{s}\Phi}T^*_pM$, for $p \in M$ near $\mathrm{mf} \cap H$, is given by dx/x^2, dx_1/x, dy_1/x, dy_2/x^2. Writing $q \in {}^{\mathrm{s}\Phi}T^*M$ as

$$q = \tau \frac{dx}{x^2} + \tau_1 \frac{dx_1}{x} + \mu_1 \cdot \frac{dy_1}{x} + \mu_2 \cdot \frac{dy_2}{x_2}$$

gives coordinates

$$(x_1,\ x_2,\ y_1,\ y_2,\ \tau,\ \tau_1,\ \mu_1,\ \mu_2) \tag{2.14}$$

on ${}^{\mathrm{s}\Phi}T^*M$ near $\mathrm{mf} \cap H$.

The differential operators of order at most k generated over $\mathcal{C}^\infty(M)$ by $\mathcal{V}_{\mathrm{s}\Phi}(M)$ are denoted $\mathrm{Diff}^k_{\mathrm{s}\Phi}(M)$. Near the interior of mf, the Lie Algebra $\mathcal{V}_{\mathrm{s}\Phi}(M)$ localizes to the scattering Lie Algebra $\mathcal{V}_{\mathrm{sc}}(\tilde{M})$, where $\tilde{M}$ denotes the noncompact manifold with boundary $M \backslash \cup_i H_i$. Consequently, we have a boundary symbol $\sigma_\partial(P)$, $P \in \mathrm{Diff}^k_{\mathrm{s}\Phi}(M)$ taking values in $S^k({}^{\mathrm{sc}}T^*_{\partial \tilde{M}}\tilde{M})$ over the interior of mf. In fact the symbol extends to an element of $S^k({}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}M)$ continuous up to the boundary of mf.

For each fibre F of H, there is a subbundle of ${}^{\mathrm{s}\Phi}TM$ consisting of all vector fields vanishing at F. The annihilator subbundle of ${}^{\mathrm{s}\Phi}T^*M$ is denoted ${}^{\mathrm{sc}}T^*(H;F)$ since it is isomorphic to the cotangent space of the fibre. The quotient bundle, ${}^{\mathrm{s}\Phi}T^*M/{}^{\mathrm{sc}}T^*(H;F)$ is denoted ${}^{\mathrm{s}\Phi}N^*Z_i$ since it is the pullback of a bundle over Z_i. The fibration π_i induces a fibration

$$\begin{aligned} \tilde{\pi}_i : {}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}\cap H}M &\to {}^{\mathrm{s}\Phi}N^*Z_i \\ (y_1, y_2, \tau, \tau_1, \mu_1, \mu_2) &\mapsto (\tau, y_1, \mu_1). \end{aligned} \tag{2.15}$$

We next describe three types of contact structures associated with the structure of M. Since $\mathcal{V}_{\mathrm{s}\Phi}(M)$ is locally the scattering structure near the interior of mf, there is an induced contact structure on ${}^{\mathrm{s}\Phi}T^*M$ over the interior of mf. In local coordinates, the contact form looks like

$$\chi = \iota_{x^2\partial_x}(\omega) = d\tau + \tau_1 dx_1 + \mu_1 dy_1 + x_1\mu_2 dy_2.$$

We see from this that at $x_1 = 0$, χ is degenerate. However, restricted to $\mathrm{mf} \cap H$, χ is the lift of a form χ_{Z_i} on ${}^{\mathrm{s}\Phi}N^*(Z_i)$, namely $d\tau + \mu_1 dy_1$, which is nondegenerate on ${}^{\mathrm{s}\Phi}N^*(Z_i)$. This determines our second type of contact structure (one for each i). The third type of contact structure is that on

${}^{\mathrm{sc}}T^*_{\partial F}(H_i; F)$ induced by $\mathcal{V}_{\mathrm{s}\Phi}(M)$ for each fibre F of H_i, since it restricts to the scattering vector fields on each fibre. In local coordinates, this looks like $d\tau_1 + \mu_2 dy_2$.

Using these three contact structures we define Legendre submanifolds and Legendre distributions.

DEFINITION 2.5. — *A Legendre submanifold G of ${}^{\mathrm{s}\Phi}T^*M$ is a Legendre submanifold of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}M$ which is transversal to ${}^{\mathrm{s}\Phi}T^*_{H_i\cap\mathrm{mf}}M$ for each H_i, for which the map (2.15) induces a fibration from ∂G to G_1, where G_1 is a Legendre submanifold of ${}^{\mathrm{s}\Phi}N^*Z_i$, whose fibers are Legendre submanifolds of ${}^{\mathrm{sc}}T^*_{\partial F}F$.*

A projectable Legendrian (one such that the projection ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}M \to \mathrm{mf}$ is a diffeomorphism when restricted to G) is always of the form

$$\operatorname{graph}\Big(d\big(\frac{\phi}{x}\big)\Big) = \Big\{\Big(\overline{y}, d\big(\frac{\phi(\overline{y})}{x}\big)\Big) \mid \overline{y} \in \mathrm{mf}\Big\}$$

for some smooth function ϕ constant on the fibres of ∂M. We then say that ϕ parametrizes G. In general, let G be a Legendre submanifold of ${}^{\mathrm{s}\Phi}T^*M$, and let $q \in G$. If q lies above the interior of mf, then a local parametrization of G near q is as described in the previous subsection, so consider $q \in \partial G$ lying in ${}^{\mathrm{s}\Phi}T^*_{\overline{y}_0}M$, where $\overline{y}_0 \in \mathrm{mf} \cap H$. A local (nondegenerate) parametrization of G near q is a function $\phi(x_1, y_1, y_2, v, w)$ of the form

$$\phi(x_1, y_1, y_2, v, w) = \phi_1(y_1, v) + x_1\phi_2(x_1, y_1, y_2, v, w), \tag{2.16}$$

defined in a neighbourhood of $q' = (\overline{y}_0, v_0, w_0) \in \mathrm{mf} \times \mathbb{R}^{k_1+k_2}$, such that $d_{v,w}\phi = 0$ at q',

$$q = \Big(\overline{y}_0, d\big(\frac{\phi(\overline{y}_0)}{x}\big)\Big) \text{ at } q'$$

in local coordinates (2.14), ϕ satisfies the nondegeneracy hypothesis at q'

$$d_{(y_1,v)}\frac{\partial\phi_1}{\partial v_j}, j = 1, \dots, k \tag{2.17}$$

$$\text{and } d_{(y_2,w)}\frac{\partial\phi_2}{\partial w_{j'}},\ j' = 1, \dots, k' \text{ linearly independent,}$$

and near q,

$$G = \Big\{\Big(\overline{y}, d\big(\frac{\phi(\overline{y})}{x}\big)\Big) \mid d_v\phi = d_w\psi = 0\Big\}. \tag{2.18}$$

A Legendre distribution of order $(m; r_1, \dots r_d)$ associated to G is a half-density such that for any $v_i \in \mathcal{C}^\infty(M)$ whose support does not intersect H_k, for $k \neq i$, $v_i u$ is of the form $u = u_0 + (\sum_{j=1}^N u_j + \sum_{j=1}^M u_j')\nu$, where ν is a smooth section of the half-density bundle ${}^{s\Phi}\Omega^{\frac{1}{2}}$ induced by ${}^{s\Phi}T^*M$, $u_0 \in \dot{\mathcal{C}}^\infty(X; {}^{s\Phi}\Omega^{\frac{1}{2}})$, and u_j, u_j' have expressions

$$(2.19)\quad u_j(x_1, x_2, y_1, y_2) = \int e^{i\phi_j(x_1,y_1,y_2,v,w)/x} a_j(x_1, x_2, y_1, y_2, v, w) \\ x_2^{m-(k+k')/2+N/4} x_1^{r_i-k/2+N/4-f_i/2}\, dv\, dw$$

with $N = \dim M$, $a_j \in \mathcal{C}^\infty{}_c([0,\epsilon) \times U \times \mathbb{R}^{k+k'})$, U open in mf, f_i the dimension of the fibres of H_i and ϕ_j a phase function parametrizing a Legendrian G on U, and

$$(2.20)\qquad u_j'(x_1, y_1, z) = \int e^{i\psi_j(y_1,w)/x} a_j(x, y_1, z, w) x^{r_i-k/2+N/4-f_i/2}\, dw$$

with $N = \dim M$, $a_j \in \mathcal{C}^\infty{}_c([0,\epsilon) \times U \times \mathbb{R}^k)$, U open in H, f_i as above, ψ_j a phase function parametrizing the Legendrian G_1.

Definition 2.6.— *A Legendre pair with conic points, $(G, G^\sharp)$, in ${}^{s\Phi}T^*M$ consists of two Legendre submanifolds G and $G^\sharp$ of ${}^{s\Phi}T^*M$ which form an intersecting pair with conic points in ${}^{sc}T^*_{\tilde{M}}\tilde{M}$ such that for each H_i the fibrations of G and $G^\sharp$ induced by (2.15) have the same Legendre submanifold G_1 of ${}^{s\Phi}N^*Z_i$ as base and for which the fibres are intersecting pairs of Legendre submanifolds with conic points of ${}^{sc}T^*_{\partial F}F$.*

The Legendrian $G^\sharp$ is required to be projectable, so it parametrized by a phase function $\phi(\overline{y})$ which is constant on the fibres of ∂M. Thus, $x' = x/\phi$ is another admissible total boundary defining function. With respect to x', $G^\sharp$ is parametrized by the function 1. Thus, without loss of generality we may assume that coordinates have been chosen so that $G^\sharp$ is parametrized by 1. This simplifies the coordinate form of the blowup (2.12). Coordinates near $\partial\hat{G}$ then are

$$(2.21)\qquad \overline{y} = (x_1, y_1, y_2),\ \tau,\ \tau_1/|\mu_2|,\ \mu_1/|\mu_2|,\ \hat{\mu}_2 = \mu_2/|\mu_2|, \text{and} |\mu_2|,$$

the last of which is a boundary defining function for $\hat{G}$ (see [4]).

As a consequence of Definition 2.6, $\hat{G}$ is a compact manifold with corners in

$$(2.22)\qquad [{}^{s\Phi}T^*_{\mathrm{mf}}X;\ \{x = 0, \mu_1 = \mu_2 = \tau_1 = 0\}],$$

with one boundary hypersurface at the intersection of $\hat{G}$ and ${}^{s\Phi}T^*_{\mathrm{mf}\cap H_i}M$ for each i (which are mutually nonintersecting, since the H_i are mutually nonintersecting), and one at the intersection of $\hat{G}$ and the front face of the blowup in (2.22). If q lies in the interior of $\hat{G}$ then the situation is as for Legendrians in the scattering setup. If q is on the boundary of $\hat{G}$, but does not lie over H_i for some i, then the situation is as for Legendrian conic pairs as in the previous subsection. If q is on the boundary of $\hat{G}$ but not in $G^\sharp$ then the situation is as above. Thus the only situation left to describe is if q is in the corner of $\hat{G}$, lying above $\overline{y}_0 \in \mathrm{mf} \cap H_i$ say.

A local parametrization of $(G, G^\sharp)$ near q (in coordinates as chosen above) is a function

$$\phi(x_1, y_1, y_2, s, w) = 1 + s x_1 \psi(x_1, y_1, y_2, s, w),$$

with ψ defined in a neighbourhood of $q' = (\overline{y}_0, 0, w_0) \subset M \times [0, \infty) \times \mathbb{R}^k$, such that $d_w\psi = 0$ at q',

$$d_{y_2}\psi \text{ and } d_{(y_2,w)}\left(\frac{\partial\psi}{\partial w_i}\right) \text{ are linearly independent at } q',$$

and such that near $q \in \hat{G}$,

$$\hat{G} = \left\{(\overline{y}, -\phi, \frac{d_{x_1}(x_1\psi)}{|d_{y_2}\psi|}, \frac{d_{y_1}(x_1\psi)}{|d_{y_2}\psi|}, \widehat{d_{y_2}\psi}, |d_{y_2}\psi|) \mid d_s\phi = 0, d_w\psi = 0, s \geqslant 0\right\},$$

in the coordinates (2.21).

A Legendre distribution of order $(m, p; r_1, \dots r_d)$ associated to $(G, G^\sharp)$ is a half-density such that for any $v_i \in \mathcal{C}^\infty(M)$ whose support does not intersect H_k, for $k \neq i$, $v_i u$ is of the form $u = u_0 + (\sum_{j=1}^N u_j + \sum_{j=1}^M u_j')\nu$, where $u_0 \in \dot{\mathcal{C}}^\infty(X; {}^{s\Phi}\Omega^{\frac{1}{2}})$, and u_j, u_j' have expressions

$$\begin{aligned} u_j(x_1, x_2, y_1, y_2) = \int_0^\infty ds \int dw\, e^{i\phi_j(x_1, y_1, y_2, v, w)/x} a_j(x_1, s, x_2/s, y_1, y_2, w) \\ \left(\frac{x_2}{s}\right)^{m-(k'+1)/2+N/4} s^{p-1+N/4} x_1^{r_i - k/2 + N/4 - f_i/2} \end{aligned} \tag{2.23}$$

with $N = \dim M$, $a_j \in {\mathcal{C}^\infty}_c([0, \epsilon) \times U \times \mathbb{R}^{k+k'})$, U open in mf, f_i the dimension of the fibres of H_i and ϕ_j a phase function parametrizing a Legendrian G on U, and where u_j' is as in (2.20).

