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ALGEBRAS OF DIFFERENTIABLE FUNCTIONS ON RIEMANN SURFACES

by K. de LEEUW and H. MIRKIL

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1. Introduction.

The structure of a C^∞ manifold is completely characterized by the algebra of C^∞ functions on it. On a Riemann surface, however, there is no obvious candidate for a characteristic algebra of globally defined functions. It is the purpose of this paper to define such an algebra, in fact denumerably many such algebras.

The present paper is a sequel to [1] and we use the notation and several of the results of that paper.

The main result established there is the classification of *rotating spaces of differentiable functions* in the plane. These are defined as follows. Denote by C_0 the space of all complex-valued continuous functions in the plane that are zero at infinity. For each set $\mathcal{A}$ of constant coefficient differential operators

$$\Sigma a_{m,n} \partial^{m+n}/\partial x^m \partial y^n,$$

define $C_0(\mathcal{A})$ to be the space of all f in C_0 that satisfy $Af \in C_0$, in the sense of the theory of distributions, for all $A \in \mathcal{A}$. Each $C_0(\mathcal{A})$ is translation invariant; those that are also rotation-invariant are the *rotating spaces of differentiable functions.*

The familiar rotating spaces are C_0^N, the space of those f in C_0 having all derivatives of order $\leqslant N$ in C_0, and $C_0^\infty = \cap C_0^N$.

In [1] we show that the only other rotating spaces are the $C_0(\mathcal{A})$ for $\mathcal{A}$ a proper subset of

$$\{\partial^{m+n}/\partial z^m \partial \bar{z}^n : m + n = N\}$$

for some positive integer N. We show that such a $C_0(\mathfrak{A})$ satisfies $C_0^N \subset C_0(\mathfrak{A}) \subset C_0^{N-1}$ and that $C_0(\mathfrak{A}_1)$ and $C_0(\mathfrak{A}_2)$ are distinct if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are distinct proper subsets of

$$\{\partial^{m+n}/\partial z^m \partial \bar{z}^n : m + n = N\}.$$

Finally we show that each of these rotating spaces is a Banach algebra in its natural topology.

The present paper introduces similar algebras on arbitrary Riemann surfaces. Our main concern is showing that these algebras determine conformal structure.

We outline our procedure. It is necessary first to introduce the algebras associated with open subsets of the plane. Let U be an open subset of the plane, C(U) the space of all complex valued continuous functions on U. For each proper subset $\mathfrak{A}$ of

$$\{\partial^{m+n}/\partial z^m \partial \bar{z}^n : m + n = N\}$$

we define $C(\mathfrak{A}, U)$ to be the subspace of C(U) consisting of those f having all derivatives of order $\leqslant N - 1$ in C(U) and which further satisfy $Af \in C(U)$, in the sense of the theory of distributions, for all $A \in \mathfrak{A}$.

In § 2 we establish several useful properties of the $C(\mathfrak{A}, U)$. In particular, we show that they are preserved under analytic mappings. That is, if U_1 and U_2 are open subsets of the plane and $\Phi : U_1 \to U_2$ is a complex analytic mapping, $f \circ \Phi$ will be in $C(\mathfrak{A}, U_1)$ if f is in $C(\mathfrak{A}, U_2)$. This allows us in § 3 to extend the definition of the spaces $C(\mathfrak{A}, U)$ to Riemann surfaces. For a Riemann surface R and $\mathfrak{A}$ a proper subset of

$$\{\partial^{m+n}/\partial z^m \partial \bar{z}^n : m + n = N\},$$

$C(\mathfrak{A}, R)$ is the space of those functions on R which are such that, if $\Phi : U \to R$ is a coordinate disk, then $f \circ \Phi$ is in $C(\mathfrak{A}, U)$. Because of the analytic invariance established in § 2, this definition agrees with the previous definition if R is a planar Riemann surface.

All the properties established for the $C(\mathfrak{A}, U)$ now extend to the $C(\mathfrak{A}, R)$. In particular if R_1 and R_2 are Riemann surfaces, $\mathfrak{A}$ a proper subset of

$$\{\partial^{m+n}/\partial z^m \partial \bar{z}^n : m + n = N\},$$

it is immediate from the definition that an analytic isomorphism of R_1 with R_2 induces an algebra isomorphism of $C(\mathcal{A}, R_1)$ with $C(\mathcal{A}, R_2)$; and if $\mathcal{A}$ is symmetric [1], an anti-analytic isomorphism of R_1 with R_2 induces an algebra isomorphism of $C(\mathcal{A}, R_1)$ with $C(\mathcal{A}, R_2)$. The remainder of the paper is devoted to establishing a converse of this assertion, that algebra isomorphisms (and even homomorphisms) of the $C(\mathcal{A}, R)$ must be induced by analytic (or possibly anti-analytic, if $\mathcal{A}$ is symmetric) mappings of the associated Riemann surfaces.

This result is established in three steps. We first show (Theorem 3.7) that the multiplicative functionals are defined by points, hence that a homomorphism of the function algebra must be induced by a point mapping. Then (Propositions 4.3 and 4.4) the homomorphism problem for function algebras on general R is reduced to the same problem on the unit disk. Finally (Propositions 5.1 through 5.8) we get at the heart of the question, and identify the homomorphisms for function algebras on the unit disk. Thus the special ideas of this paper lie mostly in Section 5, which can be read for its own sake with only minimum reference to other sections.

