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GALOIS CO-DESCENT FOR ÉTALE WILD KERNELS AND CAPITULATION

by M. KOLSTER* and A. MOVAHHEDI

Introduction.

For a number field F, the classical wild kernel - denoted by $WK_2(F)$ - is the kernel of all local power norm residue symbols on $K_2(F)$, in other words it fits into Moore's exact sequence

$$0 \to WK_2(F) \to K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0,$$

where v runs through all finite and real infinite primes of F, and $\mu(F_v)$ and $\mu(F)$ denote the groups of roots of unity of the local field F_v and of F, respectively. For a fixed prime number p, the p-primary part $WK_2(F)\{p\}$ of $WK_2(F)$ has another description in terms of étale cohomology: For any finite set S of primes in F containing the p-adic primes and the real infinite primes, we have

$$WK_2(F)\{p\} = \ker(H^2_{\text{ét}}(o_F^S, \mathbb{Z}_p(2)) \to \bigoplus_{v\in S} H^2(F_v, \mathbb{Z}_p(2))).$$

This property immediately leads to the definition of the higher *étale wild kernels* for $i \geqslant 2$:

$$WK^{\text{ét}}_{2i-2}(F) := \ker(H^2_{\text{ét}}(o_F^S, \mathbb{Z}_p(i)) \to \bigoplus_{v\in S} H^2(F_v, \mathbb{Z}_p(i))).$$

The étale wild kernels play a similar role in étale cohomology, étale K-theory and Iwasawa-theory as the p-primary parts A'_F of the S-class groups

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of F. For these, Galois co-descent is classically described by genus theory. The main result of this paper proves an analogous genus formula for the étale wild kernels of a cyclic extension L/F of degree p, p odd. Let $G = \mathrm{Gal}(L/F)$. We first show that the transfer map $WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)$ is onto except in a very special situation, and we determine its kernel as the cokernel of a certain cup-product which is obtained as follows: Let $E = F(\mu_p)$, where μ_p consists of the p-th roots of unity and let $\Delta = \mathrm{Gal}(E/F)$. We associate with the extension LE/E a certain set $T_{LE/E}$ of primes of E, consisting of all tamely ramified primes and some undecomposed p-adic primes. Let $\mathrm{Br}^T(E)$ denote the subgroup of the Brauer-group which is supported only at primes in $T_{LE/E}$, and let ${}_p\mathrm{Br}^T(E)$ denote the subgroup of all the elements in $\mathrm{Br}^T(E)$ of exponent p. The target of the cup-product is the $(1-i)-$ eigenspace ${}_p\mathrm{Br}^T(E)^{[1-i]}$, under the action of the Teichmüller character w. Now, let S be the set of primes in E consisting of the p-adic primes, the real infinite primes as well as all primes ramified in LE and denote by o_E^S the ring of S-integers in E. The étale cohomology group $H^1_{\text{ét}}(o_E^S, \mathbb{Z}_p(i))/p$ injects into the $(i-1)-$fold Tate twist of the module E^*/E^{*p} and hence is isomorphic to $D_E^{(i)}/E^{*p}(i-1)$, where $D_E^{(i)} \subset E^{*p}$ can be viewed as the analog of the Tate kernel ($i = 2$). The cup-product is now given by

$$(D_E^{(i)}/E^{*p})^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to ({}_p\mathrm{Br}^T(E))^{[1-i]}.$$

We illustrate the method by finding all Galois p-extensions of $\mathbb{Q}$ for which the p-part of the classical wild kernel is trivial.

We also discuss the Galois co-descent situation for $p = 2$ in the classical case $i = 2$.

Let E_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of E with finite layers E_n. If we assume the Gross Conjecture for E_n with n large, for instance if E is abelian over $\mathbb{Q}$, then the groups $D_{E_n}^{(i)}/E_n^{*p}$ can be described in terms of local conditions at p-adic primes, and are independent of i.

Let F_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of F with finite layers F_n and let A'_n denote the p-part of the p-class group of F_n. The classical capitulation kernel is defined as

$$\mathrm{Cap}_0(F_\infty) = \ker(A'_n \to A'_\infty) \qquad \text{for } n \text{ large.}$$

The study of capitulation kernels under Galois extensions is an essential ingredient in the more general problem of comparing Iwasawa-invariants (cp. e.g. [28]). In Section 3 we introduce similar capitulation kernels

$\mathrm{Cap}_{i-1}(F_\infty)$ for all $i \geqslant 2$ using étale K-theory, and show that they have properties similar to $\mathrm{Cap}_0(F_\infty)$.

Assume now that F is totally real, and let E^+ denote the maximal real subfield of $E = F(\mu_p)$. A conjecture of Greenberg predicts that $\varprojlim A'_n(E^+)$ is finite. Under this assumption we show that for all odd $i \geqslant 3$:

$$\mathrm{Cap}_{i-1}(F_\infty) \cong A'_n(E^+)^{[1-i]} \cong WK^{\text{ét}}_{2i-2}(F_n) \qquad \text{for } n \text{ large.}$$

Therefore the co-descent results from Section 2 imply similar results for $\mathrm{Cap}_{i-1}(F_\infty)$ and for the eigenspaces $A'_n(E^+)^{[1-i]}$, when n is large.

In Section 4, we briefly discuss how our approach can be applied to the simpler problem of Galois co-descent for étale tame kernels. This has already been studied by Assim ([1], [2]) under Leopoldt's conjecture.

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1. Preliminaries.

In this section we briefly recall some of the basic properties of étale K-theory and étale cohomology which are subsequently needed. A more detailed account can be found in [3], [20]. Let F be a number field and p a fixed prime number. Let S be a finite set of primes in F, containing the set S_p of primes above p and the set S_∞ of infinite primes. As usual, $G_S(F)$ denotes the Galois group over F of the maximal algebraic extension of F, which is unramified outside S. We note that the condition on infinite primes only intervenes if $p = 2$ and F is not totally imaginary. Let o_F^S denote the ring of S-integers of F. As is well-known, the étale cohomology groups $H^k_{\text{ét}}(\mathrm{spec}(o_F^S), \mathbb{Z}/p^n\mathbb{Z}(i))$ of $\mathrm{spec}(o_F^S)$ coincide with the Galois-cohomology groups $H^k(G_S(F), \mathbb{Z}/p^n\mathbb{Z}(i))$, and will be denoted by $H^k_{\text{ét}}(o_F^S, \mathbb{Z}/p^n\mathbb{Z}(i))$. Here, as usual, $\mathbb{Z}/p^n\mathbb{Z}(i)$ denotes the i-fold Tate twist of $\mathbb{Z}/p^n\mathbb{Z}$. Furthermore, let

$$H^k_{\text{ét}}(o_F^S, \mathbb{Z}_p(i)) = \varprojlim H^k_{\text{ét}}(o_F^S, \mathbb{Z}/p^n\mathbb{Z}(i))$$

and

$$H^k_{\text{ét}}(o_F^S, \mathbb{Q}_p/\mathbb{Z}_p(i)) = \varinjlim H^k_{\text{ét}}(o_F^S, \mathbb{Z}/p^n\mathbb{Z}(i)).$$

Assume now that p is either odd or that $p = 2$ and F contains $\sqrt{-1}$. Then for $i \geqslant 2$ and $k = 1, 2$ the étale cohomology groups $H^k_{\text{ét}}(o_F^S, \mathbb{Z}_p(i))$ are isomorphic to the higher étale K-theory groups $K^{\text{ét}}_{2i-k}(o_F^S)$, introduced by Dwyer-Friedlander ([8]). Moreover, the relation to Quillen's K-theory groups $K_{2i-k}(o_F^S)$ is provided by a Chern character, which yields split surjective maps with finite kernels

$$K_{2i-k}(o_F^S) \otimes \mathbb{Z}_p \to K^{\text{ét}}_{2i-k}(o_F^S)$$

(cp. [8], [15]), which conjecturally are isomorphisms (recall that for $p = 2$, F contains $\sqrt{-1}$). Borel's results (cp. [4]) then imply that the groups $K^{\text{ét}}_{2i-2}(o_F^S)$ are finite and that the groups $K^{\text{ét}}_{2i-1}(o_F^S)$ are finitely generated of rank $r_1 + r_2$ if i is odd, and of rank r_2 if i is even, where as usual r_1 and r_2 denote the number of real and pairs of conjugate complex embeddings of F, respectively. We note that the *odd* étale K-theory groups are independent of the choice of the set S of primes: If $H^*(F,\ \ )$ denotes the absolute Galois cohomology groups of F then, in fact, the localization sequence in étale cohomology (cp. [36, Proposition 1]) implies that

$$H^1_{\text{ét}}(o_F^S, \mathbb{Z}_p(i)) \cong H^1(F, \mathbb{Z}_p(i)) \qquad \forall i \geqslant 2.$$

We therefore simply denote the odd étale K-theory groups by $K^{\text{ét}}_{2i-1}(F)$. The torsion subgroup of $K^{\text{ét}}_{2i-1}(F)$ is isomorphic to $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(i))$.

In the special case $i = 2$ more is known: There exist isomorphisms

$$K_2(o_F^S) \otimes \mathbb{Z}_p \to H^2_{\text{ét}}(o_F^S, \mathbb{Z}_p(2))$$

and

$$K_3^{nd}(F) \otimes \mathbb{Z}_p \to H^1(F, \mathbb{Z}_p(2))$$

without any restrictions on the prime p and the number field F (cp. [36], [22]). Here $K_3^{nd}(F)$ denotes the indecomposable K_3-group of F, i.e. $K_3(F)$ divided by the image of the Milnor group $K_3^M(F)$, which is 2-torsion.

More recently, Kahn ([18]) and Rognes-Weibel ([32]) have determined the kernel and cokernel of the 2-adic Chern character

$$K_{2i-k}(o_F) \otimes \mathbb{Z}_2 \to H^k_{\text{ét}}(o_F, \mathbb{Z}_2(i)),$$

which in general are non-trivial.