2.4. The b-double space.

Here we analyze the b-double space X^2_{b}, where X is a compact manifold with boundary, from the perspective of manifolds with corners with fibred boundaries. The manifold with corners X^2_{b} has three boundary hypersurfaces: lb and rb, which are the lifts of the left and right boundaries $\partial X \times X$, $X \times \partial X$ of X^2 to X^2_{b}, and bf, coming from the blowup of $(\partial X)^2$ (see figure 1). Thus, lb and rb have natural projections to ∂X. The fibres of lb and rb meet bf transversally, so we may identify bf as the 'main face' mf of X^2_{b}. Given coordinates (x,y) or z on X, we denote the lift to X^2_{b} via the left, resp. right projection by (x',y') or z', resp. (x'',y'') or z''. We may take the distinguished total boundary defining function to be x', for $\sigma = x'/x'' < C$ and x'', for $\sigma > C^{-1}$. These are compatible since their ratio is constant on fibres on the overlap region $C^{-1} < \sigma < C$ (this is trivially true since the fibres of bf are points).

These data give X^2_{b} the structure of a manifold with corners with fibred boundary as defined above. The sΦ-vector fields then are the same as the sum of the scattering Lie Algebra $\mathcal{V}_{\mathrm{b}}(X)$ lifted to X^2_{b} from the left and right factors.

On X^2_{b}, and ${}^{\mathrm{s}\Phi}T^*X^2_{\mathrm{b}}$, it is most convenient to use coordinates lifted from X and ${}^{\mathrm{sc}}T^*X$. Near lb, but away from bf, we use coordinates (x',y',z'') and coordinates (τ',μ',ζ'') on ${}^{\mathrm{s}\Phi}T^*X^2_{\mathrm{b}}$ where we write a covector $q \in {}^{\mathrm{s}\Phi}T^*X^2_{\mathrm{b}}$ as

$$q = \tau' \frac{dx'}{x'^2} + \mu' \cdot \frac{dy'}{x'} + \zeta'' \cdot dz''.$$

Similarly near rb, but away from bf, we use coordinates $(z',x'',y''; \zeta',\tau'',\mu'')$. Near lb $\cap$ bf, we use (x'',σ,y',y'') with corresponding coordinates (τ,κ,μ',μ''), by writing $q \in {}^{\mathrm{s}\Phi}T^*X^2_{\mathrm{b}}$ as

$$q = \tau \frac{dx'}{x'^2} + \kappa \frac{d\sigma}{x'} + \mu' \cdot \frac{dy'}{x'} + \mu'' \cdot \frac{dy''}{x''}.$$

However, we may also use scattering cotangent coordinates $(\tau',\tau'',\mu',\mu'')$ lifted from ${}^{\mathrm{sc}}T^*X$, where we write

$$q = \tau' \frac{dx'}{x'^2} + \tau'' \frac{dx''}{x''^2} + \mu' \cdot \frac{dy'}{x'} + \mu'' \cdot \frac{dy''}{x''}.$$

This gives

$$\tau' = \tau + \sigma\kappa \quad \tau'' = -\kappa. \tag{2.24}$$

The coordinates (x'', σ, y', y'') hold good near bf as long as we stay away from rb, when we need to switch to $(x', \sigma^{-1}, y', y'')$. The cotangent coordinates $(\tau', \tau'', \mu', \mu'')$ are good coordinates globally near bf; notice that the roles of $(\tau, \tau_1, \mu_1, \mu_2)$ are played by $(\tau', \tau'', \mu', \mu'')$ near lb and $(\tau'', \tau', \mu'', \mu')$ near rb.

The operator H can act on half-densities on X_b^2 by acting either on the left or the right factor of X; these operators are denoted H_l and H_r respectively. For $H = \Delta + P$, where $P \in x\mathrm{Diff}^1_{sc}(X)$, the Hamilton vector field induced by H_l and the contact structure on ${}^{s\Phi}T^*_{bf}\tilde{X}^2_b$, with respect to x', takes the form

$$V_l = 2\tau'\sigma\frac{\partial}{\partial\sigma} + 2\tau'\mu'\frac{\partial}{\partial\mu'} - h'\frac{\partial}{\partial\tau'} + \left(\frac{\partial h'}{\partial\mu'}\frac{\partial}{\partial y'} - \frac{\partial h'}{\partial y'}\frac{\partial}{\partial\mu'}\right) \quad h' = h(y', \mu'). \tag{2.25}$$

Similarly, the Hamilton vector field induced by H_r and the contact structure on ${}^{s\Phi}T^*_{bf}\tilde{X}^2_b$, with respect to x'', takes the form

$$V_r = -2\tau''\sigma\frac{\partial}{\partial\sigma} + 2\tau''\mu''\frac{\partial}{\partial\mu''} - h''\frac{\partial}{\partial\tau''} + \left(\frac{\partial h''}{\partial\mu''}\frac{\partial}{\partial y''} - \frac{\partial h''}{\partial y''}\frac{\partial}{\partial\mu''}\right) \quad h'' = h(y'', \mu''). \tag{2.26}$$

Notice that V_l and V_r commute.

3. Symbol calculus for Legendre distributions.

3.1. Manifolds with boundary.

Let X be a manifold with boundary of dimension N, and let

$$u = x^q(2\pi)^{-k/2-n/4}\left(\int e^{i\phi(y,v)/x}a(x,y,v)\,dv\right)\left|\frac{dxdy}{x^{n+1}}\right|^{\frac{1}{2}} \in I^m(X, G; {}^{sc}\Omega^{\frac{1}{2}})$$

be a Legendre distribution of order m. Let $C = \{(y,v) \mid d_v\phi = 0\}$ and let λ be a set of functions in (y,v)-space such that $(\lambda, d_v\phi)$ form local coordinates near C. We temporarily define the symbol relative to the coordinate system $\mathcal{Z} = (x, y)$ and the parametrization ϕ to be the half density on G given by

$$\sigma^m_{\mathcal{Z},\phi}(u) = \big(a(0,y,v) \restriction C\big)\left|\frac{\partial(d_v\phi, \lambda)}{\partial(y,v)}\right|^{-\frac{1}{2}} |d\lambda|^{\frac{1}{2}}. \tag{3.1}$$

Here we have used the correspondence (2.11) between C and G.

If we change coordinate system, the symbol changes by

$$\sigma^m_{\mathcal{Z}',\phi}(u) = \sigma^m_{\mathcal{Z},\phi}(u) a^{n/4-m} e^{-i\rho(\mathcal{Z}',\mathcal{Z})}, \quad a = \frac{x'}{x} \tag{3.2}$$

where

$$\rho(\mathcal{Z}',\mathcal{Z}) = \Big\{ a\mu_i \frac{\partial y_i}{\partial x'} - \tau \frac{\partial a}{\partial x'} \Big\} \restriction x = 0. \tag{3.3}$$

If the parametrization is changed, then by [6], the symbol changes by

$$\sigma^m_{\mathcal{Z},\psi}(u) = \sigma^m_{\mathcal{Z},\phi}(u) e^{i\pi(\operatorname{sign} d^2_{vv}\psi - \operatorname{sign} d^2_{v'v'}\phi)/4}; \tag{3.4}$$

the exponential is a locally constant function. We use these transformation factors to define two line bundles, the E-bundle over ${}^{\mathrm{sc}}T^*X$ which is defined by the transition functions (3.3), and the Maslov bundle over G which is defined by the transition functions (3.4). (These bundles will be described in much more detail in [5].) Defining the bundle $S^{[m]}(G) = |N^*\partial X|^{m-n/4} \otimes E \otimes M(G)$ over G, we obtain an invariant symbol map from (3.1)

$$\sigma^m : I^m(X, G; {}^{\mathrm{sc}}\Omega^{\frac12}) \to \mathcal{C}^\infty(G; \Omega^{\frac12} \otimes S^{[m]}(G)).$$

The elements of the symbol calculus for Legendre distributions on manifolds with boundary have been given by Melrose and Zworski [12]:

Proposition 3.1. — *The symbol map induces an exact sequence*

$$0 \to I^{m+1}(X, G; {}^{\mathrm{sc}}\Omega^{\frac12}) \to I^m(X, G; {}^{\mathrm{sc}}\Omega^{\frac12}) \to \mathcal{C}^\infty(G, \Omega^{\frac12} \otimes S^{[m]}(G)) \to 0.$$

If $P \in \Psi^k(X; {}^{\mathrm{sc}}\Omega^{\frac12})$ and $u \in I^m(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$, then $Pu \in I^m(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$ and

$$\sigma^m(Pu) = \Big(\sigma(P) \restriction G\Big) \sigma^m(u).$$

Thus, if the symbol of P vanishes on G, then $Pu \in I^{m+1}(X, G; {}^{\mathrm{sc}}\Omega^{\frac12})$. The symbol of order $m+1$ of Pu in this case is

$$\Big(-i\mathcal{L}_{H_p} - i\big(\frac12 + m - \frac{N}{4}\big)\frac{\partial p}{\partial \tau} + p_{\mathrm{sub}}\Big)\sigma^m(u) \otimes |dx|, \tag{3.5}$$

where H_p is the Hamilton vector field of p, the principal symbol of P, and p_{sub} is the subprincipal symbol of P.

The symbol calculus for intersecting Legendre distributions is easily deduced from Melrose and Uhlmann's calculus of intersecting Lagrangian distributions. The symbol takes values in a bundle over $L_0 \cup L_1$. Let ρ_1 be a boundary defining function for ∂L_1 as a submanifold of L_0, and ρ_0 be a boundary defining function for ∂L_1 as a submanifold of L_1. To define the symbol, note that the symbol on L_0 is defined by continuity from distributions microsupported away from L_1, and takes values in

$$(3.6) \qquad \rho_1^{-1}\mathcal{C}^\infty(\Omega^{1/2}(L_0) \otimes S^{[m+1/2]}(L_0)) \\ = \rho_1^{-1/2}\mathcal{C}^\infty(\Omega_b^{1/2}(L_0 \setminus \partial L_1) \otimes S^{[m+1/2]}(L_0)),$$

while the symbol on L_1 defined by continuity from distributions microsupported away from ∂L_1 takes values in

$$\mathcal{C}^\infty(\Omega^{1/2}(L_1) \otimes S^{[m]}(L_1)) = \rho_0^{1/2}\mathcal{C}^\infty(\Omega_b^{1/2}(L_1) \otimes S^{[m]}(L_1)).$$

Melrose and Uhlmann showed that the Maslov factors were canonically isomorphic on $L_0 \cap L_1$, so $S^{[m]}(L_0)$ is naturally isomorphic to $S^{[m]}(L_1)$ over $L_0 \cap L_1$. Canonical restriction of the half-density factors to $L_0 \cap L_1$ gives terms in $\mathcal{C}^\infty(\Omega^{\frac{1}{2}}(L_0 \cap L_1) \otimes S^{[m]}(L_1) \otimes |N^*_{L_0}\partial L_1|^{-1/2} \otimes |N^*\partial X|^{1/2}$ and $\mathcal{C}^\infty(\Omega^{\frac{1}{2}}(L_0 \cap L_1) \otimes S^{[m]}(L_1) \otimes |N^*_{L_1}\partial L_1|^{1/2}$ respectively. In fact $|N^*_{L_0}\partial L_1| \otimes |N^*_{L_1}\partial L_1| \otimes |N^*\partial X|^{-1}$ is canonically trivial; an explicit trivialization is given by

$$(3.7) \qquad (d\rho_0, d\rho_1, x^{-1}) \mapsto x^{-1}\omega(V_{\rho_0}, V_{\rho_1}) \restriction L_0 \cap L_1,$$

where V_{ρ_i} are the Hamilton vector fields of the functions ρ_i extended into ${}^{\mathrm{sc}}T^*X$, and ω is the standard symplectic form. Thus the two bundles are naturally isomorphic over the intersection. We define the bundle $S^{[m]}(\tilde{L})$ to be that bundle such that smooth sections of $\Omega_b^{1/2}(\tilde{L}) \otimes S^{[m]}(\tilde{L})$ are precisely those pairs (a, b) of sections of $\rho_1^{-1}\mathcal{C}^\infty(\Omega^{1/2}(L_0) \otimes S^{[m+1/2]}(L_0))$ and $\rho_0^{1/2}\mathcal{C}^\infty(\Omega_b^{1/2}(L_1) \otimes S^{[m]}(L_1))$ such that

$$(3.8) \qquad \rho_1^{1/2} b = e^{i\pi/4}(2\pi)^{1/4}\rho_0^{-1/2} a \text{ at } L_0 \cap L_1$$

under the identification (3.7). The symbol maps of order m on L_1 and $m + 1/2$ on L_0 then extend in a natural way to a symbol map of order m on $\tilde{L}$ taking values in $\Omega_b^{1/2}(\tilde{L}) \otimes S^{[m]}(\tilde{L})$.