It is worthhile summarizing our main results.

THEOREM 1. — *Let* R_1 *and* R_2 *be connected Riemann surfaces,* $\mathcal{A}$ *a proper subset of*

$$\{\partial^{m+n}/\partial z^m \partial \bar{z}^n : m + n = N\}.$$

Each analytic mapping of R_1 *into* R_2 *induces an algebra homomorphism of* $C(\mathcal{A}, R_2)$ *into* $C(\mathcal{A}, R_1)$. *If* $\mathcal{A}$ *is symmetric, each anti-analytic mapping of* R_1 *into* R_2 *induces an algebra homomorphism of* $C(\mathcal{A}, R_2)$ *into* $C(\mathcal{A}, R_1)$. *No other algebra homomorphisms from* $C(\mathcal{A}, R_2)$ *to* $C(\mathcal{A}, R_1)$ *are possible.*

Let us also list here some notation that will be used throughout the paper.

For R an open subset of the plane, or more general any Riemann surface, we denote by C(R) the space of all continuous complex valued functions on R, $C^N(R)$ the space of functions on C(R) that are N-times continuously differentiable,

[1] We call $\mathcal{A}$ *symmetric* if $\partial^{m+n}/\partial z^m \partial \bar{z}^n$ in $\mathcal{A}$ implies $\partial^{m+n}/\partial z^n \partial \bar{z}^m$ in $\mathcal{A}$.

$C^\infty(R)$ the space of functions in $C(R)$ that are infinitely differentiable, $D(R)$ the space of functions in $C^\infty(R)$ that have compact support, $D'(R)$ the space of distributions on R, that is, the dual space of $D(R)$. If R is the plane E, these spaces will be denoted simply by C, C^N, C^∞, D and D'.

Suppose $U \subseteq V$. If f is a function defined on V, then $f|U$ denotes its restriction to U. On the other hand if f is originally defined on U, then $f|V$ denotes its extension to V by taking $f(p) = 0$ for $p \notin U$. We shall use the latter notation only when U is an open subset of V and f has compact support in U, in which case $f|V$ has the same differentiability properties as f.

2. Definition and properties of the C($\mathfrak{A}$, U).

If f is a function and A is a constant-coefficient differential operator, the statement « Af is continuous » is meant in the sense of distribution theory: there is some continuous h (unique if it exists) such that

$$\int hg = \int f A^* g$$

for g that are C^∞ with compact support, where A^* is the formal adjoint (²) of A.

Write

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We show in [1] that if $\mathfrak{B}$ is some proper subset of the operators

$$\left\{\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} : m + n = N\right\},$$

and if

$$\mathfrak{A} = \mathfrak{B} \cup \left\{\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} : 0 \leqslant m + n < N\right\},$$

then $C_0(\mathfrak{B}) = C_0(\mathfrak{A})$. From now on in the present paper we shall consider only such an $\mathfrak{A}$, which we shall call a *complete proper set of rotating operators.*

(²) If $A = \Sigma a_{m,n} \partial^{m+n}/\partial x^m \partial y^n$, $A^* = \Sigma(-1)^{m+n} a_{m,n} \partial^{m+n}/\partial x^m \partial y^n$.

Let U be an open connected subset of the plane. We define $C(\mathfrak{A}, U)$ to be the set of all functions f on U such that Af is continuous for each $A \in \mathfrak{A}$. The natural topology on $C(\mathfrak{A}, U)$ declares that f_j converges to f if, for each $A \in \mathfrak{A}$ and each compact $K \subseteq U$, Af_j converges to Af uniformly on K.

It is the purpose of this section to establish certain elementary properties of the $C(\mathfrak{A}, U)$.

PROPERTY 2.1. — *Membership in* $C(\mathfrak{A}, U)$ *can be verified locally.*

PROPERTY 2.2. — $D(U)$ *is dense in* $C(\mathfrak{A}, U)$.

PROPERTY 2.3. — $C(\mathfrak{A}, U)$ *is a topological algebra.*

PROPERTY 2.4. — *Given* $f \in C(\mathfrak{A}, U)$ *and* φ *analytic on* $f(U)$, *then* $\varphi \circ f \in C(\mathfrak{A}, U)$.

PROPERTY 2.5. — *Given* φ *analytic on* U *and* $f \in C(\mathfrak{A}, \varphi(U))$, *then* $f \circ \varphi \in C(\mathfrak{A}, U)$.

PROPERTY 2.6. — *Given* φ *anti-analytic on* U *and*

$$f \in C(\mathfrak{A}, \varphi(U)),$$

then $f \circ \varphi \in C(\overline{\mathfrak{A}}, U)$.

$$\left(\text{Definition}: \frac{\partial^{m+n}}{\partial z^n \partial \bar{z}^m} \in \overline{\mathfrak{A}} \text{ if and only if } \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} \in \mathfrak{A}.\right)$$

Each of the above properties is equivalent to, or follows easily from, the corresponding more detailed proposition below. In particular, it is clearly enough to establish properties (2.1), (2.3), (2.4), (2.5) and (2.6) for the case where $\mathfrak{A}$ consiste of the single highest-order differential operator $\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n}$ together with all $\frac{\partial^{r+s}}{\partial z^r \partial \bar{z}^s}$ of lower order. Such an $\mathfrak{A}$ we shall call $\mathfrak{A}^{m,n}$. We shall abbreviate $\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n}$ by $Z^m\overline{Z}^n$, and sometimes by $Z^{m,n}$.