The following result is due to B. Kahn (cp. [16, Theorem 2.1, Proposition 6.1]):

THEOREM 1.1. — *Let L/F be a Galois extension of number fields with Galois group G and let S be a finite set of primes in F, containing the primes which are ramified in L. There is an exact sequence*

$$0 \to H^1(G, K_3^{nd}(L)) \to K_2(o_F^S) \to K_2(o_L^S)^G \to H^2(G, K_3^{nd}(L)) \to 0.$$

The following étale analog is well-known (cp. [1], [5]), however we include a proof, since the sources are not easily accessible:

THEOREM 1.2. — *Let p be an odd prime and let L/F be a Galois p-extension of number fields with Galois group G. Let S be a finite set of primes, containing the primes above p and the primes which ramify in L. Then for $i \geqslant 2$ there is an exact sequence*

$$0 \to H^1(G, K_{2i-1}^{\text{ét}}(L)) \to K_{2i-2}^{\text{ét}}(o_F^S) \to K_{2i-2}^{\text{ét}}(o_L^S)^G \to H^2(G, K_{2i-1}^{\text{ét}}(L)) \to 0.$$

Proof. — Consider the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G, H_{\text{ét}}^q(o_L^S, \mathbb{Z}_p(i))) \Rightarrow H_{\text{ét}}^{p+q}(o_F^S, \mathbb{Z}_p(i)).$$

Since $H_{\text{ét}}^0(o_L^S, \mathbb{Z}_p(i)) = 0$ ([36, Lemme 7]), all terms E_2^{p0} vanish. On the other hand, $cd_p(G_S(F)) = 2$ and hence $H_{\text{ét}}^q(o_F^S, \mathbb{Z}_p(i)) = H_{\text{ét}}^q(o_L^S, \mathbb{Z}_p(i)) = 0$ for all $q \geqslant 3$. The spectral sequence therefore yields

$$E^1 \cong E_\infty^{01} \cong E_2^{01},$$

i.e. an isomorphism

$$K_{2i-1}^{\text{ét}}(F) \cong K_{2i-1}^{\text{ét}}(L)^G,$$

as well as the exact sequence

$$0 \to E_2^{11} \to E^2 \to E_2^{02} \to E_2^{21} \to 0,$$

which is precisely the claim. □

As a by-product, we obtained the fact that the odd étale K-groups satisfy Galois descent. Note that this, in the form

$$H^1(F, \mathbb{Z}_p(i)) \cong H^1(L, \mathbb{Z}_p(i))^G,$$

remains true for $p = 2$.

On the other hand we have Galois co-descent for the even étale K-theory groups $K_{2i-2}^{\text{ét}}(o_F^S)$:

Proposition 1.3. — *Let p be odd and L/F a Galois p-extension of number fields with Galois-group G. If S contains the primes above p and the ramified primes of L/F, then*

$$K_{2i-2}^{\text{ét}}(o_L^S)_G \cong K_{2i-2}^{\text{ét}}(o_F^S).$$

Proof. — This follows as above using the Tate spectral sequence (cp. [35], [17], [26]). □

Now $K_{2i-2}^{\text{ét}}(o_L^S)$ is finite, and hence this proposition together with Theorem 1.2 yields

Corollary 1.4. — *For any cyclic p-extension L/F (p odd) of number fields with Galois group G, the quotient*

$$\frac{|H^2(G, K_{2i-1}^{\text{ét}}(L)|}{|H^1(G, K_{2i-1}^{\text{ét}}(L)|}$$

is trivial.

Remark 1.5. — The previous results depended only upon two facts:

$$cd_p(G_S(F)) \leqslant 2 \quad \text{and} \quad H^0(G_S(F), \mathbb{Z}_p(i)) = 0.$$

Therefore analogous results also hold for example for finite extensions of local fields, thus, for a Galois p-extension E/F of local fields with Galois group G, we have an exact sequence

$$0 \to H^1(G, H^1(E, \mathbb{Z}_p(i))) \to H^2(F, \mathbb{Z}_p(i)) \to H^2(E, \mathbb{Z}_p(i))^G \to H^2(G, H^1(E, \mathbb{Z}_p(i))) \to 0,$$

and an isomorphism

$$H^2(E, \mathbb{Z}_p(i))_G \cong H^2(F, \mathbb{Z}_p(i)).$$

Again, in the case $i = 2$, more information on co-descent is available, i.e. no restrictions on F are necessary to also include results concerning the 2-primary part.

The following result is easily obtained from [16, Théorème 5.1] :

PROPOSITION 1.6.— *Let L/F be a finite Galois extension of number fields with Galois group G and let S be a finite set of primes in F, containing the primes which ramify in L/F. Then, there is a short exact sequence*

$$0 \to K_2(o_L^S)_G \to K_2(o_F^S) \to \bigoplus_{v \in S_\infty^r} \mu_2 \to 0,$$

where S_∞^r consists of the real infinite primes in F which ramify in L.

Finally, we recall the definition of the higher étale wild kernels (cp. [3], [20], [29]):

$$WK_{2i-2}^{\text{ét}}(F) = \ker\left(H_{\text{ét}}^2(o_F^S, \mathbb{Z}_p(i)) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i))\right).$$

The definition is independent of the choice of the set S containing S_p, and part of the Poitou-Tate duality sequence yields the exact sequence

$$\begin{aligned} 0 \to WK_{2i-2}^{\text{ét}}(F) \to K_{2i-2}^{\text{ét}}(o_F^S) \\ \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \to H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \to 0, \end{aligned}$$

where $*$ indicates the Pontrjagin dual. Moreover by local duality

$$H^2(F_v, \mathbb{Z}_p(i)) \cong H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*.$$

The groups $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ and $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ are finite cyclic for $i \neq 1$.

The étale wild kernels are the analogs of the p-part of the classical wild kernel $WK_2(F)$ - defined for any number field F - which occurs in Moore's exact sequence of power norm symbols (cp. [23]):

$$0 \to WK_2(F) \to K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0,$$

where v runs through all finite primes and all real infinite primes of F, and $\mu(F_v)$ and $\mu(F)$ denote the group of roots of unity of F_v and of F respectively. If S is a finite set of primes in F containing S_p and S_∞, then we obtain an exact sequence of finite groups

$$0 \to WK_2(F)\{p\} \to K_2(o_F^S)\{p\} \to \bigoplus_{v \in S} \mu(F_v)\{p\} \to \mu(F)\{p\} \to 0.$$

Here, for an abelian group A, we use the notation $A\{p\}$ for the p-primary part of A.

2. Galois co-descent for the étale wild kernel.

Let p be an odd prime and let L/F be a cyclic extension of number fields of degree p with Galois group G. In this section, for any local or global field K, we denote by K_∞ the cyclotomic $\mathbb{Z}_p$-extension of K with finite layers K_n. We also assume that $i \geqslant 2$. We obtain necessary and sufficient conditions for the étale wild kernel $WK_{2i-2}^{\text{ét}}(L)$ to satisfy Galois co-descent. This approach also yields a genus"-formula comparing the sizes of $WK_{2i-2}^{\text{ét}}(L)^G$ and $WK_{2i-2}^{\text{ét}}(F)$. Let S be the finite set of primes in F, containing the set S_p of all primes above p, as well as all primes which ramify in L. We denote by S_L the set of primes in L above S. Moreover, let $\tilde{\oplus}_{v\in S} H^2(F_v, \mathbb{Z}_p(i))$ be the kernel of the surjection

$$\underset{v\in S}{\oplus} H^2(F_v, \mathbb{Z}_p(i)) \to H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*.$$

Then by definition of the étale wild kernel from section 1, we have the following short exact sequence:

$$0 \to WK_{2i-2}^{\text{ét}}(F) \to K_{2i-2}^{\text{ét}}(o_F^S) \to \underset{v\in S}{\tilde{\oplus}} H^2(F_v, \mathbb{Z}_p(i)) \to 0.$$

By Proposition 1.3 the group $K_{2i-2}^{\text{ét}}(o_L^S)$ satisfies Galois co-descent. The following commutative diagram:

$$\begin{array}{ccccccc} & WK_{2i-2}^{\text{ét}}(L)_G & \to & K_{2i-2}^{\text{ét}}(o_L^S)_G & \to & \left(\underset{w\in S_L}{\tilde{\oplus}} H^2(L_w, \mathbb{Z}_p(i))\right)_G & \to 0 \\ & \downarrow & & \downarrow \wr & & \downarrow & \\ 0 \to & WK_{2i-2}^{\text{ét}}(F) & \to & K_{2i-2}^{\text{ét}}(o_F^S) & \to & \underset{v\in S}{\tilde{\oplus}} H^2(F_v, \mathbb{Z}_p(i)) & \to 0 \end{array}$$

then shows that

$$\operatorname{coker}(WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)) \cong \ker\left((\underset{w\in S_L}{\tilde{\oplus}} H^2(L_w, \mathbb{Z}_p(i))_G \to \underset{v\in S}{\tilde{\oplus}} H^2(F_v, \mathbb{Z}_p(i))\right)$$

and

$$\ker(WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)) \cong \operatorname{coker}\left(K_{2i-2}^{\text{ét}}(o_L^S)^G \to \left(\underset{w\in S_L}{\tilde{\oplus}} H^2(L_w, \mathbb{Z}_p(i))\right)^G \right).$$

Before we compute the first group we need a preliminary result: Let M/N be a cyclic extension of degree p, p odd, of global or local fields of characteristic $\neq p$, and let G denote the Galois group of M/N. Furthermore, let N_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of N.

