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The analyticity of $q$-concave sets of locally finite Hausdorff $(2n - 2q)$ measure


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1. Introduction.

Let $A$ be a closed subset of a complex space $X$. The question of finding reasonable assumptions on $A$ which guarantee its analyticity has been studied over the years by various authors.

Hartogs [14] considered a continuous function $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}^n$ is open, and showed that the graph $G_f$ of $f$ in $D \times \mathbb{C}$ is pseudoconcave (i.e., the complement of $G_f$ in $D \times \mathbb{C}$ is locally Stein) if and only if $f$ is holomorphic, that is $G_f$ is analytic.

Grauert revealed in his thesis [13] a new interesting aspect of the above question bringing into play thin complements of complete Kähler domains. This topic was afterwards thoroughly studied by Diederich and Fornæss ([6], [7]) and Ohsawa [19].

On the other hand, Hirschowitz [15] settled the case when $X$ is non-singular and $A$ is pseudoconcave of locally finite Hausdorff $(2n-2)$-measure, where $n$ is the complex dimension of $X$.

In this article, using $q$-convexity with corners we introduce the notion of $q$-concavity. (See §2 for definition. Note that for $q = 1$ we recover the usual pseudoconcavity as used in [15] and [18].) For instance, if $X$ is a complex manifold of pure dimension $n$ and $A \subset X$ is an analytic subset

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such that every irreducible of it has dimension $\geq n-q$, then $A$ is $q$-concave [20]. Two more examples are given at the end of Section 2.

Our main result in this note, which establishes a converse of the above result due to M. Peternell and generalizes Hirschowitz's theorem already quotes above, is the following:

**Theorem 1.** — Let $X$ be a complex space of pure dimension $n$ and $q$ a positive integer less than $n$. If $A \subset X$ is a $q$-concave subset such that its Hausdorff $(2n-2q)$-measure is locally finite, then $A$ is analytic of pure dimension $n-q$.

As an application (see also Example 2 in Section 2) we have:

**Corollary 1.** — Let $T$ be a closed positive current of bidimension $(q, q)$ on a complex manifold $M$. If the Hausdorff $2q$-measure of $\text{Supp}(T)$ is locally finite, then $\text{Supp}(T)$ is an analytic subset of $M$ of pure dimension $q$.

On the other hand, using [16], Theorem 1 yields the following removability theorem. (For $q = 1$ we recover the main result in [2].)

**Theorem 2.** — Let $M$ be a complex manifold of pure dimension $n$, $q$ a positive integer less than $n$, $E \subset M$ a closed subset of locally finite Hausdorff $(2n-2q)$-measure, and $f$ a meromorphic mapping from $M \setminus E$ into a complex space $Y$. If $E$ does not contain any $(n-q)$-dimensional analytic subset of $M$ and $Y$ possesses the meromorphic extension property in bidimension $(q^n-q)$ (e.g., if $Y$ is $q$-complete), then $f$ is continued to a meromorphic mapping from $M$ into $Y$.

The organization of this paper is as follows. After a preliminary section, we give in §3 the proofs of Theorems 1 and 2. The last section, §4, establishes connections with the $q$-pseudoconcavity notion introduced by M. Peternell [20].

2. Preliminaries.

Let $T$ be a metric space and $S$ a subset of $T$. For $p > 0$ and $\varepsilon > 0$ let $h^p_\varepsilon(S)$ denote the infimum of all (infinite) sums of the form $\sum \delta(S_n)^p$ where $S = \cup S_n$ is an arbitrary decomposition of $S$ with $\delta(S_n) < \varepsilon$ for all $n$ ($\delta =$ diameter). For $p > 0$ the Hausdorff $p$-measure $h^p$ is defined by
$h^p(S) = \sup_{\varepsilon > 0} h^p(\varepsilon S) \leq +\infty$. We define $h^0(S)$ to be equal to the cardinality of $S$. The usual notion of $k$-dimensional volume in a Riemannian manifold agrees with $h^k$ up to a constant factor depending only on $n$ (for positive integers $k$). Thus, if $A$ is a pure $k$-dimensional analytic set in a domain in $\mathbb{C}^n$, then $h^{2k}(A)$ is equal to a universal constant (depending on $k$) times the Riemannian volume of the set of regular points of $A$. For a detailed discussion on Hausdorff measure, see [11].