PROPOSITION 2.1′. — *Let f be defined on* $U = \cup U_\alpha$, *open subsets of the plane. Suppose for each* α, $f \in C(\mathfrak{A}, U_\alpha.)$ *Then* $f \in C(\mathfrak{A}, U)$.

Proof. — Differentiation and continuity are both defined locally.

PROPOSITION 2.2′. — *Let* U *be a plane domain, let* K *be a compact subset of* U, *let* $f \in C(\mathfrak{A}, U)$, *and let* $\varepsilon > 0$. *Then there is some* $g \in D(U)$ *such that for each* $A \in \mathfrak{A}$ *we have*

$$\sup_{p \in K} |Af(p) - Ag(p)| < \varepsilon.$$

PROPOSITION 2.3′. — *Let f and* $g \in C(\mathfrak{A}^{m,n}, U)$. *Then* $Z^{m,n}(fg)$ *is a fixed linear combination of terms* $(Z^{p,q}f)(Z^{r,s}g)$ *with p and* $r \leqslant m$, *q and* $s \leqslant n$.

Proof of Proposition 2.3′ in case both f and $g \in D(U)$. This is simply a version of the classical Leibniz formula. See, for instance, Lemma 6.1 of [1].

Proof of Proposition 2.3′ in case $f \in D(U)$ and $g \in C(\mathfrak{A}^{m,n}, U)$. Each side of the formula makes sense for such f and g, and indeed for $g \in D'(U)$. Fixing f, each side is continuous in g for the weak topology on $D'(U)$. Hence the present case follows from the preceding case by passage to the limit.

Proof of Proposition 2.2′. Choose in D(U) some $h \equiv 1$ on a neighborhood of K. By Property 2.1 the extended function fh/E belongs to $C_0(\mathfrak{A})$. Apply Proposition 2.2 of [1], which asserts the density of D in $C_0(\mathfrak{A})$, and restrict back to K.

Proof of Proposition 2.3′ in the general case. We need only to verify the formula near each point p. Hence multiply by some local constant h, as in the proof of Proposition 2.2′ above, pushing everything into $C_0(\mathfrak{A})$ where the formula is already established (Lemma 6.2 of [1]).

For an alternate proof, employ another passage to the limit. Start from the case where $f_j \in D(U)$ and $g \in C(\mathfrak{A}^{m,n}, U)$, then let $f_j \to f$ in the topology of $C(\mathfrak{A}^{m,n}, U)$. The right hand side of the formula converges uniformly on compacts, and the left-hand side at least in weak distribution topology.

The inductive step in each of the remaining proofs will depend on the chain rule formulas below, valid for any differentiable f and g with properly matching domain and range. We write $f \circ g$ for the composite function $(f \circ g)(p) = f(g(p))$.

Formula 2A.

$$Z(f \circ g) = (Zf \circ g)Zg + (\overline{Z}f \circ g)Z\overline{g}.$$

Formula 2B.

$$\overline{Z}(f \circ g) = (Zf \circ g)\overline{Z}g + (\overline{Z}f \circ g)\overline{Z}\overline{g}.$$

PROPOSITION 2.4′. — *Let $f \in C(\mathfrak{A}^{m,n}, U)$ and let φ be analytic on some open set containing $f(U)$. Then $Z^{m,n}(\varphi \circ f)$ is a linear combination of terms each of which is some $Z^k\varphi \circ f$ times certain factors $Z^{p,q}f$ with $p \leqslant m$ and $q \leqslant n$.*

Proof. — For $f \in D(U)$ the stated facts about the formula for $Z^{m,n}(\varphi \circ f)$ are established by straightforward induction on the order $m + n$, using Formulas 2A and 2B. The general case is now handled by passage to the limit, exactly as in the alternate proof of the general case of Proposition 2.3′.

PROPOSITION 2.5′. — *Let φ be analytic on U and let $f \in C(\mathfrak{A}^{m,n}, \varphi(U))$. Then $Z^{m,n}(f \circ \varphi)$ is a linear combination of terms each of which is some $Z^{p,q}f \circ \varphi$, with $p \leqslant m$ and $q \leqslant n$, times certain factors $Z^k\varphi$, times certain factors $\overline{Z}^l\overline{\varphi}$.*

Proof. By induction for $f \in D$. Then pass to limit as in Propositions 2.3′ and 2.4′.

PROPOSITION 2.6′. — *Let φ be anti-analytic on U (i.e. let $\overline{\varphi}$ be analytic) and let $f \in C(\mathfrak{A}^{m,n}, \varphi(U))$. Then $Z^{m,n}(f \circ \varphi)$ is a linear combination of terms each of which is some $Z^{p,q}f \circ \varphi$, with $p \leqslant n$ and $q \leqslant m$, times certain factors $Z^k\overline{\varphi}$ times certain factors $\overline{Z}^l\varphi$.*

Proof. — Same as above.

3. Definition and properties of the C($\mathfrak{A}$, R).

Throughout the following, the unit disk $\{Z : |Z| < 1\}$ will be denoted by W.

