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ON THE EXISTENCE OF EXCEPTIONAL LEAVES IN FOLIATIONS OF CO-DIMENSION ONE

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1. Introduction.

Let M be a compact n-manifold ($n \geqq 2$) with a foliated structure of co-dimension one. A leaf of such a foliation is said to be *exceptional* if it is nowhere dense in M, but its topology as a subset of M is not the same as its topology as an (n-1)-manifold. Reeb [2] has asked if it is possible for exceptional leaves to exist in sufficiently smooth foliations, and he showed in [2] that, under certain conditions, exceptional leaves do not exist. The author proved other theorems of this type in [3] and [4]. Here we shall answer Reeb's question by giving an example of a 3-manifold with a C^∞ foliated structure of co-dimension one in which there are exceptional leaves. Moreover, these leaves will be contained in a minimal set of the foliation.

2. Diffeomorphisms of S^1.

We shall first construct a group of C^∞ diffeomorphisms of S^1 which has a perfect, nowhere dense, minimal set C. Let S^1 be represented as the interval [0, 2] with its endpoints identified. The set C is defined as follows: At the first step the intervals (1/3, 2/3), (1, 4/3), and (5/3, 2) are removed from [0, 2]. At the k'th step the middle third of each closed interval which remains after the (k-1)st step is removed, as in the usual construction of a Cantor set. The set C is the set which remains after all of the steps have been completed. C is perfect and nowhere dense.

The group of diffeomorphisms of S^1 will be the group generated by the diffeomorphisms f and g defined by

$$f(x) = x + 2/3 \quad (\text{mod. } 2) \text{ for } x \in [0, 2]$$
$$g(x) = x/3 \quad \text{if} \quad 0 \leq x \leq 1$$
$$g(x) = 3x - 10/3 \quad \text{if} \quad 4/3 \leq x \leq 5/3$$

$g(x)$ is defined elsewhere in $[0,2]$ so that g is of class C^∞, $g(2) = 2$, and g^{-1} exists and is of class C^∞ on S^1. Clearly this can be done. Let G denote the group generated by f and g.

Lemma. — *C is a minimal set under the action of* G.

Proof. — It is easy to verify that C is closed and invariant under G. Let C_k denote the set which remains after the k'th step in the construction of C has been carried out. Then C_k is the union of 3.2^{k-1} disjoint closed intervals,

$$I_k^i, \ i = 1, \dots 3.2^{k-1},$$

and $C = \bigcap \{C_k : k = 1,2, \dots\}$. To verify that C is minimal it suffices to prove that any interval I_k^i is mapped onto $[0, 1/3]$ by an element of G. This is proved by induction on k. For $k = 1$, either f or f^2 will work. If $k > 1$, some power of f will map I_k^i into the interval $[0, 1/3]$, hence it can be assumed that $I_k^i \subset [0, 1/3]$. But then $g^{-1}(I_k^i) = I_{k-1}^j$ for some j, hence the induction hypothesis shows that I_k^i is mapped onto $[0, 1/3]$ by an element of G. This proves the lemma.

3. The Example.

In the example, M is the product manifold $M = S^1 \times M_2$, where M_2 is the sphere S^2 with two handles attached. M_2 is a disjoint union of three sets A, B, and C, where A is a « band » diffeomorphic to $S^1 \times [0, 1]$ passing around a handle once, and B is another such band, disjoint from A and passing around the other handle. The foliated structure of M will be defined separately on the sets $T_A = S^1 \times A$, $T_B = S^1 \times B$, and $T_C = S^1 \times C$.

Let φ be a function of ν defined for $\nu \in [0, 1]$ with the properties that: (*a*) φ is increasing and of class C^∞, (*b*) $\varphi(0) = 0$, $\varphi(1) = 1$, (*c*) all derivatives of φ vanish for $\nu = 0$ and $\nu = 1$. Again regard S^1 as the interval $[0, 2]$ with its endpoints identified. Define the C^∞ functions h, k from $S^1 \times [0, 1]$ to S^1 by:

$$h(x, \nu) = x + 2/3\,\varphi(\nu) \quad \text{mod. } 2 \text{ and}$$
$$k(x, \nu) = x + (g(x) - x)\varphi(\nu) \quad \text{mod. } 2.$$

Note that $h(x, 0) = k(x, 0) = x$ and $h(x, 1) = f(x)$,

$$k(x, 1) = g(x).$$

Let (u, ν), $u \in S^1$, $\nu \in [0, 1]$ represent a point of A, hence (x, u, ν) represents a point in T_A if $x \in S^1$. We define the foliation on T_A by agreeing that the leaf passing through $(x, u, 0)$ will contain all points $(h(x, \nu), u', \nu)$. The foliation of T_B is defined similarly except that k replaces h. The foliation on T_C is defined by the condition that $x =$ const. on each leaf.