(●) The definition of $q$-convexity is the same as in [1], namely; a function $\varphi \in C^\infty(D, \mathbb{R})$, where $D \subset \mathbb{C}^n$ is an open subset, said to be $q$-convex if its Levi form

$$L_\varphi(z)(\xi) := \sum_{i,j=1}^n \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\xi_i \xi_j, \xi \in \mathbb{C}^n,$$

has at least $n-q+1$ positive (> 0) eigenvalues for every $z \in D$. This definition can be carried over to complex spaces by local restriction.

Let $X$ be a complex space. $X$ is said to be $q$-complete if there exists a $q$-convex function $\varphi \in C^\infty(X, \mathbb{R})$ which is exhaustive, i.e., the sublevel sets $\{x \in X ; \varphi(x) < c\}, c \in \mathbb{R}$, are relatively compact in $X$. We choose the normalization such that 1-complete spaces correspond to Stein spaces.

Following [8] and [20] a function $\varphi \in C^0(X, \mathbb{R})$ is said to be $q$-convex with corners on $X$ if every point of $X$ admits an open neighborhood $U$ on which there are finitely many $q$-convex functions $f_1, \ldots, f_k$ such that $\varphi|_U = \max(f_1, \ldots, f_k)$. Denote by $F_q(X)$ the set of all functions $q$-convex with corners on $X$.

We say that $X$ is $q$-complete with corners if there exists an exhaustion function $\varphi \in F_q(X)$.

**DEFINITION 1.** — Let $X$ be a complex space. A subset $A$ of $X$ is said to be $q$-concave (in $X$) if $A$ is closed and every point of $A$ has an open neighborhood $\Omega$ such that $\Omega \setminus A$ is $q$-complete with corners.

From [24] (see also [25]) we deduce immediately:

**COROLLARY 2.** — Let $\pi : X \to Y$ be a finite surjective holomorphic map of complex spaces and $A \subset Y$ a closed subset. Then $A$ is $q$-concave in $Y$ if and only if $\pi^{-1}(A)$ is $q$-concave in $X$.

Subsequently we give some facts on $q$-completeness with corners which allow us to reduce the proof of Theorem 1 to the case when $X$ is a domain in $\mathbb{C}^n$. 

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PROPOSITION 1. — Let $Y$ be an analytic set in a complex space $X$. If $Y$ is $q$-complete with corners, then $Y$ has a neighborhood system of open sets which are $q$-complete with corners.

Proof. — By ([3], Lemma 3) if $\varphi \in F_q(Y)$ and $\eta \in C^0(Y, \mathbb{R})$, $\eta > 0$, then there exists an open neighborhood $V$ of $Y$ in $X$ and $\psi \in F_q(V)$ such that $|\psi - \varphi| < \eta$ on $Y$. The method of Colțoiu ([4], Theorem 2) or Demailly ([5], the proof of Theorem 1, p. 287) can easily be adapted to our case. □

PROPOSITION 2. — Let $X$ be a complex space and $\varphi, \psi$ be continuous exhaustion functions on $X$ such that there is an open neighborhood $\Omega$ of the set $\{\varphi = \psi\}$ in $X$ with $\varphi \in F_p(\Omega \cup \{\varphi < \psi\})$ and $\psi \in F_q(\Omega \cup \{\psi < \varphi\})$. Then $X$ is $(p + q)$-complete with corners.