Let R be a connected Riemann surface, $\mathfrak{A}$ a complete proper set of rotating operators. C($\mathfrak{A}$, R) is defined to be the subspace of C(R) consisting of all functions f that satisfy the following:

For each analytic mapping $\varphi : W \to R$, the function $f \circ \varphi$ is in C(W, $\mathfrak{A}$).

If R is an open subset of the complex plane considered as a

Riemann surface, the definition we have given of $C(R, \mathcal{A})$ agrees with that of § 2 because of Property 2.5.

In three instances $C(\mathcal{A}, R)$ can be given an equivalent definition in terms of exterior differential operators defined globally on R :

$$C(\partial/\partial z, R) = \{f : f \text{ and } \partial f \text{ continuous on } R\},$$
$$C(\partial/\partial \bar{z}, R) = \{f : f \text{ and } \bar{\partial} f \text{ continuous on } R\},$$
$$C(\partial^2/\partial z \partial \bar{z}, R) = \{f : f \text{ and } \Delta f \text{ continuous on } R\},$$

where the operators ∂, $\bar{\partial}$ and Δ, taking functions into differential forms, are defined in terms of any coordinate system by

$$\partial f = \frac{\partial f}{dz} dz, \qquad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}, \qquad \Delta f = \frac{1}{4} \frac{\partial^2 f}{\partial z \partial \bar{z}} dz\, d\bar{z}.$$

Corresponding to the Properties 2.1 through 2.6 of $C(\mathcal{A}, U)$ one can make analogous statements, call them Properties 3.1 through 3.6, about $C(\mathcal{A}, R)$. We assert that these are true properties of $C(\mathcal{A}, R)$.

Let us first examine Property 3.1. Suppose R is the union of subdomains R_α such that $f|R_\alpha \in C(\mathcal{A}, R_\alpha)$ for each α. Suppose $\varphi : W \to R$ is analytic. We can express W as the union of subdisks W_β such that each $\varphi(W_\beta)$ lies entirely in some R_α. To show $f \circ \varphi \in C(\mathcal{A}, W)$ it is enough, by Property 2.1 ,to show each $f \circ \varphi|W_\beta \in C(\mathcal{A}, W_\beta)$. And this follows by applying the definition of $C(\mathcal{A}, R_\alpha)$ with the mapping $\varphi|W_\beta$.

Property 3.2 is meaningless until we define the natural topology on $C(\mathcal{A}, R)$. We do not insist here on this (quite straightforward) definition and proof, since we shall have no need of this property. Omitting topology, 3.3 follows from 2.3 by localizing. Similarly 3.4 from 2.4.

Properties 3.5 and 3.6 follow from 2.5 and 2.6 respectively by direct use of the definition of $C(\mathcal{A}, R)$.

We conclude this section by identifying the multiplicative linear functionals on $C(\mathcal{A}, R)$.

THEOREM 3.7. — *Every multiplicative linear functional* μ *on* $C(\mathcal{A}, R)$ *is defined by evaluation at some point* $p \in R$.

Proof. — Assume that R is non-compact. Since R is connected, by the theorem of Radó [3], R is 2nd countable. Thus one can find a real C^∞ function f on R going to $+\infty$ at infinity. Adding an appropriate real constant to f, we can take $\mu(f) = 0$. Now suppose, contrary to fact, that μ is defined by no $p \in R$. Then for each p there is some $g_p \in C(\mathfrak{A}, R)$ such that $g_p(p) = 1$ but $\mu(g_p) = 0$. And multiplying g_p by an appropriate non-negative C^∞ function of compact support, we can suppose $\mathrm{Re}[g_p] \geqslant 0$ everywhere on R. Let K be the compact set on which $f \leqslant 0$, and choose $g_{p_1}, \ldots, g_{p_n}$ such that

$$\mathrm{Re}\{g_{p_1} + \cdots + g_{p_n}\} > 0$$

everywhere on K. Then for some $\varepsilon > 0$ we have

$$\mathrm{Re}\{\varepsilon f + g_{p_1} + \cdots + g_{p_n}\} > 0$$

everywhere on R. In particular, by Property 2.4,

$$\varepsilon f + g_{p_1} + \cdots + g_{p_n}$$

has its reciprocal in $C(\mathfrak{A}, R)$, contradicting the fact that $\mu(\varepsilon f + g_{p_1} + \cdots + g_{p_n}) = 0$. This proves the theorem for non-compact R. For compact R, the same argument works, omitting all mention of f.

4. Homomorphisms of $C(\mathfrak{A}, R)$.

Let R_1 and R_2 be connected Riemann surfaces. For any $\varphi : R_1 \to R_2$ write $\tilde{\varphi}$ for the transpose mapping of functions on R_2 into functions on R_1 defined by $\tilde{\varphi} f = f \circ \varphi$. Now let $\mathfrak{A}$ be a complete proper set of rotating operators. Then from Properties 3.5 and 3.6. we have immediately :

Proposition 4.1. — *If $\varphi : R_1 \to R_2$ is analytic, or possibly anti-analytic if $\mathfrak{A}$ is symmetric* (1), *then the transpose mapping $\tilde{\varphi}$ is an algebra homomorphism* (3) *from $C(\mathfrak{A}, R_2)$ into $C(\mathfrak{A}, R_1)$.*

On the other hand, from Theorem 3.7 we have :

Proposition 4.2. — *Every algebra homomorphism* (3) *from $C(\mathfrak{A}, R_2)$ into $C(\mathfrak{A}, R_1)$ is the transpose $\tilde{\varphi}$ of some point mapping $\varphi : R_1 \to R_2$.*

(3) An algebra homomorphism φ is assumed to satisfy $\varphi(1) = 1$.