PROPOSITION 3.2. — *The symbol map on $\tilde{L}$ yields an exact sequence*

$$(3.9) \quad 0 \to I^{m+1}(X, \tilde{L}; {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}) \to I^m(X, \tilde{L}; {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}) \to \mathcal{C}^\infty(\tilde{L}, \Omega_b^{\frac{1}{2}} \otimes S^{[m]}) \to 0.$$

Moreover, if we consider just the symbol map to L_1*, there is an exact sequence*

$$(3.10)\quad 0 \to I^{m+1}(X, \tilde{L}; {}^{sc}\Omega^{\frac{1}{2}}) + I^{m+\frac{1}{2}}(X, L_0; {}^{sc}\Omega^{\frac{1}{2}}) \to I^m(X, \tilde{L}; {}^{sc}\Omega^{\frac{1}{2}}) \to \mathcal{C}^\infty(L_1, \Omega^{\frac{1}{2}} \otimes S^{[m]}) \to 0.$$

If $P \in \Psi^k(X; {}^{sc}\Omega^{\frac{1}{2}})$ *and* $u \in I^m(X, \tilde{L}; {}^{sc}\Omega^{\frac{1}{2}})$*, then* $Pu \in I^m(X, \tilde{L}; {}^{sc}\Omega^{\frac{1}{2}})$ *and*

$$\sigma^m(Pu) = \Big(\sigma(P) \restriction \tilde{L}\Big)\sigma^m(u).$$

Thus, if the symbol of P *vanishes on* L_1*, then* Pu *is an element of* $I^{m+1}(X, \tilde{L}; {}^{sc}\Omega^{\frac{1}{2}}) + I^m(X, L_0; {}^{sc}\Omega^{\frac{1}{2}})$*. The symbol of order* $m+1$ *of* Pu *on* L_1 *in this case is given by (3.5).*

For a conic pair of Legendre submanifolds $\tilde{G} = (G, G^\sharp)$, with $\hat{G}$ the desingularized submanifold obtained by blowing up $G^\sharp$, the symbol is defined by continuity from the regular part of G. The symbol calculus then takes the form

PROPOSITION 3.3. — *Let* s *be a boundary defining function for* $\hat{G}$*. Then there is an exact sequence*

$$0 \to I^{m+1,p}(X, \tilde{G}; {}^{sc}\Omega^{\frac{1}{2}}) \to I^m(X, \tilde{G}; {}^{sc}\Omega^{\frac{1}{2}}) \to s^{m-p}\mathcal{C}^\infty(\hat{G}, \Omega_b^{\frac{1}{2}} \otimes S^{[m]}(\hat{G})) \to 0.$$

If $P \in \Psi^k(X; {}^{sc}\Omega^{\frac{1}{2}})$*, and* $u \in I^{m,p}(X, \tilde{G}; {}^{sc}\Omega^{\frac{1}{2}})$*, then* $Pu \in I^{m,p}(X, \tilde{G}; {}^{sc}\Omega^{\frac{1}{2}})$ *and*

$$\sigma^m(Pu) = \Big(\sigma(P) \restriction \hat{G}\Big)\sigma^m(u).$$

If the symbol of P *vanishes on* G*, then* $Pu \in I^{m+1,p}(X, \tilde{G}; {}^{sc}\Omega^{\frac{1}{2}})$*. The symbol of order* $m+1$ *of* Pu *in this case is given by (3.5).*

3.2. Codimension two corners.

When we have codimension two corners, then essentially the same results hold by continuity from the main face. The symbol is defined as a half-density on G by continuity from the interior of mf, where the scattering situation applies. We must restrict to differential operators, however, since pseudodifferential operators have not been defined in this context.

Let M be a manifold with codimension 2 corners with fibred boundaries, let $N = \dim M$, and let G be a Legendre distribution. Let ρ_i be a boundary defining function for H_i. The Maslov bundle M and the E-bundle are defined via the scattering structure over the interior of G and extend to smooth bundles over the whole of G (that is, they are smooth up to each boundary of G at ${}^{s\Phi}T^*_{H_i\cap \mathrm{mf}}M$). Let $S^{[m]}(G) = M(G)\otimes E\otimes |N^*\mathrm{mf}|^{m-N/4}\otimes |N^*H_1|^{m-N/4}\otimes\ldots\otimes|N^*H_d|^{m-N/4}$. Finally let $\mathbf{r}$ stand for $(r_1,\ldots,r_d)$, and let $\rho^{\mathbf{r}} = \prod_i \rho_i^{r_i}$.

PROPOSITION 3.4. — *There is an exact sequence*

$$0 \to I^{m+1,\mathbf{r}}(M,G;{}^{s\Phi}\Omega^{\frac12}) \to I^{m,\mathbf{r}}(M,G;{}^{s\Phi}\Omega^{\frac12}) \\ \to \rho^{m-\mathbf{r}}C^\infty(G,\Omega_b^{\frac12}\otimes S^{[m]}(G)) \to 0.$$

If $P \in \mathrm{Diff}(M;{}^{s\Phi}\Omega^{\frac12})$ *and* $u \in I^{m,\mathbf{r}}(M,G;{}^{s\Phi}\Omega^{\frac12})$, *then* $Pu \in I^{m,\mathbf{r}}(M,G;{}^{s\Phi}\Omega^{\frac12})$ *and*

$$\sigma^m(Pu) = \big(\sigma(P)\restriction G\big)\sigma^m(u).$$

Thus, if the symbol of P *vanishes on* G, *then* $Pu \in I^{m+1,\mathbf{r}}(M,G;{}^{s\Phi}\Omega^{\frac12})$. *The symbol of order* $m+1$ *of* Pu *in this case is given by (3.5).*

For a conic pair of Legendre submanifolds $\tilde G = (G, G^\sharp)$, with $\hat G$ the desingularized submanifold obtained by blowing up $G^\sharp$, the symbol calculus takes the form

PROPOSITION 3.5. — *Let* s *be a boundary defining function for* $\hat G$ *at* $\hat G \cap G^\sharp$. *Then there is an exact sequence*

$$0 \to I^{m+1,p;\mathbf{r}}(M,\tilde G;{}^{s\Phi}\Omega^{\frac12}) \to I^{m,p;\mathbf{r}}(M,\tilde G;{}^{s\Phi}\Omega^{\frac12}) \\ \to \rho^{m-\mathbf{r}}s^{m-p}C^\infty(\hat G,\Omega_b^{\frac12}\otimes S^{[m]}(\hat G)) \to 0. \tag{3.11}$$

If $P \in \mathrm{Diff}(M;{}^{s\Phi}\Omega^{\frac12})$, *and* $u \in I^{m,p;\mathbf{r}}(M,\tilde G;{}^{s\Phi}\Omega^{\frac12})$, *then* $Pu \in I^{m,p;\mathbf{r}}(M,\tilde G;{}^{s\Phi}\Omega^{\frac12})$ *and*

$$\sigma^m(Pu) = \big(\sigma(P)\restriction \hat G\big)\sigma^m(u).$$

If the symbol of P *vanishes on* G, *then* $Pu \in I^{m+1,p;\mathbf{r}}(M,\tilde G;{}^{s\Phi}\Omega^{\frac12})$. *The symbol of order* $m+1$ *of* Pu *in this case is given by (3.5).*

The proofs of these propositions are omitted, since they are easily deduced from the codimension one case.

4. Parametrix construction.

In this section, we consider self-adjoint operators H of the form $\Delta+P$, where Δ is the positive Laplacian with respect to a short-range metric on a compact manifold with boundary, X, and $P \in x^2\mathrm{Diff}^1_{\mathrm{sc}}(X)$ is a short-range perturbation of Δ. In the following section, we consider metrics and perturbations of long range gravitational type. Let $R(\sigma)$ denote the resolvent $(H-\sigma)^{-1}$ of H.

Thus, we directly construct a parametrix $G(\lambda)$ for (1.1) whose error term $E(\lambda) = (H-\lambda^2)G(\lambda) - \mathrm{Id}$ is compact. Using Fredholm theory and a unique continuation theorem we solve away the error, giving us a Schwartz kernel $\tilde{R}(\lambda)$. We then show that $\tilde{R}(\sqrt{\sigma})$ has an analytic continuation (as a distribution on X^2) to the upper half σ plane which agrees with the resolvent $R(\sigma)$ there. This proves that $\tilde{R}(\lambda)$ and $R(\lambda^2+i0)$ coincide on the real axis.

The distribution $\tilde{R}(\lambda)$ has the defining property that

$$(H-\lambda^2)\tilde{R}(\lambda) = \mathrm{Id}, \tag{4.1}$$

as an operator on $\dot{\mathcal{C}}^\infty(X; {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$, and that

$$ {}^{\mathrm{sc}}\mathrm{WF}_{\tilde{X}^2_{\mathrm{b}}}(\tilde{R}(\lambda)) \subset K_- \text{ as defined in (2.5).} \tag{4.2}$$

Here $\tilde{X}^2_{\mathrm{b}} = X^2_{\mathrm{b}} \setminus \{\mathrm{lb} \cup \mathrm{rb}\}$ is regarded as an open manifold with boundary, so that we can talk about the scattering wavefront set over the interior of bf. Equation (4.2) is the microlocal version of the outgoing Sommerfeld radiation condition. (For example, if $\lambda > 0$, $e^{i\lambda/x}$ has wavefront set in K_-, while $e^{-i\lambda/x}$ does not.)

Equation (4.1) means that the kernel of $\tilde{R}(\lambda)$, which we also denote by $\tilde{R}(\lambda)$ by an abuse of notation, satisfies

$$(H_l - \lambda^2)\tilde{R}(\lambda) = K_{\mathrm{Id}}, \tag{4.3}$$

where K_{Id} is the kernel of the identity operator, i.e., it is a delta distribution on the diagonal, and H_l is the operator H acting on the left factor of X in $X \times X$.

There are four main steps in the construction. First we find an approximation to $\tilde{R}(\lambda)$ in the scattering calculus, $G_1(\lambda) \in \Psi^{-2}_{\mathrm{sc}}(X)$, which removes the singularity on the diagonal in (4.3). This leaves an error which, when viewed on the b-double space X_b^2, is singular at the boundary of the diagonal $\partial\mathrm{diag}_b$. In fact, it is Legendrian at a Legendre submanifold lying over $\partial\mathrm{diag}_b$ which we denote $N^*\mathrm{diag}_b$ (see (4.5)). We solve this error away locally near $\partial\mathrm{diag}_b$ using an intersecting Legendrian construction which is due (in the Lagrangian setting) to Melrose and Uhlmann [11]; the singularities inside $N^*\mathrm{diag}_b \cap \Sigma(H_l - \lambda^2)$ propagate in a Legendre submanifold $L_+(\lambda)$. This Legendre submanifold intersects both lb and rb, and an 'outgoing' Legendre submanifold $L^\#(\lambda)$; $(L_+(\lambda), L^\#(\lambda))$ form a conic pair of Legendre submanifolds and we can find a conic Legendre pair which solves away the error up to an error term which is Legendrian only at $L^\#(\lambda)$, i.e., we can solve away the errors at $L_+(\lambda)$ completely. Finally, this outgoing error is solved away, using a very standard argument in scattering theory, at lb and bf, leaving an error $E(\lambda)$ which is compact on weighted L^2 spaces $x^l L^2(X)$ for all $l > 1/2$.

Thus, we seek $G(\lambda)$ (and $\tilde{R}(\lambda)$) in the class

$$(4.4)\qquad \Psi_{sc}^{2,0}(X) + I^{-\frac{1}{2}}(N^*\mathrm{diag}_b, L_+(\lambda); {}^{sc}\Omega^{\frac{1}{2}}) \\ + I^{-\frac{1}{2}, \frac{n-2}{2}; \frac{n-1}{2}, \frac{n-1}{2}}(L_+(\lambda), L^\#(\lambda); {}^{sc}\Omega^{\frac{1}{2}})$$

where the second term is an intersecting Legendrian distribution and the third is a Legendre conic pair with orders $-1/2$ at $L_+(\lambda)$, $(n-2)/2$ at $L^\#(\lambda)$ and $(n-1)/2$ at lb and rb. In this class of distributions there is a unique solution $\tilde{R}(\lambda)$ to (4.1) and (4.2).

To avoid cumbersome notation, Q will denote a generic correction to the parametrix constructed so far, and E will denote a generic error. The values of these symbols is allowed to change from line to line.

4.1. Pseudodifferential approximation.

The first step in constructing $G(\lambda)$ is a very standard argument. We seek $G_1(\lambda) \in \Psi_{sc}^{-2}(X)$ such that

$$(H - \lambda^2)G_1(\lambda) = \mathrm{Id} + E_1(\lambda), \quad E_1(\lambda) \in \Psi_{sc}^{-\infty}(X).$$

This will mean that the error term $E_1(\lambda)$ has a smooth kernel (times the standard half-density) on X_{sc}^2, so that we have solved away completely the singularity along the diagonal.

The standard elliptic argument applies here since the interior symbol of $H - \lambda^2$ is elliptic. Thus, we first choose any $Q \in \Psi_{\mathrm{sc}}^{-2}(X)$ whose interior symbol is $|\cdot|_g^{-2} = (\sigma^2(H - \lambda^2))^{-1}$. Then

$$(H - \lambda^2)Q = \mathrm{Id} + E, \quad E \in \Psi_{\mathrm{sc}}^{-1}(X).$$

Multiplying Q by a finite Neumann series $(\mathrm{Id}+E+\ldots+E^{k-1})$ thus gives an error $E^k \in \Psi_{\mathrm{sc}}^{-k}(X)$. Taking an asymptotic limit gives us a $G_1(\lambda) \in \Psi_{\mathrm{sc}}^{-2}(X)$ with the desired error term.

