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IN SEARCH OF THE INVISIBLE SPECTRUM

by Nikolai NIKOLSKI

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Dedicated to the memory of Stanislav A. Vinogradov

In this paper, we begin the study of the phenomenon of the "invisible spectrum" for commutative Banach algebras. Function algebras, formal power series and operator algebras will be considered. A quantitative treatment of the famous Wiener-Pitt-Sreider phenomenon for measure algebras on locally compact abelian (LCA) groups is given. Also, our approach includes efficient sharp estimates for resolvents and solutions of higher Bezout equations in terms of their spectral bounds. The smallest "spectral hull" of a given closed set is introduced and studied; it permits the definition of a uniformly bounded functional calculus. In this paper, the program traced above is realized for the following algebras: the measure algebras of LCA groups; the measure algebras of a large class of topological abelian semigroups; their subalgebras — the (semi)group algebra of LCA (semi)groups, the algebra of almost periodic functions, the algebra of absolutely convergent Dirichlet series. Upper and lower estimates for the best majorants and critical constants are obtained. Corresponding results for weighted Beurling-Sobolev algebras are contained in [ENZ]. The cases of radical algebras of formal power series, H^∞ quotient algebras, and some operator algebras are considered in [N2].

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0. Introduction.

0.1 Basic motivations.

To begin with, we recall three classical problems of harmonic analysis and function theory related to a phenomenon we will call invisible spectrum. Formal definitions are contained in Subsections 0.2 and 0.3 below.

• The *first problem* comes from convolution equations and is related to what is usually called the "*Wiener-Pitt phenomenon*". Namely, let G be a locally compact abelian group (LCA group) written additively, and $\mathcal{M}(G)$ the convolution algebra of all complex measures on G. The fundamental problem is to find an invertibility criterion for measures $\mu \in \mathcal{M}(G)$, that is, a criterion for the existence of $\nu \in \mathcal{M}(G)$ such that $\mu * \nu = \delta_0$, δ_0 being the unit of $\mathcal{M}(G)$ (the Dirac δ-measure at 0). An obvious obstruction for invertibility is the vanishing of the Fourier transform $\widehat{\mu}(\gamma) = 0$ at a point γ of the dual group $\widehat{G}$, since the equality $\widehat{\mu}\widehat{\nu} \equiv 1$ is equivalent to the initial

convolution equation. Generally, boundedness away from zero

(0.1.1) $$\delta = \inf_{\gamma \in \widehat{G}} |\widehat{\mu}(\gamma)| > 0$$

is necessary for μ to be invertible. N. Wiener and R. Pitt (1938), and Yu. Sreider (1950, a corrected version of Wiener and Pitt's result) discovered that, in general, this is not sufficient. Namely, there exists a measure μ on the line $\mathbb{R}$ whose Fourier transform

$$\widehat{\mu}(y) = \int_{\mathbb{R}} e^{-ixy} \, d\mu(x), \quad y \in \mathbb{R}$$

is bounded away from zero but for which there exists no measure $\nu \in \mathcal{M}(\mathbb{R})$ such that

$$\widehat{\nu}(y) = \frac{1}{\widehat{\mu}(y)}, \quad y \in \mathbb{R}.$$

In fact, this result holds true for an arbitrary LCA group G which is not discrete, see [Ru1], [GRS], [HR] for references and historical remarks. Using Banach algebra language, one can say that the dual group $\widehat{G}$, being the "visible part" of the maximal ideal space $\mathfrak{M} = \mathfrak{M}(A)$ of the algebra $A = \mathcal{M}(\mathbb{T})$, is far from being a dense subset of $\mathfrak{M}$. Nonetheless, later on we show that some quantitative precisions of (0.1.1), namely a closeness of the norm $\|\mu\|$ and δ of (0.1.1), lead to the desired invertibility, and even to a norm control of the inverse; see Subsection 0.4 below.

On the contrary, for a discrete group G, the classical Wiener theorem on absolutely convergent Fourier series says that condition (0.1.1) implies the invertibility of μ; moreover, in this case $\mathfrak{M} = \widehat{G}$. However, in this setting too, the problem of the norm control for inverses μ^{-1} is still meaningful and interesting, in spite of the Wiener theorem, since the latter does not yield any estimate. In fact, from the quantitative point of view, there is no big difference between these qualitatively polar cases (we mean the cases of nondiscrete and discrete groups). It turns out that, in both cases, one can control the norms $\|\mu^{-1}\|$ for $\delta > 0$ close enough to the norm $\|\mu\|$, but this is not the case for small $\delta > 0$; see Subsection 0.4 below.

• The *second problem* we are interested in is to distinguish, among all commutative unital Banach algebras A, those permitting an estimate of the resolvents in terms of the distance to the spectrum. More precisely, we want to know for which algebras A there exists a function φ such that

(0.1.2) $$\left\| (\lambda e - f)^{-1} \right\| \le \varphi\big(\mathrm{dist}(\lambda, \sigma(f))\big)$$

for all $\lambda \in \mathbb{C} \setminus \sigma(f)$ and all $f \in A$, $\|f\| \leq 1$. Here e stands for the unit of A, and $\sigma(f)$ for the spectrum of f in the algebra A. In fact, we treat this problem in a more general context of norm-controlled functional calculi, that is, as a partial case of the norm estimates problem for functions operating on a Banach algebra. For more details see Subsections 0.2–0.4 below, as well as Section 5.

• The *third problem* motivating this paper is a multi-element version of the previous two. Postponing precise definitions and discussions until Subsections 0.2 through 0.4 below, let us mention here the classical corona problem for the algebra $H^\infty(\Omega)$ of all bounded holomorphic functions on Ω, an open subset of $\mathbb{C}^n$ or of a complex manifold. Recall that the problem is to solve the Bezout equations

$$(0.1.3) \qquad\qquad \sum_{k=1}^{n} g_k f_k = 1$$

in the algebra $H^\infty(\Omega)$, where the data $f_k \in H^\infty(\Omega)$ satisfy an analogue of condition (0.1.1),

$$(0.1.4) \qquad\qquad \delta^2 = \inf_{z \in \Omega} \sum_{k=1}^{n} |f_k(z)|^2 > 0,$$

and to estimate solutions $g_k \in H^\infty(\Omega)$. As is well known, the Banach algebra meaning of the corona problem is the following: the existence of $H^\infty(\Omega)$ solutions for any data satisfying (0.1.4) is equivalent to the density of Ω, the "visible part" of the spectrum $\mathfrak{M} = \mathfrak{M}(H^\infty(\Omega))$, in $\mathfrak{M}$. In what follows, we consider a norm refinement of this problem for several algebras different from $H^\infty(\Omega)$.

Now, we are going to specify the statements of the problems, and the results obtained.

0.2. Visibility levels. Main problems.

Let A be a commutative unital Banach algebra continuously embedded into the space $C(X)$ of all continuous functions on a Hausdorff topological space X:

$$A \subset C(X).$$

Note that we do not suppose that the embedding is injective. However, the point evaluations δ_x, where $x \in X$,

$$\delta_x(f) = f(x), \quad f \in A,$$

are continuous complex homomorphisms of A. Hence, supposing A distinguishes points of X, we can identify X with a subset of the maximal ideal space of A (the spectrum of A), $\mathfrak{M} = \mathfrak{M}(A)$,

$$X \subset \mathfrak{M}(A).$$

Staying on X, we regard $\operatorname{clos} X$ as the visible part of $\mathfrak{M}$. We write $f \mapsto \hat{f}(\mathfrak{m})$, $\mathfrak{m} \in \mathfrak{M}$, for the Gelfand transform of an element $f \in A$, and so for $x \in X$ we identify

$$f(x) = \hat{f}(\delta_x).$$

Recall that the spectrum $\sigma(f)$ of an element $f \in A$ coincides with the range $\hat{f}(\mathfrak{M})$ of the Gelfand transform. The following definition formalizes different levels of "visibility" of the spectrum.

0.2.1. DEFINITION. — *We say that the spectrum of A is*

- *n-visible (or, n-visible from X, $n = 1, 2, \ldots$) if for all $f = (f_1, \ldots, f_n)$ in $A^n = A \times \cdots \times A$,*

$$\hat{f}(\mathfrak{M}) = \operatorname{clos}\bigl(f(X)\bigr) ;$$

- *and it is said to be* completely visible *if*

$$\mathfrak{M} = \operatorname{clos} X.$$

It is clear that $(n+1)$-visibility implies n-visibilty for any $n \geq 1$, and that complete visibility is equivalent to n-visibility for all $n \geq 1$. Moreover, the Gelfand theory of maximal ideals makes the following lemma evident.

0.2.2. LEMMA. — *For a unital Banach algebra A, the following properties are equivalent:*

(i) *the spectrum of A is n-visible;*

(ii) *for every $f = (f_1, \ldots, f_n) \in A^n$ satisfying*

$$(0.2.1) \qquad \delta^2 =: \inf_{x \in X} \sum_{k=1}^{n} |f_k(x)|^2 > 0,$$

there exists an n-tuple $g \in A^n$ solving the Bezout equation

$$(0.2.2) \qquad \sum_{k=1}^{n} g_k f_k = e.$$

In particular, the spectrum is 1-visible if and only if the following Wiener-Lévy theorem is true :

$$\inf_{x \in X} |f(x)| > 0 \text{ implies that } f \text{ is invertible in } A. \qquad \square$$

In this language, the Wiener-Pitt phenomenon is exactly the 1-invisibility of the spectrum for the measure algebra $\mathcal{M}(G)$, if we stay on $X = \widehat{G}$. Rigorously speaking, the heart of the matter is the algebra of Fourier transforms $\mathcal{F}\mathcal{M}(G) = \{\widehat{\mu} : \mu \in \mathcal{M}(G)\}$ embedded into $C(\widehat{G})$, but as we systematically identify these algebras in our setting, they are isometrically isomorphic.

The next definition specifies the previous one in the case of norm controlled invertibility instead of simple invertibility.

0.2.3. DEFINITION. — *Let A and X be as above, and let $0 < \delta \leq 1$.*

• *The spectrum of A is said to be $(\delta\text{-}n)$-visible (from X) if there exists a constant c_n such that any Bezout equation (0.2.2) with data $f = (f_1, \ldots, f_n) \in A^n$ satisfying (0.2.1) and the normalizing condition*

$$(0.2.3) \qquad \|f\|^2 =: \sum_{k=1}^{n} \|f_k\|^2 \leq 1$$

has a solution $g \in A^n$ with $\|g\| \leq c_n$.

• *The spectrum is called completely δ-visible if it is $(\delta\text{-}n)$-visible for all $n \geq 1$ and the constants c_n can be chosen in such a way that $\sup_{n \geq 1} c_n < \infty$.*

Clearly, there exist best possible constants, in the following sense. Setting

$$(0.2.4) \quad c_n(\delta) =: c_n(\delta, A) = c_n(\delta, A, X)$$

$$= \sup_{f} \left\{ \inf \left(\|g\| : \sum_{k=1}^{n} g_k f_k = e, \ g \in A^n \right) \right\},$$

where the supremum *is taken over all $f \in A^n$ satisfying (0.2.3) and (0.2.1), we get the smallest number for which $c_n = c_n(\delta) + \epsilon$ meets the requirements of Definition 0.2.3 for every $\epsilon > 0$. In particular,*

$$(0.2.5) \qquad c_1(\delta) = \sup\{\|f^{-1}\| : f \in A, \ \delta \leq |f(x)| \leq \|f\| \leq 1, \ x \in X\},$$

and, in this case, we can take $c_1 = c_1(\delta)$ in Definition 0.2.3; here we formally set $\|f^{-1}\| = \infty$ for noninvertible elements of A.