There are two maps relating the cohomology groups $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$, where we assume $k \in \mathbb{Z}$, $k \neq 0$: The natural map $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and the norm map $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$. The first one induces an isomorphism

$$H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \xrightarrow{\sim} H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))^G,$$

which implies immediately that either both groups are trivial or both groups are non-trivial. Assume now that $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ is non-trivial. Then the order of $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ is the maximal power p^m, such that the Galois group $\mathrm{Gal}(N(\mu_{p^m})/N)$ has exponent k. If $M \not\subset N_\infty$, then $[M(\mu_{p^m}) : M] = [N(\mu_{p^m}) : N]$, and therefore G acts trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$. Therefore, in this case, the natural map is an isomorphism, and hence the norm map has both kernel and cokernel of order p. On the other hand, if $M \subset N_\infty$ and say $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \cong (\mathbb{Z}/p^m\mathbb{Z})(k)$, then $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k)) \cong (\mathbb{Z}/p^{m+1}\mathbb{Z})(k)$, and - p being odd - the norm

$$(\mathbb{Z}/p^{m+1}\mathbb{Z})(k) \to (\mathbb{Z}/p^m\mathbb{Z})(k)$$

is surjective, and therefore induces an isomorphism

$$H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))_G \xrightarrow{\sim} H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)).$$

We summarize:

LEMMA 2.1.— *Let $k \in \mathbb{Z}$, $k \neq 0$ and $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \neq 0$.*

i) *If $M \not\subset N_\infty$, then G acts trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$, and hence the natural map*

$$H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$$

is an isomorphism, whereas the norm map has kernel and cokernel of order p.

ii) *If $M \subset N_\infty$, then G acts non-trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and the norm induces an isomorphism*

$$H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))_G \xrightarrow{\sim} H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)).$$

The non-vanishing of $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ can be characterized as follows: Let $d = [N(\mu_p) : N]$. Then

$$H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \neq 0 \Leftrightarrow k \equiv 0 \bmod d.$$

Let us now study the question of co-descent for $\tilde{\oplus}_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i))$. Using local duality the problem is equivalent to computing the cokernel of the map

$$\tilde{\bigoplus_{v \in S}} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to \left(\tilde{\bigoplus_{w \in S_L}} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \right)^G .$$

As we noted above, we have isomorphisms

$$H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \cong H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^G$$

and

$$\bigoplus_{v \in S} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \cong \left(\bigoplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \right)^G ,$$

hence the above cokernel is isomorphic to the kernel of

$$H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))_G \to \left(\bigoplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \right)_G .$$

We consider the commutative diagram

$$\begin{array}{ccccc} & & H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))_G & \to & \left(\bigoplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \right)_G \\ & & \downarrow & & \downarrow \\ 0 & \to & H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) & \to & \bigoplus_{v \in S} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \end{array}$$

induced by the norm maps. It is now clear that the map in the top row is *not* injective, if and only if Galois co-descent fails globally for $H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$, but holds locally for all $w \in S_L$, in which case the kernel is of order p. If $v \in S$ is decomposed in L, then obviously co-descent holds. We now define $T_{L/F}^{(i)}$ to be the set of *undecomposed* primes $v \in S$, such that Galois co-descent fails for $H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$. By Lemma 2.1, an undecomposed prime v lies in $T_{L/F}^{(i)}$ if and only if $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \neq 0$ and $L_w \not\subset F_{v,\infty}$. Let $d = [F(\mu_p) : F]$. Then it is clear from the definition that

$$T_{L/F}^{(i)} = T_{L/F}^{(j)} \quad \text{if} \quad i \equiv j \bmod d.$$

Let us analyze this set a little further:

LEMMA 2.2. — i) *$T^{(i)}_{L/F}$ contains all tamely ramified primes:*

$$S \setminus S_p \subset T^{(i)}_{L/F} \subset S.$$

ii) *Assume that $L \not\subset F_\infty$ and $i \equiv 1 \bmod d$. Then, for large n, the set $T^{(i)}_{L_n/F_n}$ contains all undecomposed p-adic primes.*

Proof. — Let v be any prime in $S \setminus S_p$. Then F_v contains the p-th roots of unity μ_p, which shows that $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \neq 0$. Moreover, $F_{v,\infty}$ is the maximal unramified pro-p-extension of F_v, which shows that $L_w \not\subset F_{v,\infty}$. This proves i). To prove ii), it suffices to choose n large enough so that no p-adic prime of L_n decomposes in L_{n+1}. □

We can now formulate our first result in terms of the set $T^{(i)}_{L/F}$.

PROPOSITION 2.3. — *The canonical map*

$$WK^{\text{ét}}_{2i-2}(L)_G \to WK^{\text{ét}}_{2i-2}(F)$$

induced by the corestriction is surjective precisely in the following situations:

i) *$T^{(i)}_{L/F} \neq \varnothing$;*

ii) *$T^{(i)}_{L/F} = \varnothing$ and either $i \not\equiv 1 \bmod d$ or $L \subset F_\infty$.*

In the exceptional case where $T^{(i)}_{L/F} = \varnothing$, $i \equiv 1 \bmod d$ and $L \not\subset F_\infty$, the cokernel of $WK^{\text{ét}}_{2i-2}(L)_G \to WK^{\text{ét}}_{2i-2}(F)$ is cyclic of order p.

Remark 2.4. — The possibility of the failure of Galois co-descent in Proposition 2.3 was already observed in [2]. The situations where this happens are easily described: First of all we must have $i \equiv 1 \bmod d$ and $L \not\subset F_\infty$, in which case, for any n, the set $T^{(i)}_{L/F} = \varnothing$ if and only if $T^{(i)}_{L_n/F_n} = \varnothing$. Now, choose n large enough, such that no p-adic prime in L_n decomposes in L_{n+1}. By Lemma 2.2, the set $T^{(i)}_{L_n/F_n} = \varnothing$ precisely when L_n/F_n is unramified and all p-adic primes of F_n split in L_n. Thus, the exceptional case occurs for L/F if and only if the following two conditions hold:

i) $i \equiv 1 \bmod d$;

ii) $\varprojlim A'_n \neq 0$ and L_∞/F_∞ is an unramified cyclic extension of degree p, in which all primes above p split.

Example 2.5. — Assume that the prime p is irregular and let $F = \mathbb{Q}(\mu_p)$. Then F possesses a cyclic extension L of degree p inside the Hilbert p-class field, which is disjoint from F_∞. Therefore the canonical map $WK^{\text{ét}}_{2i-2}(L)_G \to WK^{\text{ét}}_{2i-2}(F)$ is not surjective for any $i \geqslant 2$.

We recall that by Lemma 2.1, the natural map

$$H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

is an isomorphism for $v \in T^{(i)}_{L/F}$, and hence the norm map

$$H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

can be identified with the p-th power map on $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$, which is induced by the p-th power map on $\mathbb{Q}_p/\mathbb{Z}_p(1-i)$. Hence we have an exact sequence for $v \in T^{(i)}_{L/F}$:

$$\begin{aligned} 0 \to H^0(F_v, \mathbb{Z}/p\mathbb{Z}(1-i)) &\to H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))_G \\ &\to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))/p \to 0. \end{aligned}$$

The dual sequence then reads:

$$\begin{aligned} 0 \to {}_pH^2(F_v, \mathbb{Z}_p(i)) &\to H^2(F_v, \mathbb{Z}_p(i)) \\ &\to H^2(L_w, \mathbb{Z}_p(i))^G \to H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \to 0. \end{aligned}$$

If we compare this sequence with the one mentioned in Remark 1.5, we see that we have an isomorphism

$$H^2(G, H^1(L_w, \mathbb{Z}_p(i))^{\cdot} \cong H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

We make this isomorphism more explicit in Proposition 2.9.

Let us now consider the problem of the surjectivity of the homomorphism

$$K^{\text{ét}}_{2i-2}(o^S_L)^G \to \Big(\tilde{\bigoplus_{w \in S_L}} H^2(L_w, \mathbb{Z}_p(i))\Big)^G .$$

If $T^{(i)}_{L/F} = \varnothing$, then we assume that either $i \not\equiv 1 \bmod d$, or that $L \subset F_\infty$, so that we have Galois co-descent for $(\tilde{\oplus}_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i)))$, i.e. $WK^{\text{ét}}_{2i-2}(L)_G \to WK^{\text{ét}}_{2i-2}(F)$ is surjective. In particular this implies that the map β in the following commutative diagram is surjective:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \bigoplus_{v\in T^{(i)}_{L/F}} ({}_pH^2(F_v,\mathbb{Z}_p(i))) & \rightarrow & B & & \\
 & & \downarrow & & \downarrow & & \\
K^{\text{ét}}_{2i-2}(o^S_F) & \rightarrow & \bigoplus_{v\in S} H^2(F_v,\mathbb{Z}_p(i)) & \rightarrow & H^0(F,\mathbb{Q}_p/\mathbb{Z}_p(1-i))^* & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K^{\text{ét}}_{2i-2}(o^S_L)^G & \overset{\alpha}{\rightarrow} & \bigoplus_{v\in S}(\bigoplus_{w|v} H^2(L_w,\mathbb{Z}_p(i)))^G & \overset{\beta}{\rightarrow} & (H^0(L,\mathbb{Q}_p/\mathbb{Z}_p(1-i))^*)^G & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^2(G,K^{\text{ét}}_{2i-1}(L)) & \overset{\alpha'}{\rightarrow} & \bigoplus_{v\in T^{(i)}_{L/F}} H^2(F_v,\mathbb{Z}/p\mathbb{Z}(i)) & \overset{\beta'}{\rightarrow} & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Here we define B and C to be the kernel and cokernel of the homomorphism

$$H^0(F,\mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \rightarrow (H^0(L,\mathbb{Q}_p/\mathbb{Z}_p(1-i))^*)^G$$

respectively, hence both B and C are either trivial or of order p. More precisely, by Lemma 2.1, they are non-trivial if and only if $i \equiv 1 \bmod d$ and $L \not\subset F_\infty$. In this diagram the columns are exact and also the rows, except possibly at $\oplus_{v\in S}(\oplus_{w|v} H^2(L_w,\mathbb{Z}_p(i)))^G$ and $\oplus_{v\in T^{(i)}_{L/F}} H^2(F_v,\mathbb{Z}/p\mathbb{Z}(i))$. Note that

$$\ker\beta/\operatorname{im}\alpha = \operatorname{coker}\left(K^{\text{ét}}_{2i-2}(o^S_L)^G \rightarrow \left(\tilde{\bigoplus_{w\in S_L}} H^2(L_w,\mathbb{Z}_p(i))\right)^G\right)$$

is precisely the cokernel we want to study.

An easy diagram chase shows:

Lemma 2.6.— *The surjection*

$$\ker\beta/\operatorname{im}\alpha \rightarrow \ker\beta'/\operatorname{im}\alpha'$$

is an isomorphism if the map

$$\bigoplus_{v\in T^{(i)}_{L/F}} {}_pH^2(F_v,\mathbb{Z}_p(i)) \rightarrow B$$

is surjective (otherwise, its kernel is of order at most p).

In particular, this settles the case $T^{(i)}_{L/F} = \varnothing$:

COROLLARY 2.7. — *If* $T^{(i)}_{L/F} = \varnothing$, *then* $WK^{\text{ét}}_{2i-2}(L)_G \cong WK^{\text{ét}}_{2i-2}(F)$ *if and only if either* $i \not\equiv 1 \bmod d$ *or* $L \subset F_\infty$.

Thus, for example, in the cyclotomic $\mathbb{Z}_p$-extension, the wild kernels satisfy Galois codescent, whereas, in general, the p-class groups do not.