4.2. Intersecting Legendrian construction.

In the next step of the construction we move to X_{b}^2, and view the error $E_1(\lambda)$ from the first step of the construction on X_{b}^2 rather than X_{sc}^2. On X_{b}^2 it has a smooth kernel except at $\partial\mathrm{diag}_{\mathrm{b}}$ where it has a conic singularity. That is, at $\partial\mathrm{diag}_{\mathrm{b}} = \{x' = 0, \sigma \equiv x'/x'' = 1, y' = y''\}$, the kernel is a smooth (and compactly supported) function of x', $S = (\sigma - 1)/x'$, $Y = (y' - y'')/x'$ and y'; this is easy to see since these are smooth coordinates on $\mathrm{sf} \subset X_{\mathrm{sc}}^2$. Using the Fourier transform, we write

$$E_1(\lambda) = \left(\int_{\mathbb{R}^n} e^{i((y'-y'')\cdot\eta + (\sigma-1)t)/x'} a(x', y', \eta, t)\, d\eta\, dt \right) \nu$$

where a is smooth in all variables, and in addition Schwartz in (η, t). The phase function $(y' - y'') \cdot \eta + (\sigma - 1)t$ parametrizes the Legendrian

$$N^*\mathrm{diag}_{\mathrm{b}} = \{y' = y'', \sigma = 1, \mu' = -\mu'', \tau' = -\tau''\}. \tag{4.5}$$

Therefore, $E_1(\lambda)$ is a Legendre distribution of order 0 associated to $N^*\mathrm{diag}_{\mathrm{b}}$. (To be pedantic, $E_1(\lambda)$ does not fall strictly in the class of Legendre distributions as defined by Melrose and Zworski since its wavefront set is not a compact subset of ${}^{\mathrm{sc}}T^*_{\partial \tilde{X}_{\mathrm{b}}^2} \tilde{X}_{\mathrm{b}}^2$; from (4.5) we see that the wavefront set is a vector bundle over $\partial\mathrm{diag}_{\mathrm{b}}$. It is instead an 'extended Legendre distribution' as defined in [2]. However this is of no significance since the symbol is rapidly decreasing in each fibre of the vector bundle, hence all constructions we wish to perform here are valid in this context.)

Observe that $\sigma_\partial(H_l - \lambda^2) = {\tau'}^2 + h(y', \mu') - \lambda^2$ vanishes on a codimension one submanifold of $N^*\mathrm{diag}_{\mathrm{b}}$, and does so simply. Consider the vector field V_l which is given by (2.25). Since ${\tau'}^2 + h = \lambda^2 \neq 0$ on $\Sigma(H_l - \lambda^2)$, at least one of the coefficients of ∂_σ and $\partial_{\tau'}$ in (2.25) is nonzero, so V_l is transverse to $N^*\mathrm{diag}_{\mathrm{b}}$ at the intersection with $\Sigma(H_l - \lambda^2)$. We define $L^\circ(\lambda)$

to be the flowout Legendrian from $N^*\mathrm{diag_b} \cap \Sigma(H_l - \lambda^2)$ with respect to V_l, and $L^\circ_\pm(\lambda)$ to be the flowout in the positive, resp. negative direction with respect to V_l. Thus, at least locally near $N^*\mathrm{diag_b}$, $L^\circ_\pm(\lambda)$ are smooth manifolds with boundary. Notice that by (2.24), $N^*\mathrm{diag_b}$ is contained in $\tau = 0$ and $V_l(\tau) < 0$ at $N^*\mathrm{diag_b}$. Thus, at least locally near $N^*\mathrm{diag_b}$, $L^\circ_+(\lambda)$ is contained in $K_- = \{\tau \leqslant 0\}$. The global properties of $L^\circ_\pm(\lambda)$ are studied in the next section; in this section we only work microlocally near $N^*\mathrm{diag_b}$.

The first step in solving away the error $E_1(\lambda)$ from the previous step is to find an intersecting Legendrian $Q \in I^{-1/2}(X^2_{\mathrm{b}}, (N^*\mathrm{diag_b}, L_+(\lambda)); {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ such that

$$(H_l - \lambda^2)Q - E_1(\lambda) \in I^1(X^2_{\mathrm{b}}, N^*\mathrm{diag_b}; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}}) + I^{\frac{3}{2}}(X^2_{\mathrm{b}}, (N^*\mathrm{diag_b}, L_+(\lambda)), {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}}), \tag{4.6}$$

microlocally near $N^*\mathrm{diag_b}$. To do this we choose Q with symbol on $N^*\mathrm{diag_b}$ equal to $\sigma^0_\partial(H_l - \lambda^2)^{-1}\sigma^0(E^{(1)}_+(\lambda))$. This is an admissible symbol on $N^*\mathrm{diag_b}$ by (3.6), and (3.9), since $\sigma^0_\partial(H_l - \lambda^2)$ is a boundary defining function for $L_+(\lambda)$ on $N^*\mathrm{diag_b}$. It determines the value of the symbol on $L_+(\lambda)$ at the boundary by (3.8). We extend this symbol by requiring that the transport equation, (3.5), be satisfied. This equation is a first order linear ODE with smooth coefficients, so there is a unique solution in a neighbourhood of $N^*\mathrm{diag_b}$. Then the symbol of order $-1/2$ of $(H_l - \lambda^2)R_0 - E^{(1)}_+(\lambda)$ vanishes, and there is an additional order of vanishing on $L_+(\lambda)$ since the transport equation is satisfied. Thus by (3.10) the error term is as in (4.6).

We now show inductively that we can solve away an error E_k which is in $I^k(N^*\mathrm{diag_b}) + I^{k+1/2}(N^*\mathrm{diag_b}, L_+(\lambda))$ with a term $Q_k \in I^{k+1/2}(N^*\mathrm{diag_b}, L_+(\lambda))$, up to an error which is in $I^{k+1}(N^*\mathrm{diag_b}) + I^{k+3/2}(N^*\mathrm{diag_b}, L_+(\lambda))$. The argument is the same as above: we take the symbol of order k on $N^*\mathrm{diag_b}$ equal to $\sigma^0_\partial(H_l - \lambda^2)^{-1}\sigma^k(E_k)$, and the symbol on $L_+(\lambda)$ to solve away the symbol of order $k + 1/2$ of E_k when the transport operator is applied to it. Taking an asymptotic sum of Q and the Q_k's gives us an error term which is microlocally trivial near $N^*\mathrm{diag_b}$. By cutting off away from $\partial\mathrm{diag_b}$, we obtain an error

$$E_2(\lambda) \in I^{1/2}_c(L_+(\lambda)),$$

where the subscript c indicates that the microlocal support is compact and disjoint from the intersection with $N^*\mathrm{diag_b}$.

4.3. Structure of $L(\lambda)$.

In this section we analyze the global structure of $L^\circ(\lambda)$. This was defined as the flowout from $N^*\mathrm{diag}_{\mathrm{b}} \cap \Sigma(H_l - \lambda^2)$ by the vector field V_l. In fact, it is quite easy to see that $N^*\mathrm{diag}_{\mathrm{b}} \cap \Sigma(H_l - \lambda^2) = N^*\mathrm{diag}_{\mathrm{b}} \cap \Sigma(H_r - \lambda^2)$. Moreover, neither V_l nor V_r is tangent to $N^*\mathrm{diag}_{\mathrm{b}}$ at any point contained in $\Sigma(H_l - \lambda^2)$, but the difference $V_l - V_r$ is tangent to $N^*\mathrm{diag}_{\mathrm{b}}$. Since V_l and V_r commute, this shows that the flowout with respect to V_l is the same as the flowout with respect to V_r. We will soon see that the symbols of our parametrix on $L_+(\lambda)$, defined so as to satisfy the left transport equation, also satisfy the right transport equation.

It is convenient to write down $L^\circ(\lambda)$ explicitly. Indeed, the computation of Melrose and Zworski can be applied with a minor change (that takes care of the behavior in σ) to deduce that

$$(4.7)\quad \begin{gathered} L^\circ(\lambda) = \{(\theta, y', y'', \tau', \tau'', \mu', \mu'') : \ \exists (y, \hat\mu) \in S^*\partial X, \ s, s' \in (0, \pi), \text{ s.t.} \\ \sigma = \tan\theta = \frac{\sin s'}{\sin s}, \ \tau' = \lambda \cos s', \ \tau'' = -\lambda\cos s, \\ (y', \mu') = \lambda \sin s' \exp(s' H_{\frac{1}{2}h})(y, \hat\mu), (y'', \mu'') = -\lambda \sin s \exp(s H_{\frac{1}{2}h})(y, \hat\mu)\} \\ \cup T_+(\lambda) \cup T_-(\lambda), \quad T_\pm(\lambda) = \{(\sigma, y, y, \pm\lambda, \mp\lambda, 0, 0) : \ \sigma \in (0, \infty), \ y \in \partial X\}. \end{gathered}$$

The sets $T_\pm(\lambda)$ are, for fixed y, integral curves of both vector fields, and they appear separately only because we used the parameterization of Melrose-Zworski. The smooth structure near $T_\pm(\lambda)$ follows from the flowout description, but is not apparent in this parameterization; we discuss it below while describing the closure of $L^\circ(\lambda)$.

The closure $L(\lambda)$ of $L^\circ(\lambda)$ is ${}^{s\Phi}T^*X^2_{\mathrm{b}}$ is clear from the above description; it is

$$(4.8)\qquad \mathrm{cl}L = L(\lambda) \cup F_\lambda \cup F_{-\lambda}$$

where

$$(4.9)\quad \begin{gathered} L(\lambda) = \{(\theta, y', y'', \tau', \tau'', \mu', \mu'') : \ \exists (y, \hat\mu) \in S^*\partial X, \ s, s' \in [0, \pi], \\ (\sin s)^2 + (\sin s')^2 > 0, \text{ s.t.} \\ \sigma = \tan\theta = \frac{\sin s'}{\sin s}, \ \tau' = \lambda \cos s', \ \tau'' = -\lambda\cos s, \\ (y', \mu') = \lambda \sin s' \exp(s' H_{\frac{1}{2}h})(y, \hat\mu), (y'', \mu'') = -\lambda \sin s \exp(s H_{\frac{1}{2}h})(y, \hat\mu)\} \\ \cup T_+(\lambda) \cup T_-(\lambda), \quad \text{and} \\ F_\lambda = \{(\sigma, y', y'', -\lambda, -\lambda, 0, 0) \mid \exists \text{ geodesic of length } \pi \text{ connecting } y', \ y''\}. \end{gathered}$$

Note that the requirement $(\sin s)^2 + (\sin s')^2 > 0$ just means that s and s' cannot take values in $\{0, \pi\}$ at the same time. The set $L(\lambda) \setminus L^o(\lambda)$ comprises those points where one of s, s' takes values in $\{0, \pi\}$ while the other lies in $(0, \pi)$. The sets $T_\pm(\lambda)$ in (4.9) comprise the limit points where s and s' converge either both to 0 or both to π, whilst $F_{\pm\lambda}$ comprise the limit points as $s \to 0$ and $s' \to \pi$ or vice versa.

The smooth structure near $T_\pm(\lambda)$ becomes apparent if we note that near $\tau' = \lambda$, $\tau'' = -\lambda$, $\sigma \in [0, C)$ where $C > 1$, $L(\lambda)$ is given by

$$\begin{aligned}(4.10)\quad \{(\sigma, y', y'', \tau', \tau'', \mu', \mu'') :\ &\exists (y, \mu) \in T^*\partial X,\ |\mu| < C^{-1},\ \sigma \in [0, C) \text{ s.t.} \\ &\tau' = \lambda(1 - |\sigma\mu|^2)^{1/2},\ \tau'' = -\lambda(1 - |\mu|^2)^{1/2}, \\ (y', \mu') = \lambda \exp\left(f(\sigma\mu)V_h\right)&(y, \sigma\mu),\ (y'', \mu'') = -\lambda \exp\left(f(\mu)V_h\right)(y, \mu)\}\end{aligned}$$

where $f(\mu) = |\mu|^{-1} \arcsin(|\mu|)$ is smooth and $f(0) = 1$. Thus, the differential of the map

$$(4.11)\qquad (y, \mu) \mapsto -\lambda \exp\left(f(\mu)V_h\right)(y, \mu) = (y'', \mu'')$$

is invertible near $\mu = 0$, so it gives a diffeomorphism near $|\mu| = 0$. Hence, σ and (y'', μ'') give coordinates on $L(\lambda)$ in this region, so $L(\lambda)$ is smooth here. Away from $T_+(\lambda)$, coordinates on $L(\lambda)$ are σ, y'', $\hat{\mu}''$ and s.

In the coordinates $(y, \hat{\mu}, s, s')$, the vector field V_l is given by $\sin s'\, \partial_{s'}$ and V_r is given by $-\sin s\, \partial_s$. The intersection of $L(\lambda)$ and $N^*\mathrm{diag}_{\mathrm{b}}$ is given by $\{s = s'\}$. Thus $L_+(\lambda)$ is given by $\{s \leqslant s'\}$. On $L_+(\lambda)$, $\tau = \tau' + \sigma\tau''$ by (2.24), so

$$\tau = \lambda \frac{\sin(s - s')}{\sin s} \leqslant 0 \text{ on } L_+(\lambda).$$

Thus, any distribution in $I^m(N^*\mathrm{diag}_{\mathrm{b}}, L_+(\lambda))$ satisfies condition (4.2).