• We define the n-th critical constant $\delta_n(A, X)$ by the relation

$$\delta_n(A, X) = \inf\{\delta : c_n(\delta, A, X) < \infty\}.$$

0.2.4. Main problems. — Our main objective is to estimate from above and from below, and (if possible) to compute the critical constants $\delta_n(A, X)$ and the majorants $c_n(\delta, A, X)$ for basic Banach algebras A and, thus, to study norm controlled visibility properties for these algebras.

For $n = 1$ we deal with more general objects. Namely, we are interested in describing functions boundedly acting on a given algebra. Supposing that a "visible part" of the spectrum is fixed, $X \subset \mathfrak{M}(A)$, we can say that a function (say, φ) defined on $\sigma \subset \mathbb{C}$ acts on an algebra A if for every $a \in A$ with $\hat{a}(X) \subset \sigma$ there exists $b \in A$ such that $\varphi \circ \hat{a} = \hat{b}$ on X. A "bounded action" is defined in the same way but adding an estimate of the form $\|b\| \leq k\|\varphi\|_*$, where $k = k(\sigma, A, X)$ and $\|\cdot\|_*$ is an appropriate norm bounding $\|\varphi\|_\sigma = \sup_\sigma |\varphi|$. We mention, however, that such a definition is too broad to be useful: the *spectral inclusion $\hat{a}(X) \subset \sigma$ alone, without any norm restrictions, cannot imply the boundedness of compositions.* For example, it is easy to see that the following holds true:

If for some positive δ and C, the relation $|\hat{a}(x)| \geq \delta$ $(x \in X)$ always implies that $\|a^{-1}\| \leq C$, then A is a uniform algebra with the norm equivalent to the sup norm on X:

$$\|\hat{a}\|_X \leq \|a\| \leq (2C\delta + 1)\|\hat{a}\|_X \quad \text{for all } a \in A.$$

Indeed, we first observe that if $\inf_X |\hat{a}| = \epsilon > 0$, then $|\hat{a}(x)\delta/\epsilon| \geq \delta$, $x \in X$, and hence a is invertible and $\|a^{-1}\| \leq C\delta/\epsilon = C\delta\|\hat{a}^{-1}\|_X$. Now, for a given $b \in A$ we take $\lambda = \|\hat{b}\|_X + \epsilon$ and apply the previous statement to $a = (b - \lambda e)^{-1}$:

$$\|b\| = \|b - \lambda e + \lambda e\| \leq C\delta\|\hat{b} - \lambda\|_X + \lambda$$
$$\leq C\delta\|\hat{b}\|_X + (C\delta + 1)\lambda = (2C\delta + 1)\|\hat{b}\|_X + (C\delta + 1)\epsilon.$$

Letting $\epsilon \to 0$, we get $\|b\| \leq (2C\delta + 1)\|\hat{b}\|_X$ for all $b \in A$. $\qquad\square$

This argument shows that certain norm requirements are necessary. The classic results on functions operating on Fourier transforms show many specific examples of this kind; see [HKKR], [Ru1], and further references therein.

In this paper, after adding the normalizing condition $\|a\| \leq 1$ to the spectral inclusion $\hat{a}(X) \subset \sigma$, we look for a "minimal spectral hull" $h(\sigma) = h(\sigma, A, X)$ such that functions holomorphic on $h(\sigma)$ boundedly act on a given algebra. An approach to this problem is developed in Section 5 below.

0.3. Main results.

In what follows, we examine the (δ-n)-visibility of the measure algebras $\mathcal{M}(G)$ on LCA groups, and for their subalgebras $\mathcal{M}_d(G)$ and $L^1(G) + \mathbb{C} \cdot e$, respectively, of discrete measures and of absolutely continuous measures (with the unit added). Their semigroup counterparts $\mathcal{M}(S)$, $\mathcal{M}_d(S)$ and $L^1(S) + \mathbb{C} \cdot e$ are also considered. Here and below, S denotes a subsemigroup $S \subset G$ satisfying some natural hypotheses; see Subsection 2.2.1 for details. All frequently occurring semigroups, like $\mathbb{Z}_+^n$, $\mathbb{R}_+^n$, etc., verify these hypotheses. Our main results on these algebras can be summarized as follows.

THEOREM A. — *Let G be an LCA group, and let A be one of three algebras $A = \mathcal{M}(G)$, $\mathcal{M}_d(G)$, and $L^1(G) + \mathbb{C} \cdot e$.*

(i) *One has*

$$\delta_n(A, \widehat{G}) \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad c_n(\delta, A, \widehat{G}) \leq \frac{1}{2\delta^2 - 1} \quad \text{for} \quad \frac{1}{\sqrt{2}} < \delta \leq 1.$$

For the algebra $A = L^1(G) + \mathbb{C} \cdot e$ on a nondiscrete group G, better estimates are true for $n = 1$:

$$\delta_1(A, \widehat{G}) \leq \frac{1}{2} \quad \text{and} \quad c_1(\delta, A, \widehat{G}) \leq \frac{1}{2\delta - 1} \quad \text{if} \quad \frac{1}{2} < \delta \leq 1.$$

(ii) *If G is an infinite group, then for all three algebras A :*

$$\delta_n(A, \widehat{G}) \geq \frac{1}{2} \quad \text{and} \quad c_n(\delta, A, \widehat{G}) \geq \frac{1}{2\delta - 1} \quad \text{for} \quad \frac{1}{2} < \delta \leq 1.$$

In particular, $\delta_1(L^1(G) + \mathbb{C} \cdot e, \widehat{G}) = \frac{1}{2}$ and $c_1(\delta, A, \widehat{G}) = (2\delta - 1)^{-1}$ for $\frac{1}{2} < \delta \leq 1$, for every nondiscrete group G.

THEOREM B. — *Let S be an infinite semigroup satisfying the conditions mentioned above, and let $\widehat{S}_b$ be the semigroup of bounded*

characters on S. Then for each of the three algebras $A = \mathcal{M}(S)$, $\mathcal{M}_d(S)$, and $L^1(S) + \mathbb{C} \cdot e$ we have

$$\delta_1(A, \widehat{S}_b) = \frac{1}{2} \quad \text{and} \quad c_1(\delta, A, \widehat{S}_b) = \frac{1}{2\delta - 1} \quad \text{for} \quad \frac{1}{2} < \delta \le 1.$$

In particular,

$$\delta_1(W_+, \overline{\mathbb{D}}) = \frac{1}{2} ; \quad c_1(\delta, W_+, \overline{\mathbb{D}}) = \frac{1}{2\delta - 1} \quad \left(\frac{1}{2} < \delta \le 1 \right)$$

for the analytic Wiener algebra W_+, as well as for its multidimensional (on $\mathbb{Z}_+^n$) and continuous (on $\mathbb{R}_+^n$) analogues. The same is true for the unital group algebra of $\mathbb{R}$ (and $\mathbb{R}^n$):

$$\delta_1(L^1(\mathbb{R}) + \mathbb{C} \cdot e, \mathbb{R}) = \frac{1}{2}, \quad c_1(\delta, L^1(\mathbb{R}) + \mathbb{C} \cdot e, \mathbb{R}) = \frac{1}{2\delta - 1}.$$

For higher Bezout equations the following upper estimate holds:

$$\delta_n(L^1(\mathbb{R}) + \mathbb{C} \cdot e, \mathbb{R}) \le \frac{1}{\sqrt{2}}, \quad c_n(\delta, L^1(\mathbb{R}) + \mathbb{C} \cdot e, \mathbb{R}) \le \frac{1}{2\delta^2 - 1}$$

for $\frac{1}{\sqrt{2}} < \delta \le 1$.

On the other hand, for the symmetric Wiener algebra W we can only prove that

$$\frac{1}{2} \le \delta_1(W, \mathbb{T}) \le \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{1}{2\delta - 1} \le c_1(\delta, W, \mathbb{T}) \le \frac{1}{2\delta^2 - 1}.$$

Postponing all comments on these estimates (as well as other examples) until Sections 2 and 3, we only mention that, in fact, our results are slightly more general than as stated in Theorems A and B.

There are also a few examples of algebras of a different nature; see Theorems 2.5.1, 4.3.1 and 5.3.5.

The case of *finite groups* is briefly considered in Section 4.

The more general problem of functions boundedly operating on an algebra is considered in Section 5. Namely, given a closed subset $\sigma \subset \overline{\mathbb{D}}$ and a point $\lambda \in \mathbb{C}$, we define the resolvent majorant by

$$C(\lambda, \sigma; A, X) =: \sup \{ \|(\lambda e - f)^{-1}\| : f \in A, \ \|f\| \le 1, \ \hat{f}(X) \subset \sigma \},$$

and the (A, X)-*spectral hull* of σ by

$$h(\sigma; A, X) = \{ \lambda \in \mathbb{C} : C(\lambda, \sigma; A, X) = \infty \}.$$

The following theorem is proved.

THEOREM C. — *The (A,X)-spectral hull $h(\sigma; A, X)$ is always closed,
$\sigma \subset h(\sigma; A, X)$, and $C(\lambda, \sigma; A, X) \geq 1/\operatorname{dist}(\lambda, h(\sigma; A, X))$. The relation
$h(\sigma; A, X) = \sigma$ holds for all closed σ if and only if $\delta_1(A, X) = 0$.*

*An open set $\Omega \subset \mathbb{C}$ contains $h(\sigma; A, X)$ if and only if it is a norm
controlled calculus domain for σ, in the sense that the Riesz-Dunford calculi
$\varphi \mapsto \varphi(a)$ are well defined and uniformly continuous on $\operatorname{Hol}(\Omega)$ for all $a \in A$
such that $\|a\| \leq 1$, $\hat{a}(X) \subset \sigma$.*

*For $A = \mathcal{M}(S)$ and $X = \mathfrak{M}(A) = \widehat{S}_b$, where S stands for a semigroup
of the same class as above, the spectral hull $h(\sigma; A, X)$ coincides with the
so-called horodisc expansion of σ*

$$\operatorname{hor}(\sigma) = \bigcup_{z \in \sigma} \overline{D}\big(z, 1 - |z|\big)$$

and $C(\lambda, \sigma) = 1/\operatorname{dist}(\lambda, \operatorname{hor}(\sigma))$.

Several examples of Banach algebras A are shown, whose spectral
hulls behave very differently. For instance, we show an algebra such that
$h(\sigma, A, \mathfrak{M}(A)) = \overline{\mathbb{D}}$ for all $\sigma \subset \overline{\mathbb{D}}$, $\sigma \neq \emptyset$ (Theorem 5.3.5), and, in contrast,
other algebras satisfying $h(\sigma, A, \mathfrak{M}(A)) = \sigma$ for all $\sigma \subset \overline{\mathbb{D}}$ (Subsection 5.3.6).