Proof. — Let $\Lambda := \{\lambda \in C^\infty(\mathbb{R}, \mathbb{R}); \lambda' > 0, \lambda'' \geq 0\}$. For $\lambda \in \Lambda$ define $\Phi_\lambda : X \to \mathbb{R}$ by

$$\Phi_\lambda := 1/\left(\exp(-\lambda(\varphi)) + \exp(-\lambda(\psi))\right).$$

It is straightforward to see that $\Phi_\lambda$ is exhaustive for $X$ and it is $(p + q)$-convex with corners on $\Omega$. Now we let $\varepsilon > 0$ be continuous on $X$ such that $\{|\varphi - \psi| \leq \varepsilon\} \subset \Omega$; define $W_- = \{\varphi - \psi \leq -\varepsilon\}$ and $W_+ = \{\varphi - \psi \geq \varepsilon\}$. Clearly $W_-, W_+$ are closed subsets of $X$ and $W_- \cup W_+ \cup \Omega = X$. The proof is concluded if we show the next

CLAIM. — There is $\lambda \in \Lambda$ such that $\Phi_\lambda$ is $p$-convex with corners on $W_-$ and $q$-convex with corners on $W_+$.

But this follows by adjusting the arguments in [22]. We omit the details. □

PROPOSITION 3. — Let $U, V$ be open subsets of a complex space $X$ such that $U$ is $p$-complete with corners and $V$ is $q$-complete with corners. Then $U \cup V$ is $(p + q)$-complete with corners.

Proof. — Consider exhaustion functions $f \in F_q(U)$ and $g \in F_q(V)$ for $U$ and $V$ respectively. Let $a \in C^\infty(U, \mathbb{R})$ with $0 \leq a \leq 1$, $a(x) = 1$ if $x \in U \setminus V$ or $x \in U \cap V$ and $f(x) \leq g(x) + 1$; $a(x) = 0$ if $x \in U \cap V$ and $f(x) > g(x) + 2$. Set $D := U \cup V$. Define $\varphi$ on $D$ by setting

$$\varphi = \begin{cases} f & \text{on } U \setminus V, \\ af + (1 - a)(1 + g) & \text{on } U \cap V, \\ 1 + g & \text{on } V \setminus U. \end{cases}$$

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Then \( \varphi \) is continuous and exhaustive for \( D \).

Let \( b \in C^\infty(V, \mathbb{R}) \) with \( 0 \leq b \leq 1 \), \( b(x) = 1 \) if \( g(x) \leq \varphi(x) + 1 \) and \( b(x) = 0 \) if \( g(x) > \varphi(x) + 2 \). Define \( \psi \) on \( D \) by setting

\[
\psi = \begin{cases} 
  bg + (1 - b)(1 + \varphi) & \text{on } V, \\
  1 + \varphi & \text{on } U \setminus V.
\end{cases}
\]

Then \( \psi \) is continuous and exhaustive for \( D \).

Finally, it easy to see that \( S := \{ \psi < 1 + \varphi \} \subset V \) and \( \psi = g \) on \( S \); hence \( \psi \in F_q(S) \). Similarly, \( T := \{ \varphi < 1 + \psi \} \subset U \) and \( \varphi = f \) on \( T \); so \( \varphi \in F_p(T) \). The conclusion then follows from Proposition 2.

\[ \Box \]

**Corollary 3.** — Let \( A \) and \( B \) be \( p \)-concave and \( q \)-concave sets in the complex spaces \( X \) and \( Y \) respectively. Then \( A \times B \) is \( (p + q) \)-concave in \( X \times Y \).

**Proof.** — Since the assertion is local, we may assume that \( X \) and \( Y \) are Stein spaces, \( X \setminus A \) is \( p \)-complete with corners, and \( Y \setminus B \) is \( q \)-complete with corners. Then \( X \times Y \setminus A \times B = X \times (Y \setminus B) \cup (X \setminus A) \times Y \) is \( (p + q) \)-complete with corners by Proposition 3.

For a complex space \( X \) we introduce [20] the set \( G_q(X) \) as follows: For \( x_0 \in X \) let \( G_q(x_0) \) be the set of all functions \( g : X \rightarrow \mathbb{R} \) such that there are: an open neighborhood \( U \) of \( x_0 \) (which may depend on \( g \)) and \( f \in F_q(U) \) with \( f(x_0) = g(x_0) \) and \( f \leq g|_U \). Then put

\[
G_q(X) := C^0(X, \mathbb{R}) \cap \bigcap_{x \in X} G_q(x).
\]

Clearly \( F_q(X) \subseteq G_q(X) \subseteq C^0(X, \mathbb{R}) \).

