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FIBRATIONS OF COMPACT KÄHLER MANIFOLDS IN TERMS OF COHOMOLOGICAL PROPERTIES OF THEIR FUNDAMENTAL GROUPS

by Ngaiming MOK (*)

Let X be a compact Kähler manifold whose fundamental group Γ admits a finite-dimensional discrete Zariski-dense representation into a real semisimple Lie group of the noncompact type. Using the method of harmonic maps, the author established in [M1] that some unramified finite cover of X admits a meromorphic fibration over a projective manifold of general type. Without the discreteness assumption, one can still derive a discrete representation with similar properties, unless the representation is nonrigid, or there exists a Zariski-dense representation into some p-adic semisimple Lie group. In these cases there is the method of harmonic maps into associated Euclidean buildings of Gromov-Schoen [GS]. Using this method and the method of spectral covers, Zuo [Zu], building on a number of earlier works, established an analogous fibration theorem over projective manifolds of general type in these cases. As a consequence, the discreteness assumption can be entirely dropped.

In this article we consider fibration theorems with conditions on first cohomology groups of Γ with respect to unitary representations into (complex) Hilbert spaces. All Hilbert spaces are understood to be separable in the present article. By a holomorphic fibration we mean a proper surjective holomorphic map with connected fibers. In this terminology, a

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biholomorpic map or a modification is a holomorphic fibration. We will say for short that a compact complex manifold Y admits a meromorphic fibration over a compact complex manifold Z if there exist a modification $\rho : \widehat{Y} \to Y$ and a holomorphic fibration $\sigma : \widehat{Y} \to Z$. We prove

THEOREM 1. — *Let X be a compact Kähler manifold with fundamental group Γ. Suppose for some unitary representation Φ of Γ on a Hilbert space H we have $H^1(\Gamma; \Phi) \neq 0$. Then,*

(a) *X is of general type; or*

(b) *for some unramified finite cover X' of X, there exists a holomorphic fibration $\tau : X' \to T$ onto a compact complex torus T; or*

(c) *for some nonsingular Kähler modification $\widehat{X}$ of X there exists a holomorphic fibration $\sigma : \widehat{X} \to Z$ with positive-dimensional fibers onto a projective manifold Z such that Z is of logarithmic general type with respect to the multiplicity locus of σ.*

Here and henceforth to avoid clumsy language a compact complex torus or a projective manifold is always understood to be positive-dimensional. For (c) see Sakai [Sa] and references given there. Here we may also say that the pair $(\widehat{X}, \mathrm{mult}(\sigma))$ is of logarithmic general type, for $\mathrm{mult}(\sigma)$ denoting the reduced divisor on $\widehat{X}$ which is the multiplicity locus of σ.

In the statement of Theorem 1, the non-vanishing condition on the first cohomology group with respect to some unitary representation Φ is equivalent to saying that Γ violates Property (T) of Kazhdan (*cf.* de la Harpe-Valette [HP]). In other words, Γ satisfies the hypothesis of Theorem 1 if and only if the trivial representation of Γ is not an isolated point in the unitary dual of Γ. As an example, any Γ of subexponential growth violates Kazhdan's Property (T) since $H^1(\Gamma; \rho) \neq 0$ for the left regular representation ρ.

We aimed originally to establish a fibration theorem in which Z in (c) can be taken to be a projective manifold of general type at the expense of replacing X by some unramified finite cover. While this can always be done if Φ is finite-dimensional, in our situation (c) may arise from an infinite-dimensional representation, and the analogous statement is not known. Still, (c) yields over X nontrivial holomorphic sections of certain tensor powers of the cotangent bundle Ω_X on X. Under some additional hypotheses on the fundamental group $\Gamma = \pi_1(X)$, we obtain the following corollary to the proof of Theorem 1.

COROLLARY 1. — *Let X be a compact Kähler manifold whose fundamental group Γ violates Kazhdan's Property (T). Assume furthermore that every quotient group of Γ is residually finite. Then, some unramified finite cover X' of X admits a meromorphic fibration over a positive-dimensional compact Kähler manifold Z such that Z is either a compact complex torus or a projective manifold of general type.*

Here a discrete group H is said to be residually finite if and only if the intersection of all subgroups of finite index of H reduces to the trivial group.

An intermediate aim for proving fibration theorems on compact Kähler manifolds is to show the existence of nontrivial meromorphic functions on them. In this direction our method yields the following result on algebraic dimensions of compact Kähler manifolds.

THEOREM 2. — *Let X be a compact Kähler manifold whose fundamental group Γ violates Kazhdan's Property (T). Suppose furthermore that for every unramified finite cover X' of X the first Betti number $b_1(X') = 0$. Then, X is of algebraic dimension $\geqslant 2$.*

Theorem 2 is optimal in the sense that, irrespective of the complex dimension of X, one cannot expect the algebraic dimension of X to exceed 2. In fact, if there is a projective-algebraic manifold X_0 of dimension $m \geqslant 2$ satisfying the hypothesis, by taking hyperplane sections and applying Lefschetz's Theorem, there is always a projective-algebraic surface S satisfying the same hypothesis, and products of S with simply-connected compact Kähler manifolds of algebraic dimension 0 give examples of X of arbitrarily large complex dimensions and of algebraic dimension 2.

In order to establish fibration theorems on compact Kähler manifolds whose fundamental groups violate Kazhdan's Property (T), we resort to the method of semi-Kähler structures, *i.e.*, semi-Kähler metrics with compatible meromorphic foliations, as developed in Mok ([M1], [M3], [M4]). On compact Kähler manifolds there are two essential sources of semi-Kähler structures, constructed from harmonic maps (resp. harmonic forms). Both types of semi-Kähler structures will be used.

Our point of departure is an existence theorem (Mok [M2]) for harmonic forms with twisted coefficients on compact Riemannian manifolds with nonvanishing cohomology groups with respect to some unitary representation Φ on a Hilbert space H. In this existence theorem, the unitary representation Φ and the Hilbert space H may have to be replaced. In the case of compact Kähler manifolds X, we obtain d-closed holomorphic 1-forms with twisted coefficients. (For first cohomology groups the existence theorem also follows from the work of Korevaar-Schoen [KS] on harmonic maps.) Here and in what follows $\Phi : \Gamma \to U(H)$ will be taken to mean a unitary representation for which there exists an associated holomorphic 1-form ν with twisted coefficients. Norms of ν then lead to a semi-Kähler form ω, with a compatible meromorphic foliation $\mathcal{F}$ arising from level sets of an integral $F : \widetilde{X} \to H$ on the universal covering space $\widetilde{X}$.

As in Mok ([M1], [M4]) we distinguish two cases: the factorizable and the non-factorizable case. For definitions we refer the reader to Mok [M4], Definition-Proposition 2, with the correction that (b) there should be removed. For the ensuing discussion we restrict to the non-factorizable case. In that case there exists a nontrivial pseudogroup G of holomorphic isometries on some local cross-section of the foliated space X. We derive from this a canonical representation Θ of the fundamental group Γ on the group of Lie-algebra automorphisms $\mathrm{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G. Under the assumption of Theorem 1, in the non-factorizable case we deduce that the canonical action Θ of Γ on $\mathrm{Aut}(\mathfrak{g})$ has an infinite image. If the Zariski-closure $\overline{\Theta(\Gamma)}$ of $\Theta(\Gamma)$ is not solvable, passing to semisimple quotients of $\overline{\Theta(\Gamma)}$ the method of harmonic maps will yield on some unramified finite cover a meromorphic fibration over a projective manifold of general type. Otherwise one can derive an abelian representation of Γ with infinite image to show that X has a nontrivial Albanese map. Combining with results of Ueno [Ue] about subvarieties of Abelian varieties, Kawamata-Viehweg [KV] about ramified finite covers of Abelian varieties, and their generalizations to compact complex tori by Campana [Ca] we have obtained a meromorphic fibration of X either over a compact complex torus, or over a projective manifold of general type.

In the factorizable case we obtain a holomorphic fibration of some nonsingular Kähler modification $\widehat{X}$ of X onto a compact Kähler manifold Z. If there are no multiple fibers outside a subvariety of Z of codimension $\geqslant 2$, then the semi-Kähler form ω on X descends to a semi-Kähler form on Z which is positive-definite and of nonpositive holomorphic bisectional curvature on a Zariski-dense open subset. If the Ricci form is negative-definite at a generic point, we prove that Z is of general type. Otherwise as in Mok ([M1], [M4]) we consider a second semi-Kähler structure defined by the Ricci form. We show that this will lead to a meromorphic fibration over a projective manifold of general type.

In the presence of multiple fibers, the semi-Kähler form ω on X descends to a semi-Kähler form on $Z - A$ for some proper subvariety $A \subset Z$. In general the semi-Kähler form may have poles of fractional order along irreducible components of codimension 1 in A. There is an effective $\mathbb{Q}$-divisor D on Z such that, denoting by K_Z the canonical line bundle of Z, $K_Z \otimes [D]$ admits a singular Hermition metric of nonpositive curvature. In this case our method of proof shows that $K_Z \otimes [D]$ is big and that, as a consequence, Z is of logarithmic general type with respect to the multiplicity locus of the holomorphic fibration. The compact Kähler manifold Z is thus Moišezon and hence projective-algebraic. This gives Theorem 1. In order for the semi-Kähler form to descend to the base manifold in the factorizable case after passing to some unramified finite cover of X, it is sufficient to find a torsion-free subgroup of finite index of some discrete group acting properly discontinuously on a complex space. This leads to Corollary 1 and Theorem 2.

In the case when one can deduce that a compact Kähler manifold X is of general type directly from the existence of a holomorphic 1-form ν with coefficients twisted by a unitary representation, the semi-Kähler form ω defined by ν is positive-definite, of nonpositive holomorphic bisectional curvature and of strictly negative Ricci curvature on some Zariski-dense open subset. Heuristically, modulo some blowing-down, X should behave like a subvariety of an Abelian variety. As a by-product of our study of semi-Kähler structures we establish in the case of Kähler surfaces the Kobayashi hyperbolicity of X, under the assumption of nonexistence of rational curves and elliptic curves.

THEOREM 3. — *Let X be a compact Kähler surface with fundamental group Γ, and $\Phi : \Gamma \to U(H)$ be a unitary representation on a Hilbert space H. Suppose ν is of maximal rank at a generic point, and the induced semi-Kähler form ω is of strictly negative Ricci curvature at a generic point. Then, X is Kobayashi hyperbolic provided that there does not exist any elliptic or rational curves on X.*

Results along the line of Theorem 1 and Corollary 1 were stated in Mok [M3] with a sketch of the proof. There it was claimed that Corollary 1 holds for any X whose fundamental group violates Kazhdan's Property (T). As it turned out the author has regrettably ignored the question of finding a torsion-free subgroup of finite index of some discrete group which acts properly discontinuously on some complex space of cycles. While this is possible and well-known if the holomorphic 1-form has coefficients twisted by a finite-dimensional (unitary) representation, in the general case of unitary representations on Hilbert spaces the analogous statement is not known.

The main input of the present article, long overdue, is to introduce a method to analyse non-factorizable meromorphic foliations arising from holomorphic 1-forms with coefficients twisted by infinite-dimensional unitary representations. By this method in the non-factorizable case one can pass to finite-dimensional representations. Despite the difficulty arising from finding subgroups of finite index acting without fixed points, the author believes that the main input of the article remains intact. He also believes that to study fundamental groups of compact Kähler manifolds one has to deal with delicate questions on infinite-dimensional representations of discrete groups, as there is no reason why a "generic" Kähler group should admit nontrivial finite-dimensional representations. It is in view of the lack of results on infinite-dimensional representations on Kähler groups and the difficulty of the problem that the author ventures to present the methods and intermediate results of the present article.

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Table of Contents.

1. Harmonic 1-forms with twisted coefficients and semi-Kähler structures.

(1.1) Let (M, g) be a compact Riemannian manifold with fundamental group Γ. Let H be a Hilbert space and $\Phi : \Gamma \to U(H)$ be a unitary representation of Γ on H, and E_Φ be the locally constant bundle of Hilbert spaces on M with transition functions defined by Φ. For a nonnegative integer i we denote by $H^i(M, E_\Phi)$ the de Rham cohomology group. For $i = 1$, $H^1(M, E_\Phi)$ depends only on Γ and is isomorphic to the cohomology group $H^1(\Gamma; \Phi)$ defined as follows. A 1-cochain c is a function $c : \Gamma \to H$. The 1-cochain is called a 1-cocycle if it satisfies the identity $c_{\gamma_1\gamma_2} = \Phi(\gamma_1)(c_{\gamma_2}) + c_{\gamma_1}$ for all $\gamma_1, \gamma_2 \in \Gamma$. It is called a 1-coboundary if there exists some element $h \in H$ such that $c_\gamma = h - \Phi(\gamma)h$ for every $\gamma \in \Gamma$. Denoting the complex vector spaces of 1-cocycles (resp. 1-coboundaries) by $Z^1(\Gamma; \Phi)$ (resp. $B^1(\Gamma; \Phi)$), we have the first cohomology group $H^1(\Gamma; \Phi) := Z^1(\Gamma; \Phi)/B^1(\Gamma; \Phi)$. In general it may happen that $H^i(M, E_\Phi) = 0$ while there are no E_Φ-valued harmonic i-forms. The following result, proved in Mok [M2], derives nonetheless the existence of harmonic forms with twisted coefficients at the expense of possibly replacing the representation Φ by some other unitary representation $\Phi' : \Gamma \to H'$. The result in the case of $i = 1$ follows also from the work of Korevaar-Schoen [KS] on harmonic maps into possibly infinite-dimensional Riemannian manifolds.

THEOREM (1.1.1). — *Let (M, g) be a compact Riemannian manifold and denote by Γ its fundamental group. Let $\Phi : \Gamma \to U(H)$ be a unitary representation of Γ on a Hilbert space and denote by E_Φ the corresponding locally constant Hilbert bundle over M. Suppose $H^i(M; E_\Phi) \neq 0$ for some $i \geqslant 1$. Then, there exists a sequence $\{\xi_k\}$ of smooth Φ-equivalent i-forms on $\widetilde{M}$ such that over a fundamental domain F we have $\int_F \|\xi_k\|^2 = 1$ and $\int_F (\|d\xi_k\|^2 + \|d^*\xi_k\|^2) \leqslant \frac{1}{k}$. For any such sequence $\{\xi_k\}$ there exists a subsequence $\{\xi_{k(\ell)}\}$ and there are unitary transformations $U_{k(\ell)}$ on H such that $U_{k(\ell)}\xi_{k(\ell)}$ converges on $\widetilde{M}$ in $L^2_{\rm loc}$ to a nontrivial H-valued d-closed harmonic i-form ξ on $\widetilde{M}$. Furthermore, there exists a Hilbert subspace $H' \subset H$ and a unitary representation $\Phi' : \Gamma \to U(H')$ such that ξ is Φ'-equivariant.*

In what follows a harmonic form with values in a locally flat Hilbert bundle arising from a unitary representation $\Phi : M \to U(H)$ will simply be referred to as a harmonic form with coefficients twisted by Φ, or simply as one with twisted coefficients. (Note that Φ is used as a generic symbol for a unitary representation and carries a different meaning from that in Theorem (1.1.1).) In the special case of $i = 1$ the assumption of nonvanishing first cohomology for some unitary representations amounts to saying that Γ violates Property (T) of Kazhdan. Theorem (1.1.1) says that on a compact Riemannian manifold (M, g) with $\pi_1(M) = \Gamma$ there exists some harmonic 1-form with twisted coefficients. Since any unitary representation of Γ can be decomposed into an integral of irreducible unitary representations (*cf.* Zimmer [Zi], p. 23ff), it follows readily that there exists an *irreducible* representation $\Phi : \Gamma \to U(H)$ for which there exists a nontrivial harmonic 1-form with coefficients twisted by Φ (Mok [M4], (3.2)).