0.4. How this paper is organized.

In Section 1 we introduce two general methods: a method for upper
estimates of the visibility constants $c_n(\delta, A, X)$, and a method for their lower
estimates. The first one, described in Lemma 1.4.3 and Theorem 1.4.7, is
based on an additive splitting property, that is, on the existence of a linear
functional φ such that $\|x\| = \|x - \varphi(x)e\| + |\varphi(x)|$ for every $x \in A$. A weak
symmetry property of the algebra A (the so-called X-symmetry defined in
Subsection 1.4) is important for applications to algebras for which φ is not
an evaluation functional at a point of X.

The method for lower estimates for $c_n(\delta, A, X)$ given in Theorem 1.5.1
refers to elements of A with "almost independent" powers.

Section 2 deals with applications of the first of the above methods to
algebras $\mathcal{M}(G)$, $\mathcal{M}(S)$, and their subalgebras described in Subsection 0.3.
The class of semigroups $S \subset G$ considered in this section is described in
Subsection 2.2.1. A comparison of upper estimates for δ-visibilities from $\widehat{G}$
and from $\widehat{S}_b$ (the semigroup of bounded characters of S, which is much
larger than $\widehat{G}$), is given in Subsection 2.3. In Subsection 2.5, we use the
same method for a semigroup of different type (a nilpotent semigroup).

Section 3 is devoted to lower estimates of $c_n(\delta, A, X)$ and $\delta_1(A, X)$ for the same algebras as above. To check the hypotheses of Theorem 1.5.1, we use Sreider measures, which are defined as measures with real Fourier transforms and the spectrum filling in a disc. A specific approximation of such measures based on the Bochner characterization of the Fourier transform $\mathcal{F}\mathcal{M}(G)$ is also used. For the case where Sreider measures do not exist, that is, for discrete groups G, we move to the Bohr compactification of G. In fact, all these techniques work for a class of subalgebras of $\mathcal{M}(G)$ described axiomatically in the corresponding theorems (Theorem 3.2.2 for groups, and Theorem 3.3.7 for semigroups). This class contains the three main cases mentioned in Subsection 0.3.

Therefore, Theorem A (i) is contained in the union of Theorems 2.1.1, 2.4.2 and Remarks 2.4.3; and Theorem A (ii) coincides with Corollary 3.2.5. Theorem B splits into Theorem 2.2.2 and Theorem 3.3.7.

Section 4 contains several remarks on the case of finite groups and semigroups. Elementary arguments show that in this case the critical constant $\delta_1(\mathcal{M}(G), \widehat{G})$ is zero, and the visibility majorants $c_n(\delta, \mathcal{M}(G), \widehat{G})$ have a linear growth as $\delta \to 0$; see Lemma 4.1.1. The problem of exact values of $c_n(\delta, \mathcal{M}(G), \widehat{G})$, and even of $c_1(\delta, \mathcal{M}(G), \widehat{G})$, is still open, and we deal instead with uniform bounds for the latter constants for the cyclic groups $C_n = \mathbb{Z}/n\mathbb{Z}$, and for the finite nilpotent semigroups $\mathbb{Z}_n = \mathbb{Z}_+/(n + \mathbb{Z}_+)$. For the first of these series a uniform lower estimate is shown, and for the second one the same estimate is proved to be sharp; see Theorems 4.2.3 and 4.3.1.

Finally, in Section 5, we deal with a more general framework for $(\delta\text{-}n)$-visibility properties, namely with norm-controlled functional calculi and the so-called spectral hulls $h(\sigma, A, X)$ introduced in the same section. Some general properties of spectral hulls can be found in Theorem 5.1.2. Their relations to the norm-controlled calculus are established in Theorem 5.2.2. The full spectral hulls for the measure algebra $\mathcal{M}(S)$ on a semigroup S are described in Theorem 5.3.2.

Therefore, Theorem C is contained in the union of Theorems 5.1.2, 5.2.2 and 5.3.2.

The algebra of functions of a model operator M_Θ is shown to have an extremely invisible spectrum, namely $h(\sigma, A, X) = \overline{\mathbb{D}}$ for every $\sigma \neq \emptyset$ (Theorem 5.3.5). Examples of Beurling-Sobolev algebras on $\mathbb{Z}$ and $\mathbb{Z}_+$ having a well-visible spectrum, namely such that $h(\sigma, A, X) = \sigma$ for every closed $\sigma \subset \overline{\mathbb{D}}$, are described in Subsection 5.3.6.

0.5. Historical remarks.

As already mentioned, the prehistory of the ideas presented in this paper began with the classical theorems of Wiener-Lévy and Wiener-Pitt-Sreider quoted above.

The second wave of results, sharpening the Gelfand and Riesz-Dunford functional calculi, were devoted to functions operating on Fourier transforms, and were mostly due to H. Helson, J.-P. Kahane, Y. Katznelson, and W. Rudin. The main problem considered and resolved was to describe functions φ defined on an interval $\sigma = [a, b] \subset \mathbb{R}$ and such that $\varphi(\mathcal{F}f) \in \mathcal{F}A$ for every $f \in A$ with $\mathcal{F}f(\widehat{G}) \subset \sigma$, where $A = \mathcal{M}(G)$ or $A = L^1(G)$. See [HKKR], [Ru1], [GMG], [K1], [HR] for exhaustive presentations and further references. Nonanalytic functions operating on certain weighted algebras of Fourier transforms $\mathcal{F}\ell^1(\mathbb{Z}, w)$ occurred in the papers of J.-P. Kahane [K2] and of N. Leblanc [L]. However, no quantitative aspects similar to those of Subsections 0.1–0.4 were explicitly presented.

The third wave of results related to norm-controlled calculi can be linked with constructive proofs of the Wiener-Lévy theorem on inverses. We mention the proofs of A. Calderon, presented in [Z], P. Cohen [C] and D. Newman [New]. The Calderon approach was profoundly developed by E. Dyn'kin [D].

The problem of norm-controlled inversion (for the Wiener algebra $A = \ell^1(\mathbb{Z})$) was first mentioned by J. Stafney in [St], where the existence of $a, b, K > 0$ was proved such that $\sup\{\|f^{-1}\|_A : \|f\| \leq K, \hat{f}(\mathbb{T}) \subset [a, b]\} = \infty$. This implies that $c_1(\delta, A, \mathbb{T}) = \infty$ for some $\delta > 0$. The proof, based on Y. Katznelson's results, does not permit one to specify the value of δ. Independently and in a more constructive way, this result was obtained by H. Shapiro [Sh1] (in response to a question posed by a physicist, G. Ehrling), but also without any concrete value of δ. Several remarks were made in Shapiro's note attributing to various authors certain estimates for quantities we call the critical constants $\delta_1(\ell^1(\mathbb{Z}_+), \overline{\mathbb{D}})$ and $\delta_1(\ell^1(\mathbb{Z}), \mathbb{T})$; no precise references were given. In fact, I started this work by analyzing Shapiro's construction. In the paper of J.-E. Björk [B], a problem related to uniform functional calculi was considered. In our language, it is equivalent to an estimate for $\delta_1^0(A, \mathfrak{M}(A))$, a microlocal version of $\delta_1(A, \mathfrak{M}(A))$ studied in Section 5 below. In [B], a criterion was given in terms of another quantity which can be regarded as a "uniform spectral radius" of the algebra, but no applications were shown.

In the paper of S. Vinogradov and A. Petrov [VP], a description was

given of Banach spaces A of functions on $\mathbb{T}$ satisfying the following property: $f \in A$ and $|f| \geq \delta$ on $\mathbb{T}$ imply $1/f \in A$.

We finish these remarks by mentioning that the case of higher Bezout equations (the "corona problems") related to the constants $c_n(\delta, A, X)$ and $\delta_n(A, X)$ for $n > 1$ is considered in this paper very briefly. For its history see [Gar], [Gam], [N1], as well as [To1] and [To2].

1. Preliminaries on (δ-n)-visibility, best majorants, and critical constants.

Let A be a commutative Banach algebra with unit e, and let X be a Hausdorff topological space such that A is continuously embedded in $C(X)$. We also use other notations introduced in Section 0. The group of invertible elements of A is denoted by $\mathcal{G}(A)$. We start with simple general observations about our main subjects.

1.1. Monotonicity of (δ-n)-visibility and critical constants.

1.1.1. LEMMA. — *If the spectrum of A is (δ-n)-visible for some positive δ, it is also (δ'-n)-visible for $\delta \leq \delta' \leq 1$ and $c_n(\delta') \leq c_n(\delta)$. If the spectrum of A is (δ-n)-visible for any $\delta > 0$, it is n-visible. Also, $c_n(\delta) \leq c_{n+1}(\delta)$ for all $\delta > 0$ and $n \geq 1$.*

Proof. — Obvious from the definitions. $\qquad\qquad\square$

It is clear from this lemma that, given an integer $n \geq 1$, there exists a *critical constant* $0 \leq \delta_n \leq 1$,

$$\delta_n = \delta_n(A) = \delta_n(A, X)$$

such that for $\delta_n < \delta \leq 1$ the spectrum of A is (δ-n)-visible, and for $0 < \delta < \delta_n$ it is not. Always, $\delta_n \leq \delta_{n+1}$. We regard these quantities as one of the fundamental characteristics of a given algebra A. In Sections 2 through 4 we compute or estimate the critical constants δ_n for certain convolution Banach algebras defined on LCA groups. Several weighted convolution algebras are considered in subsequent papers [ENZ], [N2], [EZ].

1.1.2. COROLLARY. — *If the spectrum is not n-visible, there exists $\delta > 0$ such that the spectrum is not (δ-n)-visible. In this case, the constant of $(0.2.4)$ is $c_n(\delta) = \infty$, and the critical constant is positive: $\delta_n > 0$.* $\qquad\square$

1.1.3. LEMMA.

(i) *Let B be a subalgebra of A endowed with the induced norm and stable with respect to inverses, that is $\mathcal{G}(B) = \mathcal{G}(A) \cap B$ where $\mathcal{G}$ stands for the group of invertible elements. Then $c_n(\delta, A, X) \geq c_n(\delta, B, X)$ for all $n \geq 1$ and $\delta > 0$.*

(ii) *Let $Y \subset X$ be a subspace of the topological space X. If $A \subset C(X)$ and $A \subset C(Y)$, then $c_n(\delta, A, Y) \geq c_n(\delta, A, X)$ for all $n \geq 1$ and $\delta > 0$.*

Proof. — Obvious from the definitions.   □

1.1.4. Remark. — In general, no relations between $c_n(\delta, A, X)$ and $c_n(\delta, B, X)$ follow from contractive inclusions $B \subset A \subset C(X)$. For instance, for $C^{(1)}(\mathbb{T}) \subset W \subset C(\mathbb{T})$, where W stands for the Wiener algebra, we have $c_n(\delta, C^{(1)}, \mathbb{T}) < \infty$, $c_n(\delta, C, \mathbb{T}) < \infty$ for all $\delta > 0$, $n \geq 1$ (see 1.2.1 below), but $c_1(\frac{1}{2}, W, \mathbb{T}) = \infty$ (Section 3).