Proof. — For each $y \in R_1$, the mapping

$$f \to Tf(y), \qquad f \in C(\mathcal{A}, R_2),$$

is a multiplicative linear functional, so by Theorem 3.7, there is a point x_y in R_2 satisfying

$$Tf(y) = f(x_y), \qquad f \in C(\mathcal{A}, R_2).$$

If $\varphi : R_1 \to R_2$ is defined by

$$\varphi(y) = x_y, \qquad y \in R_1,$$

then φ satisfies $\tilde{\varphi}f = f \circ \varphi$. No other mapping T can satisfy $Tf = f \circ \varphi$ since $C(\mathcal{A}, R_2)$ separates points of R_2.

Our aim in the rest of this paper is to show that the φ in Proposition 4.2 must be of the sort exhibited in Proposition 4.1, namely analytic or possibly anti-analytic if $\mathcal{A}$ is symmetric. In the remainder of this section we show how to reduce this question to that where both R_1 and R_2 are disks. And this special case is settled in § 5.

The following lemma is the result that allow us to reduce our problem to that for disks.

LEMMA 4.3. — *Let R_1 and R_2 be Riemann surfaces, $\varphi : R_1 \to R_2$ a mapping with $\tilde{\varphi}(C(\mathcal{A}, R_2))$ contained in $C(\mathcal{A}, R_1)$. Let S_1 and S_2 be open subsets of R_1 and R_2 respectively with $\varphi(S_1) \subset S_2$. Then $\tilde{\varphi}(C(\mathcal{A}, S_2))$ is contained in $C(\mathcal{A}, S_1)$.*

Proof. — Let f be any function in $C(\mathcal{A}, S_2)$. We must show that $\tilde{\varphi}f = f \circ \varphi$ is in $C(\mathcal{A}, S_1)$; *i.e.*, if $\Psi : W \to S_1$ is an analytic mapping, then $f \circ \varphi \circ \Psi$ is in $C(\mathcal{A}, W)$. By Property 2.1 it is enough to show that, for any interior subdisk V of W, $(f \circ \varphi \circ \Psi)|V$ is in $C(\mathcal{A}, V)$, and this is what we shall establish. φ is a continuous mapping of R_1 into R_2, since for any h in $C^\infty(R_2)$, $h \circ \varphi$ is in $C(R_1)$. Ψ is also continuous and thus $\varphi(\Psi(\overline{V}))$ is compact in S_2. Let $k \in D(S_2)$, $k \equiv 1$ on $\varphi(\Psi(\overline{V}))$. $kf \in C(\mathcal{A}, S_2)$, so by Property 2.1, the extension $kf|E$ is in $C(\mathcal{A}, R_2)$. Denote the extension by g. Then $\tilde{\varphi}g = g \circ \varphi$ is in $C(\mathcal{A}, R_2)$, so $g \circ \varphi \circ (\Psi|V)$ is in $C(\mathcal{A}, V)$. But

$$g \circ \varphi \circ (\Psi|V) = (f \circ \varphi \circ \Psi)|V,$$

which is the function we were to show in $C(\mathfrak{A}, V)$. This completes the proof of Lemma 4.3.

We now state the main theorem of this section. We prove it under the assumption of Proposition 5.1, which is the special case of the theorem for R_1 and R_2 disks.

PROPOSITION 4.4. — *Let* R_1 *and* R_2 *be connected Riemann surfaces,* $\mathfrak{A}$ *some complete proper set of rotating operators,* $\varphi : R_1 \to R_2$ *a mapping which is such that* $\tilde{\varphi}$ *takes* $C(\mathfrak{A}, R_2)$ *into* $C(\mathfrak{A}, R_1)$. *Then if* $\mathfrak{A}$ *is non-symmetric,* φ *must be analytic. If* $\mathfrak{A}$ *is symmetric,* φ *must be either analytic or anti-analytic.*

Proof. — Let x be any point of R_1, $W_2 \subset R_2$ a coordinate disk containing $\varphi(x)$, $W_1 \subset R_1$ a coordinate disk containing x and satisfying $\varphi(W_1) \subset W_2$. By Lemma 4.3 and Proposition 5.1, φ is as claimed on W_1. We are finished in the case that $\mathfrak{A}$ is non-symmetric. In the case that $\mathfrak{A}$ is symmetric, by the connectedness of R_1, φ cannot be analytic on some coordinate disk and antianalytic on another unless it is a constant map. Thus φ must be either analytic or anti-analytic on all of R_1.

At this point, assuming Proposition 5.1 ,we have established all of the results summarized in the Theorem 1.1. The first two sentences of the theorem are the content of Proposition 4.1 and the last is Proposition 4.4.

5. Homomorphisms of $C(\mathfrak{A}, W)$.

The proof of Theorem 4.4 assumed the validity of the following special case of that theorem.