Let ν be a harmonic 1-form on M with twisted coefficients and $\widetilde{\nu}$ be the H-valued Φ-equivariant harmonic 1-form on $\widetilde{M}$. Since $\widetilde{\nu}$ is closed there exists a smooth H-valued smooth 1-form F such that $dF = \widetilde{\nu}$. We have $0 = d^*\widetilde{\nu} = d^*dF$, showing that F is a harmonic function. The Φ-equivariance of $\widetilde{\nu}$ means that for every $\gamma \in \Gamma$, we have $\widetilde{\nu}(\gamma x) = \Phi(\gamma)\widetilde{\nu}(x)$. Integrating we have the transformation rule

$$F(\gamma x) = \Phi(\gamma)F(x) + c_\gamma$$

for some constant vector $c_\gamma \in H$. The function $c : \Gamma \to H$ given by $c(\gamma) = c_\gamma$ then defines a 1-cocycle in $Z^1(M; \Phi)$. We note that if we replace the primitive F by $F + h$ for some $h \in H$, then the 1-cocycle $c = (c_\gamma)$ is replaced by $c'_\gamma = (c_\gamma + h - \Phi(\gamma)h)$, so that $c' - c$ is a 1-coboundary. For a harmonic 1-form ν with twisted coefficients, we have

LEMMA (1.1.1). — *Let (M, g) be a compact Riemannian manifold, ν be a nontrivial E_Φ-valued harmonic 1-form with values in E_Φ, and F be a primitive of the lifting $\widetilde{\nu}$ to the universal cover $\widetilde{M}$. Then the 1-cocycle c_γ is not cohomologous to 0.*

Proof. — Suppose otherwise. Replacing F by $F + h$ for some $h \in H$ we may assume without loss of generality that $c_\gamma = 0$ for every $\gamma \in \Gamma$. Then, for every $\gamma \in \Gamma$ and every $x \in M$, we have $F(\gamma x) = \Phi(\gamma)F(x)$. It follows that the smooth function $\widetilde{\varphi} := \|F\|^2$ is invariant under Γ, and descends thus to a smooth function φ on M. Let Δ be the Laplacian operator on M or on $\widetilde{M}$ on smooth functions. Since F is harmonic we have $\Delta\widetilde{\varphi} = \|dF\|^2 \geqslant 0$. Equivalently $\Delta\varphi = \|\nu\|^2 \geqslant 0$. Integrating over M we conclude the pointwise vanishing of ν, a contradiction. □

Let now X be a compact Kähler manifold. Take $\Phi : \Gamma \to U(H)$ to be irreducible such that there exists a nontrivial harmonic E_Φ-valued 1-form η. Regard H now as a real Hilbert space and write $H^{\mathbb{C}}$ for $H \otimes_{\mathbb{R}} \mathbb{C}$. We have the standard orthogonal decomposition $H^{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$ into Hilbert subspaces, where $H^{0,1} = \overline{H^{1,0}}$ and $H^{1,0}$ is isomorphic to H as a Γ-representation space. Accordingly write $E_\Phi \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{E}^{\mathbb{C}}_{\lessneqq}$. We decompose η as $\eta = \eta^{1,0} + \eta^{0,1}$, where $\eta^{1,0}$ is an $E^{\mathbb{C}}_\Phi$-valued $(1,0)$-form and $\eta^{0,1} = \overline{\eta^{1,0}}$. Then, the standard Hodge decomposition for harmonic 1-forms remains valid (*cf.* Mok [M2], Corollary (0.1)) to show that η is a d-closed holomorphic 1-form with values twisted by Φ.

An example of a unitary representation of an abstract group Γ on a Hilbert space H is given by the left regular representation ρ, as follows. Let $L^2(\Gamma)$ be the Hilbert space of all functions $f : \Gamma \to \mathbb{C}$, *i.e.*, verifying $\sum_{\gamma\in\Gamma} |f(\gamma)|^2 < \infty$. Define $\rho : L^2(\Gamma) \to L^2(\Gamma)$ by $[\rho(\gamma)(f)](g) = f(\gamma^{-1}g)$. Then,

$$\{\rho(\mu)[\rho(\gamma)(f)]\}(g) = [\rho(\gamma)(f)](\mu^{-1}g) = f(\gamma^{-1}\mu^{-1}g) = [\rho(\mu\gamma)(f)](g).$$

In other words, $\rho(\mu)\rho(\gamma) \equiv \rho(\mu\gamma)$. Clearly ρ preserves norms and is hence a unitary representation.

Let now X be a compact Kähler manifold and denote by Γ its fundamental group. Denote by $H^*_{L^2}(\widetilde{X}, \mathcal{O})$ the L^2-cohomology groups on the universal covering space $\widetilde{X}$. Then, $H^1(X, E_\rho) = 0$ if and only if $H^1_{L^2}(\widetilde{X}, \mathcal{O}) = 0$. We assert

LEMMA (1.1.2). — *$H^1_{L^2}(\widetilde{X}, \mathcal{O}) \neq 0$ whenever Γ is infinite and of subexponential growth. As a consequence, for Γ of subexponential growth we have $H^1(X, E_\rho) \neq 0$ and hence Γ violates Property (T) of Kazhdan.*

Proof. — The assumption that Γ is of subexponential growth implies the existence of geodesic balls $B_k = B(x_0; R_k)$ on $\widetilde{X}$ such that $\lim_{k\to\infty} \frac{\mathrm{Vol}(\partial B_k)}{\mathrm{Vol}(B_k)} = 0$. We proceed to prove $H^1_{L^2}(\widetilde{X}, \mathcal{O}) \neq 0$ by argument by contradiction. Choose smooth cut-off functions ρ_k such that $\rho_k \equiv 1$ on B_k; $\rho_k \equiv 0$ outside $B'_k = B(x_0; R_k + 1)$. If $H^1_{L^2}(\widetilde{X}, \mathcal{O}) = 0$, then we can solve $\overline{\partial} u_k = \overline{\partial} \rho_k$ on $\widetilde{X}$ with L^2-estimates, so that $\int_{\widetilde{X}} |u_k|^2 \leqslant C \int_{\widetilde{X}} |\overline{\partial}\rho_k|^2 \sim \mathrm{Vol}(\partial B_k)$. Here for two nonpositive sequences (a_k) and (b_k) diverging to ∞ we write $a_k \sim b_k$ to mean that for k sufficiently large $\frac{a_k}{b_k}$ is uniformly bounded above and below by positive constants. We have $\overline{\partial}(u_k - \rho_k) = 0$ so that $f_k := u_k - \rho_k$ is a square-integrable holomorphic function on $\widetilde{X}$. Since

$\int_{\widetilde{X}} |\rho_k|^2 \sim \mathrm{Vol}(B_k)$ and Γ is of subexponential growth we conclude that f_k is nontrivial for k large enough. However, by the Mean-Value Inequality and the Maximum Principle it is easy to see that there does not exist on $\widetilde{X}$ any holomorphic function of class L^2 whenever Γ is infinite. This proves Lemma (1.1.2) by contradiction. □

(1.2) We are going to obtain meromorphic fibrations by means of d-closed holomorphic 1-forms with coefficients twisted by unitary representations. In the difficult case we are led to study meromorphic foliations, with the aim of forming quotient spaces which are leaf spaces of the foliations. In order to study the possibility of forming leaf spaces we introduced the notion of semi-Kähler structures. This notion was implicitly introduced in Mok [M1] for the purpose of studying harmonic maps into Riemannian locally symmetric spaces of the noncompact type. We have

DEFINITION (1.2.1) (Semi-Kähler structures). — *Let X be a complex manifold. A semi-Kähler structure $(X,\omega,\mathcal{F},V)$ consists of*

(a) *a nontrivial closed positive (1,1)-current ω on X;*

(b) *a complex-analytic subvariety $V \subset X$ of codimension $\geqslant 2$, possibly empty, such that ω is smooth on $X - V$;*

(c) *a holomorphic foliation $\mathcal{F}$ on $X - V$ such that*

(d) *the closed semipositive (1,1)-form ω and $\mathcal{F}$ are compatible on a dense open subset U of $X - V$ in the sense that for any $x \in U$, $\mathrm{Ker}(\omega(x)) = T_x(\mathcal{F})$.*

The subvariety $V \subset X$ will sometimes be left out in the notation for a semi-Kähler structure. $\mathcal{F}$ will also be referred to as a meromorphic foliation. For the local structure of semi-Kähler structures we introduce

DEFINITION-PROPOSITION (1.2.1). — *Let $(X,\omega,\mathcal{F},V)$ be a semi-Kähler structure on an n-dimensional complex manifold. We call an open set $U \subset\subset X - V$ a distinguished polydisk if $U \cong D \times D'$ where $D \subset\subset \mathbb{C}^{k}$, $D' \subset\subset \mathbb{C}^{\ell}$, $k + \ell = n$, are polydisks centred at 0 such that the leaves of the foliation $\mathcal{F}|_U$ correspond to level sets of the canonical projection $D \times D' \to D$. Then, in terms of the corresponding Euclidean coordinates $(z_1,\ldots,z_k;w_1,\ldots,w_\ell)$ the semi-Kähler form ω is given by*

$$\omega = \sqrt{-1} \sum_{1\leqslant\alpha,\beta\leqslant k} g_{\alpha\overline{\beta}}(z)dz^{\alpha} \wedge d\overline{z}^{\beta}.$$

By Definition-Proposition (1.2.1) we have an induced quotient closed semipositive $(1,1)$ form $\overline{\omega}$ defined on D. Furthermore, $\overline{\omega}$ is positive-definite on a dense open subset of D. To form leaf spaces we introduce the notion of factorizable semi-Kähler structures, as follows.

DEFINITION-PROPOSITION (1.2.2). — *For a compact Kähler manifold X we say that a semi-Kähler structure $(X, \omega, \mathcal{F}, V)$ is factorizable if and only if all leaves L of $\mathcal{F}$ on $X - V$ are closed and the topological closures $\overline{L}$ on X are compact complex-analytic subvarieties of X. In this case replacing X by some unramified finite cover if necessary, there exists a nonsingular Kähler modification $\mu : \widehat{X} \to X$, a compact Kähler manifold Z and a surjective holomorphic map $\sigma : \widehat{X} \to Z$ with generically irreducible fibers such that for all leaves L of $\mathcal{F}$ on $X - V$, $\overline{(\mu^{-1}L)}$ is an irreducible component of a fiber $\sigma^{-1}(z)$, $z \in Z$.*

On the other hand, when the semi-Kähler structure is not factorizable, we have the following result from (Mok [M1], p. 574) producing nontrivial pseudogroups of holomorphic isometries.

PROPOSITION (1.2.1). — *Let $(X, \omega, \mathcal{F}, V)$ be a semi-Kähler structure on a compact Kähler manifold X. Then, either*

(a) *the semi-Kähler structure is factorizable; or*

(b) *for any distinguished polydisk $U \cong D \times D'$ on $X - V$ such that $\overline{\omega}$ is positive-definite on D, $(D, \overline{\omega})$ admits a nontrivial pseudogroup of holomorphic isometries.*

Here by a pseudogroup G we mean the germ at the identity map of a topological local group of homeomorphisms. In our case the pseudogroup of holomorphic isometries is a Lie pseudogroup, and any closed subset which constitutes a sub-pseudogroup is a Lie pseudogroup. By abuse of language we will say, in Case (b) of Proposition (1.2.1), that the semi-Kähler structure $(X, \omega, \mathcal{F})$ admits a nontrivial pseudogroup of holomorphic isometries. We will also say in Case (a) (resp. Case (b)) that the meromorphic foliation $\mathcal{F}$ is factorizable resp. non-factorizable.

We remark that the crux of the proof of Proposition (1.2.1) is the Bishop Extension Theorem on complex-analytic subvarieties. The idea is to measure volumes of leaves of $\mathcal{F}$ on $X - V$. Either leaves are of "bounded recurrence", in which case we prove that most leaves are of bounded volume, thus extendible to complex-analytic subvarieties on X by the Bishop Extension Theorem, or else, they are of "unbounded recurrence", in which case we produce a nontrivial pseudogroup of holomorphic isometries of $(D, \overline{\omega})$. We note that *a priori* it is possible that every leaf L of $\mathcal{F}$ on $X - V$ is closed while $\mathcal{F}$ is of "unbounded recurrence". To construct leaf spaces as implicit in Definition-Proposition (1.2.2), it is necessary for topological closures $\overline{L}$ to be complex-analytic subvarieties of X. For the notions of bounded and unbounded recurrence we refer the reader to Mok [M1], (3.3), Definition 3.

2. Fibration via the semi-Kähler structure arising from a holomorphic 1-form with twisted coefficients.

(2.1) From now on let X be an n-dimensional compact Kähler manifold and $\Phi : \Gamma \to U(H)$ be a unitary representation such that the reduced cohomology $H^1_{red}(\Gamma; \Phi) \neq 0$. Let $\nu \in H^1(X; \Omega(E_\Phi))$ be a nontrivial d-closed holomorphic 1-form with values in the associated locally constant bundle E_Φ of Hilbert spaces. Over a small open set $U \subset X$ fix a constant orthonormal basis $\{e_k\}$ of $E_\Phi|_U$ and write $\nu = \sum_{k=1}^{\infty} \nu_k e_k$ locally. Define a nonnegative real-analytic (1,1)-form ω by $\omega = \sqrt{-1} \sum_{k=1}^{\infty} \nu_k \wedge \overline{\nu}_k$. For each tangent vector η of type (1,0) at $x \in X$, $\omega\big(- \sqrt{-1}\eta \wedge \overline{\eta}\big)$ is the square of the norm of $\nu(\eta) \in E_{\Phi,x}$, so that ω is well-defined on X independent of the local choices of $\{e_k\}$. Since ν is d-closed, ω is also d-closed. It defines a semi-Kähler form on X. Suppose ω is everywhere degenerate, *i.e.*, $\omega^n \equiv 0$. From the Darboux Lemma, on the dense open set where ω is of maximal rank, the distribution defined by the kernel of ω is integrable with holomorphic leaves. Actually, the foliation is holomorphic, as follows. Let $E \subset X$ be the subset of points x where $\nu_x \in \Omega_x(E_{\Phi,x}) \cong \mathrm{Hom}(T_x, H)$ fails to be of maximal rank, and $\widetilde{E} \subset \widetilde{X}$ be the lifting $\pi^{-1}(E)$ under the universal covering map $\pi : \widetilde{X} \to X$. Let $F : \widetilde{X} \to H$ be a primitive of the lifting $\widetilde{\nu}$ of ν to $\widetilde{X}$. Then, on $X - E$ connected components of level sets of F define a holomorphic foliation $\widetilde{\mathcal{F}}$. Holomorphic tangent spaces to leaves of $\widetilde{\mathcal{F}}$ agree with kernels of $\partial F = \widetilde{\nu}$. From the definition of ω it is clear that a tangent vector η of type (1,0) lies on the kernel of ω if and only if it lies in the kernel of ν, so that $\mathrm{Ker}(\omega) = \mathrm{Ker}(\nu)$ defines an integrable holomorphic distribution, *i.e.*, a holomorphic foliation $\mathcal{F}$ outside E whose lifting to $\widetilde{X}$ agrees precisely with $\widetilde{\mathcal{F}}$. In other words, leaves of $\mathcal{F}$ on $X - E$ agree with images of leaves of $\widetilde{\mathcal{F}}$ under $\pi : \widetilde{X} \to X$. We have thus obtained a semi-Kähler structure $(X, \omega, \mathcal{F})$, which is the principal object of study in this section. We note that $\mathcal{F}$ extends in an obvious way to a meromorphic foliation on X, *i.e.*, a holomorphic foliation on $X - V$ for some complex-analytic subvariety

$V \subset E$, V being of codimension $\geqslant 2$ on X.

(2.2) From now on we assume that for the semi-Kähler structure $(X, \omega, \mathcal{F})$, $\mathcal{F}$ is a nontrivial foliation, *i.e.*, $\omega \not\equiv 0$ and $\omega^n \equiv 0$. To obtain fibration theorems on X one would naturally attempt to prove that $(X, \omega, \mathcal{F})$ is factorizable in the sense of Definition (1.2.1). From obvious examples on the compact complex torus of dimension $\geqslant 2$ one would need to rule out the trivial representation. The arguments of Mok [M4] indicate that in the case of a finite-dimensional unitary representation $\Phi : \Gamma \to U(r)$, the image of $\widetilde{X}$ under F is pseudo-invariant under a 1-parameter group of Euclidean translations. For infinite-dimensional unitary representations, we have given in Mok ([M4], §5) an example such that $(X, \omega, \mathcal{F})$ is not factorizable for another reason. In that case, arising from irreducible compact quotients of the bidisk, there is an associated representation of Γ into $SU(1,1)$ with Zariski-dense image. Our strategy for obtaining meromorphic fibrations will consist of proving that either $(X, \omega, \mathcal{F})$ is factorizable, or there exists a finite-dimensional linear representation of Γ with infinite image.

We assume now that $(X, \omega, \mathcal{F})$ is not factorizable. By Proposition (1.2.1) there exists a local cross-section $(S, \omega|_S)$ of $(X, \omega, \mathcal{F})$ and a nontrivial pseudogroup of (germs of) holomorphic isometries G. Denote by $\mathfrak{g}$ the Lie algebra of G. We proceed to define a representation Θ of Γ on the group $\mathrm{Aut}(\mathfrak{g})$ of Lie-algebra automorphisms. Then, we will show that $\Theta(\Gamma)$ must have infinite image in order to derive meromorphic fibrations from known results on compact complex tori and on harmonic maps.

Recall from (1.1) that $F : \widetilde{X} \to H$ satisfies the transformation rule $F(\gamma x) = \widehat{\Phi}(\gamma)(F(x)) = \Phi(\gamma)(F(x)) + c_\gamma$, where (c_γ) constitutes a 1-cocyle in $Z^1(\Gamma; \Phi)$. The mapping $\widehat{\Phi} : \Gamma \to U(H) \ltimes H$ is a homomorphism of groups. Fix a base point $x_0 \in \widetilde{X}$ where F is of maximal rank. Choose a local cross-section $(S_0, \widetilde{\omega}|_{S_0})$ passing through x_0. Assume that $\pi|_{S_0} : S_0 \to X$ is a biholomorphism of S_0 onto its image S, so that $(S, \omega|_S)$ is a local cross-section of $(X, \omega, \mathcal{F})$. From the way that Proposition (1.2.1) is proven in Mok [M1] the existence of a nontrivial pseudogroup of holomorphic isometries on $(S, \omega|_S)$ in the present case can be deduced as follows.