1.2. Known examples.

1.2.1. Algebras with local norms. — Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. It is clear that the algebras of all bounded functions, $A = B(\Omega)$, and of all continuous bounded functions, $A = C_b(\Omega)$, endowed with the usual supremum norm, have completely δ-visible spectra with respect to $X = \Omega$ for all $\delta > 0$, and $c_n(\delta, A, \Omega) = \delta^{-1}$ for all n and $\delta > 0$. Indeed, we can always take the standard solution of Bezout equation (0.2.2)–(0.2.1): $g_k = \bar{f}_k/|f|^2$, $k = 1, \ldots, n$, where by definition $\bar{f}_k$ stands for the complex conjugate of f_k and

$$|f|^2 = \sum_{k=1}^{n} |f_k|^2.$$

The complete δ-visibility also holds true, for all $\delta > 0$ and for all other Banach algebras with "local norms". In particular, this is the case for $A = C_b^{(\ell)}(\Omega)$, the algebra of all ℓ times differentiable functions f on Ω with $\partial^\alpha f \in C_b(\Omega)$, $|\alpha| \leq \ell$, endowed with an appropriate norm; or for $A = W_p^\ell(\Omega)$, the Sobolev space satisfying the embedding theorem $W_p^\ell(\Omega) \subset C_b(\Omega)$. One can easily check that in the latter two examples $c_n(\delta, A, \Omega) \leq \text{const}/\delta^{\ell+1}$, $n = 1, 2, \ldots$.

1.2.2. Holomorphic algebras with local norms. — Let Ω be an open subset of the complex plane $\mathbb{C}$, and $H^\infty(\Omega) = B(\Omega) \cap \mathrm{Hol}(\Omega)$, as before, the algebra of all bounded holomorphic functions on Ω. Next, by definition, let $X = \Omega$ be the visible part of the spectrum. Obviously, the spectrum $\mathfrak{M}(H^\infty(\Omega))$ is (δ-1)-visible with $c_1(\delta, H^\infty(\Omega), \Omega) = \delta^{-1}$, $\delta > 0$.

The famous Carleson corona Theorem says that for the unit disc $\Omega = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, as well as for domains Ω conformally equivalent to $\mathbb{D}$, the spectrum of $H^\infty(\Omega)$ is (δ-n)-visible for all $n \geq 1$ and all $\delta > 0$; see [Gar] for a proof based on the original L. Carleson idea with T. Wolff's improvements. Using another T. Wolff's approach, again for $\Omega = \mathbb{D}$, V. Tolokonnikov [To1] proved the complete δ-visibility with $c_n(\delta, H^\infty(\mathbb{D}), \mathbb{D}) \leq 25/\delta^4$, $\delta > 0$ (independently, M. Rosenblum [R] proved that $c_n(\delta, H^\infty(\mathbb{D}), \mathbb{D}) \leq 64/\delta^4$, $\delta > 0$). Probably, the best known estimates were presented in [N1], Appendix 3:

$$\max\left(\frac{1}{\delta}, \frac{\sqrt{1-\delta^2}}{\delta^2}\right) \leq c_n(\delta) \leq \frac{1}{\delta} + \frac{1}{\delta^2}\left[7\sqrt{\log\frac{1}{\delta}} + 20\log\frac{1}{\delta}\right].$$

For further information concerning multiconnected domains $\Omega \subset \mathbb{C}$ and some infinitely connected Ω see [Gam], [Gar]. For $\Omega \subset \mathbb{C}^N, N > 1$, the problem is open even for polydiscs and balls.

An interesting example was provided by B. Cole, see [Gam]. Namely, there exists a Riemann surface Ω such that the spectrum $\mathfrak{M}(H^\infty(\Omega))$ is 2-invisible (*i.e.* is not 2-visible). We do not know any example of a Banach algebra $A \subset C(X)$ whose spectrum is n-visible but $(n+1)$-invisible for an integer $n > 1$.

Many other holomorphic algebras with local norm, like $C_b^{(\ell)}(\Omega) \cap \mathrm{Hol}(\Omega)$, $W_p^\ell(\Omega) \cap \mathrm{Hol}(\Omega)$, $\mathrm{Lip}\,\alpha \cap \mathrm{Hol}(\Omega)$, *etc.*, were considered in [S], [To2].

1.2.3. Fourier-transformed (nonlocal) norms. — One classic example has already been mentioned: $A = \mathcal{FM}(G)$, $X = \widehat{G}$, where G is a locally compact abelian nondiscrete group. The Wiener-Pitt-Sreider theorem says that the spectrum is 1-invisible, and so the corresponding critical constant δ_1 is positive, $\delta_1 > 0$; see Corollary 1.1.2 above. The Stafney [St] and Shapiro [Sh1] results show that the same is true for $G = \mathbb{Z}$ and $X = \widehat{G} = \mathbb{T}$ ($= \mathfrak{M}$), and hence for any group G containing a copy of $\mathbb{Z}$. However, as we show later on in Section 2, for any LCA group G the spectrum of $\mathcal{FM}(G)$ is (δ-n)-visible for all $n \geq 1$, if δ is sufficiently close to 1. More precisely, we

show that any infinite LCA group satisfies $\frac{1}{2} \leq \delta_n(\mathcal{FM}(G), \widehat{G}) \leq \frac{1}{\sqrt{2}}$ for all $n \geq 1$. We also consider some subalgebras of $\mathcal{FM}(G)$; see Sections 2, 3, and 4.

1.3. Two general properties.

We start linking the resolvent estimate problem with $(\delta\text{-}1)$-visibility.

1.3.1. *Distance controlled resolvent growth.* — We say that A is an algebra of distance controlled resolvent growth if there exists a monotone decreasing function $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+^* = (0, \infty)$ such that

$$(1.3.1) \qquad \left\| (\lambda e - f)^{-1} \right\| \leq \varphi(\mathrm{dist}(\lambda, \sigma(f))), \quad \lambda \in \mathbb{C} \setminus \sigma(f)$$

for all $f \in A$, $\|f\| \leq 1$. It is easy to see that this estimate implies

$$\left\| (\lambda e - f)^{-1} \right\| \leq \frac{1}{\|f\|} \varphi\left(\frac{1}{\|f\|} \mathrm{dist}(\lambda, \sigma(f)) \right), \quad \lambda \in \mathbb{C} \setminus \sigma(f)$$

for all $f \in A$. The following lemma links the $(\delta\text{-}1)$-visibility property with the distance controlled resolvent growth. A similar relationship can be established for $(\delta\text{-}n)$-visibility and the "n-resolvent" $(\lambda e_n - f)^{-1}$, where $\lambda e_n = (\lambda_1 e, \ldots, \lambda_n e) \in A^n$, $\lambda_k \in \mathbb{C}$.

1.3.2. LEMMA. — *The spectrum of an algebra A is $(\delta\text{-}1)$-visible from $X = \mathfrak{M}(A)$ for all $\delta > 0$, that is $\delta_1(A, \mathfrak{M}(A)) = 0$, if and only if A obeys the distance controlled resolvent growth.*

Similarly, $\delta_1(A, X) = 0$ for a subset $X \subset \mathfrak{M}(A)$ if and only if the spectrum of A is 1-visible and there exists a function φ such that $\|(\lambda e - f)^{-1}\| \leq \varphi(\mathrm{dist}(\lambda, \hat{f}(X)))$ for all $\lambda \in \mathbb{C} \setminus \sigma(f)$ and all $f \in A$, $\|f\| \leq 1$.

Proof. — It is clear from the definitions that (1.3.1) implies $(\delta\text{-}1)$-visibility (with respect to $X = \mathfrak{M}(A)$) with $c_1(\delta, A, \mathfrak{M}) \leq \varphi(\delta)$.

Conversely, let $c_1(\delta, A, \mathfrak{M}) < \infty$ for all $\delta > 0$. For an element $f \in A$, $\|f\| \leq 1$, and a complex number $\lambda \in \mathbb{C} \setminus \sigma(f)$, we set $g = \lambda e - f$ and $\delta = \mathrm{dist}(\lambda, \sigma(f))$.

- If $\|g\| \leq 1$, directly from the definition of $c_1(\delta)$ we deduce that $\|g^{-1}\| \leq c_1(\delta, A, \mathfrak{M})$.

- If $\|g\| > 1$, we set $h = g/\|g\|$ and get $|h(x)| = |g(x)|/\|g\| \geq \delta(2 + \delta)^{-1}$ for all $x \in \mathfrak{M}(A)$, because $\|g\| \leq |\lambda| + \|f\| \leq \delta + 2\|f\| \leq \delta + 2$.

Hence, $\|g^{-1}\| \leq \|h^{-1}\| \leq c_1(\delta(2 + \delta)^{-1}, A, \mathfrak{M})$. Since $c_1(\delta, A, \mathfrak{M}) \leq c_1(\delta(2 + \delta)^{-1})$ (see Lemma 1.1.1 above), estimate (1.3.1) follows with $\varphi(\delta) = c_1(\delta(2 + \delta)^{-1}, A, \mathfrak{M})$.

Clearly, $\delta_1(A, X) = 0$ implies $\operatorname{clos} \hat{f}(X) = \sigma(f)$ for every $f \in A$. Hence, the second part reduces to the first one. $\qquad\square$

1.3.3. *(δ-n)-visibility and reproducing kernels.* — The next lemma links the (δ-1)-visibility property with properties of the corresponding multiplication operators. In fact, it means that the (δ-1)-visibility is equivalent to the following "reproducing kernel definiteness": for a multiplication operator, the pointwise invertibility on reproducing kernels should imply the true invertibility. Precisely, the following lemma holds.

1.3.4. LEMMA. — *Let $A \subset C(X)$ be a continuously embedded commutative unital Banach algebra, and $\delta_x \in A^*$ the reproducing kernel of A at a point $x \in X$, that is, an element of the dual space A^* such that*

$$\langle g, \delta_x \rangle = \delta_x(g) = g(x)$$

for all $g \in A$. For an element $f \in A$, denote by $\mathbf{f} : A \to A$ the multiplication operator, $\mathbf{f}g = fg$, $g \in A$, and by $\mathbf{f}^* : A^* \to A^*$ its adjoint. Let $\delta > 0$.

(i) *The lower estimate $|f(x)| \geq \delta$, $x \in X$, is equivalent to the estimate*

(1.3.2) $$\|\mathbf{f}^*\delta_x\| \geq \delta\|\delta_x\|, \quad x \in X.$$

(ii) *The invertibility of f with the inequality $\|f^{-1}\| \leq \delta^{-1}$ is equivalent to the following lower estimate for $\mathbf{f}^*$:*

(1.3.3) $$\|\mathbf{f}^*\varphi\| \geq \delta\|\varphi\|, \quad \varphi \in A^*.$$

Proof. — (i) Since

$$\langle g, \mathbf{f}^*\delta_x \rangle = \langle fg, \delta_x \rangle = f(x)g(x) = \langle g, f(x)\delta_x \rangle$$

for all $g \in A$, we have $\mathbf{f}^*\delta_x = f(x)\delta_x$, $x \in X$ (we use a bilinear duality $\langle . , . \rangle$). The assertion follows.