It is easy to check that the foliations defined on T_A, T_B, T_C fit together to define a C^∞ foliation of $M = T_A \cup T_B \cup T_C$. It is also clear that the leaves of the foliation are transversal to S^1 in product $M = S^1 \times M_2$. This transversality property implies that an arc in M_2 beginning at $b \in M_2 - A \cup B$ be « lifted » to the leaf through any point $(x, b) \in M$, $x \in S^1$. The lifted arc is uniquely determined by the initial point (x, b). If γ is a closed curve parameterized by $t (0 \leq t \leq 1)$ such that $\gamma(0) = \gamma(1) = b$, the lifted curve will end at a point $(T(x, \gamma), b) \in M$. It is easy to verify that the map $x \to T(x, \gamma)$ is of class C^∞ and depends only on the homotopy class of γ.

Suppose that the closed curve γ_A has the property that γ_A does not intersect $A \cup B$, except for one sub-arc of γ_A which is mapped homeomorphically on to the arc in A which corresponds to $u =$ const. in terms of the (u, ν) coordinates established above. Then if γ_A begins at $b \in M_2 - A \cup B$ and the mapping on the sub-arc is such that increasing t corresponds to increasing ν, $T(x, \gamma_A) = f(x)$. Similar considerations lead to a closed curve γ_B beginning at b and such that

$$T(x, \gamma_B) = g(x).$$

Finally, if γ_1 and γ_2 are closed curves which begin at b and do not meet $A \cup B$ at all, $T(x, \gamma_i) = x$. Arcs γ_A, γ_B, γ_1, γ_2 with properties described can be chosen in such a way that their homotopy classes generate the fundamental group, $\pi_1(M_2)$. The map from γ to $T(\ ,\gamma)$ induces a homorphism from $\pi_1(M_2)$ to a group of C^∞ diffeomorphisms of S^1, and it is now clear that this group is just the group G defined above.

These considerations show that if $y \in Gx \subset S^1$, (Gx is the orbit of x under G), then (x, b) and (y, b) are on the same leaf of the foliation. The converse is also easy to check, that is, if (x, b) and (y, b) are on the same leaf, $y \in Gx$.

Now if one takes x to be a point of C, the lemma implies that the closure of the points (y, b) on the leaf containing (x, b) is just $C \times \{b\}$. It follows easily that the leaf through (x, b) is exceptional and its closure is a minimal set.

4. The fundamental group of an exceptional leaf.

It was remarked in [4] that Lemma 12.1 of [4] suggests that the fundamental group of a nowhere dense leaf might be finitely generated. However, this is not the case, as will now be shown. In fact, the exceptional leaf just constructed has a fundamental group which is not finitely generated. To see this, let γ_1 be, as above, a generator of the fundamental group of M_2 which does not intersect the set $A \cup B$. Let F be an exceptional leaf. There are infinitely many points of F which project onto the initial point of γ_1. One can show that the lifts of γ_1 through these points are closed curves, which when connected to a base point, represent elements of the fundamental group of F. They cannot be represented in terms of any finite number of generators. We omit the details.

BIBLIOGRAPHY

[1] G. Reeb, Sur certaines propriétés topologiques des variétés feuilletées. *Actualités scientifiques et industrielles*, 1183. Hermann, Paris (1952).

[2] G. Reeb, Sur les structures feuilletées de co-dimension un et sur un théorème de M. A. Denjoy, *Annales de l'Institut Fourier*, 11, (1961), pp. 185-200.

[3] R. Sacksteder, Some properties of foliations, *Annales de l'Institut Fourier*, 14, 1, (1964), pp. 21-30.

[4] R. Sacksteder, Foliations and Pseudogroups, (to appear) *in The American Journal of Mathematics*).

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