We may assume that $F|_{S_0}$ is a biholomorphism of S_0 onto its image W. Denote by β the Kähler form of the Euclidean metric on the Hilbert space H. Then, $(S_0, \widetilde{\omega}|_{S_0}) = F^*(W, \beta|_W)$. We need equivalently to prove the existence of a pseudogroup G of holomorphic isometries of $(W, \beta|_W)$.

It suffices to find a sequence of distinct germs of holomorphic isometries θ_i of $(W, \beta|_W)$ at $p_0 = F(x_0)$. Denote by L the leaf of $\widetilde{\mathcal{F}}$ on $\widetilde{X} - \widetilde{E}$ passing through x_0. For any point $x \in L$ there is a local cross-section $(S_x, \omega|_{S_x})$ which agrees with the germ of $(S_0, \widetilde{\omega}|_{S_0})$ at x_0 in the sense that both germs are isomorphic to the germ of $(W, \beta|_W)$ at $p_0 = F(x_0) = F(x)$. We assume that $F|_{S_x}$ is an embedding and that $F(S_x) \subset W$. Suppose now C is a smooth curve on L joining x to some point y with the property that $\pi(y) \in \pi(S_x)$. Let T be a tubular neighborhood of C in $\widetilde{X} - \widetilde{E}$ such that the map $F|_T : T \to W \subset H$ is a proper submersion with connected fibers onto an open neighborhood $W \subset H$ of p_0 in W. The canonical isomorphism of the germ of $(S_x, \widetilde{\omega}|_{S_x})$ with the germ of $(S_y, \widetilde{\omega}|_{S_y})$ at y then induces an isomorphism of the germ of $(\pi(S_x), \omega|_{\pi(S_x)})$ at $\pi(x)$ with the germ of $(\pi(S_x), \omega|_{\pi(S_x)})$ at $\pi(y)$. There is a unique point y' on S_x such that $\pi(y') = \pi(y)$. Then, we have an isomorphism of the germs of $(S_x, \widetilde{\omega}|_{S_x})$ at x resp. y'. Via the map $F : \widetilde{X} \to H$ we have an isomorphism θ of the germs of $(W, \beta|_W)$ at the points $F(x) = F(x_0) = p_0$, resp. $F(y')$. Pulling back by $F|_{S_0}$ and then projecting by $\pi|_{S_0}$ we obtain a germ h at $\pi(x_0)$ of holomorphic isometry of $(S, \omega|_S)$ sending $\pi(x_0)$ to a point on S.

Assuming that $(X, \omega, \mathcal{F})$ is not factorizable then it is of unbounded recurrence in the sense of Mok [M1]. In this case the pseudogroup of holomorphic isometries G' of $(S, \omega|_S)$ contains germs of holomorphic isometries h_i of $(S, \omega|_S)$ of the form described, for different choices of pairs of points (x, y) lying on L, together with smooth paths C joining x to y. They correspond to germs of holomorphic isometries θ_i of $(W, \beta|_W)$ at p_0 sending p_0 to some point p_i on W. θ_i can be chosen to be distinct and converging to the identity map. In particular, the points p_i thus chosen converge to p_0. From the existence of θ_i it follows that there is a nontrivial pseudogroup of holomorphic isometries of $(W, \beta|_W)$ containing $\{\theta_i\}$ as a subset. Let G_0 to be the largest pseudogroup of such germs of holomorphic isometries. From the description above of θ and the transformation rule for $F : \widetilde{X} \to H$ it follows that θ_i is of the form $\theta_i = \widehat{\Phi}(\xi_i) = (\Phi(\xi_i), c_{\xi_i}) \in U(H) \ltimes H$ for some choices of ξ_i.

(2.3) We will consider a special pseudogroup G of germs of holomorphic isometries of $(W, \beta|_W)$ and proceed to derive a representation of Γ on the Lie algebra $\mathfrak{g}$ of G.

Proposition (2.3.1). — *There exists a nontrivial pseudogroup $G \subset G_0$ of holomorphic isometries of $(W, \beta|_W)$ (which is identified with*

$(S, \omega|_S)$) consisting of restrictions of affine unitary transformations, i.e., elements of $U(H) \ltimes H$.

Before proving Proposition (2.3.1) we study the relationship between elements of $U(H) \ltimes H$ and their restrictions to the locally closed finite-dimensional complex submanifold W. Denote by $\mathcal{B}(H)$ the algebra of bounded operators on H. We are using here the strong topology on $\mathcal{B}(H)$, where a sequence (P_i) of bounded operators converges strongly to $P \in \mathcal{B}(H)$ if and only if $\|P_i(\eta) - P(\eta)\| \to 0$ for every $\eta \in H$. In the strong topology the subgroup $U(H) \subset \mathcal{B}(H)$ is a closed subgroup.

LEMMA (2.3.1). — *Consider the restriction map ρ on $U(H) \ltimes H$ given by $\rho(T) = T|_W$. Then, ρ is injective. Furthermore, for a sequence (T_i) in $U(H) \ltimes H$, $\rho(T_i)$ converges uniformly on W to $\rho(T)$, $T \in U(H) \ltimes H$, as H-valued holomorphic maps if and only if for every $p \in H$, $T_i(p)$ converges to $T(p)$ in H, where the latter is given the norm topology.*

Proof. — Recall that $\Phi : \Gamma \to U(H)$ is irreducible. Let $V \subset H$ be the linear subspace generated by $\{\widetilde{\nu}(x)(\eta) : x \in \widetilde{X}, \eta \in T_x(X)\}$. Then, V is invariant under $\Phi(\Gamma)$, so that the topological closure $\overline{V}$ must be H, by irreducibility. Let $V' \subset V$ be the linear subspace generated by $\{\widetilde{\nu}(x)(\eta) : x \in \widetilde{S}_0, \eta \in T_x(S_0)\}$, noting that $F(S_0) = W$. V' remains unchanged if we replace S_0 by a slightly thickened neighborhood B which is a product neighborhood with base S_0 with respect to the holomorphic foliation $\widetilde{\mathcal{F}}$. In fact, $V' \subset H$ is nothing other than the linear subspace generated algebraically by vectors tangent to W. By the Identity Theorem for holomorphic functions applied to the open subset $B \subset \widetilde{X}$, a vector $\xi \in H$ is orthogonal to V' if and only if it is orthogonal to V, so that $\overline{V'} = H$. Suppose $\rho(T) = \mathrm{id}$, where $T(p) = U(p) + v, v \in H$. Then, $U(\eta) = \eta$ for any η tangent to W at some point. Since $V' \subset H$ is dense, $U \equiv \mathrm{id}$. It follows that $v = 0$ and $T \equiv 0$, so that ρ is injective.

Suppose now $\|T_i(p) - T(p)\| \to 0$ for every $p \in H$. This applies in particular to $p \in W$. Furthermore, for $p, q \in W$, we always have $\|T(p) - T(q)\| = \|p - q\|$ for any $T \in U(H) \ltimes H$. By the argument in the standard proof of Montel's theorem it follows that $\rho(T_i)$ converges uniformly to $\rho(T)$ as holomorphic maps, noting that $W \subset H$ is relatively compact.

Conversely, suppose $\rho(T_i)$ converges to $\rho(T)$ uniformly as H-valued holomorphic maps in W. Write $T_i(p) = U_i(p) + v_i$ and $T(p) = U(p) + v$.

For η tangent to W at some point q, we conclude from Cauchy estimates that $\|U_i(\eta) - U(\eta)\| = \|\partial_\eta(h_i - h)(q)\| \to 0$. Since $\overline{V'} = H$ we have $\|U_i(\eta) - U(\eta)\| \to 0$ for every $\eta \in H$. Applying this to one point $\eta = p \in W$ we conclude that $\|v_i - v\| \to 0$. Thus, $\|T_i(\eta) - T(\eta)\| = \|(U_i(\eta) - U(\eta)) + (v_i - v)\| \to 0$ for any $\eta \in H$, as desired. □

Proof of Proposition (2.3.1). — Let $G' \subset G_0$ be the subset $G' = G_0 \cap \rho((U(H) \ltimes H))$ Then, G' contains the countably infinite subset $\{\theta_i\}$. We claim that $G' \subset G$ is a closed subset. Suppose $h_i \in G_0$, $h_i = \rho(T_i)$ and h_i converges to $h \in G_0$. Writing $T_i(p) = U_i(p) + v_i$ and taking η to be any vector tangent to W at some point q, $U_i(\eta)$ converges to $\partial_\eta h(q)$ in H, by Cauchy estimates. From $\overline{V'} = H$ in the notations above it follows in particular that $\{U_i(q)\}$ is a convergent sequence in H, so that $v_i = h_i(q) - U_i(q)$ converges in H to some v. Thus $U_i|_W = h_i - v_i$ converges uniformly on W as H-valued holomorphic functions. As explained this implies that $U_i(p)$ converges in H for each $p \in H$. Being uniformly bounded, the sequence U_i admits a weak limit $L \in \mathcal{B}(H)$. Observe that for $q \in W$, $\|U_i(q) - (h(q) - v)\| \to 0$ while $U_i(q)$ converges to $L(q)$ weakly. From uniqueness of weak limits in H it follows that $L(q) = h(q) - v$, for $q \in W$. From $\overline{V'} = H$ and Cauchy estimates we conclude that U_i converges to L strongly. Since $U(H) \subset \mathcal{B}(H)$ is closed in the strong topology, $L \in U(H)$. Write now U for L. We deduce that $T_i \in U(H) \ltimes H$ converges strongly to $T \in U(H) \ltimes H$ defined by $T(p) = U(p) + v$, so that $h(q) = T(q)$ for $q \in W$, *i.e.*, $h = T|_W = \rho(T)$, and $G' \subset G_0$ is closed.

We have thus obtained a closed pseudogroup $G' \subset G_0$. Let G be the identity component of G'. Since a closed pseudogroup of a (real-analytic) Lie pseudogroup is a (real-analytic) Lie pseudogroup, G is a nontrivial pseudogroup of holomorphic isometries of $(W, \beta|_W)$. □

Remarks. — Identifying $U(H) \ltimes H$ topologically with $U(H) \times H$, where $U(H)$ is endowed the strong topology and H the norm topology, the topology on $U(H) \ltimes H$ is equivalent to the topology of uniform convergence on compact subsets of H.

We proceed to consider the action of Γ on G and hence on the Lie algebra $\mathfrak{g}$ of G. We may assume that G contains $\{\theta_i\}$. Recall that $x_0 \in S_0$ is the base point and $F(x_0) = p_0$ is the base point of W. Let $\gamma \in \Gamma$ and C_γ be a smooth path joining x_0 to $\gamma^{-1}(x_0)$. We may assume that F is an immersion on a tubular neighborhood N_γ of C_γ. We choose N_γ such that $\overline{F(N_\gamma)} \subset H$ is compact. Let $\theta \in G$, of the form $\theta = T|_W$, $T \in U(H) \ltimes H$. Given any

$\epsilon > 0$, by Lemma (2.3.1) we have $\sup\{\|T(p)-p\| : p \in \overline{F(N_j)}\} < \epsilon$ whenever θ is sufficiently close to the identity in G. $\widehat{\Phi}(\gamma)^{-1}$ maps the germ of W at x_0 to the germ of $\widehat{\Phi}(\gamma)^{-1}(W) := W_\gamma$ at $\widehat{\Phi}(\gamma)^{-1}(p_0)$. For the sake of argument we may assume that F is an embedding on N_γ, so that $F(N_\gamma)$ is a tubular neighborhood of $F(C_\gamma)$. (Otherwise we should think of $F(N_\gamma)$ as the image of a Riemann domain Ω spread over it and reason similarly.) Note that W and W_γ are open subsets of the $F(T_\gamma)$. If we choose ϵ small enough we have $T(\widehat{\Phi}_j^{-1}(p_0)) \in W_\gamma$, while $T|_{F(T_\gamma)}$, which maps the germ of W at p_0 to the germ of W at $T(p_0)$, must map the germ of W_γ at $\widehat{\Phi}(\gamma)^{-1}(p_0)$ to the germ of W_γ at $T(\widehat{\Phi}(\gamma)^{-1}(p_0))$. This is the case by the principle of analytic continuation. Now $\widehat{\Phi}(\gamma)$ maps W_γ back to W, and we have obtained an element $\widehat{\Phi}(\gamma)\theta\widehat{\Phi}(\gamma)^{-1} = \widehat{\Phi}(\gamma)T\widehat{\Phi}(\gamma)^{-1}|_W$.

Thus, the nontrivial pseudogroup G of holomorphic isometries of $(W, \beta|_W)$ is invariant (as a germ at the identity map) under conjugation by $\widehat{\Phi}$. It induces therefore a Lie-algebra automorphism $\Theta(\gamma)$ of the Lie algebra $\mathfrak{g}$ of G. Obviously $\Theta : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ is a group homomorphism, where $\mathrm{Aut}(\mathfrak{g})$ is the group of Lie algebra automorphisms of $\mathfrak{g}$. The upshot is that, whenever $(X, \omega, \mathcal{F})$ is not factorizable, we have obtained a *finite-dimensional* linear representation Θ. To make use of Θ, we need first of all to establish that Θ is nontrivial. In fact, we have

PROPOSITION (2.3.2). — *Let X be a compact Kähler manifold with fundamental group Γ, $\Phi : \Gamma \to U(H)$ be an irreducible unitary representation on some Hilbert space H such that for every subgroup $\Gamma' \subset \Gamma$ of finite index, Γ' does not fix any non-zero vector in H. Let E_Φ be the associated locally constant Hilbert bundle on X, $\nu \in \Gamma(X, \Omega(E_\Phi))$ be a non-zero E_Φ-valued d-closed holomorphic 1-form, and $\widetilde{\nu}$ be the lifting of ν to a Φ-equivariant H-valued holomorphic 1-form on $\widetilde{X}$. Let $F : \widetilde{X} \to H$ be an integral of $\widetilde{\nu}$, i.e., $dF = \widetilde{\nu}$. Assume that the associated semi-Kähler structure $(X, \omega, \mathcal{F})$ is not factorizable. Let $G \subset \rho(U(H) \ltimes H)$ be the nontrivial pseudogroup of holomorphic isometries of the semi-Kähler structure $(W, \beta|_W)$ as described, $\mathfrak{g}$ be the corresponding Lie algebra, and $\Theta : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ be the induced representation as defined. Then, the image $\Theta(\Gamma) \subset \mathrm{Aut}(\mathfrak{g})$ is infinite.*

Proof. — Suppose otherwise. Replacing Γ by a subgroup of finite index we may assume without loss of generality that Θ is trivial. It follows that there exists a sequence of elements $g_k \in \Gamma$ such that $\Phi(g_k)$ converges on compact subsets of H to id in $U(H)$, $\widehat{\Phi}(g_k) := \widehat{g}_k$ converges on compact subsets of H to $(\mathrm{id}, 0)$ in $U(H) \ltimes H$, $\widehat{g}_k \neq (\mathrm{id}, 0)$, with the property that $\widehat{\gamma}\widehat{g}_k\widehat{\gamma}^{-1} = \widehat{g}_k$ for each k. In other words,

$$\widehat{\gamma}\widehat{g}_k = \widehat{g}_k\widehat{\gamma} \quad \text{for every} \quad \gamma \in \Gamma. \tag{1}$$

Fix one $g = g_k$, and write $\widehat{\Phi}(\gamma)(h) = \Phi(\gamma)(h) + c_\gamma$ for $h \in H$. The commutativity $\widehat{\gamma}\widehat{g}_k = \widehat{g}_k\widehat{\gamma}$ translates into

$$\Phi(\gamma)\Phi(g) = \Phi(g)\Phi(\gamma), \tag{2f}$$

$$(\Phi(\gamma) - 1)c_g = (\Phi(g) - 1)c_\gamma \quad \text{for every} \quad \gamma \in \Gamma. \tag{3}$$

Let $H_0 \subset H$ be the closed proper subspace of vectors fixed by $\Phi(g)$. By (2), H_0 is invariant under Γ, contradicting the irreducibility of H unless $H_0 = 0$. In other words, $\mathrm{Ker}(\Phi(g) - 1) = 0$. Suppose for the time being $c_g = 0$. Then, it follows from (3) that $c_\gamma = 0$ for every $\gamma \in \Gamma$. By Lemma (1.2.1) F is constant, *i.e.* $\widetilde{\nu} = dF = 0$, a contradiction.