(ii) If f is invertible, the operator $\mathbf{f}$ (and hence $\mathbf{f}^*$) is invertible as well. Conversely, estimate (1.3.3) implies that $\mathbf{f}$ is a surjection, whence there exists an element $g \in A$ such that $fg = e$. Therefore, f is invertible, and then (1.3.3) means that $\|(\mathbf{f}^*)^{-1}\| \leq \delta^{-1}$. This implies that $\|f^{-1}\| \leq \delta^{-1}$. $\square$

Therefore, to get the (δ-1)-visibility of the spectrum, we need to derive the general estimate (1.3.3) from a reproducing kernel estimate (1.3.2) (on the unit ball $\|f\| \leq 1$).

1.4. Splitting X-symmetric Banach algebras for upper estimates.

As before, we consider a commutative unital Banach algebra A continuously embedded into the space $C(X)$, $A \subset C(X)$, on a Hausdorff topological space $X \subset \mathfrak{M}(A)$. Now, we need the following definition.

1.4.1. DEFINITION.

• *We say that an algebra A splits at the unit if there exists a subspace $A_0 \subset A$ such that*

(1.4.1) $$A = e \cdot \mathbb{C} + A_0$$

(*a direct sum*) *and for $f = \lambda e + f_0$, $f_0 \in A_0$ we have $\|f\| = |\lambda| + \|f_0\|$.*

• *An algebra A is said to be X-symmetric if for every $f \in A$ there exists an element $g \in A$ such that $\|g\| \leq \|f\|$ and for all $x \in X$*

$$g(x) = \overline{f(x)}.$$

As a comment, we recall that the splitting property is satisfied by the algebras A obtained by the standard adjoining of unity to a Banach algebra A_0 without unit. For instance, this is the case for the group algebra $A_0 = L^1(G)$ of a nondiscrete LCA group G. Also, it should be mentioned that the classic symmetry property of Banach algebras corresponds, in our language, to $\mathfrak{M}(A)$-symmetry. For $X \neq \mathfrak{M}(A)$, X-symmetry may happen to be a considerably weaker property. For instance, the algebra $A = \mathcal{F}\mathcal{M}(G)$ is obviously $\widehat{G}$-symmetric: for $f = \widehat{\mu} \in A$ take $g = \widehat{\mu}_*$, where

$$\mu_*(\sigma) = \overline{\mu(-\sigma)}, \quad \sigma \subset G.$$

But it is not $\mathfrak{M}$-symmetric for a nondiscrete LCA group G. This latter property essentially coincides with the Wiener-Pitt phenomenon.

For X-symmetric algebras, the $(\delta$-$n)$-visibility properties are related to each other as follows.

1.4.2. LEMMA.. — *For an X-symmetric algebra A, the $(\delta$-$1)$-visibility of the spectrum implies the $(\sqrt{\delta}$-$n)$-visibility for all $n \geq 1$, and even the complete $\sqrt{\delta}$-visibility with*

$$\sup_{n \geq 1} c_n\left(\sqrt{\delta}, A, X\right) \leq c_1(\delta, A, X).$$

Proof. — Let $f = (f_1, \ldots, f_n) \in A^n$, let

$$\delta \leq |f(x)|^2 = \sum_{k=1}^{n} |f_k(x)|^2 \leq \|f\|^2 \leq 1$$

for all $x \in X$, and let g_k be elements of A corresponding to the f_k as in the definition of X-symmetric algebras. Then $h \in A$, where

(1.4.2) $$h =: \sum_{k=1}^{n} f_k g_k,$$

and

$$\delta \leq h(x) = \sum_{k=1}^{n} |f_k(x)|^2, \quad \|h\| \leq \Big(\sum_{k=1}^{n} \|f_k\|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} \|g_k\|^2\Big)^{\frac{1}{2}} \leq 1.$$

By assumption, h is invertible and $\|h^{-1}\| \leq c_1(\delta, A, X)$. Hence,

$$g = (g_1 h^{-1}, \ldots, g_n h^{-1}) \in A^n,$$

and $\sum_{k=1}^{n} f_k(g_k h^{-1}) = e$. Moreover,

$$\|g\| = \Big(\sum_{k=1}^{n} \|g_k h^{-1}\|^2\Big)^{\frac{1}{2}}$$

$$\leq \|h^{-1}\| \Big(\sum_{k=1}^{n} \|g_k\|^2\Big)^{\frac{1}{2}} \leq \|h^{-1}\| \cdot \|f\| \leq c_1(\delta, A, X).$$

Hence, $c_n(\sqrt{\delta}, A, X) \leq c_1(\delta, A, X)$. □

The following lemma is inspired by an observation of S.A. Vinogradov.

1.4.3. LEMMA. — *Let A be an algebra splitting at the unit, and let $f = \lambda e + f_0$, $\frac{1}{2} < \delta \leq |\lambda| \leq \|f\| \leq 1$. Then f is invertible in A, and $\|f^{-1}\| \leq 1/(2\delta - 1)$.*

Proof. — One has $f = \lambda(e + \lambda^{-1} f_0)$ and

$$\|\lambda^{-1} f_0\| = |\lambda|^{-1}(\|f\| - |\lambda|) \leq |\lambda|^{-1} - 1 \leq \delta^{-1} - 1 < 1.$$

Hence, the sum $e + \lambda^{-1} f_0$ is invertible, and f is invertible as well, and

$$\|f^{-1}\| \leq \frac{|\lambda|^{-1}}{1 - \delta^{-1} + 1} \leq \frac{\delta^{-1}}{2 - \delta^{-1}} = \frac{1}{2\delta - 1}. \qquad □$$

1.4.4. *X-dominated unit evaluation functional.* — The main result of this subsection, Theorem 1.4.7 below, gives sufficient conditions for the $(\delta\text{-}n)$-visibility of the spectrum. The statement uses the splitting property (1.4.1) and the following functional φ_e evaluating the coefficient of the unit in the standard expansion of an element of the algebra:

$$(1.4.3) \qquad\qquad \varphi_e(\lambda e + f_0) = \lambda$$

for $\lambda e + f_0 \in A = \mathbb{C} \cdot e + A_0$. Note that φ_e is a norm 1 linear functional on A, not necessarily multiplicative. The following definition will be useful.

1.4.5. DEFINITION. — Let $A = e \cdot \mathbb{C} + A_0$ be a direct sum decomposition of a Banach algebra A, and let $X \subset \mathfrak{M}(A)$. We say that the unit evaluation functional (1.4.3) is *X-dominated* if

$$\bigl|\varphi_e(f)\bigr| \le \|f\|_X$$

for every $f \in A$, where $\|f\|_X = \sup_X |f|$.

The following lemma gives another form of the X-domination property. Recall that δ_x means the evaluation functional at a point $x \in X$: $\delta_x(f) = f(x)$ for $f \in A$.

1.4.6. LEMMA. — *The following assertions are equivalent:*

(i) *the functional φ_e of (1.4.3) is X-dominated*;

(ii) $\varphi_e \in \mathrm{conv}(\delta_x : x \in X)$, *where* $\mathrm{conv}(\,.\,)$ *stands for the weak-$*$closed convex hull of* $(\,.\,)$.

Proof. — (i) $\Rightarrow$ (ii). By the Riesz-Markov theorem there exists a measure μ on the closure $\overline{X} \subset \mathfrak{M}(A)$ such that $\varphi_e(f) = \int \hat{f}(x)\,\mathrm{d}\mu(x)$ for every $f \in A$, and $\|\mu\| = \mathrm{Var}(\mu) = 1$. Since $1 = \varphi_e(e) = \mu(\overline{X})$, we obtain $\mu \ge 0$, whence it follows that μ is a probability measure on $\overline{X}$. Using Riemann sum approximations to the integrals, we get a net $(\mu_i)_{i \in I}$ of discrete probability measures on X weakly convergent to φ_e. This proves property (ii).

The converse (ii) $\Rightarrow$ (i) is obvious.   □

1.4.7. THEOREM. — *Let A be a unital Banach algebra continuously embedded into the space $C(X)$, $X \subset \mathfrak{M}(A)$, and satisfying the following conditions:*

(i) A is X-symmetric;

(ii) A splits at the unit;

(iii) φ_e is X-dominated.

Then, the spectrum of A is $(\delta\text{-}n)$-visible for all $n \geq 1$ and all δ satisfying $\frac{1}{\sqrt{2}} < \delta \leq 1$, and even completely δ-visible with

$$c_n(\delta, A, X) \leq \frac{1}{2\delta^2 - 1}.$$

Proof. — Let $f = (f_1, \ldots, f_n) \in A^n$ be such that $\delta \leq |f(x)| \leq \|f\| \leq 1$ for all $x \in X$. As in Lemma 1.4.2, consider the element $h \in A$ defined by formula (1.4.2). As before, $\frac{1}{2} < \delta^2 \leq h(x) \leq \|h\| \leq 1$ for all $x \in X$. Using condition (iii), Lemma 1.4.6, and the obvious fact that the interval $[\delta^2, 1]$ is a convex set, we obtain $\delta^2 \leq \varphi_e(h) \leq 1$. Condition (ii) and Lemma 1.4.3 imply that h is invertible and $\|h^{-1}\| \leq 1/(2\delta^2 - 1)$. Hence, $g = (g_1 h^{-1}, \ldots, g_n h^{-1}) \in A^n$, and

$$\sum_{k=1}^{n} f_k(g_k h^{-1}) = e, \quad \|g\| = \left(\sum_{k=1}^{n} \|g_k h^{-1}\|^2 \right)^{\frac{1}{2}} \leq \|h^{-1}\| \cdot \|f\| \leq \frac{1}{2\delta^2 - 1}$$

as desired. □

1.4.8. Remarks. — Here, modifying (i) and (iii) of Theorem 1.4.7, we list some other sufficient conditions for norm-controlled inversion. These observations, as well as Theorem 1.4.7, are based on the obvious idea that the better we can reconstruct the coefficient $\varphi_e(f)$ of (1.4.3) using the values $f(x)$, $x \in X$, the smaller will be the constant δ allowing a norm control of the inverse of f. In particular, the convex hull in Lemma 1.4.6 can be replaced by other tools. In Theorem 1.4.7, we have assumed that φ_e is an approximate convex combination of δ_x, $x \in X$, simply because this is the most frequently used condition.

In the following assertions, we always assume condition (ii) of Theorem 1.4.7. First, we improve the estimates of Theorem 1.4.7 by sharpening condition (iii); the simplest version (a) (see below) will be useful for the group algebras and measure algebras on semigroups, see Subsection 2.2. Condition (b) means that the element f is an "outer function" with respect to X; see Subsection 2.3.4 for comments. It should be stressed that the bottom conditions are individual, that is, they are related to an individual element $f \in A$.

(i) *If*

$$\text{(a)} \quad \varphi_e(f) \in \text{clos}\big(f(X)\big)$$

or

$$\text{(b)} \quad \log\big|\varphi_e(f)\big| \in \text{conv}\big\{\log|f(x)| : x \in X\big\},$$

and if $\frac{1}{2} < \delta = \inf_X |f(x)| \leq \|f\| \leq 1$, then $\|f^{-1}\| \leq (2\delta - 1)^{-1}$. In particular, if (we mean the weak-$*$closure)

$$\varphi_e \in \text{clos}(\delta_x : x \in X),$$

then $\delta_1(A, X) \leq \frac{1}{2}$ and $c_1(\delta, A, X) \leq (2\delta - 1)^{-1}$ for $\frac{1}{2} < \delta \leq 1$.