PROPOSITION 5.1. — *Let* $\mathfrak{A}$ *be a complete proper set of rotating operators,* $\varphi : W \to W$ *a mapping which is such that* $\tilde{\varphi}$ *takes* $C(W, \mathfrak{A})$ *into itself. Then if* $\mathfrak{A}$ *as non-symmetric,* φ *must be analytic. If* $\mathfrak{A}$ *is symmetric,* φ *must be either analytic or anti-analytic.*

This section is devoted to the establishment of this result.

We shall denote $C(\mathfrak{A}, W)$ by B. By the results of § 2, B is a topological algebra containing $C^\infty(W)$ as a dense subalgebra. We denote by B^* the dual of B and by B_0 the ideal in B of those functions in B vanishing in a neighborhood of O. The annihilator $B_0^{\perp}$ of B_0 in B^* is then the set of linear functionals having support $\{O\}$. Using the density of $C^\infty(W)$ in B, we identify in the next lemma the elements in $B_0^{\perp}$ with differentiation operators at O.

For each $\partial^{r+s}/\partial z^r \partial \bar{z}^s$ in $\mathfrak{A}$ denote by $T_{r,s}$ the element of B^* defined by

$$T_{r,s}f = \left(\frac{\partial^{r+s}f}{\partial z^r \partial \bar{z}^s}\right)(0), \; f \in B.$$

Lemma 5.2. — *$\{T_{r,s}\}$ is a basis for $B_0^{\perp}$.*

Proof. — It is clear that $\{T_{r,s}\} \subset B_0^{\perp}$ and is linearly independent, so it suffices to show that it spans $B_0^{\perp}$. Let $T \in B_0^{\perp}$. By the closed graph theorem, the injection $C^\infty(W) \to B$ is continuous. Then T, restricted to $C^\infty(W)$, is a continuous linear functional on $C^\infty(W)$ having support $\{O\}$. As a consequence there is a constant-coefficient differential operator A_0 so that

$$Tf = A_0 f(0), \quad f \in C^\infty(W).$$

By the definition of the topology of B, there is a subdisk V of W and a constant K so that

$$|A_0 f(0)| = |Tf| \leqslant K \sum_{A \in \mathfrak{A}} \sup_{x \in V} |Af(x)|, \; f \in C^\infty(W).$$

In particular, for all f in D with support in W,

$$|A_0 f(0)| \leqslant K \sum_{A \in \mathfrak{A}} \|Af\|.$$

By Theorem 7.1 of [1], this is impossible unless A_0 is a linear combination of the A in $\mathfrak{A}$. Thus T agrees with a linear combination of the $T_{r,s}$ on $C^\infty(W)$, which is dense B, so T agrees with a linear combination of the $T_{r,s}$ on all of B. This completes the proof of Lemma 5.2.

We shall denote by $B_0^{\perp\perp}$ the annihilator in B of $B_0^{\perp}$, so that by Lemma 5.2 a function f in B is in $B_0^{\perp\perp}$ if and only if $T_{r,s} = 0$, all r, s. Since B_0 is a linear subspace of B, $B_0^{\perp\perp}$ is simply the closure of B_0 in B. As a consequence $B_0^{\perp\perp}$ is an ideal in B, for B_0 is an ideal and the multiplication in B is continuous.

We shall denote by z the function defined on W by $z(\zeta) = \zeta$.

For each f in B, we define the function f_0 by

$$f = \sum_{r,s} \frac{1}{r!s!} (T_{r,s}f) z^r \bar{z}^s + f_0,$$

so $T_{r,s} f_0 = 0$, all r, s, and f_0 is in the ideal $B_0^{\perp\perp}$.

The next lemma shows, in the case that $\mathcal{A}$ is not symmetric, that for certain f in B, the fact that $\partial f / \partial \bar{z} = 0$ at 0 is equivalent to an assertion about f of an algebraic nature.

Lemma 5.3. — *Assume $\mathcal{A}$ not symmetric. Choose (m, n) so that $\partial^{m+n}/\partial z^n \partial \bar{z}^m$ is in $\mathcal{A}$, $\partial^{m+n}/\partial z^m \partial \bar{z}^n$ not in $\mathcal{A}$. Let f be a function in* B *with $\bar{f}$ in* B *and $f(0) = 0$. Then the following are equivalent:*

1° $\partial f / \partial \bar{z} = 0$ at 0;

2° $f^m \bar{f}^n$ *is in the ideal* $B_0^{\perp\perp}$.

Proof. — We must treat separately the two cases where $m + n = 1$. Let $m = 1$, $n = 0$, as that $\partial / \partial \bar{z}$ is in $\mathcal{A}$, $\partial / \partial z$ is not. The assertion to be proved is that $\partial f / \partial \bar{z} = 0$ at 0 if and only if $f \in B_0^{\perp\perp}$. This is clear, since by Lemma 5.2, $B_0^{\perp}$ has as basis $\{T_{0,1}, T_{0,0}\}$.

If $m = 0$, $n = 1$, then $\partial / \partial z$ is in $\mathcal{A}$, $\partial / \partial \bar{z}$ is not. The assertion to be proved is that $\partial f / \partial \bar{z} = 0$ at 0 if and only if $f \in B_0^{\perp\perp}$. This is also clear, since by Lemma 5.2, $B_0^{\perp}$ has $\{T_{1,0}, T_{0,0}\}$ as basis.