We proceed to remove the assumption $c_g = 0$. In general, replacing F by $F + c$, c_g is changed to $c_g + c - \Phi(g)c$, which vanishes if and only if $(\Phi(g) - 1)c = c_g$. Given c_g, the latter equation is not necessarily solvable. Consider now the H-valued holomorphic function $E(x) := (\Phi(g) - 1)F(x)$ on $\widetilde{X}$. We claim that dE is Φ-equivariant, by (2). In fact, for every $\gamma \in \Gamma$,

$$\begin{aligned} E(\gamma x) &= (\Phi(g) - 1)F(\gamma x) \\ &= (\Phi(g) - 1)(\Phi(\gamma)F(x) + c_\gamma) \\ &= \Phi(\gamma)(\Phi(g) - 1)F(x) + (\Phi(g) - 1)c_\gamma \\ &= \Phi(\gamma)E(x) + (\Phi(g) - 1)c_\gamma. \end{aligned} \tag{4}$$

Thus, E is an integral of some $\mu \in \Gamma(X, \Omega(E_\Phi))$ whose lifting $\widetilde{\mu}$ to $\widetilde{X}$ agrees with dE. Write $d_\gamma = (\Phi(g) - 1)c_\gamma$. Then, the equation $(\Phi(g) - 1)c = d_g$ is trivially solvable with $c = c_g$. We have

$$\begin{aligned} E(\gamma x) + c_g &= \Phi(\gamma)E(x) + (\Phi(g) - 1)c_\gamma + c_g \\ &= \Phi(\gamma)(E(x) + c_g) + (\Phi(g) - 1)c_\gamma - (\Phi(\gamma) - 1)c_g \\ &= \Phi(\gamma)(E(x) + c_g), \end{aligned} \tag{5}$$

by (3). Thus, $E + c_g \equiv (\Phi(g) - 1)F + c_g$ is a constant vector in H, by Lemma (1.1.1). Since $\mathrm{Ker}(\Phi(g) - 1) = 0$, this contradicts with $dF = \widetilde{\nu} \not\equiv 0$. The proof of Proposition (2.3.2) is complete. □

(2.4) We have thus far established the alternatives (A) that $(X, \omega, \mathcal{F})$ is factorizable, and (B) that there is a linear representation of Γ with infinite image in a real Lie group. For alternative (B) let N be the identity component of the Zariski-closure of $\Theta(\Gamma)$. Denote by $\mathrm{Rad}(N)$ the solvable radical of N. Let $\Gamma' \subset \Gamma$ be the subgroup of finite index which is mapped into N under Θ. If N is not solvable, we obtain by composition a group homomorphism $\overline{\Theta} : \Gamma' \to N/\mathrm{Rad}(N)$ such that the image $\overline{\Theta}(\Gamma)$ is Zariski dense. By Zuo [Zu] some unramified finite cover of Γ admits a meromorphic fibration over a projective manifold of general type.

On the other hand, if N is solvable, then the commutator $[N, N] \neq N$, and we obtain by composition a nontrivial representation $\widehat{\Theta} : \Gamma' \to N/[N, N]$ into an abelian Lie group. Denote by X' the unramified finite cover corresponding to the subgroup $\Gamma' \subset \Gamma$ of finite index. From the existence of a nontrivial abelian representation of Γ' we conclude that the Albanese map α on X' is nontrivial. By the work of Ueno [Ue] on subvarieties of abelian varieties and a generalization to the compact Kähler case by Campana [Ca] we conclude that either the Albanese map $\alpha : X' \to \mathrm{Alb}(X')$ is surjective, or its image $\alpha(X') \subset \mathrm{Alb}(X')$ is a bundle of isomorphic compact complex tori over a projective variety Z_0 of general type.

Consider first the case when $\alpha : X' \to \mathrm{Alb}(X')$ is surjective. Suppose α is equidimensional. Then, either α is an isomorphism, or it is a finite map ramified somewhere. By the work of Kawamata-Viehweg [KV] on branched covers of Abelian varieties and its generalization to the compact Kähler case by Campana [Ca], some nonsingular Kähler modification $\widehat{X}$ of X' admits a holomorphic fibration over a variety of general type in such a way that generic fibers are isomorphic compact complex tori. If on the other hand α is not equidimensional, then we can form the leaf space Y of α and obtain thus a modification $\mu : \widehat{X} \to X'$ and holomorphic maps $\tau : \widehat{X} \to Y$, $\beta : Y \to \mathrm{Alb}(X')$, $\alpha \circ \mu = \beta \circ \tau$, such that τ is a holomorphic fibration and β is a finite map onto $\mathrm{Alb}(X')$. When β is ramified somewhere we can return to the equidimensional case to obtain a meromorphic fibration of Y and hence of X' over a projective manifold of general type. Otherwise we have a meromorphic fibration of X' over a compact complex torus (which is actually a holomorphic fibration by Hartogs extension).

When $\alpha' : X' \to \mathrm{Alb}(X')$ is not surjective, we have a surjective holomorphic map $\theta : X' \to Z_0$ onto some projective variety Z_0 of general type. Passing to leaf spaces of θ we obtain a meromorphic fibration over a nonsingular Kähler modification Z of some branched cover of Z_0, and Z is of general type, hence also projective.

Summing up, we conclude that

PROPOSITION (2.4.1). — *Let X be a compact Kähler manifold admitting a nontrivial holomorphic 1-form ν with coefficients twisted by an irreducible unitary representation Φ. Then, either (A) $(X, \omega, \mathcal{F})$ is factorizable; or (B) some unramified finite cover X' of X admits a meromorphic fibration over a compact complex torus or a projective manifold of general type.*

For later reference we state the following result pertaining to Case (B) in the form of a proposition, as follows.

PROPOSITION (2.4.2). — *Let X be a compact Kähler manifold whose fundamental group Γ admits a finite-dimensional linear representation Φ such that $\Phi(\Gamma)$ is infinite. Then, some unramified finite cover of X admits a meromorphic fibration over a compact complex torus or a projective manifold of general type.*

(2.5) We consider in this section the factorizable case, *i.e.*, Case (A) of Proposition (2.4.1). In this case, there is a nonsingular modification $\mu : \widehat{X} \to X$, a compact Kähler manifold Z and a surjective holomorphic map $\sigma : \widehat{X} \to Z$ such that the meromorphic foliation $\mu^*\mathcal{F}$ agrees with the meromorphic foliation on $\widehat{X}$ defined at generic points by the fibers of σ. Let ℓ be the rank of the meromorphic distribution $T(\mathcal{F})$ at a generic point x of X. Let $A \subset Z$ be a proper subvariety of Z such that σ is a regular family of connected ℓ-dimensional compact complex manifolds on $Z - A$. Let $A_1, \cdots, A_p$ be an enumeration of the finite number of irreducible components of A of codimension 1 in Z. For each k there is a Zariski-dense open subset A'_k of A_k such that (a) A'_k is smooth; (b) the fibers $\sigma^{-1}(x)$ for $x \in A'_k$ are pure $\ell-$dimensional, and (c) $\sigma|_{\sigma^{-1}(A'_k)}$ is a submersion on the set of smooth points of $\sigma^{-1}(A'_k)$, to be denoted by $\mathrm{Reg}(\sigma^{-1}(A'_k))$. Fix any A_k. Let $z_k \in A'_k$ and s be a local holomorphic defining function of A_k in a connected neighborhood W of z_k in A'_k. Then, the zero set of σ^*s has, say, n_k irreducible components. Replacing A'_k by a Zariski-dense open subset if necesary we may assume that, choosing W to be sufficiently small, each connected component of the graph of $\sigma|_{\mathrm{Reg}(\sigma^{-1}(A'_k \cap W))}$ is the graph of a regular holomorphic family of connected complex manifolds over $A'_k \cap W$. In particular, $\sigma^{-1}(z_k)$ consists precisely of n_k components for each $z_k \in A'_k$. Denote by m_k the greatest common divisor of the multiplicities of the n_k irreducible components. We say that σ has no multiple fibers outside a set of codimension $\geqslant 2$ if each m_k is equal to 1.

For the modification $\mu : \widehat{X} \to X$, $\mu_* : \pi_1(\widehat{X}) \cong \pi_1(X)$, so that Φ can be taken to be defined on $\pi_1(\widehat{X})$. To simplify notations we will from now on replace $\widehat{X}$ by X and use $\mathcal{F}$ to denote $\mu^*(\mathcal{F})$, ν to denote $\mu^*\nu$, etc. Recall that the meromorphic foliation $\mathcal{F}$ on X is defined by the meromorphic distribution which at a generic point $x \in X$ is given by $T_x(\mathcal{F}) = \mathrm{Ker}(\nu(x))$ for the holomorphic 1-form with twisted coefficients $\nu \in \Gamma(X, \Omega(E_\Phi))$. Write $B := \sigma^{-1}(A)$. Consider the regular family $\sigma|_{X-B} : X - B \to Z - A$. Denote the fibers over $Z - A$ by $E_z = \sigma^{-1}(z)$. E_z agrees with the leaf L_x of the holomorphic foliation $\mathcal{F}|_{X-B}$ through any point $x \in E_z$. Fix an arbitrary $z_0 \in Z - A$ and a sufficiently small contractible connected open neighborhood U of z_0 in $Z - A$. The regular family $\sigma|_{\sigma^{-1}(U)} : \sigma^{-1}(U) \to U$ is diffeomorphically trivial, and $\pi_1(\sigma^{-1}(U))$ is canonically isomorphic to $\pi_1(E_{z_0})$ by retraction, taking as base points for both fundamental groups an arbitrary point x_0 on the fiber E_{z_0}. Recall that $\pi : \widetilde{X} \to X$ is the universal covering map. Let $D \subset \widetilde{X}$ be a connected component of $\pi^{-1}(\sigma^{-1}(U))$ and fix a base point $\widetilde{x}_0 \in D$ such that $\pi(\widetilde{x}_0) = x_0$. Let $\Lambda \subset \Gamma = \pi_1(X; x_0)$ be the canonical image of $\pi_1(E_{z_0}; x_0)$ in Γ. Since $\pi_1(\sigma^{-1}(U)) \cong \pi_1(E_{z_0})$ canonically, Λ preserves the connected fibers $\pi^{-1}(E_z) \cap D := \widetilde{E}_z$ for any $x \in U$. Here $\widetilde{E}_z$ agrees with the leaf $\widetilde{L}_{\widetilde{x}}$ of the holomorphic foliation $\widetilde{\mathcal{F}}|_{\widetilde{X}-\widetilde{B}}$ through any point $\widetilde{x} \in \widetilde{E}_z$, where $\widetilde{B} := \pi^{-1}(B)$. For any $\lambda \in \Lambda$ and $\widetilde{x} \in D$, we have $F(\lambda\widetilde{x}) = \widehat{\Phi}(\lambda)F(\widetilde{x})$. From the definition of $\mathcal{F}$, F is constant on each $\widetilde{E}_z = \widetilde{L}_{\widetilde{x}}$, so that $\widehat{\Phi}(\lambda)$ must be an affine unitary transformation on H fixing every point of the image of $F(D)$. Since the topological closure of the linear span of $F(D)$ contains the image of $\widetilde{\nu}(z)$ for each $z \in D$ and Φ is irreducible, we conclude that $F(D)$ spans H topologically. As a consequence, $\widehat{\Phi}|_\Lambda \equiv \mathrm{id}$. We note further the following well-known fact.

LEMMA (2.5.1). — *$\Lambda \subset \Gamma$ is a normal subgroup.*

Proof. — Recall that the meromorphic foliation $\widetilde{\mathcal{F}}$ is invariant under Γ. On the other hand, for any $\lambda \in \Lambda$ and any $z \in U$, $\lambda(\widetilde{E}_z) = \widetilde{E}_z$. Since any leaf of $\widetilde{\mathcal{F}}|_{\widetilde{X}-\widetilde{B}}$ can be joined to a leaf contained in D by a continuous one-parameter family of leaves it follows that λ actually preserves each leaf of $\widetilde{\mathcal{F}}|_{\widetilde{X}-\widetilde{B}}$. As a consequence, for any $\gamma \in \Gamma$, $\lambda(\gamma(\widetilde{E}_z)) = \gamma(\widetilde{E}_z)$ and thus $\gamma^{-1}\lambda(\gamma(\widetilde{E}_z)) = \widetilde{E}_z$, showing that $\gamma^{-1}\lambda\gamma \in \Lambda$. In other words, $\Lambda \subset \Gamma$ is a normal subgroup, as asserted. □

Remarks. — Recall that a nonsingular modification of a complex manifold induces an isomorphism on fundamental groups. Lemma (2.5.1) can thus be formulated without passing to a modification $\widehat{X}$ of X. For the factorizable semi-Kähler structure $(X, \omega, \mathcal{F}, V)$ let L be a generic leaf of $\mathcal{F}|_{X-V}$. Then, Λ can be defined as the image of the fundamental group $\pi_1(\overline{L})$ for the topological closure of $\overline{L}$ of L in X.

Let $\rho : X^{\#} \to X$ be the Galois cover corresponding to the normal subgroup $\Lambda \subset \Gamma = \pi_1(X)$. Then, the quotient group $\Theta := \Gamma/\Lambda$ acts as a group of covering transformations of $X^{\#}$ over X acting transitively on the fibers $\rho^{-1}(x)$ for any $x \in X$. We may write $X = X^{\#}/\Theta$. Since $\widehat{\Phi}|_\Lambda \equiv \mathrm{id}$ the primitive $F : \widetilde{X} \to H$ of the d-closed holomorphic 1-form $\widetilde{\nu}$ with twisted coefficients descends to $F^{\#} : X^{\#} \to H$.

Consider the composite map $\sigma \circ \rho : X^{\#} \to Z$. The generic fiber of $\sigma \circ \rho$ is a countably infinite union of compact complex submanifolds. Denote by E an irreducible component of some generic fiber and let $Z^{\#}$ be the normalization of the irreducible component of the Barlet space of $X^{\#}$ whose generic point is the element $[E]$ corresponding to the compact analytic cycle E. Any covering transformation $\theta \in \Theta$ induces in a canonical way a biholomorphic automorphism of $Z^{\#}$. Since $X = X^{\#}/\Theta$ and $Z^{\#}$ is a cycle space on $X^{\#}$, as expected we have

LEMMA (2.5.2). — *Θ acts properly discontinuously on $Z^{\#}$. Furthermore, $Z^{\#}/\Theta$ is compact.*

Proof. — The lemma requires some justification since the canonical map $X^{\#} \to Z^{\#}$ is only meromorphic. To prove the lemma let K be a compact subset of $X^{\#}$ such that $\bigcup_{\theta\in\Theta} \theta K = X^{\#}$. Let $Q \subset Z^{\#}$ be the set of all cycles belonging to $Z^{\#}$ having non-empty intersection with K and $\widehat{K} \subset X^{\#}$ be the union of such cycles. Then, from the constancy of volumes of cycles belonging to $Z^{\#}$ with respect to some Kähler metric on $X^{\#}$ invariant under Θ it follows that both Q and $\widehat{K}$ are compact. Clearly $\bigcup_{\theta\in\Theta} \theta Q = Z^{\#}$, *i.e.*, $Z^{\#}/\Theta$ is compact. Moreover, there exist only a finite number of $\theta \in \Theta$ such that $\theta\widehat{K} \cap \widehat{K} \neq \emptyset$, from which it follows that there exist only a finite number of $\theta \in \Theta$ such that $\theta Q \cap Q \neq \emptyset$. In other words, Θ acts properly discontinuously on $Z^{\#}$.

Write $B^{\#} := \rho^{-1}(B)$ for the Galois cover $\rho : X^{\#} \to X$. Let $W \subset Z^{\#}$ be the Zariski-dense open subset consisting of cycles lying inside $X^{\#} - B^{\#}$. Note that we have a canonical map $X^{\#} - B^{\#} \to W$, as a regular family of compact Kähler manifolds over W. This canonical map only extends to a meromorphic map $X^{\#} \to Z^{\#}$. By Lemma (2.5.2) $\Theta = \Gamma/\Lambda$ acts properly discontinuously on $Z^{\#}$. From the construction, for $z \in Z-A$, each nontrivial $\theta \in \Theta$ permutes connected components of $(\sigma \circ \rho)^{-1}(z)$ without fixing any component, so that the action of Θ on W is free from fixed points, in such a way that the canonical map $\varphi_0 : W \to Z - A$ descends to a biholomorphism $W/\Theta \cong Z - A$. We have

LEMMA (2.5.3). — *The canonical holomorphic map $\varphi_0 : W \to Z-A$ extends holomorphically to a Θ-invariant holomorphic map $\varphi : Z^{\#} \to Z$.*

Proof. — Since $Z^{\#}$ is normal and $\varphi : W \to Z - A$ is Θ-invariant to prove the lemma it suffices to show that φ_0 extends to a continuous map on $Z^{\#}$. Let $[C] \in Z^{\#}$ be an element corresponding to a compact analytic cycle C on $X^{\#}$. The set-theoretic image $\rho(C)$ is then projected under the holomorphic fibration $\sigma : X \to Z$ to a unique point, since it is the case for a dense open subset $W \subset Z^{\#}$. This defines an extension $\varphi : Z^{\#} \to Z$. Since convergence in the space of cycles $Z^{\#}$ implies convergence as subvarieties and σ is holomorphic (hence continuous), φ is continuous, as desired. □

Since $\varphi : Z^{\#} \to Z$ is Θ-invariant we have obtained a holomorphic map $\tau : Z^{\#}/\Theta \to Z$ which maps the Zariski-dense open set $W/\Theta \subset Z^{\#}/\Theta$ onto $Z - A$. In other words, τ is a modification. Recall that the primitive $F : \widetilde{X} \to H$ of $\widetilde{\nu}$ descends to $F^{\#} : X^{\#} \to H$. Since $F^{\#} : X^{\#} \to H$ is constant on the support of each cycle $[C]$ belonging to $Z^{\#}$, it descends to a continuous map $G : Z^{\#} \to H$ which is holomorphic on the Zariski-dense open subset $W \subset Z^{\#}$. Since $Z^{\#}$ is normal, G is holomorphic.