We also define

$$(4.12)\qquad L^{\#}(\lambda) = \{(\theta, y', y'', -\lambda, -\lambda, 0, 0) :\ y', y'' \in \partial X,\ \theta \in [0, \pi/2]\},$$

so $L^{\#}(\lambda)$ is a Legendrian submanifold of ${}^{\mathrm{sc}}T^*_{\mathrm{bf}}\tilde{X}^2_{\mathrm{b}}$, and

$$(4.13)\qquad \mathrm{cl}L \cap L^{\#}(\pm\lambda) = F_{\pm\lambda}.$$

Proposition 4.1. — *The pair*

$$(4.14)\qquad \tilde{L}(\lambda) = (L(\lambda), L^{\#}(\lambda) \cup L^{\#}(-\lambda))$$

is a pair of intersecting Legendre manifolds with conic points.

Proof. — We must show that when the set $\{tq \mid t > 0, q \in L^{\#}(\lambda)\}$ is blown up inside ${}^{s\Phi}T^*X^2_{\mathrm{b}}$, the closure of $L(\lambda)$ is a smooth manifold with corners which meets the front face of the blowup transversally. Let us restrict attention to a neighbourhood of $L^{\#}(\lambda)$; the case of $L^{\#}(-\lambda)$ is similar. Consider the vector field $V_l + V_r$. By (2.25) and (2.26), in $\Sigma(H_l - \lambda^2) \cap \Sigma(H_r - \lambda^2)$ this is given by

$$2(\tau' - \tau'')\sigma\partial_\sigma + 2\tau'\mu' \cdot \partial_{\mu'} + 2\tau''\mu'' \cdot \partial_{\mu''} + \partial_{\mu'}h' \cdot \partial_{y'} \\ - \partial_{y'}h' \cdot \partial_{\mu'} + \partial_{\mu''}h'' \cdot \partial_{y''} - \partial_{y''}h'' \cdot \partial_{\mu''}.$$

This is equal to -2λ times the b-normal vector field $\mu' \cdot \partial_{\mu'} + \mu'' \cdot \partial_{\mu''}$ plus a sum of vector fields which have the form ρV, where ρ vanishes at $L^{\#}(\lambda)$ and V is tangent to lb and $L^{\#}(\lambda)$ (all considerations taking place inside $\Sigma(H_l - \lambda^2) \cap \Sigma(H_r - \lambda^2)$). Thus, under blowup of $\{tq \mid t > 0, q \in L^{\#}(\lambda)\}$, $V_l + V_r$ lifts to a vector field of the form

$$V_l + V_r = -2\lambda s\partial_s + sW, \tag{4.15}$$

where W is smooth and tangent to the boundary of $\hat{L}(\lambda)$, and so dividing by s yields a nonvanishing normal vector field plus a smooth tangent vector field. As above, such a vector field has a continuation across the boundary to the double of $\hat{L}(\lambda)$ (across the front face) as a smooth nonvanishing vector field. This holds true smoothly up to the corner with lb, so $\hat{L}(\lambda)$ is a smooth manifold with corners. □

4.4. Smoothness of symbols.

In the next stage of the construction, we solve away the error $E_2(\lambda)$ which is microsupported in the interior of $L_+(\lambda)$. This involves solving the transport equation globally on $L_+(\lambda)$. In view of Proposition 4.1, we can expect the construction to involve Legendrian conic pairs with respect to $(L_+(\lambda), L^{\#}(\lambda))$. In order for the symbol to be quantizable to such a conic pair, we need to show regularity of the symbol on $\hat{L}_+(\lambda)$, so that it lies in the symbol space of the exact sequence from Proposition 3.5. That is, the symbol of order $j - 1/2$ on $L_+(\lambda)$ should lie in

$$\rho_{\mathrm{lb}}^{n/2-j}\rho_{\mathrm{rb}}^{n/2-j}\rho_{\#}^{(n-1)/2-j}\mathcal{C}^\infty(\hat{L}_+(\lambda); \Omega_b^{1/2} \otimes S^{[j-1/2]}(\hat{L}_+(\lambda))). \tag{4.16}$$

(We will ignore the symbol bundle $S^{[j-1/2]}$ in the rest of this section.)

To do this, we observe that the symbol on $L_+(\lambda)$ automatically satisfies the transport equation for the right Hamilton vector field. To see this, let $G_2(\lambda)$ be the approximation to $\tilde{R}(\lambda)$ constructed so far, with

$$(H_l - \lambda^2)G_2(\lambda) - K_{\mathrm{Id}} = E_2(\lambda),$$

and

$$^{\mathrm{sc}}\mathrm{WF}(E_2(\lambda)) \subset L_+(\lambda) \cap \{\tau < -c\}$$

for some $c > 0$. Consider applying $H_r - \lambda^2$ to $E_2(\lambda)$. Since H_l and H_r commute, and $H_l K_{\mathrm{Id}} = H_r K_{\mathrm{Id}}$, we have

$$\begin{aligned}(H_l - \lambda^2)\big((H_r - \lambda^2)G_2(\lambda) - \mathrm{Id}\big) &= (H_r - \lambda^2)\big((H_l - \lambda^2)G_2(\lambda) - \mathrm{Id}\big)\\ &= (H_r - \lambda^2)E_2(\lambda).\end{aligned}$$

We claim that $^{\mathrm{sc}}\mathrm{WF}((H_r - \lambda^2)G_2(\lambda) - K_{\mathrm{Id}})$ is contained in $\{\tau < -c\}$. For if there is a point where $\tau \geqslant -c$, then by (2.8), the maximal bicharacteristic ray in

$$\begin{aligned}\Sigma(H_l - \lambda^2) \setminus {}^{\mathrm{sc}}\mathrm{WF}\Big((H_l - \lambda^2)\big((H_r - \lambda^2)G_2(\lambda) - K_{\mathrm{Id}}\big)\Big)&\\ = \Sigma(H_l - \lambda^2) \setminus {}^{\mathrm{sc}}\mathrm{WF}((H_r - \lambda^2)E_2(\lambda))&\end{aligned} \tag{4.17}$$

lies in $^{\mathrm{sc}}\mathrm{WF}((H_r - \lambda^2)G_2(\lambda) - K_{\mathrm{Id}})$. Such rays always propagate into $\tau > 0$. But

$$^{\mathrm{sc}}\mathrm{WF}(G_2(\lambda)) \subset \{\tau \leqslant 0\}, \quad {}^{\mathrm{sc}}\mathrm{WF}(K_{\mathrm{Id}}) \subset \{\tau \leqslant 0\},$$

so this is impossible. Consequently, $(H_r - \lambda^2)G_2(\lambda) - K_{\mathrm{Id}}$ has no scattering wavefront set for $\{\tau \geqslant -c\}$, and so the symbols of $G_2(\lambda)$ must obey the right transport equations in this region. By cutting off the symbols closer and closer to the boundary of $L_+(\lambda)$, we see that the right transport equations must be satisfied everywhere on $L_+(\lambda)$.

Let us examine the form of these transport equations at the boundary of $L_+(\lambda)$. Near lb, we have coordinates (y'', μ'', σ) near $T_+(\lambda)$ and $(y'', \hat{\mu}'', \sigma, s)$ away from $T_+(\lambda)$, which are valid coordinates for $\sigma < 2$, say. The situation near rb is similar so the argument will be omitted.

In either set of coordinates, the left vector field, restricted to $L_+(\lambda)$, takes the form

$$V_l = 2\tau'\sigma\partial_\sigma.$$

Also, by Lemma 2.3, the subprincipal symbol of $H_l - \lambda^2$, which is equal to the subprincipal symbol of $H - \lambda^2$ in the singly-primed coordinates, vanishes where μ' vanishes, and $\mu' = 0$ at lb on $L_+(\lambda)$. Therefore, by (3.5), the transport equation for the symbol of order $-1/2$ takes the form

$$(4.18) \qquad \left(-i\left(\mathcal{L}_{V_l} - n\tau'\right) + \sigma f\right) a_0 \left|\frac{d\sigma}{\sigma} dy'' d\mu''\right|^{\frac{1}{2}} = 0, \ f \in \mathcal{C}^\infty(L_+(\lambda)),$$

near $T_+(\lambda)$, or

$$(4.19) \quad \left(-i\left(\mathcal{L}_{V_l} - n\tau'\right) + \sigma f\right) a_0 \left|\frac{d\sigma}{\sigma} \frac{ds}{s} dy'' d\hat{\mu}''\right|^{\frac{1}{2}} = 0, \ f \in \mathcal{C}^\infty(L_+(\lambda)),$$

away from $T_+(\lambda)$, which gives an equation for a_0 of the form

$$(4.20) \qquad -i\tau'\left(\partial_\sigma + f\right)(\sigma^{-n/2} a_0) = 0, \quad f \in \mathcal{C}^\infty(L_+(\lambda)).$$

This shows that $\sigma^{-n/2} a_0$ is smooth across $\sigma = 0$.

To show regularity near $L^\#(\lambda)$, we use the fact that the symbol satisfies both the right and left transport equation. We take the sum of the transport equations that obtain when we use the total boundary defining function x' for H_l, and x'' for H_r. The right transport operator with respect to x'' takes the form

$$-i\left(\mathcal{L}_{V_r} - n\tau''\right) + p_{\text{sub}}.$$

However, by (3.2) the symbol written in terms of x'' is equal to $(x''/x')^{-1/2-n/2}$ times the symbol written in terms of x'. Since we are writing the symbol in terms of x', we must include a factor $\sigma^{-1/2-n/2}$ to be consistent with (4.19). This gives an equation for a_0 of the form
(4.21)

$$\left(-i\left(\mathcal{L}_{V_r} - n\tau''\right) + p_{\text{sub}}(y'', \mu'', \tau'')\right) \left(\sigma^{-1/2-n/2} a_0 \left|\frac{d\sigma}{\sigma} \frac{ds}{s} dy'' d\hat{\mu}''\right|^{\frac{1}{2}}\right) = 0.$$

In view of the term $-2\tau''\sigma\partial_\sigma$ in the formula (2.26) for V_r, and since p_{sub} vanishes at $s = 0$ since $\mu'' = 0$ there, we get an equation for a_0

$$(4.22) \qquad \left(V_r + \tau'' + sf'\right) a_0 = 0 \quad f' \in \mathcal{C}^\infty(\hat{L}_+(\lambda)).$$

Combining with the left transport equation gives an equation which, using (4.15) and the fact that $\tau' = \tau'' = -\lambda$ at $s = 0$ takes the form

$$2\lambda\left(s\partial_s - \frac{n-1}{2} + sW + s\tilde{f}\right) a_0 = 0,$$

where W is tangent to the boundary of $\hat{L}_+(\lambda)$ and $\tilde{f}$ is smooth on $\hat{L}_+(\lambda)$. This may be written

$$\big(\partial_s + W + \tilde{f}\big)\big(s^{-(n-1)/2}a_0\big) = 0.$$

This together with (4.20) shows that $a_0 \in \sigma^{n/2}s^{(n-1)/2}\mathcal{C}^\infty(\hat{L}_+(\lambda))$.