Indeed, the hypotheses imply $|\varphi_e(f)| \geq \delta$, and the assertion follows from Lemma 1.4.3. □

(ii) *Supposing conditions (ii) and (iii) of Theorem 1.4.7, we let $f \in A$ be such that $\|f\| \leq 1$ and*

$$\sup\big\{\inf_X(\text{Re}(fg)) : \|g\| \leq 1\big\} =: \gamma > \frac{1}{2}.$$

Then $f \in \mathcal{G}(A)$, and

$$\|f^{-1}\| \leq (2\gamma - 1)^{-1}.$$

In particular, this is the case if f ranges in a disc segment, that is, there exists a rotation $z \mapsto e^{-i\theta} z$ such that $\text{Re}(e^{-i\theta} f(x)) \geq \gamma > \frac{1}{2}$ for all $x \in X$.

Indeed, let $\epsilon > 0$, and let $\|g\| \leq 1$ be such that $\text{Re}(fg) \geq \gamma(1 - \epsilon) > \frac{1}{2}$ everywhere on X. Since $\{z \in \mathbb{C} : \text{Re}(z) \geq \gamma(1 - \epsilon)\}$ is a convex set, Lemma 1.4.6 implies the inequality $\text{Re}(\varphi_e(fg)) \geq \gamma(1 - \epsilon)$. Lemma 1.4.3 finishes the proof. □

Observe that always $\gamma \leq \delta = \inf_X |f|$.

1.5. A method for lower estimates.

The method to get a lower estimate for $c_1(\delta, A, X)$ we are considering in this subsection is inspired by Shapiro's example [Sh1] mentioned above. Essentially, it reduces to the existence of elements $a \in A$ whose normalized powers $a^k / \|a\|^k, 0 \leq k \leq p$, for a given p are ϵ-equivalent to the standard basis of an ℓ^1-space, whereas asymptotically they tend to zero faster than a given exponential. We use this method in Section 3; see also [ENZ].

1.5.1. THEOREM. — *Let A be a unital commutative Banach algebra continuously embedded into the space $C(X)$, X being a Hausdorff topological space. Let $\epsilon > 0$, and let A^ϵ denote the set of all elements $a \in A$ such that $\|a\|_{C(X)} < \epsilon$ and $\|a\| = 1$. Suppose that*

(1.5.1)
$$\sup_{a \in A^\epsilon} \left\| \sum_{k=0}^{p} b_k a^k \right\| \geq \sum_{k=0}^{p} b_k$$

for all $p \geq 0$, $\epsilon > 0$, and $b_k \geq 0$. Then

$$c_1(\delta, A, X) \geq \frac{1}{2\delta - 1}$$

for all δ such that $\frac{1}{2} < \delta < 1$. In particular,

$$\delta_1(A, X) \geq \frac{1}{2}.$$

Proof. — Let $\frac{1}{2} < \delta < 1$ and $t > 0$ be such that $\dfrac{1}{1+t} > \delta$, that is $t < \delta^{-1} - 1$. Setting

$$f = \frac{1}{1+t}(e - ta)$$

for an element $a \in A^\epsilon$, we get $\|f\| \leq 1$ and

$$|f(x)| \geq \frac{1}{1+t}(1 - t\epsilon) = \delta'$$

for all $x \in X$. Moreover,

$$\|f^{-1}\| = (1+t)\left\| \sum_{k \geq 0} t^k a^k \right\|,$$

which implies

$$\|f^{-1}\| \geq (1+t)\left(\left\| \sum_{k=0}^{p} t^k a^k \right\| - \sum_{k>p} t^k \right) = (1+t)\left\| \sum_{k=0}^{p} t^k a^k \right\| - \frac{t^{p+1}(1+t)}{1-t}.$$

Choosing $\epsilon > 0$ in such a way that $\delta' > \delta$, and passing in the last inequality to the supremum over all $a \in A^\epsilon$, we obtain

$$c_1(\delta, A, X) \geq c_1(\delta', A, X) \geq (1+t)\frac{1 - t^{p+1}}{1-t} - \frac{t^{p+1}(1+t)}{1-t}$$

$$= \frac{1+t}{1-t} - (1+t)\frac{2t^{p+1}}{1-t}.$$

Passing to the limit first as $p \to \infty$, then as $t \to t_\delta = \delta^{-1} - 1$, we get

$$c_1(\delta, A, X) \geq \frac{1 + t_\delta}{1 - t_\delta} = \frac{1}{2\delta - 1}. \qquad \square$$

The following property will be useful in Section 3.

1.5.2. LEMMA. — *Let A and X be as in Theorem 1.5.1. If the unit evaluation functional φ_e of the algebra A is $C(X)$-continuous (in particular, if it is X-dominated, as defined in 1.4.5), then (1.5.1) implies*

$$\sup_{a \in A_0^\epsilon} \left\| \sum_{k=0}^p b_k a^k \right\| \geq \sum_{k=0}^p b_k$$

for all $\epsilon > 0$ and $b_k \geq 0$, where

$$A_0^\epsilon = \left\{ a \in A : \|a\|_{C(X)} < \epsilon, \ \|a\| = 1, \ \varphi_e(a) = 0 \right\}.$$

Proof. — By assumption, there exists a constant $C > 0$ such that $|\varphi_e(a)| \leq C\|a\|_X$ for all $a \in A$. Let $\eta > 0$. By (1.5.1), for every $b_k \geq 0$ and every $\epsilon > 0$ there exists an element $a \in A^\epsilon$ such that

$$\left\| \sum_{k=0}^p b_k a^k \right\| > (1 - \eta) \sum_{k=0}^p b_k.$$

Clearly, the latter inequality is stable under small perturbations of a. Thus, replacing a by $a' = \dfrac{a - \varphi_e(a)e}{\|a - \varphi_e(a)e\|}$ we obtain a similar inequality with $1 - \eta$ replaced by $1 - 2\eta$ and with an element satisfying

$$\|a'\| = 1, \quad \varphi_e(a') = 0 \quad \text{and} \quad \|a'\|_{C(X)} < (C+1)\epsilon/(1 - C\epsilon) = \epsilon'.$$

Hence, $a' \in A_0^{\epsilon'}$, and the result follows. $\qquad\qquad \square$

2. Upper estimates for measure algebras on groups and semigroups.

2.1. The measure algebra $\mathcal{M}(G)$.

Let G be an LCA group and $\mathcal{M}(G)$ the convolution algebra of all complex Borel measures on G endowed with the standard variation norm $\|\mu\| = \mathrm{Var}(\mu)$. The Fourier transforms $\mathcal{F}\mu = \widehat{\mu}$,

$$\mathcal{F}\mu(\gamma) = \widehat{\mu}(\gamma) = \int_G (-x, \gamma)\, d\mu(x), \quad \gamma \in \widehat{G},$$

form an algebra of functions on the dual group $\widehat{G}$. We denote this algebra by $\mathcal{F}\mathcal{M}(G)$ and endow it with the range norm $\|\mathcal{F}\mu\| = \|\mu\|$. Clearly, $\mathcal{F}(\mu * \nu) = (\mathcal{F}\mu) \cdot (\mathcal{F}\nu)$ for all $\mu, \nu \in \mathcal{M}(G)$, and so the Banach algebras $\mathcal{M}(G)$ and $\mathcal{F}\mathcal{M}(G)$ are isometrically isomorphic. The term "the measure algebra of G" will be used to refer to both.

We regard the inclusion $\mathcal{F}\mathcal{M}(G) \subset C(X)$, where $X = \widehat{G}$, as the natural injective embedding defining the *visible spectrum* of $\mathcal{M}(G)$ in the sense of the preceding sections. In particular, $\widehat{G} \subset \mathfrak{M}(\mathcal{M}(G))$, and $\operatorname{clos} \widehat{\mu}(\widehat{G}) \subset \sigma(\mu)$ for any $\mu \in \mathcal{M}(G)$. The Wiener-Pitt-Sreider Theorem implies (see Introduction) that $\mathfrak{M}(\mathcal{M}(G)) = \operatorname{clos} \widehat{G}$ if and only if G is a discrete group; in fact, if this is the case, the maximal ideal space is simply equal to $\widehat{G}$: $\mathfrak{M}(\mathcal{M}(G)) = \widehat{G}$. However, even in this latter case, the problem of norm-controlled inversions, as described in previous sections, is still of interest. Moreover, from the *quantitative point of view*, there is no advantage to discrete groups, for which the spectrum $\widehat{G} = \mathfrak{M}(\mathcal{M}(G))$ is completely visible (in the sense of Definition 0.2.1), as compared with the general LCA groups, for which $X = \widehat{G} \neq \mathfrak{M}(\mathcal{M}(G))$ and the spectrum is even 1-invisible. This is an essential distinction between the concepts of n-visibility (without any norm control) and $(\delta$-$n)$-visibility. Speaking informally, the 1-visibility of $\mathfrak{M}(\mathcal{M}(G))$ for discrete groups, guaranteed by the classical Wiener-Lévy theorem, is illusory because it does not endure quantitative specifications by $(\delta$-1$)$-estimates of inverses.

In this section, we give an upper estimate for the norm-controlling constants

$$c_n\big(\delta, \mathcal{M}(G)\big) = c_n\big(\delta, \mathcal{M}(G), \widehat{G}\big),$$

as defined in 0.2.3.

2.1.1. THEOREM. — *For every LCA group G, the spectrum of $\mathcal{M}(G)$ is completely δ-visible for $\frac{1}{\sqrt{2}} < \delta \leq 1$, and, consequently, $\delta_n(\mathcal{M}(G), \widehat{G}) \leq \frac{1}{\sqrt{2}}$ for all $n \geq 1$. Moreover, for $\frac{1}{\sqrt{2}} < \delta \leq 1$ and all $n \geq 1$*

$$(2.1.1) \qquad c_n\big(\delta, \mathcal{M}(G), \widehat{G}\big) \leq \frac{1}{2\delta^2 - 1}.$$

Proof. — We simply check three conditions of Theorem 1.4.7.

• Condition (i) is obvious, and already mentioned: the algebra $\mathcal{F}\mathcal{M}(G)$ is $\widehat{G}$-symmetric; see comments to Definition 1.4.1.

