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ON INFINITESIMAL TRANSFORMATIONS PRESERVING THE CURVATURE TENSOR FIELD AND ITS COVARIANT DIFFERENTIALS

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We shall say that a transformation φ of a Riemannian manifold M is *strongly curvature-preserving* if it preserves the curvature tensor field R and all its successive covariant differentials $\nabla^m R$. Similarly, an infinitesimal transformation X on M is strongly curvature-preserving if

$$L_X(\nabla^m R) = 0, \quad m = 0, 1, 2, \ldots,$$

where L_X denotes Lie differentiation with respect to X and $\nabla^0 R = R$.

Of course, an affine transformation or an infinitesimal affine transformation is strongly curvature-preserving. In the present note, we shall prove the converse in the following form. Recall that an infinitesimal transformation X is conformal, homothetic, or Killing according as $L_X g = fg$ (f: function), $L_X g = cg$ (c: constant), or $L_X g = 0$, respectively, where g denotes the metric tensor.

THEOREM 1 (2). — *Let* M *be an irreducible analytic Riemannian manifold of dimension* $\geqslant 2$. *Then a strongly curvature-preserving infinitesimal transformation is necessarily homothetic. If* M *is furthermore complete, then* X *is Killing.*

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(2) We have since extended theorem 1 to the case of a global transformation; this result will appear elsewhere.

Note that the additional assertion is a consequence of a result of Kobayashi [2]. The proof of Theorem 1 will depend on the following results.

THEOREM 2. — *Let* M *be an irreducible Riemannian manifold of dimension* > 2. *An infinitesimal conformal transformation* X *is homothetic if* $L_X R = 0$.

THEOREM 3. — *Let* M *be an irreducible analytic Riemannian manifold of dimension* 2. *An infinitesimal transformation* X *is homothetic if* $L_X R = 0$ *and* $L_X(\nabla R) = 0$.

The proof of Theorem 2 makes use of a result of Guillemin and Sternberg [1] on the prolongations of the conformal algebra.

Finally, we shall prove the following generalization of Theorem 1.

THEOREM 4. — *Let* M *be a connected, complete and analytic Riemannian manifold which has no Euclidean part (i.e., the restricted homogeneous holonomy group* Ψ^0 *has no non-zero fixed vector). Then any strongly curvature-preserving infinitesimal transformation* X *is a Killing vector field.*

1. Preliminaries.

For an arbitrary infinitesimal transformation X on M, we shall define a tensor field K of type (1, 2) which measures the deviation of X from being affine; X is affine if and only if $K = 0$. For any vector field Y, consider the derivation

$$K(Y) = [L_X, \nabla_Y] - \nabla_{[X, Y]} \tag{1}$$

of the algebra of tensor fields. It is easy to verify that K(Y) is actually a tensor field of type (1, 1) and that $K(fY) = fK(Y)$ for any differentiable function f. This means that K is a tensor field of type (1, 2) which associates to a vector field Y the tensor field K(Y) of type (1, 1).

Using the formula $L_X = A_X + \nabla_X$, where A_X is the tensor field of type (1, 1) defined by $A_X Y = -\nabla_Y X$ (cf. [3], p. 235), we may express K(Y) as follows:

$$K(Y) = R(X, Y) - \nabla_Y(A_X). \tag{2}$$

In fact, we have

$$\begin{aligned} K(Y) &= [A_X + \nabla_X, \nabla_Y] - \nabla_{[X, Y]} \\ &= [A_X, \nabla_Y] + [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \\ &= - \nabla_Y(A_X) + R(X, Y). \end{aligned}$$

We now prove

LEMMA 1. — *The tensor field* K *corresponding to a vector field* X *has the following properties*:

1) $K(Y)Z = K(Z)Y$ *for any vector fields* Y *and* Z;

2) $(\nabla_U K)(Y)Z = (\nabla_U K)(Z)Y$ *for any vector fields* Y, Z, *and* U;

3) *If* $L_X R = 0$, *then* $(\nabla_Y K)(Z) = (\nabla_Z K)(Y)$ *for any vector fields* Y *and* Z;

4) *If* X *is conformal*: $L_X g = fg$, *then*

$$K(Y)g = - \alpha(Y)g \tag{3}$$

for any vector field Y, *where* $\alpha = df$.

5) *If* X *is conformal, then, for the form* α *in* 4), *we have*

$$(\nabla_U K)(Y)g = - (\nabla_U \alpha)(Y)g$$

for any vector fields Y *and* U.