We may now assume $m + n \geqslant 2$. We show first that 1° implies 2°. If 1° holds for f, then

$$f = cz + (\text{terms in } z^r \bar{z}^s,\ r + s > 1) + f_0,$$
$$\bar{f} = c\bar{z} + (\text{terms in } z^r \bar{z}^s,\ r + s > 1) + (\bar{f})_0.$$

As a consequence

$$f^m = c^m z^m + (\text{terms of order} > m) + (\text{element of } B_0^{\perp\perp}),$$
$$\bar{f}^n = \bar{c}^n \bar{z}^n + (\text{terms of order} > n) + (\text{element of } B_0^{\perp\perp}),$$

using the fact that $B_0^{\perp\perp}$ is an ideal. Thus, since $z^u z^v \in B_0^{\perp\perp}$ if $u + v > m + n$,

$$f^m \bar{f}^n = c^m \bar{c}^m z^m \bar{z}^n + (\text{element of } B_0^{\perp\perp}).$$

But $z^m \bar{z}^n \in B_0^{\perp\perp}$ by Lemma 5.2 because of the choice of (m, n). Thus $f^m \bar{f}^n \in B_0^{\perp\perp}$.

We show next that 2° must be false if 1° is false. We may assume $\partial f/\partial \bar{z} = 1$ at 0. Then

$$f = az + \bar{z} + (\text{terms of order} > 1) + f_0,$$
$$\bar{f} = \bar{a}\bar{z} + z + (\text{terms of order} > 1) + (\bar{f})_0.$$

There are two cases to discuss, depending on whether or not a is 0.

Case I. $a = 0$. Then

$$f^m\bar{f}^n = \bar{z}^m z^n + (\text{element of } B_0^{\perp\perp}).$$

$\bar{z}^m z^n$ is not in $B_0^{\perp\perp}$ since $\partial^{m+n}\partial z^n \partial \bar{z}^m$ is in $\mathfrak{A}$. Thus $f^m\bar{f}^n$ is not in $B_0^{\perp\perp}$.

Case II. $a \neq 0$. Then

$$f^m = (az + \bar{z})^m + (\text{terms of order} > m) + (\text{element of } B_0^{\perp\perp}),$$
$$\bar{f}^n = (\bar{a}\bar{z} + z)^n + (\text{terms of order} > n) + (\text{element of } B_0^{\perp\perp}),$$

so

$$(5.1) \quad f^m\bar{f}^n = (az + \bar{z})^m(\bar{a}\bar{z} + z)^n + \text{element of } B_0^{\perp\perp}).$$

But

$$(5.2) \qquad (az + \bar{z})^m(\overline{az} + z)^n = \sum_{-m}^{+n} c_t z^{n-t}\bar{z}^{m+t},$$

where

$$c_t = \sum_{\substack{0 \leq r \leq m \\ 0 \leq s \leq n \\ s-r=t}} \binom{m}{r}\binom{n}{s} a^r \bar{a}^s = a^t \sum_{\substack{0 \leq r \leq m \\ 0 \leq s \leq n \\ s-r=t}} \binom{m}{r}\binom{n}{s} |a|^{2r}.$$

In particular $c^t \neq 0$, all t, so by Lemma 5.2, $\sum_{-m}^{+n} c_t z^{n-t}\bar{z}^{m+t}$ cannot be in $B_0^{\perp\perp}$. Thus, because of (5.1) and (5.2), $f^m\bar{f}^n$ is not in $B_0^{\perp\perp}$. This completes the proof of Lemma 5.3.

We next state the analogue of Lemma 5.3 for symmetric $\mathfrak{A}$.

LEMMA 5.4. — *Assume $\mathfrak{A}$ symmetric. Choose (m, n) so that $m + n = N$ and $\partial^{m+n}/\partial z^n \partial \bar{z}^m$ is not in $\mathfrak{A}$. Let f be a function in* B *with $\bar{f}$ in* B *and $f(0) = 0$. Then the following are equivalent:*

1° *Either $\partial f/\partial \bar{z} = 0$ at 0 or $\partial f/\partial z = 0$ at 0;*

2° *$f^m\bar{f}^n$ is in the ideal $B_0^{\perp\perp}$.*

Proof. — Here we must have $m+n \geqslant 2$. That 1° implies 2° is proved exactly as in Lemma 5.3. And that 1° false implies 2° false is proved as in Case II in Lemma 5.3.

Suppose now that $\varphi : W \to W$ is a mapping that satisfies $\tilde{\varphi}(B) \subset B$. We will use the two preceding lemmas to show that φ must be analytic, or possibly anti-analytic if $\mathcal{A}$ is symmetric.

First observe that φ must be at least C^1. For z and $\bar{z}$ are in B, $\tilde{\varphi}(z) = \varphi$ and $\tilde{\varphi}(\bar{z}) = \bar{\varphi}$, so φ and $\bar{\varphi}$ are both in B. If

$$N > 1, B \subset C^1(W),$$

so φ must be C^1. If $N = 1$, one of the operators $\partial/\partial z$ or $\partial/\partial\bar{z}$ is in $\mathcal{A}$, and since both φ and $\bar{\varphi}$ are in B, φ must be C^1.