As $\widehat{\Phi}|_{\Lambda} \equiv \mathrm{id}$, Φ induces a representation $\overline{\Phi} : \Theta = \Gamma/\Lambda \to U(H)$, and $\widehat{\Phi}$ induces a representation $\widehat{\overline{\Phi}} : \Theta = \Gamma/\Lambda \to U(H) \ltimes H$ such that $G(\theta z) = \widehat{\Psi}_0(G(z))$ for any $\theta \in \Theta$. Assume now that Θ acts on $Z^{\#}$ without fixed points. Let $\alpha : Z_1 \to Z^{\#}/\Theta$ be a nonsingular Kähler modification. Since $\alpha_* : \pi_1(Z_1) \to \pi_1(Z^{\#}/\Theta)$ is surjective and $(\tau \circ \alpha)_* : \pi_1(Z_1) \to \pi(Z)$ is an isomorphism, α_* is actually an isomorphism. Let Z' be the normal cover of Z_1 corresponding to the normal subgroup $\Lambda \subset \pi_1(Z) \cong \pi_1(Z_1)$. $G : Z^{\#} \to H$ lifts to $G' : Z' \to H$, in such a way that it satisfies the transformation rule $G'(\theta z') = \widehat{\overline{\Phi}}(\theta)(G'(z'))$ for any $\theta \in \Theta$ and any $z' \in Z'$. We have readily the following intermediate result.

PROPOSITION (2.5.1). — *Let X be a compact Kähler manifold admitting a nontrivial holomorphic 1-form ν with coefficients twisted by an irreducible unitary representation $\Phi : \Gamma := \pi_1(X) \to U(H)$ on a Hilbert space H. Suppose the semi-Kähler structure $(X, \omega, \mathcal{F}, V)$ defined by ν is* *factorizable. Denote by L a generic leaf on $X - V$ and $\overline{L}$ its topological closure in X. Suppose for the image Λ of $\pi_1(\overline{L})$ in $\pi_1(X)$, the quotient group $\Theta := \Gamma/\Lambda$ is residually finite. Then, replacing X by some unramified finite cover if necessary, some nonsingular Kähler modification $\widehat{X}$ of X admits a holomorphic fibration $\sigma : \widehat{X} \to Z$ onto a compact Kähler manifold Z and there exists a representation $\Psi : \pi_1(Z) \to U(H)$ such that $\Phi(\gamma) = \Psi(\sigma_*(\gamma))$ for every $\gamma \in \Gamma$. Furthermore, there exists a holomorphic 1-form κ on Z with coefficients twisted by Ψ such that $\nu = \sigma^*\kappa$.*

Proof. — Suppose for the time being that Θ acts on $Z^{\#}$ without fixed points. In what follows we identify $\pi_1(Z_1)$ with $\pi_1(Z)$ canonically. For some nonsingular Kähler modification $\widehat{X}$ of X, we have a holomorphic fibration $\sigma_1 : \widehat{X} \to Z_1$. In what follows X will stand for $\widehat{X}$. Define now $\Psi : \pi_1(Z_1) \to U(H)$ by $\Psi = \overline{\Phi} \circ \delta$ for the canonical homomorphism $\delta : \pi_1(Z_1) \to \Theta$. Let $\beta : \widetilde{X} \to Z'$ be the canonical map induced by σ. Recall the holomorphic map $G' : Z' \to H$ defined before which satisfies $G' \circ \beta = F$. Then dG' is an H-valued holomorphic 1-form on Z' which is the lifting $\widetilde{\kappa}$ of a holomorphic 1-form $\kappa \in \Gamma(Z_1, \Omega(E_\Psi))$. We now take Z in the proposition to be Z_1, and σ to be σ_1. From $G' \circ \beta = F$ it follows that $\nu = \sigma^*\kappa$, as desired.

Consider now the general case when Θ may act on Z' with fixed points. Since Θ acts properly discontinuously on Z' and $Z = Z'/\Theta$ is compact, elements of finite order in Θ break into at most a finite number of conjugacy classes, with representatives $\theta_1 = \mathrm{id}, \theta_2, \cdots, \theta_m$, say. By assumption Θ is residually finite. Let now $\Theta_0 \subset \Theta$ be a subgroup of finite index such that $\theta_2, \cdots, \theta_m \notin \Theta_0$. Replacing Θ_0 by the intersection of $\gamma\Theta_0\gamma^{-1}$ for a full set of representatives γ of the finite coset space Θ/Θ_0, we may assume without loss of generality that $\Theta_0 \subset \Theta$ is a *normal* subgroup. Then, Θ_0 does not contain any nontrivial elements of finite order. Let $\Gamma_0 \subset \Gamma$ be the normal subgroup of finite index which is $q^{-1}(\Theta_0)$ for the quotient homomorphism $q : \Gamma \to \Theta$. Then, replacing X by the unramified finite cover corresponding to Γ_0, Θ by Θ_0 and Z by a nonsingular Kähler modification of $Z^{\#}/\Theta_0$, we are back to the case where Θ acts on $Z^{\#}$ without fixed points. The proof of Proposition (2.5.1) is complete. □

Remarks. — The procedure for getting factorization theorems by passing to torsion-free subgroups of finite index is well-known. This was for instance used in Kollár ([Ko], Theorem 4.5) in relation to Shafarevich maps for projective manifolds with residually finite fundamental groups. As any finitely generated matrix group is residually finite, there is no difficulty in finding suitable torsion-free subgroups of finite index in the context of fibration theorems arising from finite-dimensional representations, (*cf.* Zuo [Zu], Theorem 1).

We note further the following topological fact on $Z^{\#}$.

LEMMA (2.5.4). — *The cycle space $Z^{\#}$ is simply connected.*

Proof. — Let $\alpha : P \to X^{\#}$ be a nonsingular modification such that the canonical meromorphic map $X^{\#} \to Z^{\#}$ lifts to a holomorphic map $h : P \to Z^{\#}$. Note that $\alpha_* : \pi_1(P) \cong \pi_1(X^{\#}) \cong \Lambda$. Since h is a surjective proper holomorphic map with connected fibers, $h_* : \pi_1(P) \to \pi_1(Z^{\#})$ is surjective. On the other hand, for a generic fiber E of $h : P \to Z^{\#}$, the canonical homomorphism $\pi_1(E) \to \pi_1(P) \cong \Lambda$ is an isomorphism, so that $h_*(\pi_1(P)) = \{1\}$. Thus, $\pi_1(Z^{\#}) = \{1\}$, *i.e.*, $Z^{\#}$ is simply connected, as desired. □

(2.6) When Θ acts on $Z^{\#}$ with fixed points, the holomorphic 1-form with twisted coefficients $\nu \in \Gamma(X, \Omega(E_\Phi))$ may not descend to a holomorphic 1-form with twisted coefficients. Even the representation $\Phi : \Gamma \to U(H)$ may not descend. Recall in the notations of the first paragraph of (2.5) that $\sigma|_{X-B} : X - B \to Z - A$ is a regular family of compact Kähler manifolds and that $A_1, \cdots, A_p$ is an enumeration of the finite number of irreducible components of A of codimension 1 in Z. Let $\Phi_1 : \pi_1(X - B) \to U(H)$ be the unitary representation defined by $\Phi_1 = \Phi \circ \chi$, where $\chi : \pi_1(X - B) \to \pi_1(X) = \Gamma$ is the canonical homomorphism, which is surjective. Then, Φ_1 descends to a representation $\Psi_1 : \pi_1(Z-A) \to U(H)$, and $\nu|_{X-B} \in \Gamma(X-B, \Omega(E_{\Phi_1}))$ descends to some holomorphic 1-form with twisted coefficients $\xi \in \Gamma(Z - A, \Omega(E_{\Psi_1}))$. The fundamental group $\pi_1(Z-A)$ is generated by $\pi_1(Z)$ and a finite number of fundamental loops $\epsilon_1, \cdots, \epsilon_p$, where each $\epsilon_k, 1 \leqslant k \leqslant p$, wraps around the divisor A_k.

Fix $k, 1 \leqslant k \leqslant p$. Let $B_{k,1}, \cdots, B_{k,n_k}$ be the irreducible components of $\sigma^{-1}(A_k)$. Let $z_k \in A_k$ be a generic smooth point and $(z; w)$, $w = (w_1, \cdots, w_{s-1})$, be a system of holomorphic local coordinates on a neighborhood U_k of z_k such that $A_k \cap U_k$ is the zero-divisor of z. Let $x_{k,\ell} \in B_{k,\ell}$, $1 \leqslant \ell \leqslant n_k$, be generic smooth points on $B_{k,\ell}$. Suppose $\sigma^* z$ vanishes at $x_{k,\ell}$ to the order $q_{k,\ell}$. Let $\zeta_{k,\ell}$ be a holomorphic function on a sufficiently small contractible coordinate neighborhood $V_{k,\ell}$ of $x_{k,\ell}$ such that $\zeta_{k,\ell}^{q_{k,\ell}} = \sigma^* z$. Writing w for $\sigma^* w$ we may assume that $(\zeta_{k,\ell}; w)$ constitutes a system of holomorphic local coordinates on a holomorphic cross-section $S_{k,\ell}$ of $V_{k,\ell}$ at $x_{k,\ell}$ with respect to the foliation $\mathcal{F}$, which is holomorphic at $x_{k,\ell}$. Recall the regular covering $\rho : X^{\#} \to X$ corresponding to the normal subgroup $\Lambda \subset \Gamma$ and $\Theta = \Gamma/\Lambda$. Let $W_{k,\ell} \subset \widetilde{X}$ be a connected component of $\rho^{-1}(V_{k,\ell})$. Write $\zeta = \zeta_{k,\ell}$. Use also $(\zeta; w)$ as holomorphic coordinates on $W_{k,\ell}$ and complete them to a system of holomorphic local coordinates $(\zeta; w')$, $w = (w_1, \cdots, w_{n-1})$ by adding holomorphic fiber coordinates $(w_s, \cdots, w_{n-1})$ for the holomorphic foliation $\mathcal{F}^{\#}|_{W_{k,\ell}}$, where $\mathcal{F}^{\#}$ is the lifting of $\mathcal{F}$ to $X^{\#}$. Recall that $\overline{\Phi} : \Theta \to U(H)$ resp. $\widehat{\overline{\Phi}} : \Theta \to U(H) \ltimes H$ are induced by $\Phi : \Gamma \to U(H)$ and $\widehat{\Phi} : \Gamma \to U(H) \ltimes H$. There is a canonical element $\theta_{k,\ell} \in \Theta$ such that $F^{\#}(e^{\frac{2\pi i}{q_{k,\ell}}}\zeta; w') = \widehat{\overline{\Phi}}(\theta_{k,\ell})F^{\#}(\zeta; w')$. We have $\theta_{k,\ell}^{q_{k,\ell}} = \mathrm{id}$. In relation to the representation $\Psi_1 : \pi_1(Z - A) \to U(H)$ and the associated representation $\widehat{\Psi}_1 : \pi_1(Z - A) \to U(H) \ltimes H$ we have $\widehat{\overline{\Phi}}(\theta_{k,\ell}) = \widehat{\Psi}_1(\epsilon_k)$. It follows that $\widehat{\Psi}_1(\epsilon_k^{m_k}) = \mathrm{id}$, where m_k denotes the multiplicity of the fiber $\sigma^{-1}(z_k)$, which is the greatest common divisor of $(q_{k,1}, \cdots, q_{k,n_k})$. As a consequence, if $m_k = 1$ for all $k, 1 \leqslant k \leqslant p$, *i.e.*, if there exist no multiple fibers of $\sigma : X \to Z$ outside a subvariety of codimension $\geqslant 2$, then $\widehat{\Psi}_1(\epsilon_k) = \mathrm{id}$ for each k. Since the kernel of the canonical homomorphism $\tau : \pi_1(Z - A) \to \pi_1(Z)$ is generated as a normal subgroup by $\epsilon_1, \cdots, \epsilon_p$, the homomorphism $\Psi_1 : \pi_1(Z - A) \to U(H)$ descends to $\Psi : \pi_1(Z) \to U(H)$, and $\widehat{\Psi}_1 : \pi_1(Z - A) \to U(H)$ descends to $\widehat{\Psi} : \pi_1(Z) \to U(H) \ltimes H$. Recall that we have $\xi \in \Gamma(Z - A, \Omega(E_{\Psi_1}))$, such that $\sigma^*\xi = \nu|_{X-B}$. The locally constant bundle E_{Ψ_1} of Hilbert spaces on $Z - A$ extends to the locally constant bundle E_Ψ on Z. We claim that ξ extends to a holomorphic section in $\Gamma(Z, \Omega(E_{\Psi_1}))$, as follows. $\xi|_{U_k - A_k}$ is now a holomorphic section of the trivial bundle $E_\Psi|_{U_k}$ over $U_k - A_k$. Take any ℓ, $1 \leqslant \ell \leqslant n_k$, then $\nu|_{S_{k,\ell} - B_{k,\ell}} = \sigma^*\xi|_{S_{k,\ell} - B_{k,\ell}}$. In terms of the holomorphic coordinates $(z; w)$ on U_k and $(\zeta; w)$ on $S_{k,\ell}$, writing ζ for $\zeta_{k,\ell}$, q for $q_{k,\ell}$, we have $z = \zeta^q$, $dz = q\zeta^{q-1}d\zeta$, *i.e.*, $d\zeta = \frac{1}{q}\frac{dz}{z^{\frac{q-1}{q}}}$ (where $z^{\frac{1}{q}}$ means some choice of the q-th root for $z \neq 0$). It follows in particular that absolute values of each coefficient of $\xi|_{U_k - A_k}$ can have at worst a fractional pole order (strictly less than 1). As a consequence, $\xi|_{U_k - A_k}$ extends holomorphically to U_k for each k, $1 \leqslant k \leqslant n_k$. By Hartogs extension ξ extends holomorphically to Z, and we will write $\xi \in \Gamma(Z, E_\Phi)$.

In general it is difficult to remove multiple fibers by passing to an unramified finite cover of X. Note that even when there are multiple fibers it may happen that ν vanishes on the divisors $B_{k,\ell}$ to the right orders so that ν descends to Z. As explained in (2.5) to show that ν descends to Z it is enough that Θ acts on $Z^{\#}$ without fixed points. In this case the real-analytic semi-Kähler form ω on X descends to a real-analytic semi-Kähler form on Z which is positive-definite on a Zariski-dense open subset. When there are multiple fibers we can *a priori* only assert that the semi-Kähler form $\omega|_{X-B}$ descends to a semi-Kähler form $\overline{\omega}$ on $Z - A$, positive-definite on a Zariski-dense open subset, and that $\overline{\omega}^m$ extends to Z as a singular Hermitian metric on the inverse of $K_Z \otimes [D]$ for some effective $\mathbb{Q}$-divisor D on Z. More precisely, we have

PROPOSITION (2.6.1). — *Let X be a compact Kähler manifold with fundamental group Γ. Let $\Phi : \Gamma \to U(H)$ be a unitary representation on a Hilbert space H. Suppose there exists a nontrivial holomorphic 1-form $\nu \in \Gamma(X, \Omega(E_\Phi))$ with twisted coefficients and that the semi-Kähler structure $(X, \omega, \mathcal{F})$ defined by ν is factorizable. Let $\sigma : X \to Z$ be a holomorphic fibration onto a compact Kähler manifold Z whose relative tangent bundle agrees with $\mathcal{F}$, replacing X by some nonsingular Kähler modification if necessary. Let $A \subset Z$ be the smallest subvariety of Z such that $\sigma|_{X-B} : X - B \to Z - A$ is a regular family of compact Kähler manifolds for $B = \sigma^{-1}(A)$. Then the semi-Kähler form $\omega|_{X-B}$ on X descends to a semi-Kähler form $\overline{\omega}$ on $Z - A$. Let $A_1, \cdots, A_p$ be an enumeration of all irreducible components of A of codimension 1. For $1 \leqslant k \leqslant p$ write m_k for the multiplicity of the fiber over a generic point of A_k. Let $[D]$ denote the $\mathbb{Q}$-divisor line bundle $[A_1]^{\frac{m_1-1}{m_1}} \otimes \cdots \otimes [A_p]^{\frac{m_p-1}{m_p}}$. Then the volume form $\overline{\omega}^s$, $s = \dim_{\mathbb{C}} Z$, extends to Z as a singular Hermitian metric of nonpositive curvature in the generalized sense on $(K_Z \otimes [D])^{-1}$.*

Remarks. — We call the support of $[D]$ the multiplicity locus of the fibration $\sigma : X \to Z$. For the $\mathbb{Q}$-divisor line bundle $[D]$, the formal tensor power $[D]^r$ for some positive integer r is a divisor line bundle. To say that h is a (singular) Hermitian metric on $(K_Z \otimes [D])^{-1}$ we mean that formally h^r is a (singular) Hermitian metric on the holomorphic line bundle $(K_Z \otimes [D])^{-r}$. The precise form of $[D]$ in the proposition, with explicit exponents, will actually not be used in the sequel, except for the fact that the exponents are less than 1.