• Condition (ii) is also obvious because if $\mu = \lambda\delta_0 + \nu$ and $\nu(\{0\}) = 0$, then

$$\|\mu\| = Var(\mu) = |\lambda| + \|\nu\|.$$

Hence, to fit Definition 1.4.1, we may set

$$\mathcal{M}(G)_0 = \big\{\nu \in \mathcal{M}(G) : \nu(\{0\}) = 0\big\}.$$

• To verify condition (iii), we choose a base of compact symmetric neighbourhoods of zero of the form $V_\alpha = W_\alpha - W_\alpha$, and set

$$k_\alpha = f_\alpha * \widetilde{f}_\alpha,$$

where $f_\alpha = (mW_\alpha)^{-\frac{1}{2}} \chi_{W_\alpha}$, $\widetilde{f}(x) = \overline{f(-x)}$ for $x \in G$, $m = m_G$ stands for Haar measure on G, and χ_E for the characteristic function of a set E. Then $\widehat{k}_\alpha(\gamma) \geq 0$ for $\gamma \in \widehat{G}$, $0 \leq k_\alpha(x) \leq 1 = k_\alpha(0)$ for $x \in G$, and $k_\alpha(x) = 0$ for $x \notin V_\alpha$ (the k_α are called *triangular functions* on G). Set

$$\varphi_\alpha(\mu) =: \int_G k_\alpha \, \mathrm{d}\mu$$

for every $\mu \in \mathcal{M}(G)$. Since

$$\varphi_\alpha(\mu) = \int_{\widehat{G}} \widehat{k}_\alpha(\gamma)\widehat{\mu}(\gamma) \, \mathrm{d}m_{\widehat{G}}(\gamma),$$

the functional φ_α belongs to the hull $\mathrm{conv}(\delta_\gamma : \gamma \in \widehat{G})$, $\delta_\gamma(\mu) = \widehat{\mu}(\gamma)$. Moreover, from the properties of k_α mentioned above, it follows that

$$\lim_\alpha \varphi_\alpha(\mu) = \mu(\{0\}) = \varphi_e(\mu)$$

for all $\mu \in \mathcal{M}(G)$, whence $\varphi_e \in \mathrm{conv}(\delta_\gamma : \gamma \in \widehat{G})$.

Now, inequality (2.1.1) follows from Theorem 1.4.7.   □

2.1.2. Remark. — So, the last part of the proof shows that the algebra $\mathcal{M}(G)$ and, therefore, all its subalgebras have $\widehat{G}$-dominated unit evaluation functional in the sense of Definition 1.4.5.

2.2. Measure algebras on semigroups.

Here we consider the convolution measure algebras $\mathcal{M}(S)$ on semigroups. Since harmonic analysis for these algebras is not well advanced and not commonly known, and even the basic language of the theory is not canonically fixed, we clarify, first, what objects we are dealing with. In fact, we restrict ourselves to Borel subsemigroups of LCA groups, but all results are correct for abstract LCA semigroups as well; the only difference is that the latter case requires more prerequisites, for instance, as presented in [T].

2.2.1. Basic definitions. — By a *semigroup* S we mean the following:

(i) S is a Borel subset of a LCA group G such that $x, y \in S$ implies $x + y \in S$,

(ii) $0 \in S$.

• A *bounded character* (also called *semicharacter*) on S is a bounded continuous function $\gamma : S \to \mathbb{C}$ such that $\gamma(0) = 1$ and $\gamma(x + y) = \gamma(x)\gamma(y)$ for all $x, y \in S$. It is clear that every such function is bounded by 1: $|\gamma(x)| \le 1$, $x \in S$.

• The set of *all bounded characters* of S is denoted by $\widehat{S}_b$. Obviously, $\widehat{G} \subset \widehat{S}_b$, in the sense that the restriction $\gamma_{|S}$ of a character $\gamma \in \widehat{G}$ is a bounded character of S.

In what follows, we assume that the following separation property holds.

(iii) For every $x \in S$, $x \ne 0$, there exists a bounded character $\gamma \in \widehat{S}_b$ such that $|\gamma(x)| < 1$.

In particular, $S \cap (-S) = \{0\}$ if a semigroup S satisfies condition (iii).

Let
$$\mathcal{M}(S) = \{\mu \in \mathcal{M}(G) : \mu_{|(G \setminus S)} \equiv 0\}$$
be the subspace of $\mathcal{M}(G)$ consisting of all measures supported by S. An immediate verification shows that $\mathcal{M}(S)$ is a (closed) subalgebra of $\mathcal{M}(G)$; indeed, for μ, $\nu \in \mathcal{M}(S)$ and for any bounded Borel function f on G vanishing on S, we have

$$\int_G f \, d(\mu * \nu) = \int_{G \times G} f(x + y) \, d\mu(x) \, d\nu(y)$$

$$= \int_{S \times S} f(x + y) \, d\mu(x) \, d\nu(y) = 0;$$

therefore, $(\mu * \nu)_{|(G \setminus S)} \equiv 0$.

Now, we define the *Fourier-Laplace transformation* on $\widehat{S}_b$ setting for $\mu \in \mathcal{M}(S)$

$$\mathcal{L}\mu(\gamma) = \widehat{\mu}(\gamma) = \int_S \gamma(x) \, d\mu(x), \quad \gamma \in \widehat{S}_b.$$

Clearly, the functional

(2.2.1) $$\mu \longmapsto \mathcal{L}\mu(\gamma), \quad \mu \in \mathcal{M}(S),$$

is a norm continuous homomorphism of the algebra $\mathcal{M}(S)$ for every $\gamma \in \widehat{S}_b$. Moreover, $\mathcal{L}\mu(\gamma) = \mathcal{F}\mu(-\gamma)$ for all $\gamma \in \widehat{G}$. The Fourier-Laplace transform $\mathcal{L}\mu$ is a continuous function on $\widehat{S}_b$ endowed with the weak*topology $\sigma((\mathcal{M}(S))^*, \mathcal{M}(S))$, precisely as for the dual group $\widehat{G}$. Hence,

$$\mathcal{L}\mathcal{M}(S) \subset C(\widehat{S}_b),$$

and it is natural to consider the space of bounded characters $\widehat{S}_b$ as the visible part of $\mathfrak{M}(\mathcal{M}(S))$.

The following theorem gives an upper estimate for the $(\delta\text{-}1)$-visibility constant of $\mathcal{M}(S)$ (if we stay on $\widehat{S}_b$). Examples and comments are presented in Subsection 2.3 below.

2.2.2. THEOREM. — *Let S be a semigroup satisfying conditions (i)–(iii), and $\widehat{S}_b$ the space of bounded characters of S. Then,*

(2.2.2)
$$\delta_1\big(\mathcal{M}(S), \widehat{S}_b\big) \leq \frac{1}{2},$$

(2.2.3)
$$c_1\big(\delta, \mathcal{M}(S), \widehat{S}_b\big) \leq \frac{1}{2\delta - 1},$$

for all δ, $\frac{1}{2} < \delta \leq 1$

Proof. — As in Theorem 2.1.1, it is obvious that the algebra $\mathcal{M}(S)$ splits at the unit. Let $\mu = \lambda\delta_0 + \nu \in \mathcal{M}(S)$, $\|\mu\| \leq 1$, and $|\widehat{\mu}(\gamma)| \geq \delta > \frac{1}{2}$ for $\gamma \in \widehat{S}_b$.

It suffices to show that $\lambda = \mu(\{0\}) \in \text{clos}(\widehat{\mu}(\widehat{S}_b))$, and to use Remark 1.4.8 (i). To this end, we take $\epsilon > 0$ and a compact set $K \subset S \setminus \{0\}$ satisfying

$$\int_{S\setminus(K\cup\{0\})} \mathrm{d}|\mu| < \epsilon.$$

Such a compact set exists since the variation $|\mu|$ is a regular finite Borel measure on G supported by S. For a point $x \in K$, we take a bounded character γ such that $|\gamma(x)| < 1$; see condition (iii) above. Then, $|\gamma_x(x)| < \epsilon$ for $\gamma_x = \gamma^n$ and for a sufficiently large integer n. Being continuous, the character $|\gamma_x|$ is strictly less than ϵ on a neighbourhood V_x of x. Taking a finite covering for K, say, V_{x_i}, $i = 1, \ldots, m$, we get a bounded character $\gamma = \prod_{i=1}^{m} \gamma_{x_i}$ such that $|\gamma(x)| < \epsilon$ for every $x \in K$. This implies

$$\left|\widehat{\mu}(\gamma) - \mu(\{0\})\right| = \left|\int_{S\setminus\{0\}} \gamma(x)\,\mathrm{d}\mu(x)\right|$$

$$\leq \int_{S\setminus(K\cup\{0\})} \mathrm{d}|\mu| + \int_K |\gamma(x)|\,\mathrm{d}|\mu| < \epsilon + \epsilon = 2\epsilon.$$

The required inclusion (and, therefore, the theorem) follows. □

2.3. Examples and comments.

2.3.1. Symmetric and analytic Wiener algebras. — Let $G = \mathbb{Z}$ be the additive group of integers, and let $S = \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$. Then

$$\mathcal{M}(\mathbb{Z}) = \ell^1(\mathbb{Z}), \quad \mathcal{M}(\mathbb{Z}_+) = \ell^1(\mathbb{Z}_+),$$

and

$$W = \mathcal{F}\mathcal{M}(\mathbb{Z}) = \left\{\widehat{\mu} = \sum_{n\in\mathbb{Z}} \mu(n)\zeta^n : \mu \in \ell^1(\mathbb{Z})\right\}$$

is the Wiener algebra of absolutely converging Fourier series on the circle group $\widehat{\mathbb{Z}} = \mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. The bounded characters of $\mathbb{Z}_+$ fill in the closed unit disc

$$\widehat{S}_b = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

The corresponding Fourier-Laplace transformation is $\mu \mapsto \widehat{\mu}(z) = \sum_{n\geq 0} \mu(n)z^n$, and

$$W_+ = \mathcal{L}\mathcal{M}(\mathbb{Z}_+) = \left\{\widehat{\mu} = \sum_{n\geq 0} \mu(n)z^n : \mu \in \ell^1(\mathbb{Z}_+)\right\}$$

is the analytic Wiener algebra on $\overline{\mathbb{D}}$. Our theorems say that

$$\left(\mu \in W, \; \frac{1}{\sqrt{2}} < \delta \leq |\widehat{\mu}(\zeta)| \leq \|\mu\|_{\mathcal{M}(\mathbb{Z})} \leq 1 \text{ for } |\zeta| = 1\right)$$

$$\Longrightarrow \|\mu^{-1}\| \leq \frac{1}{2\delta^2 - 1},$$

and

$$\left(\mu \in W_+, \; \frac{1}{2} < \delta \leq |\widehat{\mu}(z)| \leq \|\mu\|_{\mathcal{M}(\mathbb{Z}_+)} \leq 1 \text{ for } |z| \leq 1\right)$$

$$\Longrightarrow \|\mu^{-1}\| \leq \frac{1}{2\delta - 1}.$$

Similar norm-controlled inversion theorems hold for the multivariate Wiener algebra

$$W = \mathcal{FM}(\mathbb{Z}^N) = \left\{ \widehat{\mu} = \sum_{n \in \mathbb{Z}^N} \mu(n) \zeta^n : \mu \in \ell^1(\mathbb{Z}^N) \right\}$$

on the torus $\mathbb{T}^N$, and for analytic Wiener algebra

$$W_+ = \mathcal{LM}(\mathbb{Z}_+^N) = \left\{ \widehat{\mu} = \sum_{n \in \mathbb{Z}_+^N} \mu(n) z^n : \mu \in \ell^1(\mathbb{Z}_+^N) \right\}$$

on the polydisc $\overline{\mathbb{D}}^N$.