Let us assume now that $T^{(i)}_{L/F} \neq \varnothing$. Then L is disjoint from F_∞, and therefore the kernel B is non-trivial if and only if $i \equiv 1 \bmod d$. In this case B is clearly isomorphic to ${}_pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$, and we can characterize the surjectivity of the map $\oplus_{v\in T^{(i)}_{L/F}} {}_pH^2(F_v, \mathbb{Z}_p(i)) \to {}_pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$ as follows:

LEMMA 2.8. — *If* $T^{(i)}_{L/F} \neq \varnothing$ *and* $i \equiv 1 \bmod d$, *then*

$$\bigoplus_{v\in T^{(i)}_{L/F}} {}_pH^2(F_v, \mathbb{Z}_p(i)) \to {}_pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$$

is surjective if and only if at least one of the primes in $T^{(i)}_{L/F}$ *is undecomposed in the first layer* F_1 *of the cyclotomic* $\mathbb{Z}_p$*-extension* F_∞/F.

Proof. — It is clear that the map in question is surjective if and only if $|H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))| = |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))|$ for at least one prime $v \in T^{(i)}_{L/F}$. On the other hand $|H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))| > |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))|$ if and only if v splits in F_1. □

We note that any finite place v in F is finitely decomposed in F_∞. Therefore, if n is large enough, all the primes in $T^{(i)}_{L_n/F_n}$ will be undecomposed in F_{n+1}. If $i \equiv 1 \bmod d$, we will assume that $T^{(i)}_{L/F}$ contains at least one prime, which is undecomposed in F_1. We are then left with the determination of $|\ker \beta'/\mathrm{im}\,\alpha'|$.

The order of ker β' is clearly equal to

$$|\ker \beta'| = \frac{p^{|T^{(i)}_{L/F}|}}{|H^0(F, \mathbb{Z}/p\mathbb{Z}(1-i))|}.$$

To determine the order of im α' we construct a canonical homomorphism

$$H^2(G, H^1(L, \mathbb{Z}_p(i))) \to H^2(F, \mathbb{Z}/p\mathbb{Z}(i)),$$

which gives rise to a commutative diagram

$$\begin{array}{ccc} H^2(G, H^1(L, \mathbb{Z}_p(i))) & \to & \bigoplus_{v\in T^{(i)}_{L/F}} H^2(G, H^1(L_w, \mathbb{Z}_p(i))) \\ \downarrow & & \wr\downarrow \\ H^2(F, \mathbb{Z}/p\mathbb{Z}(i)) & \to & \bigoplus_{v\in T^{(i)}_{L/F}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \end{array}$$

and will factor the map α'.

PROPOSITION 2.9. — *Let M/N be a cyclic extension of degree p of local or global fields of characteristic $\neq p$, where p is an arbitrary prime. Let $G = \mathrm{Gal}(M/N)$. There is a canonical map*

$$H^2(G, H^1(M, \mathbb{Z}_p(i))) \to H^2(N, \mathbb{Z}/p\mathbb{Z}(i))$$

with kernel isomorphic to

$$(H^1(N, \mathbb{Z}_p(i))/p \cap N_{M/N}(H^1(M, \mathbb{Z}/p\mathbb{Z}(i)))) \,/\, N_{M/N}(H^1(M, \mathbb{Z}_p(i))/p).$$

Proof. — We first note that the exact sequence

$$0 \to \mathbb{Z}_p(i) \to \mathbb{Z}_p(i) \to \mathbb{Z}/p\mathbb{Z}(i) \to 0$$

induces an injection

$$H^1(N, \mathbb{Z}_p(i))/p \hookrightarrow H^1(N, \mathbb{Z}/p\mathbb{Z}(i)),$$

and therefore we can view $H^1(N, \mathbb{Z}_p(i))/p$ as a subgroup of $H^1(N, \mathbb{Z}/p\mathbb{Z}(i))$, and similarly for M. Since G is cyclic, we have a canonical isomorphism

$$H^2(G, H^1(M, \mathbb{Z}_p(i))) \cong \hat{H}^0(G, H^1(M, \mathbb{Z}_p(i))) \otimes H^2(G, \mathbb{Z}_p)$$

given by the cup-product. Here $\hat{H}$ denotes Tate-cohomology. Now the group $H^1(M, \mathbb{Z}_p(i))$ satisfies Galois descent as we have seen in the proof of Theorem 1.2, even in the case $p = 2$. Hence

$$\begin{aligned}\hat{H}^0(G, H^1(M, \mathbb{Z}_p(i))) &\cong H^1(N, \mathbb{Z}_p(i))/N_{M/N}(H^1(M, \mathbb{Z}_p(i)))) \\ &\cong (H^1(N, \mathbb{Z}_p(i))/p)/N_{M/N}(H^1(M, \mathbb{Z}_p(i))/p).\end{aligned}$$

Now $H^2(G, \mathbb{Z}_p) \cong H^1(G, \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1(G, \mathbb{Z}/p\mathbb{Z})$, since G is cyclic of order p, and we have the cup-product

$$H^1(N, \mathbb{Z}/p\mathbb{Z}(i)) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to H^2(N, \mathbb{Z}/p\mathbb{Z}(i))$$

whose kernel is equal to

$$N_{M/N}(H^1(M, \mathbb{Z}/p\mathbb{Z}(i))) \otimes H^1(G, \mathbb{Z}/pZ).$$

To see this, we may assume without loss of generality that N contains μ_p, in which case this product is just a twisted version of the standard cup-product into the Brauer group of F. Restricting the last morphism to the subgroup $H^1(N, \mathbb{Z}_p(i))/p \otimes H^1(G, \mathbb{Z}/p\mathbb{Z})$ yields the result. □

Let us return now to the situation considered before: p is odd and L/F is a cyclic extension of number fields of degree p with Galois group G. We are going to compare the global and local maps constructed in Proposition 2.9. Let $C_v := \operatorname{coker}(H^2(F_v, \mathbb{Z}_p(i)) \to (\oplus_{w|v} H^2(L_w, \mathbb{Z}_p(i)))^G$. Then by definition

$$C_v = 0 \Leftrightarrow v \notin T^{(i)}_{L/F}$$

and $C_v = H^2(G, H^1(L_w, \mathbb{Z}_p(i)) \cong H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ if $v \in T^{(i)}_{L/F}$. The following commutative diagram:

$$\begin{array}{ccccc} & & H^2(G, K^{\text{ét}}_{2i-1}(L)) & \to & \prod_v C_v \\ & & \downarrow & & \downarrow \\ 0 & \to & H^2(F, \mathbb{Z}/p\mathbb{Z}(i)) & \to & \prod_v H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \end{array}$$

then shows that the image of $H^2(G, K^{\text{ét}}_{2i-1}(L))$ in $\prod_v H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ is in fact contained in $\oplus_{v\in T^{(i)}_{L/F}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$. The injectivity of the localization map $H^2(F, \mathbb{Z}/p\mathbb{Z}(i)) \to \prod_v H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ is proved for instance in [33, Section 2, Lemma 7]. We can now conclude:

LEMMA 2.10. — *The canonical map $H^2(G, K^{\text{ét}}_{2i-1}(L)) \to H^2(F, \mathbb{Z}/p\mathbb{Z}(i))$ induces the map*

$$\alpha' : H^2(G, K^{\text{ét}}_{2i-1}(L)) \to \bigoplus_{v\in T^{(i)}_{L/F}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

Furthermore:

$$\ker \alpha' \cong [K^{\text{ét}}_{2i-1}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i))]/N_{L/F}(K^{\text{ét}}_{2i-1}(L)/p)$$

and

$$|\operatorname{im} \alpha'| = [K^{\text{ét}}_{2i-1}(F)/p : K^{\text{ét}}_{2i-1}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i)))].$$

Combining this with the calculation of ker β' provides the main result of this section, a "genus formula" for the étale wild kernels for cyclic extensions of degree p:

THEOREM 2.11. — *Let L/F be a cyclic extension of number fields of degree p, p odd, with Galois group G. Assume that $T^{(i)}_{L/F} \neq \varnothing$ and that some $v \in T^{(i)}_{L/F}$ is undecomposed in F_1 if $i \equiv 1 \bmod d$. Then the natural map $WK^{\text{ét}}_{2i-2}(L)_G \to WK^{\text{ét}}_{2i-2}(F)$ is surjective and its kernel has order*

$$\frac{p^{|T^{(i)}_{L/F}|}}{|H^0(F, \mathbb{Z}/p\mathbb{Z}(1-i))| \cdot [K^{\text{ét}}_{2i-1}(F)/p : K^{\text{ét}}_{2i-1}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i)))]}.$$

Remark 2.12. — Let us consider the special case that $i \equiv 1 \bmod d$, and that all p-adic primes of L are undecomposed in L_∞. Then $T^{(i)}_{L/F}$ contains all undecomposed p-adic primes as well as all tamely ramified primes. Hence we can rewrite the formula in the preceding theorem as

$$\frac{|WK^{\text{ét}}_{2i-2}(L)^G|}{|WK^{\text{ét}}_{2i-2}(F)|}$$
$$= \frac{\prod_{\mathfrak{p}|p} d_{\mathfrak{p}}(L/F) \cdot \prod_{\mathfrak{p}\nmid p} e_{\mathfrak{p}}(L/F)}{[L:F]\cdot[K^{\text{ét}}_{2i-1}(F)/p \,:\, K^{\text{ét}}_{2i-1}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i)))]}.$$

Here $d_{\mathfrak{p}}(L/F)$ and $e_{\mathfrak{p}}(L/F)$ denote the local degrees and the ramification indices, respectively. If we replace the étale K-theory index by the index $[U'_F : U'_F \cap N_{L/F}(L^*)]$ for the p-units U'_F, then this becomes precisely the genus formula for the p-class groups. We will return to this peculiarity later on.