Note that given an open set \( D \subseteq X \), an \( \varepsilon > 0 \), and a function \( g \in G_q(X) \), there is a function \( h \in F_q(D) \) such that \( |h - g| < \varepsilon \) on \( D \). See [20], Lemma 1. But we cannot use this fact and the classical perturbation procedure (see for instance [8]) to get a globally defined \( h \) since we do not know that given \( v \in G_q(X) \) and \( \theta \in C_a^\infty(X, \mathbb{R}) \) there is \( \varepsilon_o > 0 \) such that \( v + \lambda \theta \in G_q(X) \) for every \( \lambda \in \mathbb{R}, |\lambda| < \varepsilon_o \). However we can avoid this difficulty since we show:

**Lemma 1.** — The set \( F_q(X) \) is dense in \( G_q(X) \) in the sense that given an arbitrary \( g \in G_q(X) \) and \( \eta \in C^0(X, \mathbb{R}), \eta > 0 \), there is \( f \in F_q(X) \) such that \( |f - g| < \eta \).
Proof. — We do this in three steps.

Step 1). Fix $x \in X$ and $\varepsilon > 0$. By definition there is an open neighborhood $\Omega$ of $x$ and $\varphi \in F_q(\Omega)$ with $\varphi(x) = g(x)$ and $\varphi \leq g$ on $\Omega$. Let $W, U$ be open neighborhoods of $x$, $W \subset U \subset \Omega$, such that $\varphi \geq g - \varepsilon$ on $U$; then let $\theta \in C_0^\infty(U, \mathbb{R})$, $\theta = -1$ on $\partial W$ and $\theta(x) = 1$. If $c > 0$ is small enough, then $\psi := \varphi + c\theta \in F_q(U)$, $\psi < g$ on $\partial W$, $\psi \geq g$ on a neighborhood of $x$ in $W$, and $|\psi - g| < 2\varepsilon$ on $U$.

Step 2). The above step shows that for all compact subsets $K, L$ of $X$, there are: a finite set of indices $I$ (which depends on $K$ and $L$), open sets $V_i \subset W_i \subset U_i \subset L$ such that $\{V_i\}_{i \in I}$ cover $K$, functions $f_i \in F_q(U_i)$ with $|f_i - g| < 2\varepsilon$ on $W_i$, $f_i \geq g$ on $V_i$ and $f_i < g$ on $\partial W_i$.

Step 3). Let $\{K_\nu\}_{\nu \in \mathbb{N}}$ be an exhaustion sequence for $X$ by compact sets, $K_0 = \emptyset$ (by convention set $K_{-1} = \emptyset$), and $K_\nu \subset \operatorname{int}(K_{\nu+1})$ for all $\nu$. For each $\nu$ apply Step 2 to $K = K_\nu \setminus \operatorname{int}(K_{\nu-1})$, $L = K_{\nu+1} \setminus \operatorname{int}(K_{\nu-2})$, and $\varepsilon = (\min L, n)/2$. We therefore obtain open sets $V_{i\nu} \subset W_{i\nu} \subset U_{i\nu}$ such that the family $\{W_{i\nu}\}$ is locally finite, $\{V_{i\nu}\}$ is a covering of $X$, and functions $f_{i\nu} \in F_q(U_{i\nu})$ as in Step 2 from above. Then define $f : X \to \mathbb{R}$ by $f(x) = \max\{f_{i\nu}(x); x \in W_{i\nu}\}$, where the maximum is taken over all indices $i, \nu$ such that $W_{i\nu} \ni x$. It is straightforward to see that $f$ is continuous, $f \in F_q(X)$, and $g < f < g + \eta$.

Remark. — It can be shown that for $q > \dim(X)$ the set $F_q(X)$ is dense in the above sense even in $C^0(X, \mathbb{R})$.