It follows that there is a Legendre distribution $I^{-\frac{1}{2},\frac{n-2}{2};\frac{n-1}{2},\frac{n-1}{2}}(L_+(\lambda),\hat{L}_+(\lambda))$ which has the correct symbol of order $-1/2$ at $L_+(\lambda)$. Thus it solves the equation

$$(4.23)\qquad (H_l - \lambda^2)Q - E(\lambda) \in I^{\frac{3}{2},\frac{n}{2};\frac{n+3}{2},\frac{n-1}{2}}(X^2_{\mathrm{b}}, L_+(\lambda), L^\#(\lambda); {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}}),$$

where $(n+3)/2$ is the order of vanishing at lb and $(n-1)/2$ is the order of vanishing at rb. The order of improvement at lb is two since not only is the Legendrian G_1 of Definition 2.5 at lb characteristic for $H_l - \lambda^2$, but the transport operator for symbols of order $(n-1)/2$ vanishes, so we automatically get two orders of improvement there. At rb however we can expect no improvement. As shown above, Q will automatically satisfy the equation

$$(H_r - \lambda^2)Q - E_2(\lambda) \in I^{\frac{3}{2},\frac{n}{2};\frac{n-1}{2},\frac{n+3}{2}}(X^2_{\mathrm{b}}, L_+(\lambda), L^\#(\lambda); {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}}).$$

Let us assume by induction that we have a kernel which solves the left equation above up to an error in

$$(4.24)\qquad I^{k+\frac{1}{2},\frac{n}{2};\frac{n+3}{2},\frac{n-1}{2}}(X^2_{\mathrm{b}}, L_+(\lambda), L^\#(\lambda); {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$$

and hence the right equation up to an error in

$$(4.25)\qquad I^{k+\frac{1}{2},\frac{n}{2};\frac{n-1}{2},\frac{n+3}{2}}(X^2_{\mathrm{b}}, L_+(\lambda), L^\#(\lambda); {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}}).$$

We wish to improve this by one order at $L_+(\lambda)$. To do this, we choose $Q_k \in I^{k-\frac{1}{2},\frac{n-2}{2};\frac{n-1}{2},\frac{n-1}{2}}(L_+(\lambda),\hat{L}_+(\lambda))$ to have the symbol of order $k-1/2$ on $L_+(\lambda)$ which solves the left transport equation (and therefore the right transport equation) on $L_+(\lambda)$. We need to investigate the regularity of this symbol to see if it extends to a Legendrian conic pair. The argument is very analogous to the one above, but now we have error terms on the right hand side. In the first region, after removing the half-density factor, we get an equation of the form

$$(4.26)\qquad -2i\tau'\Big(\sigma\partial_\sigma + \Big(-\frac{n}{2}+k\Big) + \sigma f\Big)a_k = b_k, \quad f \in \mathcal{C}^\infty(L_+(\lambda)).$$

The term b_k comes from the error to be solved away. Since the error term is of order $k+1/2$ at $L_+(\lambda)$ and order $(n+3)/2$ at lb, $b_k \in$

$\sigma^{n/2-k+1}\mathcal{C}^\infty(L_+(\lambda))$. This shows that $a_k \in \sigma^{n/2-k}\mathcal{C}^\infty(L_+(\lambda))$, as desired. Similarly, in the second region, near the corner $\mathrm{lb} \cap L^\#(\lambda)$, by combining the vector fields $V_l + V_r$ we get an equation of the form

$$(4.27)\quad -2i\tau''\Big(-s\partial_s + sW + (-\frac{n-1}{2} + k) + s\sigma f\Big)a_k = b_k, \quad f \in \mathcal{C}^\infty(L_+(\lambda))$$

with b_k again the error to be solved away. To calculate its order of vanishing at $s = 0$, consider the transport equation for symbols of order $(n-2)/2$ at $L^\#(\lambda)$. Noting that the subprincipal symbols vanish identically on $L^\#(\lambda)$, the left transport operator is

$$-i\big(\mathcal{L}_{V_l} - \tau'\big)$$

whilst the right transport operator with respect to x'' is

$$-i\big(\mathcal{L}_{V_r} - \tau''\big).$$

To write this with respect to x' we must conjugate by σ (by (3.2)). In view of the term $-2\tau''\sigma\partial_\sigma$, this changes the operator to

$$-i\big(\mathcal{L}_{V_r} + \tau''\big).$$

The sum of these two operators vanishes on $L^\#(\lambda)$ so actually the right hand side in (4.27) comes from a term in $I^{k+\frac{1}{2}, \frac{n+2}{2}; \frac{n+1}{2}, \frac{n-1}{2}}(X_{\mathrm{b}}^2, L_+(\lambda), L^\#(\lambda); {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$. From Proposition 3.5 we see that $b_k \in s^{(n-1)/2-k+1}\sigma^{n/2-k}$ $\mathcal{C}^\infty(\hat{L}_+(\lambda))$, one power in s better than might be expected. This shows that $a_k \in s^{(n-1)/2-k}\sigma^{n/2-k}\mathcal{C}^\infty(\hat{L}_+(\lambda))$, as desired. Therefore, one can find a Legendre conic pair with symbol of order $k - 1/2$ on $L_+(\lambda)$ equal to a_k which solves away the error term of order $k + 1/2$ at $L_+(\lambda)$. This completes the inductive step. By asymptotically summing these correction terms, we end up with an approximation $G_3(\lambda)$ to the resolvent kernel with an error $E_3(\lambda)$ in $I^{\frac{n-2}{2}; \frac{n+3}{2}, \frac{n-1}{2}}(X_{\mathrm{b}}^2, L^\#(\lambda); {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$. That is, we have solved away the scattering wavefront set of the error term at $L_+(\lambda)$ completely.

4.5. Solving away outgoing error.

In the last step of the construction of the parametrix, we solve away the error to infinite order at bf and lb. We begin by considering the expansion at rb. By construction, the parametrix $G_3(\lambda)$ has an expansion at rb (here we are interested in what happens at $\mathrm{rb} \cap \mathrm{bf}$, where we start the error removal process)

$$G_3(\lambda) \sim e^{i\lambda/x''} x''^{(n-1)/2} \sum_{j \geqslant 0} x''^j g_j(z', y'') \cdot \nu' \cdot \nu'',$$

where $g_j(z', y'') \cdot \nu' \in I^{-n/4-j, n/4-1/2-j}(G_{y''}(\lambda), G^\sharp(\lambda))(X; {}^{sc}\Omega^{\frac{1}{2}})$, $G_{y''}(\lambda)$ is the fibre Legendrian of Definition 2.5 and y'' is regarded as a smooth parameter. (This expansion is an immediate consequence of the definition of fibred Legendre distributions.) The factors ν', ν'' are the Riemannian half-density factors on X lifted to X^2_{b} via the left and right projections, respectively. We will ignore the half-density factors from here on; since $\Delta(a \cdot \nu' \cdot \nu') = \Delta(a) \cdot \nu' \cdot \nu'$, this only has the effect of changing $H = \Delta + P$ to $\Delta + P'$ for some P' with the same properties as P.

The error term after applying $H - \lambda^2$ to $G_3(\lambda)$ has the form

$$E_3(\lambda) \sim e^{i\lambda/x''} x''^{(n-1)/2} \sum_{j \geqslant 0} x''^j e_j(z', y''), \tag{4.28}$$

where $e_j \in x'^{(n+1)/2-j} e^{i\lambda/x'} \mathcal{C}^\infty(X \times Y)$. Again we regard y'' as a parameter. Thus we have

$$(H - \lambda^2) g_j = e_j. \tag{4.29}$$

Consider the problem of solving away errors of the form e_j, to infinite order at bf (of course we cannot solve the errors away exactly without being able to solve $(H - \lambda^2)u = f$ exactly, which we cannot do until we have constructed the resolvent kernel!). If we apply $H - \lambda^2$ to a series of the form

$$e^{i\lambda/x'} (x')^{(n-1)/2-k} \sum_{j \geqslant 0} (x')^j b_j, \quad b_j \in \mathcal{C}^\infty(X), \tag{4.30}$$

we get a series of the form

$$e^{i\lambda/x'} (x')^{(n+1)/2-k} \sum_{j \geqslant 0} (x')^j c_j, \quad c_j \in \mathcal{C}^\infty(X), \quad c_0 = 2i\lambda k b_0. \tag{4.31}$$

(Here we suppressed y'' in the notation since we regard it as a parameter.) Thus, we can add to g_j a series of the form (4.30) to solve away the powers greater than $(n+1)/2$, but the power $(n+1)/2$ presents a problem (without introducing logarithmic terms), because of the vanishing of c_0 in (4.31) when $k = 0$. We need the following results.

Lemma 4.2. — *If $g \in e^{i\lambda/x'} (x')^{(n-1)/2-k} \mathcal{C}^\infty(X)$, $k = 1, 2, \ldots$, satisfies*

$$(H - \lambda^2) g = (x')^{(n+1)/2} e^{i\lambda/x'} \mathcal{C}^\infty,$$

then

$$(H - \lambda^2) g = (x')^{(n+3)/2} e^{i\lambda/x'} \mathcal{C}^\infty.$$

Proof. — It follows inductively using (4.31) that the coefficient of order $(n+1)/2 - l$ vanishes, for $l = k, k-1, \ldots$. Thus, actually $g \in e^{i\lambda/x'}(x')^{(n-1)/2}\mathcal{C}^\infty(X)$. Then (4.31) shows that the next coefficient also vanishes. □

COROLLARY 4.3. — *The same result holds if the condition* $g \in e^{i\lambda/x'}(x')^{(n-1)/2-k}\mathcal{C}^\infty(X)$ *is replaced by* $g \in I^{p,(n-1)/2-n/4-k}(K, G^\sharp)$ *for any* p *and any Legendre conic pair* $(K, G^\sharp)$.

Proof. — Apply the above argument to the symbol at $G^\sharp$. □

Thus, for each j, we can modify g_j by a series of the form (4.30) until the error term is of the form $x'^{(n+1)/2}e^{i\lambda/x'}\mathcal{C}^\infty(X \times Y)$. Then applying the corollary to g_j, we find that unsolvable term of order $x'^{(n+1)/2}$ vanishes. Therefore, we can solve away the e_j to infinite order at bf. Thus, we may assume that our error in $E_3(\lambda)$ vanishes to infinite order at the corner bf $\cap$ rb.

Next, we solve the error away at $L^\#$. This involves solving the transport equation

$$i\lambda\left(\sigma\partial_\sigma + \left(\frac{1}{2} + j\right)\right)a_j = b_j. \tag{4.32}$$

The equation for a_0 then is

$$\left(\sigma\partial_\sigma + \frac{1}{2}\right)a_0 = b_0,$$

and b_0 is rapidly decreasing at rb and is in $\sigma^{3/2}\mathcal{C}^\infty(L^\#(\lambda))$ at lb. There is a unique solution which is rapidly decreasing at rb and in $\sigma^{-1/2}\mathcal{C}^\infty$ at lb. We can thus find a correction term which reduces the error to $I^{(n+2)/2;(n+3)/2,(n-1)/2}(L^\#(\lambda))$, with infinite order vanishing at bf $\cap$ rb. Inductively, assume that we have reduced the error to $I^{n/2+k;(n+3)/2,(n-1)/2}(L^\#(\lambda))$, with infinite order vanishing at bf $\cap$rb. The transport equation for a_k is then

$$\left(\sigma\partial_\sigma + \frac{1}{2} + k\right)a_k = b_k,$$

where inductively, b_k is rapidly decreasing at rb and is in $\sigma^{1/2-k}\mathcal{C}^\infty(L^\#(\lambda))$ at lb. There is a unique solution rapidly decreasing at rb and in $\sigma^{-1/2+k}\mathcal{C}^\infty$ $(L^\#(\lambda))$ at lb. A Legendre distribution in $I^{n/2+k-1,(n-1)/2,(n-1)/2}(L^\#(\lambda))$ with a_j as symbol then reduces the error to $I^{n/2+k+1;(n+3)/2,(n-1)/2}(L^\#(\lambda))$, with infinite order vanishing at bf $\cap$ rb, so this completes the inductive step. Taking an asymptotic sum of such correction terms yields a parametrix $G_4(\lambda) = G(\lambda)$ leaving an error which is the sum of a term supported away from rb of the form

$$e^{i\lambda/x'} x'^{(n+3)/2} a(y', z'')$$

with a smooth and rapidly decreasing at bf, plus a term supported away from lb of the form

$$e^{i\lambda/x''} x''^{(n-1)/2} b(y'', z')$$

at rb, with b rapidly decreasing at bf. The error at lb can be solved away using (4.30)-(4.31), leaving an error term $E_4(\lambda)$ which can be expressed on the blown-down space X^2 as

$$E_+(\lambda) = e^{i\lambda/x''} x''^{(n-1)/2} b(z', z'')$$

with b smooth on X^2 and rapidly decreasing at $x' = 0$. Such an error term is compact on the weighted L^2 space $x^l L^2(X)$ for any $l > 1/2$ (where L^2 is taken with respect to the metric density). This completes the construction of the parametrix.

5. Long range case.

The case of long range metrics and long range perturbations, $P \in x\mathrm{Diff}^1_{\mathrm{sc}}(X)$, requires only minor modifications in the parametrix construction until the last step (removing the outgoing error). In particular, there is no change in the construction of the pseudodifferential approximation. In the intersecting Legendrian construction, as well as solving the transport equations on $L^\circ(\lambda)$, the only difference is in the structure of the subprincipal symbol, which no longer obeys Lemma 2.3. Thus, the arguments in Section 4.4 and Section 4.5 have to be modified. Let q denote the boundary subprincipal symbol of H. Notice that in the gravitational long range case, q is a constant, but in the general long range case, q is an arbitrary smooth function of y which is a quadratic on each fibre of K over Y. Let q_l and q_r denote the lift of q to X^2_{b} via the left, respectively right, projection.