Example 2.13. — 1) Take $p = i = 3$ and $F = \mathbb{Q}$ the field of rationals. Since $K_4(\mathbb{Z})$ is trivial (cp. [30], [31], [32]), so is $WK^{\text{ét}}_4(\mathbb{Z})$. We are going to give an infinite family of cubic fields L such that $WK^{\text{ét}}_4(L) = 0$. For this, consider the set of primes (see also [37, Remarks page 182])

$$\begin{aligned} P &= \{\ell \,;\, \ell \equiv 1 \bmod 3 \text{ and } 3^{\frac{\ell-1}{3}} \equiv 1 \bmod \ell\} \\ &= \{\ell \,;\, \ell \equiv 1 \bmod 3 \text{ and } \sqrt[3]{3} \in \mathbb{Z}/\ell\mathbb{Z}\}. \end{aligned}$$

Obviously, by Hensel's lemma, we have

$$\begin{aligned} P &= \{\ell \,;\, \mu_3 \subset \mathbb{Q}_\ell \text{ and } \sqrt[3]{3} \in \mathbb{Q}_\ell\} \\ &= \{\ell \,;\, \ell \text{ splits in } \mathbb{Q}(\mu_3, \sqrt[3]{3})\}. \end{aligned}$$

We are interested in the infinite family (of density $\frac{1}{6} - \frac{1}{18}$) of the primes ℓ in P which do not split in $\mathbb{Q}(\mu_9, \sqrt[3]{3})$. Now let L be the cubic extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\mu_\ell)$ and $G = G(L/\mathbb{Q})$. Then $T^{(3)}_{L/\mathbb{Q}} = \{\ell\}$ and, according to Theorem 2.11, the wild kernel $WK^{\text{ét}}_4(L) = 0$.

2) In this example, we are going to determine the Galois p-extensions M of $\mathbb{Q}$, for which the p-part of the classical wild kernel is trivial. The two cases $p = 3$ and $p \geqslant 5$ are completely different due to the fact that in the latter case the considered fields do not contain the maximal real subfield $\mathbb{Q}(\mu_p)^+$of the cyclotomic field $\mathbb{Q}(\mu_p)$. For $p \geqslant 5$, the Galois p-extensions M of $\mathbb{Q}$ for which $WK_2(M)\{p\} = 0$ are exactly the layers $\mathbb{Q}_n$ of the $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$. Indeed, since $\mathbb{Q}_\infty$ is the maximal p-ramified

pro-p-extension of $\mathbb{Q}$, we see that the maximal p-ramified extension of $\mathbb{Q}$ contained in M is a layer $\mathbb{Q}_n$ of $\mathbb{Q}_\infty$. If $M = \mathbb{Q}_n$ then, by Corollary 2.7, $WK_2(M)\{p\} = 0$. Otherwise, choose a tower of degree p cyclic extensions

$$\mathbb{Q}_n = M_0 \subset M_1 \subset \cdots \subset M_r = M.$$

Since $T^{(2)}_{M_1/\mathbb{Q}_n} \neq \varnothing$, we have $WK_2(M_1)\{p\} \neq 0$ (Theorem 2.11). Moreover, for each intermediate extension $M_{\nu+1}/M_\nu$, the canonical map

$$WK_2(M_{\nu+1})\{p\}_{G(M_{\nu+1}/M_\nu)} \to WK_2(M_\nu)\{p\}$$

is surjective (Proposition 2.3), which shows that $WK_2(M)\{p\} \neq 0$. A number field M for which $H^2_{\text{ét}}(o_M, \mathbb{Z}/p\mathbb{Z}) = 0$, is called p-rational [25], [24]. Moreover, if M contains $\mathbb{Q}(\mu_p)^+$, then it is also called p-regular [10]. The p-regularity of M is simply expressed by the triviality of the p-part of the tame kernel $K_2(o_M)$. As the $\mathbb{Q}_n$ are not the only p-extensions of $\mathbb{Q}$ which are p-rational, we notice that, for $p \geqslant 5$, among the p-extensions M of $\mathbb{Q}$, some are p-rational but have a non-trivial $WK_2(M)\{p\}$. Now take $p = 3$. Then by Moore's exact sequence $WK_2(M)\{3\} = 0$ if and only if the tame kernel $K_2(o_M)$ has no 3-torsion. Hence the number field M is 3-rational or 3-regular. In this case, $WK_2(M)\{3\} = 0$ if and only if outside the prime 3, the 3-extension $M/\mathbb{Q}$ is at most ramified at one prime l, which is inert in the $\mathbb{Z}_3$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ (cp. [10], [25], [24]).

Let us have a closer look at the cup-product

$$K^{\text{ét}}_{2i-1}(F)/p \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to H^2(F, \mathbb{Z}/p\mathbb{Z}(i)),$$

which occurred in the proof of Proposition 2.9, and describe the maps α' and β'. Let $E = F(\mu_p)$ and $\Delta = \mathrm{Gal}(E/F)$. Over E we have

$$H^2(E, \mathbb{Z}/p\mathbb{Z}(i)) \cong H^2(E, \mathbb{Z}/p\mathbb{Z}(1))(i-1) \cong {}_p\mathrm{Br}(E)(i-1),$$

where $\mathrm{Br}(E)$ stands for the Brauer group of E. The set $T^{(i)}_{LE/E}$ is independent of i, and we simply denote it by $T_{LE/E}$. Obviously, every prime in E which lies above a prime in $T^{(i)}_{L/F}$ belongs to $T_{LE/E}$. Conversely, let v_E be a prime in $T_{LE/E}$, and let v denote the prime of F below v_E. Then $v \in T^{(i)}_{L/F}$ if and only if $H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \neq 0$. Let $\mathrm{Br}^T(E)$ denote the subgroup of $\mathrm{Br}(E)$ of all isomorphism classes of central simple E-algebras split outside $T_{LE/E}$. It is now easy to see that

$$\ker \beta' \cong ({}_p\mathrm{Br}^T(E)(i-1))^\Delta \cong ({}_p\mathrm{Br}^T(E))^{[1-i]},$$

where ω denotes the Teichmüller character of Δ, and $A^{[j]}$ denotes the j-th eigenspace of ω acting on a Δ-module A.

Since $K_{2i-1}^{\text{ét}}(E)/p$ is contained in $H^1(E,\mathbb{Z}/p\mathbb{Z}(i)) \cong (E^*/E^{*p})(i-1)$, there exists a subgroup $D_E^{(i)}$ of E^* containing E^{*p} - the analog of the Tate-kernel in case $i=2$ - such that

$$K_{2i-1}^{\text{ét}}(E)/p \cong (D_E^{(i)}/E^{*p})(i-1),$$

and hence

$$K_{2i-1}^{\text{ét}}(F)/p \cong ((D_E^{(i)}/E^{*p})(i-1))^{\Delta} \cong (D_E^{(i)}/E^{*p})^{[1-i]}.$$

Note that for $i \equiv 1 \bmod d$ we can similarly define $D_F^{(i)}$, and clearly in this case $(D_E^{(i)}/p)^{\Delta} \cong D_F^{(i)}/p$. The considerations after Proposition 2.9 now show that the cup-product over E is explicitly given as

$$(D_E^{(i)}/E^{*p})(i-1) \otimes H^1(G,\mathbb{Z}/p\mathbb{Z}) \to {}_p\mathrm{Br}^T(E)(i-1),$$

where

$$D_E^{(i)}/E^{*p} \otimes H^1(G,\mathbb{Z}/p\mathbb{Z}) \to {}_p\mathrm{Br}^T(E),$$

is the classical cup-product $x \otimes \chi \mapsto (\chi, x)$ (cp. [34, Chap. XIV]). Descending to F, we see that the image of α' is precisely the image of the cup-product

$$(D_E^{(i)}/E^{*p})^{[1-i]} \otimes H^1(G,\mathbb{Z}/p\mathbb{Z}) \to ({}_p\mathrm{Br}^T(E))^{[1-i]}.$$

We can therefore reformulate the condition for Galois co-descent of the wild kernel as follows:

THEOREM 2.14. — *Let L/F be a cyclic extension of number fields of degree p, p odd, with Galois group G. Assume that $T_{L/F}^{(i)} \neq \varnothing$ and that some $v \in T_{L/F}^{(i)}$ is undecomposed in F_1 if $i \equiv 1 \bmod d$. Then the étale wild kernel $WK_{2i-2}^{\text{ét}}(L)$ satisfies Galois co-descent if and only if the cup-product*

$$(D_E^{(i)}/E^{*p})^{[1-i]} \otimes H^1(G,\mathbb{Z}/p\mathbb{Z}) \to ({}_p\mathrm{Br}^T(E))^{[1-i]}$$

is surjective.

In the special case where $i \equiv 1 \bmod d$, the condition can be reformulated as: The cup-product

$$D_F^{(i)}/F^{*p} \otimes H^1(G,\mathbb{Z}/p\mathbb{Z}) \to {}_p\mathrm{Br}^{T_{L/F}^{(i)}}(F)$$

is surjective. Moreover, in this special case the genus-formula simplifies to:

$$\frac{|WK^{\text{ét}}_{2i-2}(L)^G|}{|WK^{\text{ét}}_{2i-2}(F)|} = \frac{p^{|T^{(i)}_{L/F}|-1}}{[D^{(i)}_F : D^{(i)}_F \cap N_{L/F}(L^*)]}.$$

Since $i \geqslant 2$, we can reinterpret the cup-product as a Galois symbol: Let $EL = E(\sqrt[p]{\delta})$. Then

$$(D^{(i)}_E/E^{*p})(i-1) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) = (D^{(i)}_E/E^{*p}) \otimes H^1(G, \mu_p)(i-2),$$

and therefore the cup-product over E is the $(i-2)$-th twist of the Galois symbol

$$D^{(i)}_E/E^{*p} \otimes H^1(G, \mu_p) \to \mu_p \otimes {}_p\mathrm{Br}^T(E).$$

The Kummer radical $H^1(G, \mu_p)$ is generated by δ, and the map is given by (cp. [23])

$$x \otimes \delta \mapsto \zeta_p \otimes \left[\left(\frac{x,\delta}{E}\right)\right],$$

where ζ_p is a primitive pth root of unity and $[(\frac{x,\delta}{E})]$ denotes the isomorphism class of the cyclic algebra $(\frac{x,\delta}{E})$, with generators u, v and relations: $u^p = x, v^p = \delta, vu = \zeta_p uv$.