From ([20], Lemma 4) we quote:

**Lemma 2.** — Let $U$ be a complex space, $V$ a complex manifold of pure dimension $r$, and $f \in F_{q+r}(U \times V)$ such that $\sup f < \infty$. Consider $g : U \to \mathbb{R}$ defined by

$$g(x) = \sup\{f(x, y); y \in V\}, \quad x \in U.$$  

Assume that for some $x_0 \in U$ there is $y_0 \in V$ with $g(x_0) = f(x_0, y_0)$. Then $g \in G_q(x_0)$.

The key proposition for the proof of Theorem 1 is:

**Proposition 4.** — Let $X$ and $Y$ be complex manifolds such that $Y$ is of pure dimension $r$ and $p$-complete with corners. Let $A$ be a $(q + r)$-concave subset in $X \times Y$ such that the natural projection $\pi : A \to X$ is...
proper. Then \( \pi(A) \) is \((q+p-1)\)-concave in \( X \). In particular, if \( Y \) is Stein (i.e. \( p = 1 \)), then \( \pi(A) \) is \( q \)-concave.

**Proof.** — Set \( m := q + p - 1 \). We may assume without any loss in generality that \( X \) is Stein. The statement of the proposition follows from the next claim.

**CLAIM.** — For every relatively compact Stein open subset \( U \) of \( X \), the set \( U \setminus \pi(A) \) is \( m \)-complete with corners.

In order to show this, consider a relatively compact open subset \( V \) of \( Y \) which is \( p \)-complete with corners and such that \( \pi^{-1}(\overline{U} \times \pi(A)) \subseteq \overline{U} \times V \). Then \( K := \overline{U} \times \partial V \) is compact and disjoint from \( A \). Now, since \( U \times Y \setminus A \) is \((m+r)\)-complete with corners by [20], there exists an exhaustion function \( \psi \in \mathcal{F}_{m+r}(U \times Y \setminus A) \).

Let \( \lambda := \max_K \psi \) and define \( \sigma : U \setminus \pi(A) \to \mathbb{R} \) by setting

\[
\sigma(x) = \max\{\psi(x,y), y \in V\}, x \in U \setminus \pi(A).
\]

Clearly \( \sigma \) is continuous. Consider \( \theta \) be a 1-convex exhaustion function on \( U \) and then define \( \varphi : U \setminus \pi(A) \to \mathbb{R} \) by setting

\[
\varphi = \theta + \max(\lambda, \sigma).
\]

Then \( \varphi \) is continuous and exhaustive. To conclude the proof, in view of Lemma 1, it suffices to show that \( \varphi \in G_m(x) \) for ever \( x \in U \setminus \pi(A) \). Indeed, two cases may occur:

a) If \( \sigma(x) > \lambda \), then \( \sigma \in G_m(x) \) by Lemma 2. Since \( \varphi = \sigma + \theta \) on a neighborhood of \( x \), we get \( \varphi \in G_m(x) \).

b) If \( \sigma(x) \leq \lambda \), then \( \varphi(x) \) and since \( \lambda + \theta \leq \varphi \) on \( U \setminus \pi(A) \), \( \varphi \in G_1(x) \), a fortiori, \( \varphi \in G_m(x) \).

The proof is complete. \( \square \)

(*) Denotes by \( \Delta^k(t) \) the open polydisc in \( \mathbb{C}^k \) of polyradius \( (t, \ldots, t) \) centered at the origin. Let \( n \) and \( q \) be positive integers such that \( q < n \). We define the \((q, n-q)\) Hartogs figure in \( \mathbb{C}^n = \mathbb{C}^q \times \mathbb{C}^{n-q} \) to be the open set \( H_q \subset \mathbb{C}^n \) given by

\[
H_q := \left( (\Delta^q(1) \setminus \overline{\Delta^q(t)}) \times \Delta^{n-q}(1) \right) \cup \left( \Delta^q(1) \times \Delta^{n-q}(s) \right)
\]

where \( 0 < t, s < 1 \). Put \( \tilde{H}_q := \Delta^n(1) \), i.e. the envelope of holomorphy of \( H_q \).
Following [16] we say that a complex space \( Y \) possesses the meromorphic extension property (in bidimension \( (q, n-q) \)) if every meromorphic map \( f : H_q \to Y \) extends to a meromorphic map \( \tilde{f} : \tilde{H}_q \to Y \).