Proof. — Recall that in the notations of preceding paragraphs we have $d\zeta = \frac{1}{q}\frac{dz}{z^{\frac{q-1}{q}}}$, where $(z; w)$ are holomorphic local coordinates at $z_k \in A_k$ and $(\zeta; w)$ are holomorphic local coordinates at $x_{k,\ell} \in B_{k,\ell}$ for a local holomorphic section $S_{k,\ell}$ at $x_{k,\ell}$ of the fibration $\sigma : X \to Z$. Suppose $\sigma^{-1}(z_k)$ is irreducible. Then, $m_k = q$. In this case the volume form $\overline{\omega}$ is of the form $|z|^{\frac{2(1-q)}{q}} \cdot \alpha \cdot i^n dz \wedge d\overline{z} \wedge dw^1 \wedge \overline{dw^1} \wedge \cdots \wedge dw^{s-1} \wedge \overline{dw^{s-1}}$, where, for some positive integer t, α^t is of the form $\sum_j |f_j|^2$ for a possibly infinite sequence of holomorphic functions f_j such that $(f_1, \cdots, f_j, \cdots)$ is defined and square-integrable on some neighborhood of z_k. This implies in particular that $\overline{\omega}$ extends across z_k to give a singular Hermitian metric h on $(K_Z \otimes [A_k]^{\frac{q-1}{q}})^{-1}$ and that h is of nonpositive curvature in the generalized sense at z_k.

It remains to consider the case when there are $n_k > 1$ irreducible components, in which case m_k is the common divisor of $q_{k,1}, \cdots, q_{k,n_k}$. Fix any ℓ, $1 \leqslant \ell \leqslant n_k$, and write $\zeta = \zeta_{k,\ell}, q = q_{k,\ell}, \theta = \theta_{k,\ell}, \epsilon = \epsilon_k, m = m_k$. In the coordinates $(\zeta; w')$ we may take $W_{k,\ell}$ to be the unit polydisk $\triangle^n$, and $S_{k,\ell}$ to be $\triangle^s \times \{0\}$. Recall that $\widehat{\overline{\Phi}}(\theta) = \widehat{\Psi}_1(\epsilon), F^{\#}(e^{\frac{2\pi i}{q}}\zeta, w') = \widehat{\overline{\Phi}}(\theta) F^{\#}(\zeta, w')$. Since $\widehat{\overline{\Phi}}(\theta^m) = \widehat{\Psi}_1(\epsilon^m) = id$ we have $F^{\#}(e^{\frac{2\pi i m}{q}}\zeta, w') = F^{\#}(\zeta, w')$. Expanding in power series this forces $F^{\#}(\zeta, w') = g(\eta, w')$ for $\eta = \zeta^{qm}$. Restricting to $S_{k,\ell} \cong \triangle^s$ this means that for $h : \triangle^s \to \triangle^s$ given by $h(\zeta, w) = (\zeta^{qm}, w), \nu|_{\triangle_s} = h^*\lambda$ for some holomorphic 1-form λ on $\triangle^s$. Since $z = \eta^m$ we are back to the situation treated at the beginning of the proof, giving the exponent $\frac{m_k - 1}{m_k}$ for the divisor line bundle $[A_k]$, as desired. □

3. Fibration via the semi-Kähler structure arising from the Ricci form.

(3.1) To prove Theorem 1 it remains to consider the base manifold Z in the case (A) of Proposition (2.4.1), where $(X, \omega, \mathcal{F})$ is factorizable. To streamline the presentation we make first of all the simplifying assumption that

(†) in the statement of Proposition (2.5.2), Θ acts on $Z^{\#}$ without fixed points.

In this case we take Z_1 to be a nonsingular modification of $Z^{\#}$ and replace Z by Z_1 for the purpose of proving Theorem 1. By the simplifying assumption (†) we have by Proposition (2.5.1) a representation $\Psi : \pi_1(Z) \to U(H)$ such that $\Phi \equiv \sigma^*\Psi$, and a holomorphic 1-form $\kappa \in \Gamma(Z, \Omega(E_\Psi))$ with twisted coefficients such that $\sigma^*\kappa = \nu$.

To avoid introducing new notations in place of writing Z, Ψ, κ etc., we will write X, Φ, ν etc with the additional assumption that at a generic point $p \in X$, $\nu(p) \in \Omega_p(E_\Phi) \cong \mathrm{Hom}(T_p(X), H)$ is of maximal rank. In other words, $F : \widetilde{X} \to H$ is a holomorphic immersion at a generic point, where $dF = \widetilde{\nu}$. Under this assumption we proceed to prove that some unramified finite cover of X can be meromorphically fibered over a compact complex torus or a projective manifold of general type. In this case, following Mok ([M1], [M4]), we will consider a second semi-Kähler structure arising from the Ricci form of $\widetilde{\omega} = F^*\beta$ for the Kähler form β of the Euclidean metric on H. Note that ω is the semi-Kähler form on X whose lifting to $\widetilde{X}$ is $\widetilde{\omega}$.

By assumption, the semi-Kähler form ω on X is positive-definite outside a complex-analytic subvariety E, consisting of points p where $\nu(p) \in \Omega_p(E_\Phi) \cong \mathrm{Hom}(T_p(X), E)$ fails to be of rank n. Writing $F = (f_1, f_2, \ldots, f_k, \ldots)$ in terms of an orthonormal basis (e_k) of H, the volume form of (X, ω) is given by

$$\frac{\omega^n}{n!} = \frac{1}{n!}\Big(\sqrt{-1}\sum_{k=1}^{\infty} \partial f_k \wedge \overline{\partial f_k}\Big)^n$$
$$= (\sqrt{-1})^{n^2} {\sum}'(\partial f_{\sigma_1} \wedge \cdots \wedge \partial f_{\sigma_n}) \wedge (\overline{\partial f_{\sigma_1} \wedge \cdots \wedge \partial f_{\sigma_n}}),$$

where $\sum'$ is the summation over $(\sigma_1, \ldots, \sigma_n)$ satisfying $1 \leqslant \sigma_1 < \sigma_2 < \cdots < \sigma_n$. Denote by I the set of such n-tuples $\sigma = (\sigma_1, \ldots, \sigma_n)$. The n-th exterior product $\wedge^n H$ of H is in a natural way a pre-Hilbert space. Denote by H' the completion of $\wedge^n H$ with respect to the inner product derived from H. Define $g : \widetilde{X} \dashrightarrow \mathbb{P}(\mathbb{H}')$ by $g = [g_\sigma]_{\sigma \in I}$, where each g_σ is given by

$$\partial f_{\sigma_1} \wedge \cdots \wedge \partial f_{\sigma_n} = g_\sigma d\zeta^1 \wedge \cdots \wedge d\zeta^n,$$

$(\zeta_1, \ldots, \zeta_n)$ being local holomorphic coordinates on $\widetilde{X}$, $[g_\sigma(x)]_{\sigma \in I}$ serving as homogeneous coordinates of a point in the possibly infinite-dimensional projective space $\mathbb{P}(\mathbb{H}')$ whenever $\widetilde{\nu}(x)$ is of rank n. Then, the Ricci form $\widetilde{\rho}$ of $(\widetilde{X}, \widetilde{\omega})$ is given by

$$\widetilde{\rho} = -\sqrt{-1}\partial\overline{\partial} \log\Big(\sum_{\sigma \in I} |g_\sigma|^2\Big)$$

on $\widetilde{X} - \widetilde{E}$, $\widetilde{E}$ being the lifting of E to $\widetilde{X}$. We note that $(g_\sigma(x))_{\sigma \in I}$ is of class L^2 since $\widetilde{\nu}(x) = \sum_{j=1}^n \widetilde{\nu}_j d\zeta^j$, with $\widetilde{\nu}_j = \sum \widetilde{\nu}_{j,k} e_k$, where $(\widetilde{\nu}_{j,k})_{k \geqslant 1}$ is of class L^2 and hence of class L^{2n}. Furthermore, $-\widetilde{\rho}$ agrees on $\widetilde{X} - \widetilde{E}$ with the pull-back of the canonical Fubini-Study metric θ on the infinite-dimensional projective space $\mathbb{P}(\mathbb{H}')$, where H' is the Hilbert space as defined. Suppose $\widetilde{\rho}$ is everywhere degenerate on $\widetilde{X} - \widetilde{E}$, i.e., $\widetilde{\rho}^n \equiv 0$. Since $\widetilde{\rho} \leqslant 0$, from the

Darboux Lemma the distribution $\mathrm{Ker}(\widetilde{\rho})$ is integrable on the open set where $\widetilde{\rho}$ is of maximal rank. The formula for $\widetilde{\rho}$ says that the leaves of $\mathrm{Ker}(\widetilde{\rho})$ agree with connected components of level sets of g. Thus, the level sets of g define a meromorphic foliation $\widetilde{\mathcal{E}}$ on $\widetilde{X}$ compatible with $(\widetilde{X}, -\widetilde{\rho})$, when $-\widetilde{\rho} \geqslant 0$ is considered a semi-Kähler metric. In other words, $(\widetilde{X}, -\widetilde{\rho}, \widetilde{\mathcal{E}})$ defines a semi-Kähler structure on $\widetilde{X}$. From the definition it is clear that the semi-Kähler structure is invariant under the action of $\Gamma = \pi_1(X)$ on $\widetilde{X}$. It descends to a semi-Kähler structure $(X, -\rho, \mathcal{E})$ on X which is the main object of study of this section. We will maintain the simplifying assumption (†).

(3.2) In Mok ([M4], (3.4)) we studied $(X, -\rho, \mathcal{E})$ in the case of finite-dimensional unitary representations (for the purpose of proving a generalized Castelnuovo-de Franchis theorem). In the notations of the present article we considered a divergent sequence (γ_i) in Γ inducing germs of holomorphic isometries σ_i of $(S, -\rho|_S)$ belonging to G and converging to the identity. The proof there that $(X, -\rho, \mathcal{E})$ is factorizable proceeded by argument by contradiction. It consisted of showing under the assumption that $(X, -\rho, \mathcal{E})$ is not factorizable that (i) the sequence $\widehat{\Phi}(\gamma_i) \in U(r) \ltimes \mathbb{C}^r$ is discrete and (ii) that discreteness of $\widehat{\Phi}(\gamma_i)$ implies the pseudo-invariance of $F(\widetilde{X})$ under a 1-parameter group of Euclidean translations. When H is allowed to be infinite-dimensional we cannot expect to prove discreteness of $\widehat{\Phi}(\Gamma) \subset U(H) \ltimes H$, as illustrated by the example of Mok ([M4], (3.5)) arising from irreducible compact quotients X of the bidisk. Instead we are going to establish the alternatives (A) that $(X, -\rho, \mathcal{E})$ is factorizable; (B) that Γ admits a finite-dimensional representation with infinite image and (C) that $F(\widetilde{X})$ is pseudo-invariant under a 1-parameter group of Euclidean translations. (By "alternatives" we mean an exhaustive list of possible cases which are not necessarily mutually exclusive.) In all cases we will obtain meromorphic fibrations of X after passing to an unramified finite cover if necessary.

Let $(X, -\rho, \mathcal{E})$ be as in (3.1) and assume that $(X, -\rho, \mathcal{E})$ is not factorizable. By Proposition (1.2.1) there exists a local holomorphic section $(S, -\rho|_S)$ of $(X, -\rho, \mathcal{E})$, S of dimension $m < n$, together with a nontrivial pseudogroup G of germs of holomorphic isometries at $x_0 \in S$. Recall that the meromorphic foliation $\widetilde{\mathcal{E}}$ on $\widetilde{X}$ is induced by a meromorphic map $g : \widetilde{X} \dashrightarrow \mathbb{P}(\mathbb{H}')$, where $\pi : \widetilde{X} \to X$ is the universal cover. We may choose S such that $\widetilde{S} := \pi^{-1}(S)$ is a disjoint union of connected components γS_0, $\gamma \in \Gamma$, such that $\pi|_{S_0} : S_0 \to S$ is a biholomorphism. We may also assume that $g|_{S_0}$ is holomorphic and is a biholomorphism onto its image $V = g(S_0)$.

We will sometimes identify $(S, -\rho|_S)$, with $(V, \theta|_V)$. $\Phi : \Gamma \to U(H)$ induces a representation $\Phi' : \Gamma \to U(H')$ and hence $[\Phi'] : \Gamma \to \mathbb{PU}(\mathbb{H}')$. By the same argument as in (2.2), we have

PROPOSITION (3.2.1). — *Suppose $(X, -\rho, \mathcal{E})$ is not factorizable. Then, there exists a divergent sequence γ_i of elements of Γ such that under the identification of $(S, \rho|_S)$ with $(V, \theta|_V)$, the sequence $\sigma_i = [\Phi'(\gamma_i)] \in \mathbb{P}(\mathbb{H}')$ lies on the pseudogroup G of germs of holomorphic isometries (at some base point) and converges to the identity.*

Consider now $\widehat{\Phi}(\gamma_i) \in U(H) \ltimes H$. We may assume that S is disjoint from the exceptional set $E \subset X$. Replacing S by some sufficiently small relatively compact open subset containing the base point, there exist $n - m$ linearly independent H-valued holomorphic vector fields $\alpha_1, \ldots, \alpha_n$ on S_0 such that at $x \in S_0$ the leaf of $\widetilde{\mathcal{E}}$ at x is mapped into the affine $(n-m)$-plane on H given by

$$\{F(x) + t_1\alpha_1(x) + \cdots + t_{n-m}\alpha_{n-m}(x) : t_1, \ldots, t_{n-m} \in \mathbb{C}\} := \not\succ_{\frown}.$$

Define $h : S_0 \times \mathbb{C}^{\ltimes - \triangleright} \to \mathbb{H}$ by $h(x; t_1, \ldots, t_{n-m}) = F(x) + t_1\alpha_1(x) + \ldots + t_{n-m}\alpha_{n-m}(x)$.

Regarding $\widehat{\Phi}(\gamma_i) \in U(H) \ltimes H$ there are two alternatives: (a) that $(\widehat{\Phi}(\gamma_i))$ constitutes a bounded sequence in $U(H) \ltimes H$; (b) that replacing $\widehat{\Phi}(\gamma_i)$ by a subsequence if necessary and writing $\widehat{\Phi}(\gamma_i)(p) = U_i(p) + \eta_i$ with $U_i \in U(H)$ and $\eta_i \in H$, the norms $\|\eta_i\|$ diverge to infinity.

Recall that (X, ω) is positive-definite outside a proper complex-analytic subvariety E. We may also consider (X, ω) as a semi-Kähler structure with the trivial 0-dimensional distribution. A local cross-section then simply means $(W, \omega|_W)$, where W is a nonempty connected open subset of X on which ω is positive-definite. With this understanding and in the language of (1.2) under alternative (a) we proceed to establish

PROPOSITION (3.2.2). — *Assume that $(\widehat{\Phi}(\gamma_i))$ constitutes a bounded sequence in $U(H) \ltimes H$. Then, (X, ω) admits a nontrivial pseudogroup of holomorphic isometries which are restrictions of affine unitary transformations on H. As a consequence, Γ admits a finite-dimensional linear representation with infinite image.*

Proof. — We have

$$F(\gamma_i x) = \widehat{\Phi}(\gamma_i)F(x) = U_i(F(x)) + \eta_i = h(\sigma_i(x); t_{1,i}(x), \ldots, t_{n-m,i}(x)),$$

where σ_i are germs at a base point x_0 of holomorphic isometries of $(S_0, -\widetilde{\rho}|_{S_0})$. Without loss of generality we assume here that σ_i are defined on some neighborhood of $\overline{S}_0$ with images in some local cross-section S_0' which contain S_0 as a relatively compact subset. Consider $\Theta_i : S_0 \to S_0' \times \mathbb{C}^{\ltimes - >}$ given by $\Theta_i(x) = (\sigma_i(x); t_{1,i}(x), \ldots, t_{n-m,i}(x))$. By the hypothesis (Θ_i) constitutes a uniformly bounded sequence of holomorphic maps. From this it follows that $\{\widehat{\Phi}(\gamma_i)|_{F(S_0)}\}$ is a relatively compact set of holomorphic maps. The same applies to a slight thickening W of S_0 given by the image under F of $\{(x; t_1, \ldots, t_{n-m}) : x \in S_0; |t_i|, \ldots, |t_{n-m}| < \epsilon\}$ for some sufficiently small $\epsilon > 0$. Passing to a subsequence $\widehat{\Phi}(\gamma_i)|_W$ converge uniformly to a biholomorphism φ of W onto its image. It follows that there exists a divergent sequence of elements μ_k such that $\widehat{\Phi}(\mu_k)|_W$ converges to the identity map. One can take μ_k to be of the form $\mu_k = \gamma_{i(k)}^{-1}\gamma_{j(k)}$, with $i(k), j(k)$ diverging to infinity and suitably chosen so that for $\varepsilon_k := \sup\{\|\widehat{\Phi}(\mu_k)(p) - p\| : p \in \overline{W}\}$, ε_k decreases monotonically to 0. The upshot is that the Kähler manifold $(W, \beta|_W)$ admits a nontrivial pseudogroup of holomorphic isometries which are restrictions of affine unitary transformations of H, noting that $F^*\beta = \widetilde{\omega}$. By the argument of Proposition (2.3.1), Γ admits a finite-dimensional linear representation with infinite image, as desired. □

For the alternative (b) regarding the sequence $\widehat{\Phi}(\gamma_i) \in U(H) \ltimes H$ we proceed to establish

PROPOSITION (3.2.3). — *Assume that writing $\widehat{\Phi}(\gamma_i)(p) = U_i(p) + \eta_i$ with $U_i \in U(H)$ and $\eta_i \in H$, the norm $\|\eta_i\|$ diverges to infinity. Let $W_0 \subset \widetilde{X} - \widetilde{E}$ such that F is a holomorphic embedding of some neighborhood of W_0 into H. Then the locally closed subvariety $F(W_0)$ is pseudo-invariant under a 1-parameter group of Euclidean translations.*

Proof. — Recall that

$$F(\gamma_i x) = \widehat{\Phi}(\gamma_i)(F(x)) = U_i(F(x)) + \eta_i = h(\sigma_i(x); t_{1,i}(x), \ldots, t_{n-m,i}(x)).$$

Hence for any $x \in S_0$, $U_i(F(x)) + \eta_i$ lies on the affine $(n-m)$-plane Π_{y_i} where $y_i = F(\sigma_i(x))$. Consider the unit vector $\xi_i := (U_i(F(x)) + \eta_i - y_i)/\|(U_i(F(x)) + \eta_i - y_i)\|$, which is tangent to Π_{y_i} at y_i. Since $\{\Pi_y : y \in F(W_0)\}$ constitutes a holomorphic (hence continuous) family of affine $(n-m)$-planes, ξ_i must converge strongly to a unit vector $\psi_{F(x)}$ at $F(x)$ tangent to $\Pi_{F(x)}$. Since $\{y_i\}$ is uniformly bounded, and U_i preserves lengths, while $\|\eta_i\|$ diverges to infinity, it follows that $\eta_i/\|\eta_i\|$

converges strongly to the unit vector $\psi_{F(x)}$. But then this shows that $\psi_{F(x)}$ is a constant vector, independent of x. In other words, $F(W_0)$ is pseudo-invariant under a 1-parameter group of Euclidean translations, as desired. □

(3.3) For the semi-Kähler structure $(X, -\rho, \mathcal{E})$ assuming that the meromorphic distribution $\mathcal{E}$ is nontrivial by Propositions (3.2.1) and (3.2.2) we have now established the alternatives (A) that $(X, -\rho, \mathcal{E})$ is factorizable; (B) that Γ admits a finite-dimensional linear representation with infinite image; and (C) that $F(\widetilde{X})$ is pseudo-invariant under a 1-parameter group of Euclidean translations.