In general, not much is known about the higher "Tate-kernels" $D^{(i)}_E$ defined by $K^{\text{ét}}_{2i-1}(E)/p \cong (D^{(i)}_E/E^{*p})$. However, for n large, the groups $D^{(i)}_{E_n}/E^{*p}_n$ can be characterized in terms of local conditions, if we assume the Gross conjecture, which we describe next: Let F be any number field. For a finite prime v in F let $\hat{F}_v = \varprojlim F^*_v/F^{*p^k}_v$, let $\hat{U}_v = \mathbb{Z}_p \otimes U_v$ and let $\mathcal{N}_v \subset \hat{F}_v$ denote the group of norms from the cyclotomic $\mathbb{Z}_p$-extension of F_v. Thus $\mathcal{N}_v = \hat{U}_v$ if $v \notin S_p$, and for $v \in S_p$ we have the following characterization:

$$a \in \mathcal{N}_v \Leftrightarrow \log_p(N_{F_v/\mathbb{Q}_p}(a)) = 0,$$

where $\log_p$ denotes the p-adic logarithm normalized by $\log_p(p) = 0$ (cp. [9], [19]). There is a natural homomorphism

$$g_F : \mathbb{Z}_p \otimes U'_F \to \bigoplus_{v|p} \hat{F}^*_v/\mathcal{N}_v$$

and the Gross kernel $GK(F) := \ker g_F$ has $\mathbb{Z}_p$-rank $r_1(F) + r_2(F) + \delta_F$, where $\delta_F \geqslant 0$ is the Gross defect. $GK(F)$ is therefore characterized by the following local conditions:

$$\epsilon \in GK(F) \Leftrightarrow \epsilon \in \mathbb{Z}_p \otimes U'_F \quad \text{and} \quad \log_p(N_{F_v/\mathbb{Q}_p}(\epsilon)) = 0 \quad \forall v \in S_p.$$

The Gross Conjecture postulates that $\delta_F = 0$, which is true for instance for abelian fields F. Let - as before - $E = F(\mu_p)$ and $\Delta = \mathrm{Gal}(E/F)$. The following result was proved for $i = 2$ in [19, Theorem 2.5]. The method was extended to higher étale K-theory in [5].

THEOREM 2.15. — *For n large there is an exact sequence*

$$0 \to K^{\text{ét}}_{2i-1}(E_n)/p \to (\ker g_{E_n}/p)(i-1) \to (\mathbb{Z}/p\mathbb{Z})^{\delta_n} \to 0,$$

where δ_n denotes the Gross defect for the field E_n.

COROLLARY 2.16. — *Assume that the Gross Conjecture holds for E_n, n large. Then*

$$D^{(i)}_{E_n}/E_n^{*p} \cong GK(E_n)/p \qquad \text{for } n \text{ large.}$$

In particular, for n large, the groups $D^{(i)}_{E_n}/E_n^{*p}$ are independent of i.

So far in this section we have ignored the prime 2. Let us briefly discuss the case $p = 2$ in the classical situation $i = 2$, where special attention has to be paid to real infinite primes in F. Let $L = F(\sqrt{\delta})$ be a quadratic extension of number fields with Galois group G. Denote by $T_{L/F}$ the set of finite primes in F which consists of all ramified non-dyadic primes and of all undecomposed dyadic primes v of F, for which either $\mu(L_w)\{2\} = \mu(F_v)\{2\}$ or L_w is not contained in the cyclotomic $\mathbb{Z}_2$-extension of F_v, where w is the prime above v in L. Also, denote by D_F the subgroup of F^* of all elements x, such that $\{-1, x\} = 1$ in $K_2(F)$. This is the classical Tate-kernel. Then the following results can be proved along the same lines as for odd p:

PROPOSITION 2.17. — *The canonical map $WK_2(L)\{2\}_G \to WK_2(F)\{2\}$ is surjective precisely in the following situations, and has cokernel of order 2 otherwise:*

i) $|\mu(L)\{2\}| > |\mu(F)\{2\}|$ *and* $L \subset F_\infty$.

ii) $|\mu(L)\{2\}| > |\mu(F)\{2\}|$, $L \not\subset F_\infty$ *and* $\mu(L_w)\{2\} = \mu(L)\{2\}$ *for some* $w \mid v$, $v \in T_{L/F}$.

iii) $\mu(L)\{2\} = \mu(F)\{2\}$ *and* $\mu(L_w)\{2\} = \mu(F_v)\{2\}$ *for some* $v \in T_{L/F}$.

We note in particular that the map $WK_2(L)\{2\}_G \to WK_2(F)\{2\}$ is always surjective if a non-dyadic prime of F is ramified in L.

THEOREM 2.18.— *Let L/F be a relative quadratic extension with Galois group G.*

a) *If $|\mu(L)\{2\}| > |\mu(F)\{2\}|$ and $L \subset F_\infty$, then $WK_2(L)\{2\}_G \cong WK_2(F)\{2\}$.*

b) *If either $|\mu(L)\{2\}| = |\mu(F)\{2\}|$ or $L \not\subset F_\infty$, and if either a real infinite prime of F ramifies in L or if $|\mu(F_v)\{2\}| = |\mu(F)\{2\}|$ for some prime $v \in T_{L/F}$, then*

$$\frac{|WK_2(L)\{2\}_G|}{|WK_2(F)\{2\}|} = \frac{2^{|T_{L/F}|-1}}{[D_F : D_F \cap N_{L/F}(L^*)]}.$$

In [6], Browkin and Schinzel computed the 2-rank of the wild kernel of a quadratic number field and obtained a complete list of quadratic number fields with trivial 2-primary wild kernels. A combination of their results with the genus formula in Theorem 2.18 and methods of [12] yield a complete list of bi-quadratic fields with trivial 2-primary wild kernels. Details will appear elsewhere.

3. Capitulation kernels.

Let p be an odd prime and let F_∞/F be an arbitrary $\mathbb{Z}_p$-extension of F with finite layers F_n. Let $A'_n = A'(F_n)$ denote the p-part of the p-class group of F_n and $A'_\infty = \varinjlim A'_n$. We define the capitulation kernel $\mathrm{Cap}_0(F_\infty/F_n) = \ker(A'_n \to A'_\infty)$. As is well-known (cp. [13]) these kernels stabilize, more precisely, the norm $N_{F_m/F_n} : \mathrm{Cap}_0(F_\infty/F_m) \to \mathrm{Cap}_0(F_\infty/F_n)$ is an isomorphism for n large and $m \geqslant n$ and we set $\mathrm{Cap}_0(F_\infty) = \varprojlim \mathrm{Cap}_0(F_\infty/F_n)$.

Remark 3.1.— Let A_n denote the p-part of the (usual) class group of F_n and let $A_\infty = \varinjlim A_n$. Once again, the capitulation kernels $\ker(A_n \to A_\infty)$ stabilize, and we can consider $\widetilde{\mathrm{Cap}}(F_\infty) = \varprojlim \ker(A_n \to A_\infty)$. We note that in general $\widetilde{\mathrm{Cap}}(F_\infty) \neq \mathrm{Cap}_0(F_\infty)$. Indeed, from the explicit examples elaborated by Greenberg in [11, section 8], it is not hard to see that if we take $F = \mathbb{Q}(\sqrt{142})$, $p = 3$, and let F_∞ be the cyclotomic $\mathbb{Z}_3$-extension of F, then $\widetilde{\mathrm{Cap}}(F_\infty) \cong \mathbb{Z}/3\mathbb{Z}$, whereas $\mathrm{Cap}_0(F_\infty)$ is trivial. From a K-theoretic point of view, $\mathrm{Cap}_0(F_\infty)$ is the appropriate object to study.

We want to consider the analog of these kernels in higher étale K-theory.

Let again S be a finite set of primes in F containing S_p. To simplify notation, we put

$$\tilde{K}^{\text{ét}}_{2i-1}(F_\infty) = \varinjlim K^{\text{ét}}_{2i-1}(F_n)$$

and

$$\tilde{K}^{\text{ét}}_{2i-2}(o^S_\infty) = \varinjlim K^{\text{ét}}_{2i-2}(o^S_n),$$

where o^S_n denotes the ring of S-integers in F_n, i.e. the integral closure of o^S_F in F_n. We now define for $i \geqslant 2$:

$$\mathrm{Cap}_{i-1}(F_\infty/F_n) = \ker(K^{\text{ét}}_{2i-2}(o^S_n) \to \tilde{K}^{\text{ét}}_{2i-2}(o^S_\infty)).$$

The following result implies in particular that the definition is independent of the choice of the finite set S containing S_p. Let Γ_n denote the Galois group of F_∞/F_n with the usual convention $\Gamma_0 = \Gamma$.

Proposition 3.2. — *For $i \geqslant 2$ there is a short exact sequence*

$$0 \to H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) \to K^{\text{ét}}_{2i-2}(o^S_n) \to \tilde{K}^{\text{ét}}_{2i-2}(o^S_\infty)^{\Gamma_n} \to 0.$$

Proof. — For each $m \geqslant n$, Theorem 1.2 gives an exact sequence

$$0 \to H^1(\Gamma_n/\Gamma_m, K^{\text{ét}}_{2i-1}(F_m)) \to K^{\text{ét}}_{2i-2}(o^S_n) \\ \to K^{\text{ét}}_{2i-2}(o^S_m)^{\Gamma_n/\Gamma_m} \to H^2(\Gamma_n/\Gamma_m, K^{\text{ét}}_{2i-1}(F_m)) \to 0.$$

From Corollary 1.4 we see that the orders of the groups $H^2(\Gamma_n/\Gamma_m, K^{\text{ét}}_{2i-1}(F_m))$ are bounded independently of m by the order of $K^{\text{ét}}_{2i-2}(o^S_n)$, and therefore the limit

$$H^2(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) = \varinjlim H^2(\Gamma_n/\Gamma_m, K^{\text{ét}}_{2i-1}(F_m))$$

is finite. On the other hand, this group is divisible, since $cd_p(\Gamma_n) = 1$, hence trivial. □

In the classical case $i = 1$, it was shown by Iwasawa (cp. [13, Theorem 12]) that

$$\mathrm{Cap}_0(F_\infty/F_n) \cong H^1(\Gamma_n, U'_\infty),$$

where $U'_\infty = \varinjlim U'_n$ and U'_n denotes the group of p-units of F_n. Therefore Proposition 3.2 gives, in particular, the following higher-dimensional analog of this result:

COROLLARY 3.3. — *For $i \geqslant 2$*

$$\mathrm{Cap}_{i-1}(F_\infty/F_n) \cong H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)).$$

To go further, we quote the following general result of Kahn (cp. [16, Proposition 6.2]), which he attributes to Nguyen Quang Do:

LEMMA 3.4. — *Let A be a discrete torsion free Γ-module. Assume that for all integers $n \geqslant 0$:*

i) *$H^0(\Gamma_n, A)$ is finitely generated;*

ii) *$H^1(\Gamma_n, A)$ is finite;*

iii) *$H^2(\Gamma_n, A) = 0$.*

Then the groups $H^1(\Gamma_n, A)$ stabilize, in particular $\varprojlim H^1(\Gamma_n, A)$ is finite.