By [16] every \( q \)-complete complex space possesses a meromorphic extension property in bidimension \( (q, n-q) \) for every integer \( n > q \).

**DEFINITION 2.** — \( M \) be a complex manifold of pure dimension \( n \). We say that a closed subset \( A \subset M \) is pseudoconcave of order \( q \) if for every injective holomorphic map \( f : H_q \to M \) such that \( f(H_q) \cap A = \emptyset \), the set \( f(H_q) \cap A \) is also empty.

In this set-up, a variant of Proposition 4 for \( Y = \mathbb{C}^r \) is straightforward. See ([10], Lemma 3.6).

Also by ([24], Corollary 5) one has: A closed subset \( A \) of a pure dimensional complex manifold is pseudoconcave of order \( q \) if and only if \( A \) is \( q \)-concave.

Pseudoconcavity of order \( q \) is easier to handle; though it does not suit to complex spaces. One has the next examples:

1) Let \( M \) be a Stein manifold of pure dimension \( n \) and \( K \subset M \) a compact set. Then \( \hat{K} \setminus K \) is \((n-1)\)-concave in \( X \setminus K \). (See [23].)

2) The support of a closed positive current of bidegree \((q, q)\) on a pure dimensional complex manifold is \( q \)-concave. (This follows by [12], Corollary 2.6 and the above remark.)

3. Proof of Theorems 1 and 2.

**Proof of Theorem 1.**

We remark that it suffices to show that \( A \) is analytic and for this we distinguish three steps.

**Step 1.** — Here we reduce the proof to the case when \( X \subset \mathbb{C}^n \) is open. For this we need:

**Lemma 3.** — Let \( Z \) be a complex space, \( X \subset Z \) an analytic subset, and \( A \subset X \) a closed subset (not necessarily analytic). If \( A \) is \( q \)-concave in \( X \) and \( X \) is \( r \)-concave in \( Z \), then \( A \) is \((q+r)\)-concave in \( Z \).
Proof. — Let $x_0 \in A$ and $U$ be a Stein open neighborhood of $x_0$ in $Z$ such that $U \setminus X$ is $r$-complete with corners and $(U \setminus A) \cap X$ is $q$-complete with corners. Since $(U \setminus A) \cap X$ is analytic in $U \setminus A$, there is by Proposition 1 an open subset $\Omega$ of $U \setminus A$ which is $q$-complete with corners and contains $(U \setminus A) \cap X$. Therefore $U \setminus A = (U \setminus X) \cup \Omega$ is $(q+r)$-complete with corners by Proposition 3.

To complete Step 1, we let $x \in A$, then take a coordinate patch $\iota : U \to D \subset \mathbb{C}^n$ around $x \in X$ with $D$ Stein; hence $U$ is isomorphic to the closed analytic subset $\iota(U)$ of $D$, hence $\iota(U)$ is $q$-concave in $\iota(U)$. Put $p := q + N - n$. Note that $N - p = n - q$. Therefore $\iota(A \cap U)$ is $p$-concave in $D$ by Lemma 3 since $\iota(U)$ is $(N-n)$-concave in $D$. On the other hand, $\iota(A \cap U)$ as a closed subset of $D$ has its Hausdorff $(2N - 2p)$-measure locally finite.

Step 2). — We give here some general facts for further reduction of the proof of Theorem 1.

Let $E \subset \mathbb{C}^n$ be a locally closed set with $h^{2n-2q+1}(E) = 0$ and suppose $0 \in E$. Then there is a complex $(n-q)$-plane $\Gamma$ through $0$ such that $h^1(E \cap \Gamma) = 0$ ([21], Lemma 2). Hence for a suitable unitary transformation $\sigma$ of $\mathbb{C}^n$ we have $h^1(\sigma(E) \cap (\mathbb{C}^{n-q} \times \{0\})) = 0$. By ([21], Corollary 2), $\sigma(E) \cap (\partial B(r) \times \{0\})$ is empty for $(h^1)$-almost all $r > 0$. (Here $B(r)$ denotes the open unit ball in $\mathbb{C}^{n-q}$ of radius $r$.) Since $\sigma(E)$ is also locally closed in $\mathbb{C}^n$ and $0 \in \sigma(E)$, there is $r > 0$ arbitrary small and a polydisc $P$ in $\mathbb{C}^q$ centered at the origin such that $\sigma(E) \cap (\overline{B(r)} \times P)$ is closed in $\overline{B(r)} \times P$ and $\sigma(E) \cap (\partial B(r) \times P)$ is empty. In particular, the canonically induced projection map $\pi$ from $\sigma(E) \cap (B(r) \times P)$ into $B(r)$ is proper.