Let us now discuss the necessary modifications to Sections 4.4 and 4.5. Equation (4.18) becomes

$$-i\big(\mathcal{L}_{V_l} - n\tau' + iq_l\big)a_0 \left|\frac{d\sigma}{\sigma} dy'' d\mu''\right|^{\frac{1}{2}} = 0, \tag{5.1}$$

near $T_+(\lambda)$. Thus, (4.20) is replaced by

$$-i\tau'\big(\partial_\sigma + f\big)\big(\sigma^{-n/2-i\frac{q_l}{2\lambda}}a_0\big) = 0, \quad f \in \mathcal{C}^\infty(L_+(\lambda)). \tag{5.2}$$

Thus, now we conclude that in this region a_0 is of the form

$$\sigma^{\frac{n}{2}+i\frac{q_l}{2\lambda}}\mathcal{C}^\infty(L_+(\lambda)).$$

Next, in the second region, at the corner $\mathrm{lb}\cap L^\sharp(\lambda)$, the right transport equation (4.22) becomes

$$\big(V_r + \tau'' + iq_r + sf'\big)a_0 = 0 \quad f' \in \mathcal{C}^\infty(\hat{L}_+(\lambda)). \tag{5.3}$$

Adding this to the left transport equation yields

$$2\lambda\Big(s\partial_s - \frac{n-1}{2} - i\frac{q_l}{2\lambda} - i\frac{q_r}{2\lambda} + sW + s\tilde{f}\Big)a_0 = 0,$$

which now gives that a_0 is of the form

$$\sigma^{\frac{n}{2}+i\frac{q_l}{2\lambda}}s^{\frac{n-1}{2}+i\frac{q_l+q_r}{2\lambda}}\mathcal{C}^\infty(\hat{L}_+(\lambda)).$$

Combining this with the other similar results at rb and the interior of $L^\sharp(\lambda)$, we deduce that

$$a_0 \in \rho_{\mathrm{lb}}^{n/2+i\frac{q_l}{2\lambda}}\rho_{\mathrm{rb}}^{n/2+i\frac{q_r}{2\lambda}}\rho_{\#}^{(n-1)/2+i\frac{q_l+q_r}{2\lambda}}\mathcal{C}^\infty(\hat{L}_+(\lambda);\Omega_b^{1/2}\otimes S^{[-1/2]}).$$

In the general long range case, the dependence of q on y, and its appearance in the exponent of the boundary defining functions ρ_{lb}, etc., means that differential operators from the left factor, acting on a Legendre function with principal symbol a_0, introduce logarithmic terms. For example, in a neighborhood of lb in $L_+(\lambda)$ the error term b_k in (4.26) for $k=1$ will take the form

$$b_1 = \sigma^{\frac{n}{2}+i\frac{q_l}{2\lambda}}\big((\log\sigma)^2 c_2 + \log\sigma\, c_1 + c_0\big), \quad c_j \in \mathcal{C}^\infty(L_+(\lambda)),\ j=0,1,2.$$

Then the transport equation for a_1 takes the form

$$\begin{aligned}\big(-i\big(\mathcal{L}_{V_l} + (-n+2)\tau' + iq_l\big) + \sigma f\big)\, a_1\left|\frac{d\sigma}{\sigma}dy''d\mu''\right|^{\frac{1}{2}}& \\ = 0,\ f \in \mathcal{C}^\infty(L_+(\lambda)),&\end{aligned} \tag{5.4}$$

$$-i\tau'\Big(\sigma\partial_\sigma - \frac{n}{2} + 1 - iq_l/(2\lambda) + \sigma f\Big)a_1 = b_1, \quad f \in \mathcal{C}^\infty(L_+(\lambda)). \tag{5.5}$$

Hence, near lb but away from $L^\sharp(\lambda)$, a_1 will take the form

$$a_1 = \sigma^{\frac{n}{2}-1+i\frac{q_l}{2\lambda}}((\log\sigma)^2 c_2' + \log\sigma\, c_1' + c_0'), \quad c_j' \in \mathcal{C}^\infty(L_+(\lambda)).$$

A similar discussion works at the other boundary faces of $\hat{L}^+(\lambda)$, with up to quadratic factors in each of $\log\rho_{\mathrm{lb}}$, $\log\rho_{\mathrm{rb}}$, $\log_{\rho_\sharp}$, and can be repeated (with progressively higher powers of logarithms) for all a_k's.

Since the most important long-range case is the gravitational case where the subprincipal symbol is constant, and since it makes the discussion more transparent, in what follows we make the assumption that

$$\Delta \text{ and } P \text{ are of long range gravitational type,}$$

which implies that q is constant. Let

$$\alpha = \frac{q}{2\lambda}.$$

The point is that in this case the powers of ρ_{lb}, etc., above are constant, thus no logarithmic factors arise when we apply $H - \lambda^2$ to such Legendre functions. Then

$$a_k \in \rho_{\mathrm{lb}}^{n/2+i\alpha}\rho_{\mathrm{rb}}^{n/2+i\alpha-k}\rho_{\#}^{(n-1)/2+2i\alpha-k}\mathcal{C}^\infty(\hat{L}_+(\lambda);\Omega_b^{1/2}\otimes S^{[-1/2]}),$$

and asymptotic summation gives an outgoing error

$$E_+(\lambda) \in e^{i\lambda/x'}e^{i\lambda/x''}x''^{(n-1)/2+i\alpha}x'^{(n+1)/2+i\alpha}\mathcal{C}^\infty(X_b^2;{}^{\mathrm{sc}}\Omega^{\frac{1}{2}}).$$

Since α is a constant, (4.30)-(4.31) are still true (k need not be an integer; it suffices that it is a constant), except that now

$$c_0 = (2i\lambda k + 2\lambda\alpha)b_0.$$

Since now we are taking $k = l + i\alpha$, where l is an integer, we can solve away the series, provided that the coefficient of the $l = 0$ term vanishes, which is assured just as in Lemma 4.2. The rest of the argument requires only similar modifications as compared to the short-range case, so we conclude, as there, that we can modify the parametrix to obtain an error term of the form

$$E_4(\lambda) = e^{i\lambda/x''}(x'')^{(n-1)/2+i\alpha}b(z', z'')$$

with b smooth on X^2 and rapidly decreasing at $x' = 0$.

6. Resolvent from parametrix.

In the previous two sections, we constructed a parametrix $G(\lambda)$ for $\tilde{R}(\lambda)$ which satisfies

$$(H_l - \lambda^2)G(\lambda) = K_{\mathrm{Id}} + E(\lambda),$$

where $E(\lambda)$ has a kernel which is of the form

$$e^{i\lambda/x''}x''^{(n-1)/2}x'^{\infty}\mathcal{C}^{\infty}(X_{\mathrm{b}}^2;{}^{\mathrm{sc}}\Omega^{\frac{1}{2}}).$$

Thus, it is a Hilbert-Schmidt kernel on $x^l L^2(X)$ for every $l > 1/2$, and in particular is compact. In fact, we also see directly from its form that $E(\lambda) : x^l L^2(X) \to \dot{\mathcal{C}}^{\infty}(X)$ for $l > 1/2$. Crude estimates (such as Schur's Lemma) show that $G(\lambda)$ acts as a bounded operator from $x^l L^2$ to $x^{-l} L^2$ for large enough $l > 1/2$; more refined estimates, which we do not need here, show that in fact this is true for any $l > 1/2$. (The more refined estimates reflect the improvement arising from the oscillatory behavior of the kernel, i.e., are analogous to the standard FIO mapping property estimates, except that now the kernel is not simply Legendrian. A different proof of these optimal estimates is given by the limiting absorption principle; however, this is exactly what we do not wish to use here to maintain the constructive nature of our arguments!) Thus, the equation above becomes an operator equation

$$(H - \lambda^2)G(\lambda) = \mathrm{Id} + E(\lambda)$$

from $x^l L^2$ to $x^{-l} L^2$.

6.1. Finite rank perturbation.

To correct $G(\lambda)$ to the actual $\tilde{R}(\lambda)$, we must solve away the error term $E(\lambda)$. Thus, we would like $\mathrm{Id} + E(\lambda)$ to be invertible. However, this is certainly not necessarily the case as things stand; if for example we modified $G(\lambda)$ by subtracting from it the rank one operator $G(\lambda)(\phi)\langle\phi,\cdot\rangle$, for some $\phi \in \dot{\mathcal{C}}^{\infty}(X;{}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$, then the modified $G(\lambda)$ would be microlocally indistinguishable from the old one, but would annihilate ϕ, so the modified $\mathrm{Id} + E(\lambda)$ would not be invertible.

Since $\mathrm{Id} + E(\lambda)$ is compact, it has a null space and cokernel of the same finite dimension N. To make $\mathrm{Id} + E(\lambda)$ invertible, we try to correct

$G(\lambda)$ by adding to it a finite rank term

$$\sum_{i=1}^{N} \phi_i \langle x^{2l}\psi_i, \cdot\rangle. \tag{6.1}$$

Here $\langle, \rangle$ denotes L^2 pairing, ψ_i should lie in L^2, and the factor of x^{2l} is included to ensure that it acts on $x^l L^2$. We require that ϕ_i are in $x^{-l}L^2$ so that (6.1) is bounded from $x^l L^2 \to x^{-l}L^2$. We wish to choose ϕ_i and ψ_i so that

$$\mathrm{Id} + E(\lambda) + \sum_{i=1}^{N}((H-\lambda^2)\phi_i)\langle x^{2l}\psi_i, \cdot\rangle$$

is invertible. This is possible if we can choose $x^l\psi_i$ to span the null space of $\mathrm{Id} + E(\lambda)$ and $(H-\lambda^2)\phi_i$ to span a subspace supplementary to the range. Note that if $(\mathrm{Id} + E(\lambda))u = 0$ and $u \in x^l L^2$, then $u = -E(\lambda)u$, so the mapping properties of $E(\lambda)$ imply that $u \in \dot{\mathcal{C}}^\infty(X)$. Thus, we automatically have $\psi_i \in \dot{\mathcal{C}}^\infty(X)$ above. To proceed, we need the following lemma.

LEMMA 6.1. — *Let $l > 1/2$. Then the image of $H - \lambda^2$ on the sum of $\dot{\mathcal{C}}^\infty(X)$ and the range of $G(\lambda)$ applied to $\dot{\mathcal{C}}^\infty(X)$ is dense in $x^l L^2$.*

Remark 6.2. — Note that $(H-\lambda^2)G(\lambda)g = (\mathrm{Id} + E(\lambda))g \in \dot{\mathcal{C}}^\infty(X)$ if $g \in \dot{\mathcal{C}}^\infty(X)$, and for $u \in \dot{\mathcal{C}}^\infty(X)$, $(H-\lambda^2)u \in \dot{\mathcal{C}}^\infty(X)$, so the image of $H-\lambda^2$ on the space in the statement of the lemma is a subspace of $\dot{\mathcal{C}}^\infty(X)$.

Proof. — To proceed, we give the proof for short range H; the proof for long range H requires only minor modifications.

Let $\mathcal{M}$ be the subspace of $x^l L^2$ given by the image of $H - \lambda^2$ on the sum of $\dot{\mathcal{C}}^\infty(X)$ and the range of $G(\lambda)$ applied to $\dot{\mathcal{C}}^\infty(X)$. If $\mathcal{M}$ is not dense, then there is a function $f \in x^l L^2$ orthogonal to $\mathcal{M}$. Since $u \in \dot{\mathcal{C}}^\infty(X)$ implies $(H-\lambda^2)u \in \mathcal{M}$, f satisfies

$$\begin{aligned} &\langle x^{-l}f, x^{-l}(H-\lambda^2)u\rangle = 0 \quad \forall\, u \in \dot{\mathcal{C}}^\infty(X) \\ &\Rightarrow \langle (H-\lambda^2)x^{-2l}f, u\rangle = 0 \quad \forall\, u \in \dot{\mathcal{C}}^\infty(X) \\ &\Rightarrow (H-\lambda^2)h = 0, \quad h = x^{-2l}f, \end{aligned} \tag{6.2}$$

where we used that H is symmetric on $\dot{\mathcal{C}}^\infty(X)$. On the other hand, $G(\lambda)$ maps $\dot{\mathcal{C}}^\infty(X) \to x^{(n-1)/2}e^{i\lambda/x}\mathcal{C}^\infty(X)$, and for any $g \in \dot{\mathcal{C}}^\infty(X)$, $(H-\lambda^2)G(\lambda)g = (\mathrm{Id} + E(\lambda))g \in \dot{\mathcal{C}}^\infty(X)$, hence $(\mathrm{Id} + E(\lambda))g \in \mathcal{M}$. In addition, $E(\lambda)^*$, with kernel $E(\lambda)^*(z', z'') = \overline{E(\lambda, z'', z')}$, maps $\mathcal{C}^{-\infty}(X) \to$

$x^{(n-1)/2}e^{-i\lambda/x}\mathcal{C}^\infty(X)$, so we have

$$\begin{aligned}(6.3)\qquad &\langle x^{-l}f, x^{-l}(H-\lambda^2)G(\lambda)g\rangle \\ &= \langle x^{-l}f, x^{-l}(\mathrm{Id}+E(\lambda))g\rangle = 0 \quad \forall\, g\in\dot{\mathcal{C}}^\infty(X) \\ &\Rightarrow \langle(\mathrm{Id}+E(\lambda)^*)x^{-2l}f, g\rangle = 0 \quad \forall\, g\in\dot{\mathcal{C}}^\infty(X) \\ &\Rightarrow (\mathrm{Id}+E(\lambda)^*)h = 0.\end{aligned}$$

If $h = -E(\lambda)^*h$, then h has the form $x^{(n-1)/2}e^{-i\lambda/x}\mathcal{C}^\infty(X)$, i.e., it is incoming. A standard argument, similar to the one presented in [10], then implies that $h\equiv 0$. Indeed, let $h = x^{(n-1)/2}e^{-i\lambda/x}h_0(y)+\tilde{h}$, where $\tilde{h}\in x^{(n+1)/2}e^{-i\lambda/x}\mathcal{C}^\infty(X)$. Green's formula yields

$$\begin{aligned}(6.4)\qquad 0 &= \int_X \overline{h}(H-\lambda^2)h - h(H-\lambda^2)\overline{h} \\ &= 2i\lambda\lim_{\epsilon\to 0}\int_{\{x=\epsilon\}} \overline{h}x^2\partial_x h - hx^2\partial_x\overline{h} = 2i\lambda\int_Y |h_0(y)|^2,\end{aligned}$$

so $h_0\equiv 0$. It then follows iteratively from (4.30) and (4.31) that the expansion of h at the boundary of X vanishes identically, that is, that $h\in\dot{\mathcal{C}}^\infty(X)$. Finally a unique continuation theorem, see e.g. [7, Chapter XVII], shows $h=0$ identically. This means that $\mathcal{M}$ is indeed dense in x^lL^2. □

Thus, we can choose the $\phi_i\in x^{(n-1)/2}e^{i\lambda/x}\mathcal{C}^\infty(X)$ above so that $(H-\lambda^2)\phi_i\in\dot{\mathcal{C}}^\infty(X)$ span a supplementary subspace of range $\mathrm{Id}+E(\lambda)$. The modified parametrix then satisfies

$$(H-\lambda^2)G_5(\lambda) = \mathrm{Id}+E_5(\lambda),$$

where $E_5(\lambda)$ has the same form as $E(\lambda)$ but in addition $\mathrm{Id}+E_5(\lambda)$ is invertible on x^lL^2 for all $l>1/2$.