Corollary 5.5. — *Let* $\varphi : W \to W$ *satisfy* $\varphi(0) = 0$ *and* $\tilde{\varphi}(B) \subset B$. *If* $\mathcal{A}$ *is not symmetric, then* $\partial\varphi/\partial\bar{z} = 0$ *at* 0. *If* $\mathcal{A}$ *is symmetric, then either* $\partial\varphi/\partial\bar{z} = 0$ *or* $\partial\varphi/\partial z = 0$ *at* 0.

Proof. — $\tilde{\varphi}: B \to B$ is continuous by the closed graph theorem. Since $\varphi(0) = 0$, $\tilde{\varphi}(B_0) \subset B_0$, and thus by the continuity of $\tilde{\varphi}$, $\tilde{\varphi}(B_0^{\perp\perp}) \subset B_0^{\perp\perp}$. Choose (m, n) as in the preceding two lemmas. $z^m\bar{z}^n \in B_0^{\perp\perp}$ so $\tilde{\varphi}(z^m\bar{z}^n) \in B_0^{\perp\perp}$. But

$$\tilde{\varphi}(z^m\bar{z}^n) = (\tilde{\varphi}z)^m(\tilde{\varphi}\bar{z})^n = \varphi^m\bar{\varphi}^n,$$

so $\varphi^m\bar{\varphi}^n \in B_0^{\perp\perp}$, and the conclusion follows from Lemmas 5.3 and 5.4.

It is simple now to obtain the analogue of the above corollary for any point of the disk.

Corollary 5.6. — *Let* $\varphi : W \to W$ *satisfy* $\tilde{\varphi}(B) \subset B$. *If* $\mathcal{A}$ *is not symmetric, then* $\partial\varphi/\partial\bar{z} = 0$ *at all points of* W. *If* $\mathcal{A}$ *is symmetric, then at each point of* W *either* $\partial\varphi/\partial\bar{z} = 0$ *or* $\partial\varphi/\partial z = 0$.

Proof. — Choose any point x of W. Let $\psi : W \to W$ be a mapping of the form $\psi(z) = az + b$, $a > 0$, with $\psi(O) = x$. Let $\theta : W \to W$ be a mapping of the form $\theta(z) = cz + d$, $c > 0$, with $\theta(\varphi(x)) = 0$. Define η to be $\theta \circ \varphi \circ \psi$. Then

$$\eta(W) \subset W, \eta(0) = 0$$

and $\tilde{\eta}(B) = \tilde{\psi}(\tilde{\varphi}(\tilde{\theta}(B))) \subset B$, using Proposition 2.5 twice. Assume $\mathcal{A}$ non-symmetric. Then, by Corollary 5.5 applied to η,

$$\partial\eta/\partial\bar{z} = 0$$

at 0, so $\partial\varphi/\partial\bar{z} = 0$ at x. Assume $\mathfrak{A}$ symmetric. Then, by Corollary 5.5 applied to η, either $\partial\eta/\partial\bar{z} = 0$ or $\partial\eta/\partial z = 0$ at 0, so either $\partial\varphi/\partial\bar{z} = 0$ or $\partial\varphi/\partial z = 0$ at x.

The proof of Proposition 5.1 is now almost complete. Corollary 5.6 contains the part of Proposition 5.1 concerned with non-symmetric $\mathfrak{A}$. And to finish the symmetric case it is clear that we need only one additional fact, which is Lemma 5.8 below. For the proof of this lemma we use the following theorem of Radó. See [2].

Proposition 5.7. — *Let g be a continuous function defined in* W. *Suppose that g is analytic on*

$$\{z : z \in W,\ g(z) \neq 0\}.$$

Then g is analytic throughout W.

Lemma 5.8. — *Let φ be a* C^1 *function with domain* W. *Suppose that at each point of* W *either* $\partial\varphi/\partial\bar{z} = 0$ *or* $\partial\varphi/\partial z = 0$. *Then either* $\partial\varphi/\partial\bar{z} \equiv 0$ *or* $\partial\varphi/\partial z \equiv 0$.

Proof. — We shall denote the functions $\partial\varphi/\partial z$ and $\partial\varphi/\partial\bar{z}$ by φ_z and $\varphi_{\bar{z}}$. Let

$$U = \{x : x \in W,\ \varphi_z(x) \neq 0\}.$$

We first show φ_z analytic in U. On U we have $\varphi_{\bar{z}} = 0$. Thus the distribution

$$\left(\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\partial}{\partial z}\right)\varphi = \left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial\bar{z}}\right)\varphi$$

is zero in U, so $\dfrac{\partial}{\partial z}\varphi$ is an analytic distribution in U. But an analytic distribution must be an analytic function, proving φ^z analytic in U. By Proposition 5.7, φ_z is analytic in all of W. The same type of argument shows $\overline{\varphi_{\bar{z}}}$ analytic in W. Since W is the union of the two sets $\{x : \varphi_z(x) = 0\}$ and $\{x : \varphi_{\bar{z}}(x) = 0\}$, which are closed in W, one of them must contain a disk. Thus either φ_z or $\overline{\varphi_{\bar{z}}}$ vanishes on a disk. But they are both analytic in W, so either φ_z or $\overline{\varphi_{\bar{z}}}$ vanishes on all of W.

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