Proof. — 1) By using (2), we have

$$\begin{aligned} K(Y)Z &= R(X, Y)Z - [\nabla_Y(A_X)]Z \\ &= R(X, Y)Z - \nabla_Y(A_X Z) + A_X(\nabla_Y Z) \end{aligned}$$

and hence

$$K(Y)Z = R(X, Y)Z + \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X$$

by definition of A_X. Thus alternating with respect to Y and Z, we have

$$\begin{aligned} & K(Y)Z - K(Z)Y \\ &= R(X, Y)Z - R(X, Z)Y + ([\nabla_Y, \nabla_Z] - \nabla_{[Y, Z]})X = 0 \end{aligned}$$

by virtue of Bianchi's identity:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$

and the definition of the curvature tensor:

$$[\nabla_Y, \nabla_Z] - \nabla_{[Y, Z]} = R(Y, Z).$$

2) We take ∇_U of 1) and obtain

$$(\nabla_U K)(Y)Z + K(\nabla_U Y)Z + K(Y)\nabla_U Z \\ = (\nabla_U K)(Z)Y + K(\nabla_U Z)Y + K(Z)\nabla_U Y,$$

from which, using 1) again, we find

$$(\nabla_U K)(Y)Z = (\nabla_U K)(Z)Y.$$

3) By using (2), we have

$$(\nabla_Y K)(Z) = \nabla_Y(K(Z)) - K(\nabla_Y Z) \\ = (\nabla_Y R)(X, Z) + R(\nabla_Y X, Z) + R(X, \nabla_Y Z) - \nabla_Y \nabla_Z(A_X) \\ - R(X, \nabla_Y Z) - \nabla_{\nabla_Y Z} Z(A_X)$$

or

$$(\nabla_Y K)(Z) = (\nabla_Y R)(X, Z) - R(A_X Y, Z) - (\nabla_Y \nabla_Z - \nabla_{\nabla_Y Z})(A_X).$$

Alternating with respect to Y and Z, we find

$$(\nabla_Y K)(Z) - (\nabla_Z K)(Y) \\ = (\nabla_Y R)(X, Z) - (\nabla_Z R)(X, Y) - R(A_X Y, Z) + R(A_X Z, Y) \\ - ([\nabla_Y, \nabla_Z] - \nabla_{[Y, Z]})(A_X) \\ = (\nabla_X R)(Y, Z) - R(A_X Y, Z) - R(Y, A_X Z) - R(Y, Z)A_X \\ = [(\nabla_X + A_X)R](Y, Z) = (L_X R)(Y, Z) = 0,$$

by virtue of Bianchi's identity:

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$$

and the assumption $L_X R = 0$.

4) By definition of $K(Y)$, we have

$$K(Y) = L_X \nabla_Y - \nabla_Y L_X - \nabla_{[X, Y]}.$$

Applying this derivation to g, we find

$$K(Y)g = -\nabla_Y L_X g.$$

Thus if $L_X = fg$, then we have

$$K(Y)g = -\alpha(Y)g,$$

where $\alpha = df$.

5) Taking ∇_U of the equation in 4), we have

$$(\nabla_U K)(Y)g + K(\nabla_U Y)g = -(\nabla_U \alpha)(Y)g - \alpha(\nabla_U Y)g,$$

which implies

$$(\nabla_U K)(Y)g = -(\nabla_U \alpha)(Y)g,$$

since $K(\nabla_U Y)g = -\alpha(\nabla_U Y)g$ by 4).

We shall now interpret Lemma 1 above in terms of the prolongations of the conformal algebra [1]. By the conformal algebra over an n-dimensional real vector space V with inner product, we mean the following. Let co(V) be the set of all linear endomorphisms A of V such that

$$(AX, Y) + (X, AY) = c(X, Y)$$

for all X, Y in V, where c is a constant which depends on A. With respect to the usual bracket $[A, B] = AB - BA$, co(V) forms a Lie algebra.

Suppose X is conformal. Property 4) means that for any Y in the tangent space $T_x(M)$ at a point $x \in M$, the endomorphism K(Y) is in the conformal algebra co(x) over $T_x(M)$, of course, with respect to the metric g_x. Property 1) means that the linear mapping $K : Y \in T_x(M) \to K(Y) \in co(x)$ is an element of the first prolongation $co(x)^{(1)}$. Property 5) means that for any $U \in T_x(M)$, the endomorphism $(\nabla_U K)(Y)$ belongs to co (x) for any $Y \in T_x(M)$. Property 2) means that the linear mapping $\nabla_U K : Y \in T_x(M) \to (\nabla_U K)(Y) \in co(x)$ is an element of co $(x)^{(1)}$. Now assume that $L_X R = 0$. Property 3) means that the linear mapping $\nabla K : U \in T_x(M) \to \nabla_U K \in co(x)^{(1)}$ is actually an element of the second prolongation $co(x)^{(2)}$. It is known [1], however, that $co(x)^{(2)} = 0$ when dim $M > 2$. Thus we arrive at the following consequence of the lemma above:

If X *is conformal and* $L_X R = 0$, *then the corresponding tensor field* K *satisfies* $\nabla K = 0$.