Consider the alternative (A) that $(X, -\rho, \mathcal{E})$ is factorizable. Let L be a generic leaf of $\mathcal{E}$ on $X - E$. L is of dimension $n - m$. By assumption, $L \subset X - E$ is closed, and the topological closure $\overline{L} \subset X$ is a complex-analytic subvariety. Denote by $\pi_{\overline{L}}$ the image of $\pi_1(\overline{L})$ in $\Gamma = \pi_1(X)$. Let Λ be a connected component of the lifting of $\overline{L}$ and x be a generic smooth point of Λ. Then, the holomorphic map $F : \widetilde{X} \to H$ sends Λ into an affine $(n - m)$-plane Π_y, where $y = f(x)$. $\widehat{\Phi}(\pi_{\overline{L}}) \subset U(H) \ltimes H$ consists of affine unitary transformations preserving Π_y and inducing isometries on Π_y. Write $\Pi_y = y + P_y$, where $P_y \subset H$ is an $(n-m)$-dimensional linear subspace. Then $\Phi(\pi_{\overline{L}})$ is conjugate (by the translation by $-y$) to a subgroup Θ_y of $U(P_y) \ltimes P_y$. If Θ_y is not discrete, as in the proof of Proposition (3.2.1) we can obtain a nontrivial pseudogroup of holomorphic isometries of $(X, -\rho)$, leading to alternative (B). From the Maximum Principle for holomorphic functions it follows that Θ_y is infinite. If Θ_y is discrete then some subgroup of finite index consists entirely of translations. For leaves L' sufficiently close to L and hence leaves Λ' sufficiently close to Λ, $\pi_{\overline{L}'}$ are in a canonical way identical subgroups of Γ. We have thus obtained in particular an affine unitary transformation $T = \widehat{\Phi}(\gamma)$ on H, for some $\gamma \in \Phi(\pi_{\overline{L}})$, such that $\widehat{\Phi}(\gamma)$ induces a nontrivial translation on each $\Pi_{y'}$ for y' sufficiently close to y on $F(\widetilde{X})$. In the notations of the proof of Proposition (3.2.3), taking $\gamma_i = \gamma^i$ so that $\widehat{\Phi}(\gamma_i) = T^i$, we obtain vectors η_i with norms diverging to infinity, such that the unit vector $\eta_i/\|\eta_i\|$ converges strongly to a unit vector ξ tangent to each $\Pi_{y'}$. From this we deduce that $F(\widetilde{X})$ is pseudo-invariant under a 1-parameter group of Euclidean translations, leading to alternative (C).

It remains therefore to consider the alternatives (B) and (C). In case of alternative (B), by Proposition (2.4.2) some unramified finite cover of

X admits a meromorphic fibration over a compact complex torus or over a projective manifold of general type. In the case of alternative (C), $F(\widetilde{X})$ is pseudo-invariant under a nontrivial finite-dimensional vector subspace $H_0 \subset H$. Take H_0 to be the maximal vector subspace with this property. For a vector $\xi \in H$ denote by T_ξ the Euclidean translation given by $T_\xi(p) = p+\xi$. For any $\xi \in H_0$ and any $\gamma \in \Gamma$, $\widehat{\Phi}(\gamma)T_\xi\widehat{\Phi}(\gamma)^{-1}$ is a Euclidean translation T_η, $\eta = \eta(\gamma, \psi)$, which leaves $F(\widetilde{X})$ pseudo-invariant. More precisely, we have

$$\begin{aligned}
\widehat{\Phi}(\gamma)(x) &= \Phi(\gamma)x + c_\gamma \\
\widehat{\Phi}(\gamma)^{-1}(x) &= \Phi(\gamma)^{-1}x - \Phi(\gamma)^{-1}c_\gamma \\
\widehat{\Phi}(\gamma)T_\xi\widehat{\Phi}(\gamma)^{-1}(x) &= \Phi(\gamma)(\Phi(\gamma)^{-1}x - \Phi(\gamma)^{-1}c_\gamma + \xi) + c_\gamma \\
&= x - c_\gamma + \Phi(\gamma)\xi + c_\gamma = x + \Phi(\gamma)\xi.
\end{aligned}$$

In other words, $H_0 \subset H$ is invariant under the action of $\Phi(\Gamma)$, contradicting with the irreducibility of $\Phi : \Gamma \to U(H)$, unless of course $H_0 = H$. Since $F(\widetilde{X})$ is pseudo-invariant under all translations on H, the latter must be m-dimensional for some m, $1 \leqslant m \leqslant n$. The subgroup $\Phi(\Gamma) \subset U(H) \cong U(m)$ is either infinite or finite. In the former case by Proposition (2.4.1), some unramified finite cover of X can be meromorphically fibered over a compact complex torus or a projective manifold of general type. In the latter case passing to some unramified finite cover we may take Φ to be the trivial representation (which is no longer irreducible if $m \geqslant 2$). Then, using the Albanese map as in (2.4) some finite cover of X admits a meromorphic fibration over a compact complex torus or over a projective manifold of general type.

PROPOSITION (3.3.1). — *Let X be a compact Kähler manifold with fundamental group Γ. Let $\Phi : \Gamma \to U(H)$ be a unitary representation of Γ into a Hilbert space for which there exists a nontrivial holomorphic 1-form $\nu \in \Gamma(X, \Omega(E_\Phi))$ with coefficients twisted by Φ such that $\nu(p) \in \Omega_p(E_\Phi) \cong \mathrm{Hom}(T_p(X), H)$ is of maximal rank at a generic point $p \in X$. Then, some unramified finite cover of X can be meromorphically fibered over a compact complex torus or a projective manifold of general type.*

Proof. — Assuming that the meromorphic distribution $\mathcal{E}$ is nontrivial we have proven that in all circumstances some unramified finite cover of X admits a meromorphic fibration such that the base manifold Z is either a compact complex torus or a projective manifold of the general type. If on the other hand the meromorphic distribution $\mathcal{E}$ is trivial, then the semi-Kähler metric ω on X is of strictly negative Ricci curvature at generic points. It remains to prove that X must itself be of general type in this case. This follows already from the arguments of Mok [M1] basing on an easy case of the Grauert-Riemennschneider Conjecture as given in Siu [Si]. Alternatively one can also show this using L^2-estimates of $\overline{\partial}$, as follows. The volume form ω^n on X gives a real-analytic singular Hermitian metric h on the canonical line bundle K_X of nonnegative curvature in the generalized sense. Furthermore, (K_X, h) is non-degenerate and of strictly positive curvature on a Zariski-dense open subset $X - E$. Since X is Kähler and $E \subset X$ is a subvariety, replacing X by a nonsingular Kähler modification if necessary we may assume that E has at worst normal-crossing singularities. From this and standard constructions using potential functions and the background Kähler metric on X we see that $X - E$ admits a complete Kähler metric g. One can then solve $\overline{\partial}$ with L^2-estimates on the Hermitian holomorphic line bundle $\left(K^p_{X-E}, h^p\right) \otimes \left(K_{X-E}, \frac{1}{\det g}\right)$, for p sufficiently large, to find $n+1$ holomorphic sections $(s_0, \cdots, s_n)$ which define a local biholomorphism into $\mathbb{P}^\kappa$ at some point $x \in X - E$. Since ω^n is smooth on X, h dominates some smooth Hermitian metric on K_X, the holomorphic section $s_i, 0 \leqslant i \leqslant n$, must extend holomorphically to X by square-integrability. This shows that X is of Kodaira dimension equal to n *i.e.*, X is of general type, hence also Moišezon. As X is Kähler and Moišezon, it must be projective-algebraic, as desired. □

(3.4) We are now ready to complete the proof of the main results.

Proof of Theorem 1. — By Proposition (2.4.1) it remains to treat the case where $(X, \omega, \mathcal{F})$ is factorizable. As in (3.1) assume that

(†) in the statement of Proposition (2.5.1), Θ acts on $Z^{\#}$ without fixed points.

Let Z_1 be a nonsingular Kähler modification of $Z^{\#}/\Theta$ and $\sigma_1 : \widehat{X} \to Z_1$ be a holomorphic fibration on some nonsingular Kähler modification $\widehat{X}$ of X compatible with the foliation $\mathcal{F}$. By Proposition (2.5.1) the semi-Kähler structure $(X, \omega, \mathcal{F})$ descends to Z_1. For the purpose of proving Theorem 1 one can therefore replace X by Z_1, and reduce to the case where the semi-Kähler structure on ω is positive-definite almost everywhere. In this case we have been considering in (3.2) and (3.3) the semi-Kähler structure $(X, -\rho, \mathcal{E})$, where ρ is the Ricci form of ω at generic points of X. Then, by Propositions (3.2.1) and (3.2.2) we have established the alternatives (A) that $(X, -\rho, \mathcal{E})$ is factorizable; (B) that Γ admits a finite-dimensional linear representation with infinite image; and (C) that $F(\widetilde{X})$ is pseudo-invariant under a 1-parameter group of Euclidean translations for the primitive $F : \widetilde{X} \to H$ for the lifting of the holomorphic 1-form ν on X to the universal covering $\widetilde{X}$. By Proposition (3.3.1) we have established that some unramified finite cover of X admits a meromorphic fibration over a compact complex torus or a projective manifold of general type.

For the general case, *i.e.*, without the extra assumption (†), starting with the semi-Kähler structure $(X, \omega, \mathcal{F})$, in the notations of Proposition (2.6.1) the semi-Kähler form $\omega|_{X-B}$ descends to a semi-Kähler form $\overline{\omega}$ on $Z - A$ for some divisor A on Z whose support is the multiplicity locus of the holomorphic fibration $\sigma : X \to Z$ as in Proposition (2.6.1). Assuming for the time being that the Ricci form ρ of $\overline{\omega}$ is negative-definite at a generic point $Z - A$. Then, the proof of Proposition (3.3.1) works equally well to show that the $K_Z \otimes [D]$ is a big holomorphic $\mathbb{Q}$-line bundle on Z. In other words, the formal tensor power $(K_Z \otimes [D])^r$ is a big holomorphic line bundle on Z for some positive integer r. Recall that $[D] = [A_1]^{\frac{m_1-1}{m_1}} \otimes \cdots \otimes [A_p]^{\frac{m_p-1}{m_p}}$ with all exponents < 1. In particular, $K_Z \otimes [A_1] \otimes \ldots \otimes [A_p]$ is big, *i.e.*, Z is of logarithmic general type with respect to the multiplicity locus of σ.

To complete the proof of Theorem 1, it remains to consider the case where the initial semi-Kähler structure $(X, \omega, \mathcal{F})$ is factorizable, and where, for the induced semi-Kähler form $\overline{\omega}$ on $Z - A$, the Ricci form may *a priori* not be of strictly negative curvature at generic points. We note that, although $\overline{\omega}$, considered as a (1,1) form on Z, has poles along the multiplicity locus of σ, we can still define a meromorphic foliation $\mathcal{E}$ on Z which agrees on $Z - A$ with the meromorphic foliation defined by the Ricci form ρ. This can be seen by lifting to $Z^{\#}$. In fact, on $Z^{\#}$ we have $F^{\#} : Z^{\#} \to H$ and one can define a meromorphic foliation $\mathcal{E}^{\#}$ on $Z^{\#}$ which agrees at generic points with the level set of some associated meromorphic map $g^{\#} : Z^{\#} \to \mathbb{P}(\mathbb{H}')$, in the same way as was discussed in (3.1). Under the action of Θ on $Z^{\#}$ the meromorphic foliation $\mathcal{E}^{\#}$ descends to a meromorphic foliation $\mathcal{E}$ on Z. Note that the nonnegative closed $(1,1)$ form $-\rho$ on $Z - A$ extends trivially to a nonnegative closed $(1,1)$ form on Z, smooth outside a subvariety of codimension $\geqslant 2$, from the local expression of the volume form $\overline{\omega}^n$ across a generic point of each irreducible component of the multiplicity locus D as given in the proof of Proposition (2.6.1). In what follows we will write $-\rho$ for the extended semi-Kähler form on Z. The triple $(Z, -\rho, \mathcal{E})$ is then a semi-Kähler structure on Z.

We have to prove again the alternatives (A) that $(Z, -\rho, \mathcal{E})$ is factorizable; (B) that Γ admits a finite-dimensional linear representation with infinite image; and (C) that $F^{\#}(Z^{\#})$ is pseudo-invariant under a 1-parameter group of Euclidean translations for the primitive $F^{\#} : Z^{\#} \to H$. But the proof of the alternatives follows almost *verbatim* from the proofs of Propositions (3.2.1) to (3.2.3), replacing $(X, -\rho, \mathcal{E})$ there by $(Z, -\rho, \mathcal{E})$ and arguing with $F^{\#} : Z^{\#} \to H$ is place of $F : \widetilde{X} \to H$ there.

Having established the alternatives (A), (B), (C), the arguments in (3.3) are applicable. Only the arguments for alternative (A) require some modification. Let $E \subset Z$ be the subvariety of codimension $\geqslant 2$ such that $\mathcal{E}$ is holomorphic on $Z - E$. Assume that $(Z, -\rho, \mathcal{E})$ is factorizable. Let $L \subset Z - E$ be a generic leaf. Since $Z = Z^{\#}/\Theta$ and Θ may act with fixed points, in place of subgroups $\pi_{\overline{L}} \subset \Gamma$ as in (3.3) we should talk about subgroups $\Theta_{\overline{L}} \subset \Theta$ of the discrete cocompact group $\Theta \subset \mathrm{Aut}(Z^{\#})$ of biholomorphic automorphisms. Let $\Lambda \subset Z^{\#}$ be an irreducible component of $\varphi^{-1}(\overline{L})$ for the canonical map $\varphi : Z^{\#} \to Z$. We define $\Theta_{\overline{L}} \subset \Theta$ to be the subgroup of all $\theta \in \Theta$ which leave Λ invariant as a set. Then, $\Theta_{\overline{L}} \subset \Theta$ plays the role of $\pi_{\overline{L}} \subset \Gamma$. The arguments of (3.3) then apply to complete the proof of Theorem 1. □

Proof of Corollary 1. — Corollary 1 (of the proof of Theorem 1) pertains to the case where every quotient group of $\Gamma = \pi_1(X)$ is residually finite. In particular, $\Theta = \Gamma/\Lambda$ is residually finite. In this case, by Proposition (2.5.1), replacing X by some unramified finite cover if necessary, X admits a meromorphic fibration over a compact Kähler manifold Z on which there exists a nontrivial holomorphic 1-form with coefficients twisted by some unitary representation such that the associated semi-Kähler form ω is positive-definite at a generic point. Then, by Proposition (3.3.1) some unramified finite cover of Z can be meromorphically fibered over a compact complex torus or a projective manifold of general type, as desired. □

Proof of Theorem 2. — Let X be a compact Kähler manifold whose fundamental group Γ violates Kazhdan's Property (T). Assume furthermore that $b_1(X') = 0$ for any unramified finite cover X' of X. We proceed to prove that X is of algebraic dimension $\geqslant 2$. From the two assumptions clearly X is of complex dimension $\geqslant 2$. Since the algebraic dimensions remain unchanged when X is replaced by an unramified finite cover or by a modification, by Theorem 1 we may assume without loss of generality that either X is of general type, or X can be holomorphically fibered over a positive-dimensional projective manifold Z. It remains to show that for $\sigma : X \to Z$ arising from Theorem 1, alternative (c), Z must be of complex dimension $\geqslant 2$. From the proof of Theorem 1, either (i) $\pi_1(Z)$ admits a finite- dimensional linear representation with infinite image, of (ii) Z is of the form Z'/Θ as in the proof of Proposition (2.5.1). If Z is of complex dimension 1 and (i) occurs, then $b_1(Z) \neq 0$, so that $b_1(X) \neq 0$, violating the hypothesis of Theorem 2. Assume now that Z is of complex dimension 1 and (ii) occurs. By Lemma (2.5.1), Z' (which is the same as $Z^{\#}$ in the notations of the lemma as $\dim_{\mathbb{C}} Z = 1$) is simply connected. Then, $\mathrm{Aut}(Z')$ is a linear group, and any finitely generated discrete subgroup of $\mathrm{Aut}(Z')$ admits a torsion-free subgroup of finite index. Thus, replacing Θ by a torsion-free subgroup $\Theta_0 \subset \Theta$ of finite index, Z can be taken to be a compact Riemann surface of genus $\geqslant 1$. But this contradicts the assumption that $b_1(X') = 0$ for any unramified finite cover of X. We have thus proven that the base manifold Z of $\sigma : X \to Z$ must be of complex dimension $\geqslant 2$, *i.e.*, X is of algebraic dimension $\geqslant 2$, as desired. □

In Mok [M3] we formulated a conjecture regarding meromorphic fibrations of (unramified finite covers of) compact Kähler manifolds over projective manifolds of general type, and claimed that the conjecture is valid for an n-dimensional compact Kähler manifold with fundamental group Γ of subexponential growth, provided that Γ does not admit a subgroup of finite index isomorphic to a free Abelian group of rank $\leqslant 2n$. The latter statement was based first of all on the method of proof leading to Theorem 1 of the present article. Because of the difficulty of finding torsion-free discrete subgroups of finite index in relation to alternative (c) of Theorem 1, the claim remains conjectural.