6.2. Resolvent.

The inverse $\mathrm{Id}+S(\lambda)$ of $\mathrm{Id}+E_5(\lambda)$ is Hilbert-Schmidt on x^lL^2 since this is true of $E_5(\lambda)$. Moreover, since

$$S(\lambda) = -E_5(\lambda)+E_5(\lambda)^2+E_5(\lambda)S(\lambda)E_5(\lambda),$$

it is easy to see that also $S(\lambda)\in e^{i\lambda/x''}x''^{(n-1)/2}x'^\infty\mathcal{C}^\infty(X^2_{\mathrm{b}};{}^{\mathrm{sc}}\Omega^{\frac12})$. Our solution for the kernel $\tilde{R}(\lambda)$ is then

$$\tilde{R}(\lambda) = G_5(\lambda)(\mathrm{Id}+S(\lambda)).$$

It is not hard to show that $G_5(\lambda)S(\lambda)$ has the form

$$x'^{(n-1)/2}x''^{(n-1)/2}e^{i\lambda/x'}e^{i\lambda/x''}\mathcal{C}^\infty(X^2_{\mathrm{b}};{}^{\mathrm{sc}}\Omega^{\frac{1}{2}}),$$

so $\tilde{R}(\lambda)$ has the desired microlocal regularity (4.4).

Remark. — Lemma 6.1 directly shows the absence of positive eigenvalues of H. Suppose that $(H-\lambda^2)u=0$ and $u\in x^sH^2(X)$ for some $s>-1/2$. This would certainly be the case of an eigenfunction since H has elliptic interior symbol, so u would lie in $H^k(X)$ for every k. We need to show that for all functions $g\in x^{(n-1)/2}e^{i\lambda/x}\mathcal{C}^\infty(X)$ the equation

$$\langle (H-\lambda^2)u,g\rangle = \langle u,(H-\lambda^2)g\rangle = 0 \tag{6.5}$$

holds. Indeed, this implies that u is L^2-orthogonal to the image of $H-\lambda^2$ acting on $x^{(n-1)/2}e^{i\lambda/x}\mathcal{C}^\infty(X)$, or equivalently that $x^{2l}u\in x^lL^2$ is orthogonal in x^lL^2 to the image of $H-\lambda^2$ acting on $x^{(n-1)/2}e^{i\lambda/x}\mathcal{C}^\infty(X)$ in x^lL^2. Then Lemma 6.1 shows that $u\equiv 0$.

To deduce (6.5) for $g\in x^{(n-1)/2}e^{i\lambda/x}\mathcal{C}^\infty(X)$, let $\phi\in\mathcal{C}^\infty(\mathbb{R})$, identically 1 on $[1,\infty)$, 0 near the origin. Then

$$\begin{aligned} \langle (H-\lambda^2)u,g\rangle &= \lim_{t\to 0}\langle (H-\lambda^2)u,\phi(x/t)g\rangle \\ &= \lim_{t\to 0}\langle u,(H-\lambda^2)\phi(x/t)g\rangle \\ &= \lim_{t\to 0}\langle u,\phi(x/t)(H-\lambda^2)g\rangle + \lim_{t\to 0}\langle u,[H,\phi(x/t)]g\rangle \\ &= \langle u,(H-\lambda^2)g\rangle + \lim_{t\to 0}\langle u,[H,\phi(x/t)]g\rangle. \end{aligned} \tag{6.6}$$

Note that $[H,\phi(x/t)]$ is uniformly bounded (i.e., with bounds independent of t) as a map $x^lH^1\to x^{l+1}L^2$, and in fact $[H,\phi(x/t)]\to 0$ strongly as $t\to 0$. Applying this with $l=-1/2-\epsilon$, $\epsilon>0$ sufficiently small, we see that the last term goes to 0 as $t\to 0$, proving (6.5).

6.3. Analytic continuation.

It is not hard to show that the kernel $\tilde{R}(\lambda)$ constructed above continues analytically (as a distribution on X^2_{sc}) into $\operatorname{Im}\lambda\geqslant 0, \operatorname{Re}\lambda>0$. We complete the proof of Theorem 1.1 by showing that this analytic continuation coincides with the outgoing resolvent, $R(\lambda^2)$, for $\operatorname{Im}\lambda>0$.

An interesting question, raised by the referee, is whether under stronger assumptions our methods yield a precise description, as well as a new proof of existence, of the analytic continuation of $R(\lambda^2)$ into $\operatorname{Im}\lambda \leqslant 0, \operatorname{Re}\lambda > 0$ as well. The existence of the analytic continuation to a sector containing the real axis, under appropriate analyticity conditions near ∂X, was shown by Wunsch and Zworski [14]. As can be seen below, there are certainly delicate issues (e.g. the combination of exponentially growing Legendrians and asymptotic summation) which would require methods of different character from those employed elsewhere in the paper. Nonetheless, one would expect that under suitably strong assumptions on H one can indeed construct an analytic continuation to a neighborhood of the real axis. Since it appears that such a construction would take us far afield, we do not pursue this issue here.

Our parametrix is defined as an asymptotic sum of symbols, which is really a sum with cutoff functions inserted (see [7, Proposition 18.1.3] for an explicit construction). The cutoffs depend on C^k norms of a finite number of symbols and ensure that the sum converges in C^k for all k. If the symbols are holomorphic in λ then the C^k norms may be taken uniform on compact subsets of λ. Since holomorphy is preserved under uniform limits, we need only check that the phase and symbols analytically continue in some explicit parametrization of the Legendrians.

It is standard that the pseudodifferential approximation $G_1(\lambda)$ analytically continues. Blowing down sf, we solve away the error as an intersecting Legendrian, see Section 4.2. Let ϕ be a local parametrization of the Legendrian $L(1)$ near $L(1) \cap N^*\mathrm{diag}_{\mathrm{b}}$. Then it is easily checked that the phase function

$$k\phi + s(\lambda - k)$$

locally parametrizes $(N^*\mathrm{diag}_{\mathrm{b}}, L^+(\lambda))$. Since the variable s takes nonnegative values, the function $e^{i(k\phi+s(\lambda-k))/x}$ continues to $\operatorname{Im}\lambda > 0$.

Away from sf, the value of τ is strictly negative on the Legendrian, and so the phase is of the form

$$e^{i\lambda\phi/x}$$

where ϕ is positive on the Legendrian, independent of λ. By restricting the support of the symbol sufficiently, therefore, we may assume that ϕ is positive everywhere on the support of the integral. Thus this also analytically continues to the upper half plane with uniform bounds.

The symbols are defined by iteratively solving transport equations of the form

$$\Big(- i\mathcal{L}_{H_p} - i\big(\frac{1}{2} + m - \frac{N}{4}\big)\frac{\partial p}{\partial \tau} + p_{\mathrm{sub}}\Big) a_j = b_j,$$

where $b_0 = 0$. These equations are solved along $L^\circ(\lambda)$, i.e., if we consider amplitudes in an explicit parameterization of the Legendrian, then along the critical submanifold $C_\phi = \{(0,y,u) : d_u\phi = 0\}$, where $\lambda\phi/x$ is the phase function as above. Note that C_ϕ is independent of λ, and it is identified with $L^\circ(\lambda)$ via the map $C_\phi \ni (0,y,u) \mapsto (0,y,d_{(x,y)}(\lambda\phi/x)|_{(0,y,u)})$. Along C_ϕ the transport equation becomes an ODE whose coefficients depend on λ polynomially, since the only λ dependence of the coefficients arises from this identification map, and H_p, $\frac{\partial p}{\partial \tau}$, p_{sub} are polynomial in the fiber variables. Thus, the solution a_j of the transport equation, as a function on $D \times C_\phi$, D a neighborhood of the positive real axis in $\mathbb{C} = \mathbb{C}_\lambda$, is holomorphic (in λ), provided b_j is (here we identify C_ϕ with $L^\circ(\lambda)$). Note that the b_j's arise because solving the transport equations only guarantees that the 'error term' $E_3(\lambda)$, arising from the application of $H - \lambda^2$ to $R_3(\lambda)$, is one order lower than expected, so for each λ, $b_j|_{C_\phi}$ depends on a_i, $i < j$ near C_ϕ, and not just on $a_i|_{C_\phi}$. (In fact, $b_j|_{C_\phi}$ depends on a finite number of terms of the Taylor series of a_i, $i < j$, at C_ϕ.) To ensure that the b_j are holomorphic in λ, we define the a_i *near* $D \times C_\phi$, rather than *at* $D \times C_\phi$, e.g. by introducing a local product decomposition $C_\phi \times U$, $U \subset \mathbb{R}^k$, of the parameter space near C_ϕ, and pull-back the a_i, first defined on $D \times C_\phi$, by the projection. Then, having constructed a_i, $i < j$, b_j will be holomorphic in λ near, hence at, $D \times C_\phi$, so a_j is also holomorphic at $D \times C_\phi$, hence it extends to be holomorphic near $D \times C_\phi$. If we express the amplitudes a_j with respect to a different parameterization of $L^\circ(\lambda)$, which is still of the form $\lambda\tilde{\phi}/x$, then the new amplitudes $\tilde{a}_j$ will still be holomorphic functions of λ, so holomorphy is preserved at the overlap of parameterizations of different parts of $L^\circ(\lambda)$. This completes the inductive argument.

Therefore, our parametrix constructed above may be assumed holomorphic in some set $B(\epsilon, \lambda_0) \cap \{\operatorname{Im}\lambda \geqslant 0\}$, for some $\lambda_0 > 0$. It is easy to see that for non-real λ, the parametrix is in the small calculus, since the positivity of ϕ implies that the exponent of $e^{i\lambda\phi/x}$ has negative real part, and is therefore rapidly decreasing at bf, lb and rb. The finite rank correction may be taken independent of λ if we chose $\epsilon > 0$ sufficiently small. Then, we have

$$(\Delta - \lambda^2)G_5(\lambda) = \mathrm{Id} + E_5(\lambda), \quad \operatorname{Im}\lambda \geqslant 0, \ \operatorname{Re}\lambda > 0,$$

where all terms are holomorphic in some small open set as above, $E_5(\lambda)$ is invertible on $x^l L^2$ for all $l > 1/2$, and off the real axis, $G_5(\lambda)$ and $E_5(\lambda)$ are in the small calculus. Define $\mathrm{Id} + S(\lambda)$ to be the inverse of $\mathrm{Id} + E_5(\lambda)$ on $x^l L^2$ for some fixed l. By the symbolic functional calculus [3], for $\mathrm{Im}\,\lambda > 0$, S is a family of scattering pseudodifferential operators which is clearly holomorphic. Then $\tilde{R}(\lambda) \equiv G_5(\lambda)(\mathrm{Id} + S(\lambda))$ satisfies $(H - \lambda^2)\tilde{R}(\lambda) = \mathrm{Id}$ on $x^l L^2$. But by self-adjointness, and the symbolic functional calculus, for $\mathrm{Im}\,\lambda > 0$, $(H - \lambda^2)$ has a pseudodifferential inverse on L^2. Since $R(\lambda)$ is a bounded operator on L^2 for $\mathrm{Im}\,\lambda > 0$ it must be the inverse. Therefore we have shown the inverse on the real axis constructed above continues as a Schwartz kernel to the upper half plane and agrees with the resolvent there. This completes the proof of Theorem 1.1.

Remark 6.3. — The only place where we used that $\mathrm{Im}\,\lambda \geqslant 0$ is to make our parametrix act on, and its error compact on, weighted Sobolev spaces. Namely, in the last step of the construction, i.e., when we add a finite rank perturbation to remove the error $E(\lambda)$, we need $E(\lambda)$ to be a compact operator on $x^l L^2$ for $l > 1/2$. However, the kernel of $E(\lambda)$ is of the form $e^{i\lambda/x''} x''^{(n-1)/2} x'^{\infty} \mathcal{C}^\infty(X^2_{\mathrm{b}}; {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$, and for $\mathrm{Im}\,\lambda < 0$ the real part of the exponent is positive, so the kernel of $E(\lambda)$ is not even a tempered distribution on X^2_{b}. In particular, it does not even map $\dot{\mathcal{C}}^\infty(X)$ to $\mathcal{C}^{-\infty}(X)$. The same statement holds for $G(\lambda)$ as well.

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