2. Proof of Theorem 2.

From the preceding interpretation of the Lemma, we see that $\nabla K = 0$. Let γ be the 1-form defined by $\gamma(Y)$ = trace of K(Y). We have then $\nabla \gamma = 0$. Since M is irreducible, we have $\gamma = 0$, that is, trace $K(Y) = 0$ for any Y. Since K(Y) is in co(x), it follows that K(Y) is skew-symmetric. In equation (3), we have $K(Y)g = -\alpha(Y)g = 0$ for any Y, which means that $\alpha = 0$. Since $\alpha = df$ in the proof of equation (3), we see that f is a constant, that is X is homothetic.

3. Proof of Theorem 3.

In a two-dimensional irreducible Riemannian manifold, the Ricci tensor S has the form

$$S = \lambda g,$$

where λ is a function which is not identically zero. From this we have

$$\nabla_Y S = (Y\lambda)g$$

for any vector Y.

If the infinitesimal transformation X satisfies $L_X R = 0$ and $L_X(\nabla R) = 0$, then it satisfies $L_X S = 0$ and $L_X(\nabla S) = 0$. From $S = \lambda g$ and $L_X S = 0$, we obtain

$$(X\lambda)g + \lambda(L_X g) = 0. \tag{4}$$

From $\nabla_Y S = (Y\lambda)g$ and $L_X(\nabla S) = 0$, we obtain

$$\begin{aligned} 0 = L_X \nabla_Y S - \nabla_{[X,Y]} S &= (XY\lambda)g + (Y\lambda)L_X g - ([X, X]\lambda)g \\ &= (YX\lambda)g + (Y\lambda)L_X g, \end{aligned}$$

that is,

$$(YX\lambda)g + (Y\lambda)(L_X g) = 0. \tag{5}$$

Taking ∇_Y of (4) and taking (5) into account, we get

$$\lambda \nabla_Y (L_X g) = 0.$$

Since our manifold is real analytic, the set of zero points of λ is nowhere dense. Hence we have

$$\nabla L_X g = 0.$$

Since the manifold is irreducible, we get

$$L_X g = cg,$$

where c is a constant.

4. Proof of Theorem 1.

Since M is an analytic Riemannian manifold, the holonomy algebra h_x (Lie algebra of the restricted holonomy group at x) is generated by all endomorphisms of the form

$$R(Y, Z), (\nabla_U R)(Y, Z), \ldots, (\nabla^m R)(Y, Z; U_1; \ldots; U_m), \ldots,$$

where $Y, Z, U_1, \ldots, U_m$ are arbitrary vectors at x

(cf. [3, p. 152]). From the assumption $L_X(\nabla^m R) = 0$, it follows that $A_X(\nabla^m R) = -\nabla_X(\nabla^m R)$. It is easy to see that

$$[A_X, (\nabla^m R)(Y, Z; U_1; \dots; U_m)] \in h_x$$

and hence

$$[A_X, h_x] \subset h_x.$$

The tensor $L_X g = A_X g$ at x is then invariant by h_x. In fact, for any $B \in h_x$, we have

$$B(A_X g) = A_X(Bg) + [A_X, B]g = 0,$$

since B and $[A_X, B]$ are skew-symmetric as elements in h_x. Since h_x is irreducible, $A_X g$ at x is a scalar multiple of the tensor g_x. This being the case at every point x of M, we have $A_X g = fg$, that is, $L_X g = fg$, where f is a function. This means that X is conformal.

Thus, if the dimension of $M > 2$, then Theorem 2 implies that X is homothetic.

If the dimension of M is 2, then Theorem 1 is as pecial case of Theorem 3.

5. Proof of Theorem 4.

We may assume that M is simply connected. Let $M = M_1 \times \cdots \times M_k$ be the de Rham decomposition, where $M_1, \dots, M_k$ are irreducible, complete and analytic Riemannian manifolds. We shall show that the vector field X decomposes naturally, that is, there exists a strongly curvature-preserving infinitesimal transformation X_i on M_i, $1 \leqq i \leqq k$, such that

$$X_{(x_1, \dots, x_k)} = (X_1)_{x_1} + \cdots + (X_k)_{x_k}$$

for any point $x = (x_1, \dots, x_k) \in M_1 \times \cdots \times M_k$. Once this is shown, we see that X_i is Killing on M_i by Theorem 1 and hence X is Killing on M.