If furthermore $h^{2n-2q}(E) < \infty$, then $\pi^{-1}(z)$ is finite for $(h^{2n-2q})$-almost all $z \in B(r)$ ([21], Corollary 4).

Recall that a set $\Gamma \subset \mathbb{C}^n$ is said to be locally pluripolar if for every $a \in \Gamma$ there is a connected neighborhood $U \ni a$ and a plurisubharmonic function $\varphi$ on $U$, $\varphi \neq -\infty$, such that $\Gamma \cap U \subset \{ \varphi = -\infty \}$. In fact, if $\Gamma$ is locally pluripolar then by [17] one can take $U = \mathbb{C}^n$, so $\Gamma$ is pluripolar. Note that for $n = 1$ pluripolarity of a set in $\mathbb{C}$ means that it is of zero-capacity as used in [18]. Also it is easy to check that for $U \subset \mathbb{C}^n$ open and $S \subset \mathbb{C}^n$ of zero Lebesgue measure, the set $U \setminus S$ is not pluripolar.

Step 3). — Here we conclude the proof.

By Steps 1, 2, and Proposition 4 it remains to show the next lemma.
LEMMA 4. — Let $U \subset \mathbb{C}^{n-q}$ be an open set, $\Delta$ the open unit disc in $\mathbb{C}$, and $A \subset U \times \Delta^q$ a closed subset such that the canonical projection $\pi : A \to U$ is proper. If $A$ is $q$-concave and $\pi^{-1}(z)$ is finite for $z$ in a non pluripolar subset of $U$, then $A$ is analytic of pure dimension $n-q$.

Proof. — For $q = 1$ this is precisely the lemma due to Hartogs-Oka-Nishino [18]. For $q > 1$ we proceed as follows. Notice that it suffices to show the analyticity of $A$. In order to do this we let $p_j : \Delta^q \to \Delta$, $j = 1, \ldots, q$, denote the projection onto the $j^{th}$ component of $\Delta^q$, then let $\sigma_j : A \to U \times \Delta$ naturally induced by $p_j$. Then $\sigma_j$ is proper and Proposition 4 implies that $\sigma_j(A)$ is 1-concave in $U \times \Delta$ for all indices $j = 1, \ldots, q$. Furthermore if we consider $\pi_j : \sigma_j(A) \to U$ canonically induced, we arrive at the case $q = 1$. So the sets $\sigma_j(A)$ are analytic for all $j$.

Now, if $\iota : U \times \Delta^q \to (U \times \Delta) \times \cdots \times (U \times \Delta)$ (the product is taken $q$-times) is given by $\iota(z, t_1, \ldots, t_q) = ((z, t_1), \ldots, (z, t_q))$, then $A = \iota^{-1}(\sigma_1(A) \times \cdots \sigma_q(A))$, whence the lemma. Thus the proof of Theorem 1.

Proof of Theorem 2.

Denote by $A^0 :=$ the set of points $x \in A$ such that $f$ extends meromorphically onto a neighborhood of $x$. Then $A' := A \setminus A^0$ is closed and as the complement to $A$ is locally connected in $M$ these local meromorphic continuations of $f$ in points of $A^0$ glue together to a unique meromorphic map from $M \setminus A'$ into $Y$.

Now, we assert that $A'$ is pseudoconcave of order $q$. For this we let $\Phi : \hat{H}_q \to M$ be an injective holomorphic map with $\Phi(H_q) \cap A' = \emptyset$. Then $f \circ \Phi$ is meromorphic from $H_q$ into $Y$, hence it extends to $\hat{H}_q$; therefore $f$ extends over $\Phi(\hat{H}_q)$, and by definition $\Phi(\hat{H}_q) \subset A^0$; whence the desired assertion.