Let

$$\bar{K}^{\text{ét}}_{2i-1}(F_n) = K^{\text{ét}}_{2i-1}(F_n)/\mathit{torsion}$$

and

$$\bar{K}^{\text{ét}}_{2i-1}(F_\infty) = \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)/\mathit{torsion}.$$

We want to apply the previous lemma with $A = \bar{K}^{\text{ét}}_{2i-1}(F_\infty)$. From the exact sequence

$$0 \to H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)) \to \tilde{K}^{\text{ét}}_{2i-1}(F_\infty) \to \bar{K}^{\text{ét}}_{2i-1}(F_\infty) \to 0,$$

we deduce the exact sequence

$$\begin{aligned} 0 \to \bar{K}^{\text{ét}}_{2i-1}(F_n) \to \bar{K}^{\text{ét}}_{2i-1}(F_\infty)^{\Gamma_n} \to H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i))) \\ \to H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) \to H^1(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty)) \to 0, \end{aligned}$$

as well as an isomorphism

$$H^2(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) \cong H^2(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty)).$$

The proof of Proposition 2.2 showed that $H^2(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) = 0$, and hence we see that $\bar{K}^{\text{ét}}_{2i-1}(F_\infty)$ satisfies the assumptions of the previous lemma. We obtain the fact that the groups $H^1(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty))$ stabilize and therefore that $\varprojlim H^1(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty))$ is finite. To obtain the same result for the groups $H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty))$ and their limit, we look at the term $H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)))$ in the above exact sequence: The group $H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i))$ is either $\mathbb{Q}_p/\mathbb{Z}_p(i)$ or finite. In the first case, Tate's Lemma implies that $H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i))) = 0$, hence

$$H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) \cong H^1(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty)).$$

In the second case, $H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)))$ stabilizes for n large, and hence in any case we obtain:

PROPOSITION 3.5. — *The groups* $\mathrm{Cap}_{i-1}(F_\infty/F_n)$ *stabilize; more precisely, the corestriction maps*

$$\mathrm{Cap}_{i-1}(F_\infty/F_{n+1}) \to \mathrm{Cap}_{i-1}(F_\infty/F_n)$$

are surjective for all n *and* $\varprojlim \mathrm{Cap}_{i-1}(F_\infty/F_n)$ *is finite.*

We now define

$$\mathrm{Cap}_{i-1}(F_\infty) = \varprojlim \mathrm{Cap}_{i-1}(F_\infty/F_n).$$

Now let us specialize and take F_∞/F to be the *cyclotomic* $\mathbb{Z}_p$-extension. As in the case $i = 1$, the finite groups $\mathrm{Cap}_{i-1}(F_\infty)$ then have various characterizations in terms of Iwasawa-theory. Let $E = F(\mu_p)$, let $E_\infty = F(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of E and identify Γ_n with the Galois group of E_∞/E_n. We first describe $\mathrm{Cap}_{i-1}(E_\infty)$. Let $\mathcal{X}_\infty$ denote the standard Iwasawa-module for E_∞, i.e. the Galois group over E_∞ of the maximal abelian p-ramified pro-p-extension of E_∞. Denote by $\mathrm{tor}_\Lambda \mathcal{X}_\infty$ the torsion part of $\mathcal{X}_\infty$ as a module over $\Lambda = \mathbb{Z}_p[[\Gamma]]$. As is well-known, there exists an injective homomorphism ([11, Theorem 3])

$$\mathcal{X}_\infty/\mathrm{tor}_\Lambda \mathcal{X}_\infty \to \Lambda^{r_2(E)}$$

with finite cokernel H. The following result is due to Iwasawa ([13]) for $i = 1$, to Coates ([7]) for $i = 2$ and to Nguyen Quang Do([27, section 4]) in general:

THEOREM 3.6. — *For all* $i \geqslant 1$ *and all* $n \geqslant 0$, *there are canonical isomorphisms*

$$\mathrm{Cap}_{i-1}(E_\infty/E_n) \cong H^*(i)_{\Gamma_n}.$$

Since H is finite, the group Γ_n acts trivially on $H^*(i)$ for all i provided n is large enough. Therefore, as abstract groups, all capitulation kernels $\mathrm{Cap}_{i-1}(E_\infty)$ are isomorphic to H.

Let $\Delta = \mathrm{Gal}(E/F)$ and let d denote the order of Δ. Now clearly

$$\mathrm{Cap}_{i-1}(F_\infty) = \mathrm{Cap}_{i-1}(E_\infty)^\Delta.$$

Theorem 3.6 shows that $\mathrm{Cap}_{i-1}(E_\infty)$ and $\mathrm{Cap}_{j-1}(E_\infty)$ are isomorphic as Δ-modules for $i \equiv j \bmod d$. Therefore we obtain the following periodicity result:

COROLLARY 3.7.— *Let p be odd and let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension of F. Then*

$$\mathrm{Cap}_{i-1}(F_\infty) \cong \mathrm{Cap}_{j-1}(F_\infty)$$

for all $i, j \geqslant 1$, $i \equiv j \bmod d$.

Next we would like to discuss another well-known relation between capitulation kernels and Iwasawa-theory: We continue to assume that $E_\infty = F(\mu_{p^\infty})$ is the cyclotomic $\mathbb{Z}_p$-extension of $E = F(\mu_p)$, and that p is odd. As usual, let X'_∞ denote the Galois group over E_∞ of the maximal abelian unramified pro-p-extension of E_∞, in which all primes above p are completely decomposed. Thus $X'_\infty \cong \varprojlim A'_n(E)$. The co-invariants $(X'_\infty)_\Gamma$ have been described by Jaulent as a group of logarithmic classes $\tilde{cl}(E)$ which can be interpreted as the class field theory analog of the wild kernels corresponding to the case $i = 1$. The Galois co-descent for these modules $\tilde{cl}(E)$ has been studied in [14]. Now, let $(X'_\infty)^0$ denote the maximal finite submodule of X'_∞. It is well-known (cp. [21]) that

$$\mathrm{Cap}_0(E_\infty) \cong (X'_\infty)^0.$$

On the other hand, we have for all $n \geqslant 0$ and all $i \geqslant 2$, an isomorphism

$$(X'_\infty(i-1))_{\Gamma_n} \cong WK^{\text{ét}}_{2i-2}(E_n)$$

(cp. 33, section 6, Lemma 1]), and therefore

$$\begin{aligned}\ker\left(WK^{\text{ét}}_{2i-2}(E_n) \to WK^{\text{ét}}_{2i-2}(E_m)\right)&\\ &\cong \ker\left((X'_\infty(i-1))_{\Gamma_n} \to (X'_\infty(i-1))_{\Gamma_m}\right)\\ &\cong (X'_\infty)^0(i-1)\end{aligned}$$

for n large and m sufficiently larger than n. If we define

$$\tilde{W}K_{2i-2}^{\text{ét}}(E_\infty) = \varinjlim WK_{2i-2}^{\text{ét}}(E_n),$$

then we obtain

PROPOSITION 3.8. — *For $i \geqslant 2$ and n sufficiently large we have:*

$$\mathrm{Cap}_{i-1}(E_\infty) \cong \ker\left(WK_{2i-2}^{\text{ét}}(E_n) \to \tilde{W}K_{2i-2}^{\text{ét}}(E_\infty)\right) \cong (X'_\infty)^0(i-1)$$

as Δ-modules.

For the original field F and the cyclotomic $\mathbb{Z}_p$-extension F_∞/F this implies:

$$\mathrm{Cap}_{i-1}(F_\infty) = \ker\left(WK_{2i-2}^{\text{ét}}(F_n) \to \tilde{W}K_{2i-2}^{\text{ét}}(F_\infty)\right) \cong ((X'_\infty)^0(i-1))^\Delta.$$

Again let ω denote the Teichmüller character on Δ. We have

$$((X'_\infty)^0(i-1))^\Delta \cong ((X'_\infty)^0)^{[1-i]} \cong (X'_\infty{}^{[1-i]})^0,$$

and hence

$$\mathrm{Cap}_{i-1}(F_\infty) \cong (X'_\infty{}^{[1-i]})^0 \cong \mathrm{Cap}_0(E_\infty)^{[1-i]}$$

for all $i \geqslant 1$. We therefore obtain a decomposition of $\mathrm{Cap}_0(E_\infty)$ into eigenspaces:

$$\underset{0}{\mathrm{Cap}}(E_\infty) \cong \bigoplus_{j=0}^{d-1} \underset{j}{\mathrm{Cap}}(F_\infty)$$

with $\mathrm{Cap}_j(F_\infty)$ being isomorphic to the $(d-j)$-th eigenspace of $\mathrm{Cap}_0(E_\infty)$. The following result gives the connection with Section 2:

PROPOSITION 3.9. — *For $i \geqslant 2$, the following statements are equivalent:*

i) $\mathrm{Cap}_{i-1}(F_\infty) \cong WK_{2i-2}^{\text{ét}}(F_n)$ *for large n.*

ii) $X'_\infty{}^{[1-i]}$ *is finite.*

Proof. — As already mentioned we have for $i \geqslant 2$:

$$(X'_\infty(i-1))_{\Gamma_n} \cong WK_{2i-2}^{\text{ét}}(E_n),$$

hence

$$X'_\infty(i-1) \cong \varprojlim WK_{2i-2}^{\text{ét}}(E_n),$$

and therefore

$$X'^{[1-i]}_{\infty} \cong \varprojlim WK^{\text{ét}}_{2i-2}(F_n).$$

The equivalence of i) and ii) is now obvious. □

Let us assume now that the base field F is totally real. Then E is a CM-field with maximal real subfield E^+. Since obviously the plus-part of the group H is trivial in this situation, Theorem 3.6 implies that $\mathrm{Cap}_{i-1}(F_\infty) = 0$ for all even $i \geqslant 2$, hence that the minus-part of $\mathrm{Cap}_0(E_\infty)$ vanishes: $\mathrm{Cap}_0(E_\infty)^- = 0$. Let X_∞ denote the Galois group of the maximal abelian unramified pro-p-extension of E_∞. Greenberg's Conjecture (cp. [11]) for the cyclotomic $\mathbb{Z}_p$-extension F_∞ of the totally real field F is equivalent to the fact that X_∞^Δ is finite. Clearly this implies that $(X'_\infty)^\Delta$ is also finite, and the converse implication is true if one assumes for example that Leopoldt's Conjecture holds for the layers F_n of F_∞/F. We will refer to Greenberg's Conjecture in the form: $(X'_\infty)^\Delta$ *is finite*. In fact we will consider Greenberg's Conjecture for the field E^+. Using Proposition 3.9, we can summarize:

Proposition 3.10. — *Let F be a totally real number field, p an odd prime, $E = F(\mu_p)$ and E^+ the maximal real subfield of E. Furthermore, let F_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of F and E_∞ the cyclotomic $\mathbb{Z}_p$-extension of E. Then:*

i) $\mathrm{Cap}_0(E_\infty)^- = 0$, *i.e.* $\mathrm{Cap}_{i-1}(F_\infty) = 0$ *for all even* $i \geqslant 2$.

ii) $\mathrm{Cap}_{i-1}(F_\infty) \cong WK^{\text{ét}}_{2i-2}(F_n)$ *for large n and all odd $i \geqslant 3$, if and only if Greenberg's Conjecture holds for E^+.*

As an immediate consequence of part ii), we obtain that under Greenberg's Conjecture the étale wild kernels $WK^{\text{ét}}_{2i-2}(F_n)$ show the same periodic behaviour as the capitulation kernels for n large and $i \geqslant 3$ odd. On the other hand, under Greenberg's Conjecture for E^+, we also have $\mathrm{Cap}_0(E^+_\infty) = \mathrm{Cap}_0(E_\infty)^+ = A'_n(E)^+$ for n large; hence for all $i \geqslant 3$ odd:

$$\mathrm{Cap}_{i-1}(F_\infty) \cong A'_n(E)^{[1-i]} \cong WK^{\text{ét}}_{2i-2}(F_n) \qquad \text{for } n \text{ large.}$$

Therefore, the Galois co-descent results of Section 2 also apply to both $\mathrm{Cap}_{i-1}(F_\infty)$ and the eigenspaces $A'_n(E)^{[1-i]}$ of $A'_n(E^+)$ for n large. In particular:

Theorem 3.11. — *Let L/F be a cyclic extension of totally real number fields of degree p, p odd, with Galois group G and let $E = F(\mu_p)$. Assume Greenberg's conjecture holds for E^+, LE^+ and the Gross* *conjecture holds for* E_n*, n large. Then for* $i \geqslant 3$ *odd, n large and* $T_{L_n/F_n} \neq \varnothing$*, Galois co-descent holds for* $\mathrm{Cap}_{i-1}(L_\infty)$ *and* $A'_n(LE)^{[1-i]}$ *if and only if the cup-product*

$$(GK(E_n)/p)^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to {}_p\mathrm{Br}^T(E_n)^{[1-i]}$$

is surjective.

Remark 3.12. — If $i \equiv 1 \bmod d$, then, under the assumptions of Theorem 3.11, we can compare the genus formulae for $WK^{\text{ét}}_{2i-2}(L_n)$ and $A'_n(L)$ to obtain for large n:

$$[U'_n : U'_n \cap N_{L_n/F_n}(L^*_n)] = [GK(F_n) : GK(F_n) \cap N_{L_n/F_n}(L^*_n)],$$

a result which one can also prove directly.

4. Galois co-descent for the étale tame kernel.

In this final section we briefly discuss how the methods of Section 2 can be used to study the much easier problem of Galois co-descent for the étale tame kernels again for cyclic extensions L/F of degree p, p odd. Results for arbitrary finite Galois p-extensions have been obtained by Assim (cp. [1], [2]) in terms of primitive ramification, however under the assumption that Leopoldt's Conjecture holds for the fields $L(\mu_{p^n})$ for all n. Let S be the finite set of primes of F, consisting of the set S_p and the tamely ramified primes in L/F. We have the following exact sequence:

$$0 \to K^{\text{ét}}_{2i-2}(o_F) \to K^{\text{ét}}_{2i-2}(o_F^S) \to \bigoplus_{v \in S \setminus S_p} H^2(F_v, \mathbb{Z}_p(i)) \to 0,$$

which, combined with Proposition 1.3, shows that the canonical map

$$K^{\text{ét}}_{2i-2}(o_L)_G \to K^{\text{ét}}_{2i-2}(o_F)$$

is always surjective and that the kernel of this map is isomorphic to the cokernel of the map

$$K^{\text{ét}}_{2i-2}(o_L^S)^G \to \left(\bigoplus_{w \in S'_L} H^2(L_w, \mathbb{Z}_p(i))\right)^G,$$

where S'_L consists of the primes in L above $S \setminus S_p$. We recall that $S \setminus S_p$ is always contained in $T^{(i)}_{L/F}$. The following is now clear from the results in Section 2:

THEOREM 4.1. — *The kernel of the surjective map* $K^{\text{ét}}_{2i-2}(o_L)_G \to K^{\text{ét}}_{2i-2}(o_F)$ *is isomorphic to the cokernel of the map*

$$K^{\text{ét}}_{2i-1}(F)/p \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{v \in S \setminus S_p} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

BIBLIOGRAPHY

[1] J. ASSIM, Sur la p-nullité de certains noyaux de la K-théorie, Thèse, Université de Franche-Comté, 1994.

[2] J. ASSIM, Codescente en K-théorie étale et corps de nombres, Manuscripta Math., 86 (1995) 499-518.

[3] G. BANASZAK, Generalization of the Moore exact sequence and the wild kernel for higher K-groups, Compositio Math., 86 (1993), 281-305.

[4] A. BOREL, Cohomologie de SL_n et valeurs de fonctions zêta aux points entiers, Ann. Scuola Normale Sup. Pisa, Ser. 4, 4 (1977), 613-636.

[5] B. BRAUCKMANN, Étale K-theory and Iwasawa-theory of number fields, Thesis McMaster University, 1993.

[6] J. BROWKIN, A. SCHINZEL, On Sylow 2-subgroups of K_2O_F for quadratic number fields F, J. Reine Angew. Math., 331 (1982), 104–113.

[7] J. COATES, p-adic L-functions and Iwasawa's theory in: Algebraic Number Fields (ed. by A. Fröhlich), Academic Press, London, 1977, 269-353.

[8] W.-G. DWYER, E.M. FRIEDLANDER, Algebraic and étale K-theory, Trans. AMS 292, No. 1 (1985), 247-280.

[9] L. FEDERER, B.H. GROSS (with an appendix by W. Sinnott), Regulators and Iwasawa modules, Invent. Math., 62 (1981), 443-457.

[10] G. GRAS, J.-F. JAULENT, Sur les corps de nombres réguliers, Math. Z., 202 (1989), 343-365.

[11] R. GREENBERG, On the Iwasawa invariants of totally real fields, Amer. J. Math., 98, (1976), 263-284.

[12] J. HURRELBRINK, M. KOLSTER, Tame kernels under relative quadratic extensions and Hilbert symbols, J. Reine Angew. Math., 499 (1998) 145-188.

[13] K. IWASAWA, On $\mathbb{Z}_l$-extensions of algebraic number fields, Ann. of Math., 98 (1973), 246-326.

[14] J.-F. JAULENT, Classes logarithmiques des corps de nombres, J. Théor. Nombres Bordeaux, 6 (1994), 301-325.

[15] B. KAHN, On the Lichtenbaum-Quillen Conjecture in: Algebraic K-theory and Algebraic Topology (ed. by J.F. Jardine), Nato Proc. Lake Louise 407, Kluwer 1993, 147-166

[16] B. KAHN, Descente galoisienne et K_2 des corps de nombres, K-theory, 7 (1993), 55-100.

[17] B. KAHN, Deux théorèmes de comparaison en cohomologie étale; applications, Duke Math. J., 69 (1993) 137-165.

[18] B. KAHN, The Quillen-Lichtenbaum Conjecture at the prime 2, preprint, 1997.

[19] M. KOLSTER, An idelic approach to the wild kernel, Invent. Math., 103 (1991), 9-24.

[20] M. KOLSTER, Remarks on étale K-theory and Leopoldt's Conjecture in: Séminaire de Théorie des Nombres, Paris, 1991-1992, Progress in Mathematics 116, Birkhäuser 1993, 37-62.

[21] L. V. KUZ'MIN, The Tate module for algebraic number fields, Math., USSR Izv., 6, No. 2 (1972), 263-321.

[22] A. S. MERKURJEV and A. A. SUSLIN, The group K_3 for a field, Math., USSR Izv., 36, No. 3 (1991), 541-565.

[23] J. MILNOR, Introduction to Algebraic K-Theory, Annals of Mathematics Studies 72, Princeton University Press, Princeton, 1971.

[24] A. MOVAHHEDI, Sur les p-extensions des corps p-rationnels, Thèse Paris 7, 1988.

[25] A. MOVAHHEDI et T. NGUYEN QUANG DO, Sur l'arithmétique des corps de nombres p-rationnels in Séminaire de Théorie des nombres, Paris 1988-89, Birkhäuser 1990, 155-200.

[26] T. NGUYEN QUANG DO, Sur la $\mathbb{Z}_p$-torsion de certains modules galoisiens, Ann. Inst. Fourier, 36-2 (1986), 27-46.

[27] T. NGUYEN QUANG DO, Sur la cohomologie de certains modules galoisiens p-ramifiés, Théorie des nombres, J.-M. De Koninck et C. Levesque (éd.), C. R. Conf. Int., Quebec/Can. 1987, 740-754 (1989).

[28] T. NGUYEN QUANG DO, K_3 et formules de Riemann-Hurwitz p-adiques, K-Theory, 7 (1993), 429-441.

[29] T. NGUYEN QUANG DO, Analogues supérieurs du noyau sauvage in Séminaire de Théorie des Nombres, Bordeaux, 4 (1992), 263-271.

[30] J. ROGNES, Approximating $K_*(\mathbb{Z})$ through degree five, K-Theory, 7 (1993), 175-200.

[31] J. ROGNES, $K_4(\mathbb{Z})$ is the trivial group, preprint, 1998.

[32] J. ROGNES, C. WEIBEL, Two-primary Algebraic K-Theory of rings of integers in number fields, J. Amer. Math. Soc., to appear.

[33] P. SCHNEIDER, Über gewisse Galoiskohomologiegruppen, Math. Z., 168 (1979), 181-205.

[34] J.-P. SERRE, Corps Locaux, Hermann, Paris, 1968.

[35] J.-P. SERRE, Cohomologie Galoisienne, LNM 5, Springer, 1964.

[36] C. SOULÉ, K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale, Inv. Math., 55 (1979), 251-295.

[37] K. WINGBERG, On the product formula in Galois groups, J. Reine Angew. Math., 368 (1986), 172-183.

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