In order to prove a natural decomposition of X, we proceed as follows. Let $(T_1), \dots, (T_k)$ be the parallel distributions corresponding to the de Rham decomposition $M_1 \times \cdots \times M_k$.

Lemma 2. — $L_X(T_i) \subset (T_i)$ *for each* i, *in the sense that if* Y *is a vector field belonging to the distribution* (T_i), *then*

$$L_X(Y) = [X, Y]$$

belongs to (T_i).

Proof. — Since $L_X = \nabla_X + A_X$ and since $\nabla_X(T_i) \subset (T_i)$ because (T_i) is parallel, it is sufficient to show that $A_X(T_i) \subset (T_i)$. Let x be an arbitrary point. In the proof of Theorem 1, we have seen that $(A_X)_x$ lies in the normalizor of the holonomy algebra h_x. Thus the 1-parameter group of linear transformations $\exp tA_X$ of $T_x(M)$ lies in the normalizor of the holonomy group Ψ_x. It follows that, for each t, $(\exp tA_X).(T_i)_x$ coincides with some $(T_j)_x$ by virtue of the uniqueness of the de Rham decomposition

$$T_x(M) = (T_1)_x + \cdots + (T_k)_x$$

(cf. Theorem 5.4, (4), p. 185, and Lemma, p. 186, in [3]). By continuity, we see that $(\exp tA_X).(T_i)_x = (T_i)_x$ for every t. This implies $A_X(T_i)_x \subset (T_i)_x$.

LEMMA 3. — *Let* Δ *be a differentiable distribution on a differentiable manifold* M. *If a vector field* X *on* M *satisfies* $L_X(\Delta) \subset \Delta$, *then a local 1-parameter group* φ_t *of local transformations generated by* X *preserves the distribution.*

Proof. — Let $Y_1, \ldots, Y_r$ be a local basis for Δ in a neighborhood of x. It is sufficient to show that $(\varphi_t.(Y_i))_x$ belongs to Δ_x for every t. We recall the formula

$$\frac{d(\varphi_t.Y_i)_x}{dt} = -(\varphi_t.[X, Y_i])_x$$

(Corollary 1.10, p. 16, [3]).

Since $[X, Y_i]$ belongs to Δ, we have

$$[X, Y_i] = \sum_{j=1}^{r} f_{ij} Y_j,$$

where f_{ij} are certain functions. Therefore

$$\frac{d(\varphi_t Y_i)_x}{dt} = -\left(\varphi_t.\left(\sum_{j=1}^{r} f_{ij} Y_j\right)\right)_x$$
$$= -\sum_{j=1}^{r} (f_{ij} \circ \varphi_t^{-1}).(\varphi_t Y_j)_x.$$

If we denote $(\varphi_t Y_i)_x$ by $Y_i(t)$, then the functions $Y_i(t)$ with values in $T_x(M)$ satisfy a system of differential equations

$$\frac{dY_i(t)}{dt} = \sum_{j=1}^{r} g_{ij}(t) Y_j(t), \tag{6}$$

where $g_{ij}(t) = -f_{ij}(\varphi_t^{-1}(x))$. The initial conditions are $Y_i(0) = (Y_i)_x$. It follows that $Y_i(t)$ has to be a linear combination

$$Y_i(t) = \sum_{j=1}^{r} F_{ij}(t)(Y_j)_x$$

of the vectors $(Y_1)_x, \ldots, (Y_r)_x$, that is, $Y_i(t) \in \Delta_x$. ($F(t) = [F_{ij}(t)]$ is the matrix function which is a unique solution of

$$\frac{dF}{dt} = G(t)F(t)$$

with initial condition $F(0) = [\delta_{ij}]$. The existence of such a solution is a special case of Lemma, p. 69, [3].) This proves Lemma 3.

Now we can prove that X decomposes naturally. Let φ_t be a local 1-parameter group of local transformations generated by X in a neighborhood of a point x. By Lemma 2,

$$L_X(T_i) \subset (T_i).$$

By Lemma 3, φ_t preserves each distribution (T_i) and hence its maximal integral manifold. It follows, by an argument similar to the proof of Theorem 3.5, p. 240, in [3], that there exists, for each t a local transformation $\varphi_t^{(i)}$ of M_i such that

$$\varphi_t(x_1, \ldots, x_k) = (\varphi_t^{(1)}(x_1), \ldots, \varphi_t^{(k)}(x_k)).$$

Each $\varphi_t^{(i)}$ is a local 1-parameter group and defines a vector field X_i on M_i. It is clear that $X = X_1 + \ldots + X_k$. Since the curvature tensor R and its successive covariant differentials $\nabla^m R$ decompose naturally, it is obvious that each X_i is strongly curvature-preserving on M_i.

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