Finally, by Theorem 1, if $A'$ is not the empty set, then $A'$ is analytic of pure dimension $n-q$. But this contradicts the hypothesis, whence the proof.
4. A final remark.

Motivated by M. Peternel's work ([20], §7) we give:

**DEFINITION 3.** Let $X$ be a complex space of pure dimension $n$. A closed subset $A$ of $X$ is said to be $q$-pseudoconcave if there is an analytic subset $B \subset X$ such that

1) $A \setminus B = A$.

2) For each point $x \in A \setminus B$ there is a locally closed analytic subset $Y$ of $X$ which passes through $x$, $Y \subset A$, and $Y$ is a complex manifold of dimension $n-q$.

As an example, if $A$ is analytic and $\dim_x A \geq n-q$, $\forall x \in A$, then $A$ is $q$-pseudoconcave.

Let now $r$ be a non-negative integer and suppose $X$ is purely dimensional. We say that $X$ has property $(E_r)$, if there is $\varphi \in F_{n+r}(X \times X \setminus \Delta_X)$, where $\Delta_X$ is the diagonal set of $X \times X$, such that $\varphi(x,\nu) \to +\infty$ if $x_\nu \to x$, $x_\nu \neq x$, $\forall x \in X$. Condition $(E_r)$ holds locally on $X$ if every point of $X$ admits an open neighborhood $U$ which satisfies $(E_r)$.

The next proposition is an easy consequence of ([20], Lemma 9).

**PROPOSITION 5.** Let $X$ be a pure dimensional complex space such that $(E_r)$ holds locally. Then every $q$-pseudoconcave subset of $X$ is $(q+r)$-concave.

The importance of the condition $(E_r)$ resides in the fact that, for example, if a Stein space $X$ fulfils $(E_0)$, then every locally Stein open subset of $X$ is Stein. It is easy to check for a Stein manifold that $(E_0)$ holds. However, this fails, in general, if we allow singularities. For example, we let $X$ be the Segre cone in $\mathbb{C}^4$, $X = \{xy = zw\}$. Clearly the hypersurface $A = \{x = z = 0\}$ is 1-pseudoconcave. Now, if $(E_0)$ would hold locally on $X$, then $A$ will be 1-concave; and as $X$ has isolated singularities $X \setminus A$ will be Stein. But this is absurd since $X \setminus A$ is biholomorphic to $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$.

**COROLLARY 4.** If $X$ is a complex manifold, then every $q$-pseudoconcave subset of $X$ is also $q$-concave.

**Example 3.** For every positive integer $q$ there is an open subset $X$ of $\mathbb{C}^{q+1}$ and a $q$-concave subset $A \subset X$ which is **not** $q$-pseudoconcave.
To do this we consider a compact subset $K$ of $\mathbb{C}^2$ such that $\hat{K} \setminus K$ contains no analytic disc. See [26] for the existence of $K$. Put $X := (\mathbb{C}^2 \setminus K) \times \mathbb{C}^{q-1}$ and $A := (\hat{K} \setminus K) \times \{0\}$. Then $A$ is not $q$-pseudoconcave in $X$; however, by Example 1 in §2 and Corollary 3 it is easily seen that $\hat{K} \setminus K$ is $q$-concave in $X$.

The corresponding version of Theorem 1 reads:

**Theorem 3.** — *Let $A$ be a closed subset of a pure $n$-dimensional complex space $X$ such that $A$ is $q$-pseudoconcave and its Hausdorff $(2n-2q)$-measure is locally finite. Then $A$ is analytic of pure dimension $n-q$.*

**Proof.** — If $\iota : U \rightarrow D$ is a local path of $X$, where $D$ is an open subset of $\mathbb{C}^N$, then $\iota(A \cap U)$ is $(N-n+q)$-pseudoconcave in $D$. Now we conclude by the above corollary and Theorem 1.

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**BIBLIOGRAPHY**


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