4. Brody maps into certain Kähler surfaces admitting holomorphic 1-forms with twisted coefficients.

(4.1) Let X be a compact Kähler manifold admitting some nontrivial holomorphic 1-form ν with coefficients twisted by a unitary representation. Suppose on some Zariski-dense open subset of X the semi-Kähler form ω defined by ν is positive-definite, and of strictly negative Ricci curvature. Note that (X, ω) is of nonpositive holomorphic bisectional curvature wherever ω is positive-definite. Heuristically, modulo some blowing-down, X should behave like a subvariety of an Abelian variety. Along this line we know that X is of general type, just as a subvariety of an Abelian variety of negative Ricci curvature at a generic point. Theorem 3 is a step further in that direction in the case of compact Kähler surfaces. It asserts that X is Kobayashi hyperbolic under the assumption of nonexistence of rational curves and elliptic curves. In relation to the heuristic analogy to subvarieties of Abelian varieties, we note that under the hypotheses of Theorem 3, any chain of rational curves on X is exceptional. One can then blow down maximal chains of rational curves on X, if they exist, to obtain a possibly singular compact complex surface X'. In this context the proof of Theorem 3 would say that X' is Kobayashi hyperbolic if and only if X' contains no elliptic curves just as in the case of a (possibly singular) complex surface in an Abelian variety of negative Ricci curvature at a generic point. We will not make the modification here.

We will give a proof basing on techniques of meromorphic fibrations via the Ricci form as in §3. Let now X be a compact Kähler manifold with fundamental group Γ, $\Phi : \Gamma \longrightarrow H$ be a unitary representation into a Hilbert space and $\nu \in \Gamma(X, \Omega(E_\Phi))$ be a nontrivial holomorphic 1-form with coefficients twisted by Φ. Suppose $\nu(x) \in \Omega(E_{\Phi,x}) \cong \mathrm{Hom}(T_x(X), E_{\Phi,x})$ is generically injective, which is the same as saying that the induced Kähler semi-metric ω is positive-definite at generic points. Suppose furthermore that the Ricci form of ω is strictly negative at generic points. This is the case for instance for the base manifold Z in the generalized theorem of Castelnuovo-de Franchis as in [M4], (3.1), Theorem 2. The assumptions imply that X is of general type and hence Moišezon. Together with the Kähler assumption this implies that X is projective-algebraic.

(4.2) Proceeding to prove Theorem 3 we specialize now to the case of Kähler surfaces X. Suppose X is not Kobayashi hyperbolic. There exists then a Brody map $\beta_0 : \mathbb{C} \longrightarrow \mathbb{X}$, when X is endowed the given Kähler metric. Since $\mathbb{C}$ is simply-connected $\beta_0 : \mathbb{C} \longrightarrow \mathbb{X}$ lifts to some $\beta : \mathbb{C} \longrightarrow \widetilde{\mathbb{X}}$. The composition $P := F \circ \widetilde{\beta} : \mathbb{C} \longrightarrow \mathbb{H}$ then gives a holomorphic map with uniformly bounded first derivative. Writing $P = (p_1, p_2, \ldots, p_k, \ldots)$ with respect to an orthonormal basis of H we conclude that P is a linear map even when H is infinite-dimensional. Recall that $E \subset X$ is the subvariety where ν fails to be of maximal rank. If the Zariski-closure $\overline{\beta(\mathbb{C})}$ lies on E, then it must lie on an irreducible component which is either an elliptic curve or a rational curve. These possibilities are ruled out by the assumptions of Theorem 3. Thus $\beta(\mathbb{C})$ does not lie on E, so that P is nontrivial.

For any point x on $X - E$ consider a non-zero tangent vector η at x such that there exists a smooth contractible holomorphic curve C at x, $T_x(C) = \mathbb{C}\eta$, with the property that $F(\widetilde{C})$ is an open subset of an affine line in H for any irreducible component $\widetilde{C}$ of $\pi^{-1}(C)$. We will sometimes write $f = F \circ \pi^{-1}$ for $\pi^{-1} : C \to \widetilde{C}$ the inverse map of $\pi|_{\widetilde{C}} : \widetilde{C} \xrightarrow{\cong} C$. At any point $x \in X - E$ denote by $\mathcal{S}_x \subset \mathbb{PT}_{\curvearrowright}(\mathbb{X})$ the subset consisting of all $[\eta]$ with the above property. Let $\widetilde{x} \in \widetilde{C}$ be such that $\pi(\widetilde{x}) = x \in C$. The condition of being able to pass the germ of an "affine line" through x in the direction of η, in terms of the lifting to $\widetilde{X}$ at $\widetilde{x}$, can be formulated as follows. Let L be the affine line in H at $F(\widetilde{x})$ in the direction of $\widetilde{\nu}(\widetilde{\eta})$, $d\pi(\widetilde{\eta}) = \eta$, and W be the germ of a complex submanifold at $F(\widetilde{x})$ which is the image of the germ of $\widetilde{X}$ at $\widetilde{x}$ under F. Then, $[\eta] \in \mathcal{S}_x$ if and only if L is tangent to W at $\widetilde{x}$ to the order k for any positive integer k. This gives a countable set of conditions on $[\eta]$, holomorphic on $[\eta]$ as the latter varies, so that over $X - E$ we obtain a complex-analytic subvariety of $\mathcal{S} \subset \mathbb{PT}(\mathbb{X}-\mathbb{E})$. We have

PROPOSITION (4.2.1). — *$\mathcal{S} \subset \mathbb{PT}(\mathbb{X} - \mathbb{E})$ contains an irreducible component which dominates $X - E$.*

Proof. — Since $P = F \circ \widetilde{\beta}$ is a nontrivial linear map, so that $\|dP(\frac{\partial}{\partial z})\|$ is a constant $a \neq 0$, and $d\beta$, $\widetilde{\nu} = dF$ are uniformly bounded from the above, it follows that $0 < b < \|d\beta(\frac{\partial}{\partial z})\| < c$ on $\mathbb{C}$ for some constants b and c. For $t \in \mathbb{C}$; $\varepsilon > 0$ write $\triangle(t;\varepsilon) = \{z \in \mathbb{C} : |F - \approx| < \varepsilon\}$ and $\triangle(\varepsilon) = \triangle(0;\varepsilon)$. From Cauchy estimates, the holomorphic maps $\beta_t : \mathbb{C} \to \mathbb{X}$ defined by $\beta_t(\zeta) = \beta(t + \zeta)$ form a normal family as t ranges over $\mathbb{C}$. From the lower bound $\|d\beta(\frac{\partial}{\partial z})\| > b > 0$ it follows that any limiting map must be nontrivial. Suppose now $\beta : \mathbb{C} \to \mathbb{X}$ is not an immersion onto an elliptic curve. Then, there exists some $x \in X$, a sequence $s_i \in \mathbb{C}$ diverging to infinity, a neighborhood U_x of x, and $\varepsilon > 0$ such that (i) $\beta(s_i) := x_i$ converges to x; (ii) $\beta_{t_i}(\triangle_\varepsilon) \cap U_x := C_i$ is a closed connected smooth holomorphic curve on U_x and the curves C_i are distinct. Write $\eta_i = d\beta(s_i)(\frac{\partial}{\partial z})$. Reparametrizing the Brody map with the origin being mapped to x_i we can extract a convergent subsequence to get a new Brody map β' such that $\beta'(0) = x$ and such that $\eta := d\beta'(0)(\frac{\partial}{\partial z}) \neq 0$. From (i) and (ii) it follows readily that $\beta'(\mathbb{C})$ must contain a local holomorphic curve on $X - E$ which consists of points of accumulation of a countably infinite family of local curves on $X - E$ belonging to $\beta(\mathbb{C})$. Since the canonical projection $\rho : \mathbb{PT}(\mathbb{X}) \longrightarrow \mathbb{X}$ is proper $\mathcal{S} \subset \mathbb{PT}(\mathbb{X} - \mathbb{E})$ must project to a subvariety of $X - E$, it follows from the above that $\mathcal{S}$ must surject to $X - E$. Since $F(\widetilde{X})$ is not an affine plane on H, it follows that $\mathcal{S}$ must contain as an irreducible component a hypersurface of $\mathbb{PT}(\mathbb{X} - \mathbb{E})$ which dominates

$X - E$, as desired. □

From now on we will replace $\mathcal{S}$ by the union of 2-dimensional irreducible components which dominate $X - E$.

(4.3) We proceed now to prove Theorem 3 by using arguments related to the study of semi-Kähler structures defined by Ricci forms, especially Proposition (3.2.3). The projection $\rho : \mathcal{S} \longrightarrow X - E$ is k-to-one for some k outside a subvariety A of $X - E$. On $X - E - A$ we have a multi-foliation by k distinct holomorphic families of local holomorphic curves which are mapped onto open subsets of affine lines by f. From the proof of Proposition (4.2.1) reparametrizations of the original Brody map $\beta : \mathbb{C} \longrightarrow \mathbb{X}$ by translations on $\mathbb{C}$ converge to some Brody map $\beta' : \mathbb{C} \longrightarrow \mathbb{X}$. Note that β' is a holomorphic immersion such that $\|d\beta'\|$ is bounded between two positive constants, where $\|\cdot\|$ is measured with respect to the Euclidean metric on $\mathbb{C}$ and a Kähler metric on X. As a consequence, under the assumption of nonexistence of (immersed) elliptic curves, $\beta'(\mathbb{C}) \cap (\mathbb{X} - \mathbb{E})$ is not closed in the complex topology, so that a generic point of the latter will avoid A. Choose such a point x which we may take to be $\beta'(0)$. Choose a contractible neighborhood D of x in $X - E$ which is a product neighborhood for one branch $\mathcal{H}$ of the k-sheeted multi-foliation defined near x. Reparametrize β by translation so that $\beta(0)$ lies on D and β maps a neighborhood of 0 onto a leaf of $\mathcal{H}$ on D. From the choices of s_i, $\beta_{s_i} \to \beta'$, it follows that β must map a neighborhood of s_i on $\mathbb{C}$ to a leaf of $\mathcal{H}$ on D for i sufficiently large. Let $\widetilde{\beta}$ resp. $\widetilde{\beta}'$ be liftings of β resp. β' to $\widetilde{X}$. The liftings can be chosen in such a way that both $\beta(0)$ and $\beta'(0)$ are lifted to the same contractible open set $\widetilde{D}$ which is a fixed connected component of $\pi^{-1}(D)$. Recall that $P = F \circ \widetilde{\beta}$. The accumulation at $\beta'(0)$ can only happen if there exists elements $\gamma_i \in \Gamma$ such that $\widehat{\Phi}(\gamma_i)(\beta(s_i)) = P(s_i) = P(0) + s_i\xi$ for some nonzero vector ξ in H.

The foliation $\mathcal{H}$ on D determines a family of affine lines Π_y whose intersection with $\widetilde{D}$ correspond to liftings of leaves of D. Denote by T the union of these affine lines. Consider now the holomorphic map g at smooth points of T defined by assigning to y the complex line Λ_y in H passing through the origin and parallel to Π_y. We may assume that the lines Λ_y are not all identical, otherwise D will be locally the isometric product of a Hermitian Riemann surface with a flat Riemann surface, so that the Ricci curvature is everywhere degenerate, contradicting with the assumption in Theorem 3. We can also define a multi-valued holomorphic map h on $\widetilde{X} - \pi^{-1}(E \cup A)$ as in the definition of g. The map h leads to a multi-valued meromorphic map into the projective space $\mathbb{P}(\mathbb{H})$ of lines in H. The action of Γ via Φ induces projective unitary transformations on $\mathbb{P}(\mathbb{H})$ which preserve the Fubini-Study metric. Consider the subset $\widetilde{B}$ of $\widetilde{X}-\pi^{-1}(E\cup A)$ where one of the k branches of h fails to be a holomorphic map of rank 1 into $\mathbb{P}(\mathbb{H})$. Then, $\widetilde{B}$ is invariant under Γ and descends to a complex-analytic subvariety B of $X-E-A$. Without loss of generality we may assume that $\beta'(0)$ has been chosen to avoid B. The upshot is that we have a multi-semi-Kähler structure defined by pulling back the Fubini-Study metric by branches of h such that the sequence $\{\widehat{\Phi}(\gamma_i)\}$ of elements of $U(H)\ltimes H$ induces a countably infinite set of holomorphic isometries of a branch $\mathcal{H}$ of the multi-foliation on D. The relation $\widehat{\Phi}(\gamma_i)(\beta(s_i)) = P(s_i) = P(0)+s_i\xi$ and the argument in Proposition (3.2.3) then show that there exists a non-zero vector η tangent to each point of $\widetilde{D}$, implying that g is constant, a contradiction. The proof of Theorem 3 is complete. $\square$

BIBLIOGRAPHY

[Ca] F. CAMPANA, Réduction d'Albanese d'un morphisme propre et faiblement kählérien, (French), C.R. Acad. Sci. Paris Sér. I Math., 296 (3) (1983), 155–158.

[GS] M. GROMOV and R. SCHOEN, Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one, Publi. Math. IHES, 76 (1992).

[HV] P. de la HARPE and A. VALETTE, La Propriété (T) de Kazhdan pour les Groupes Localement Compacts, Astérisque, Vol. 175, Paris 1989.

[Ko] J. KOLLÁR, Shafarevich maps and plurigenera of algebraic varieties, Invent. Math., 113 (1993), 177–215.

[KS] N. J. KOREVAAR and R. M. SCHOEN, Sobolev spaces and harmonic maps for metric space targets. Comm. Anal. Geom., 1 (3-4) (1993), 561–659.

[KV] Y. KAWAMATA and E. VIEHWEG, On a characterisation of Abelian varieties in the classification theory of algebraic varieties, Comp. Math., 41 (1980), 355–360.

[M1] N. MOK, Factorization of semisimple discrete representations of Kähler groups, Invent. Math., 110 (1992), 557–614.

[M2] N. MOK, Harmonic forms with values in locally constant Hilbert bundles, Proceedings of the conference in honor of J.-P. Kahane (Orsay 1993), Journal of Fourier Analysis and Applications (1995), Special Issue, 433–455.

[M3] N. MOK, Fibering compact Kähler manifolds over projective-algebraic varieties of the general type, Proceedings of the International Congress of Mathematicians (ICM) (ed. S.D. Chatterji), Zürich 1994, Birkhäuser 1995.

[M4] N. MOK, The generalized Theorem of Castelnuovo-de Franchis for unitary representations, in Geometry from the Pacific Rim (ed. A.J. Berrick, B. Loo and H.-Y. Wang, de Gruyter), Berlin-New York, 1997, pp. 261–284.

[Sa] F. SAKAI, Semi-stable curves on algebraic surfaces and logarithmic pluricanonicial map, Math. Ann., 254 (1980), 89–120.

[Si] Y.-T. SIU, Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems, J. Diff. Geom., 17 (1982), 55–138.

[Ue] K. UENO, Classification theory of algebraic varieties and compact complex spaces, Springer Lecture Notes, Vol. 439, 1975.

[Zi] R. L. ZIMMER, Ergodic Theory and Semisimple Groups, Birkhäuser, Boston, 1985.

[Zu] K. ZUO, Kodaira dimension and Chern hyperbolicity of the Shafarevich maps for representations of π_1 of compact Kähler manifolds, J. reine angew. Math., 472 (1996), 139–156.

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