2.3.2. Continuous versions of Wiener algebras. — These algebras correspond to $G = \mathbb{R}$, $S = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, and – in several variables – to $G = \mathbb{R}^N$, $S = \mathbb{R}_+^N$. Here $\widehat{G} = \mathbb{R}$, and

$$\widehat{S}_b = \overline{\mathbb{C}}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\},$$

the closed upper half-plane. The Fourier-Laplace transformation is the classical one,

$$\mathcal{L}\mu(z) = \int_{\mathbb{R}_+} e^{ixz}\, d\mu(x), \quad z \in \overline{\mathbb{C}}_+.$$

Similarly, for several variables, $\widehat{G} = \mathbb{R}^N$, $\widehat{S}_b = \overline{\mathbb{C}}_+^N$.

2.3.3. Cones in $\mathbb{Z}^N$ and $\mathbb{R}^N$. — Instead of the cones $\mathbb{R}_+^N \subset \mathbb{R}^N$ and $\mathbb{Z}_+^N = \mathbb{R}_+^N \cap \mathbb{Z}^N$, we may consider an arbitrary subsemigroup S of $\mathbb{R}^N$ containing 0 and such that $S \setminus \{0\}$ is contained in an open halfspace (this requirement guarantees the separation property (iii) of Subsection 2.2.1). Usually, S is a convex cone satisfying the latter property, or, in its discrete version, the intersection of such a cone with $\mathbb{Z}^N$. For the continuous version, the characters are $x \mapsto e^{i(x \cdot z)}$ with $z \in \widehat{S}_b$,

$$\widehat{S}_b = \{z \in \mathbb{C}^N : \mathrm{Im}(x \cdot z) \geq 0 \text{ for all } x \in S\},$$

where $x \cdot z = \sum_{k=1}^N x_k z_k$. The Fourier-Laplace transformation is again the classical one,

$$\mathcal{L}\mu(z) = \int_S e^{i(x \cdot z)}\, d\mu(x), \quad z \in \widehat{S}_b.$$

For instance, for

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, \ -kx_1 \leq x_2 \leq kx_1\}$$

where $k > 0$, we have

$$\widehat{S}_b = \{(z_1, z_2) \in \mathbb{C}^2 : k|\mathrm{Im}(z_2)| \leq \mathrm{Im}(z_1)\}.$$

Another example is the halfspace

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\} \cup \{0\},$$

with the corresponding dual set of characters

$$\widehat{S}_b = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_2) = 0, \ \mathrm{Im}(z_1) \geq 0\} = \mathbb{C}_+ \times \mathbb{R}.$$

2.3.4. *Comment: $\widehat{G}$ versus $\widehat{S}_b$.* — We would like to emphasize once more that $(\delta\text{-}n)$-visibility properties and visibility constants $c_n(\delta, A, X)$ depend essentially on the choice of the "visible spectrum" X.

For instance, the algebra $\mathcal{FM}(S) \subset \mathcal{FM}(G)$ for a semigroup S described in Subsection 2.2.1 can be viewed as being embedded into $C(\widehat{G})$, and into $C(\widehat{S}_b)$ as well. The preceding results say that $\delta_1(\mathcal{M}(S), \widehat{S}_b) \leq \frac{1}{2}$ and $c_1(\delta, \mathcal{M}(S), \widehat{S}_b) \leq (2\delta - 1)^{-1}$, and, moreover, one can derive from these results and Corollary 3.3.8 below that $\delta_1(\mathcal{M}(S), \widehat{S}_b) = \delta_1(\mathcal{M}(S), \{0\}) = \frac{1}{2}$. But no restrictions on the behaviour of $|\widehat{\mu}|$ on $\widehat{G}$ can guarantee that an element $\mu \in \mathcal{M}(S)$ is invertible in $\mathcal{M}(S)$. Take $G = \mathbb{Z}, S = \mathbb{Z}_+$ and $\widehat{G} = \mathbb{T}$, $\widehat{S}_b = \overline{\mathbb{D}}$ with $\widehat{\mu}(\zeta) = \zeta, \zeta \in \mathbb{T}$, as an example. However, some specific value disributions of $\widehat{\mu}(\gamma), \gamma \in \widehat{G}$, can be rewarded by the invertibility of μ in $\mathcal{M}(S)$, see 2.3.5 below.

Moreover, even for the latter classical case, taking the smaller set $\mathbb{T} = \widehat{G}$ for visible spectrum, we can guarantee the norm-controlled invertibility of a function $\widehat{\mu} \in \mathcal{FM}(S) = W_+$, as well as of any other function belonging to $\mathcal{FM}(G) = W$, under more restrictive hypotheses than in the previous paragraph, and only in the larger algebra $\mathcal{FM}(G) = W$, and with cruder estimates. Namely, we can only say that the relations

$$\frac{1}{\sqrt{2}} < \delta \leq |\widehat{\mu}(\zeta)| \leq \|\mu\| \leq 1 \quad (\text{for } \zeta \in \mathbb{T})$$

imply the invertibility in W and the estimate $\|\mu^{-1}\|_W \leq (2\delta^2 - 1)^{-1}$. Later on, in Subsection 3.3, we show that even staying on $\overline{\mathbb{D}} = \widehat{S}_b$, we have $\sup\{\|\mu^{-1}\| : \widehat{\mu} \in W_+, \delta \leq |\widehat{\mu}(z)| \leq \|\mu\| \leq 1, z \in \overline{\mathbb{D}}\} = \infty$ for $\delta \leq \frac{1}{2}$. If we stay on $\widehat{G}$, the interval $\frac{1}{2} < \delta \leq \frac{1}{\sqrt{2}}$ still represents an open problem.

In return for these difficulties of the case $X = \widehat{G}$, caused by a restricted accessibility to the unit coefficient $\mu(\{0\}) = \varphi_e(\mu)$ from the visible values $\widehat{\mu}(\gamma)$, $\gamma \in \widehat{G}$, we profit from $\widehat{G}$-symmetry of $\mathcal{M}(G)$ in getting an easy route to all $(\delta\text{-}n)$-visibility constants.

By contrast, staying on $\widehat{S}_b = \overline{\mathbb{D}}$, we know more about $c_1(\delta)$ but cannot say anything about the bounds $c_n(\delta, W_+, \overline{\mathbb{D}})$ for solutions of Bezout equations for $n \geq 2$ (at least at the same elementary level as before; cf. our Theorem B and the results of [EZ]). The same is true for general subsemigroups $S \subset G$. We can give at least two reasons for this difference. First, the algebra $\mathcal{M}(S)$ is not inversion stable: indeed, the measure $\delta_x \in \mathcal{M}(S)$ is invertible in $\mathcal{M}(G)$ but the inverse $\delta_x^{-1} = \delta_{-x}$ does not belong to $\mathcal{M}(S)$ for $x \in S \setminus \{0\}$. Second, the algebra $\mathcal{M}(S)$ is neither $\widehat{G}$- nor $\widehat{S}_b$-symmetric.

However, the estimates $\delta_n(W, \mathbb{T}) \leq \frac{1}{\sqrt{2}}$ and $c_n(\delta, W, \mathbb{T}) \leq (2\delta^2 - 1)^{-1}$ are still valid for all n. They ensure the existence of norm-controlled solutions for Bezout equations, but in W, not in W_+. The same is true for general subsemigroups $S \subset G$. The next remark shows that under a more specific hypothesis on the localization of the range $\widehat{\mu}(\widehat{G})$ we can relax the lower bound $\frac{1}{\sqrt{2}}$ up to the value $\frac{1}{2}$, as if we stayed on $\widehat{S}_b$.

2.3.5. *Comment: "outer measures" on S and disc segment ranges.* Applying Remarks 1.4.8 for the case $A = \mathcal{M}(S)$, we can improve the preceding results. Namely, let $S \subset G$ be a semigroup as in Theorem 2.2.2, and let $\mu \in \mathcal{M}(S)$, $\|\mu\| \leq 1$.

Supposing $|\widehat{\mu}(\gamma)| \geq \delta > \frac{1}{2}$ for $\gamma \in \widehat{G}$ (not for $\widehat{S}_b$!), we deduce from Remark 1.4.8 (i), part (b), that if μ is an "outer element" of $\mathcal{M}(S)$, that is,

$$\log|\varphi_e(\mu)| \in \operatorname{conv}\{\log|\widehat{\mu}(\gamma)| : \gamma \in \widehat{G}\},$$

then $\mu \in \mathcal{G}(\mathcal{M}(S))$ and $\|\mu^{-1}\| \leq (2\delta - 1)^{-1}$. The reason to call such an element μ outer is that, for the classical case $G = \mathbb{Z}$, $S = \mathbb{Z}_+$ and $\widehat{G} = \mathbb{T}$, $\widehat{S}_b = \overline{\mathbb{D}}$, an outer function $\mathcal{F}\mu$ in the standard Hardy space sense obviously verifies condition (b) of 1.4.8 (i). Moreover,

$$\log|\varphi_e(\mu)| = \log|\mathcal{F}\mu(0)| = \int_{\mathbb{T}} \log|\mathcal{F}\mu(\zeta)|\,dm(\zeta).$$

Another specific hypothesis implying invertibility is the disc segment distibution of the values $\widehat{\mu}(\gamma)$, $\gamma \in \widehat{G}$, as described in Remark 1.4.8 (ii).

Namely, suppose that there exists a measure $\nu \in \mathcal{M}(S)$ such that $\|\nu\| \leq 1$ and

$$\mathrm{Re}\big(\widehat{\nu}(\gamma)\widehat{\mu}(\gamma)\big) \geq \delta > \frac{1}{2} \quad \text{for} \quad \gamma \in \widehat{G}.$$

The main special case is when $\nu = e^{-i\theta}\delta_0$, and so $e^{-i\theta}\widehat{\mu}(\widehat{G})$ is contained in the disc segment $\Delta = \{z \in \mathbb{C} : \delta \leq \mathrm{Re}(z) \leq |z| \leq 1\}$ (see Figure 1).
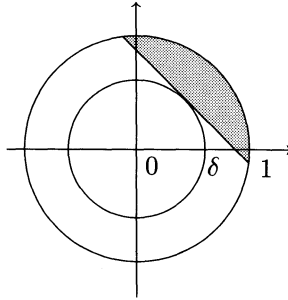

Figure 1: $\mathrm{Re}(\alpha\widehat{f}) \geq \delta > \frac{1}{2}$.

Then, by 1.4.8 (ii), the measure μ is invertible in $\mathcal{M}(S)$ and $\|\mu^{-1}\| \leq (2\delta - 1)^{-1}$. $\square$

2.4. Some subalgebras of $\mathcal{M}(G)$ and $\mathcal{M}(S)$.

In this subsection, we briefly consider two classical subalgebras of $\mathcal{M}(G)$ (or $\mathcal{M}(S)$), namely, the algebras of absolutely continuous and discrete measures on G (respectively on S).

2.4.1. *The group algebra $L^1(G)$.* — Let G be an LCA group, and $L^1(G)$ the convolution group algebra on G. By the standard identification of a function $f \in L^1(G)$ with the measure $f \, dm$, we can view the algebra $L^1(G)$ as a subalgebra of $\mathcal{M}(G)$. These algebras coincide if and only if G is a discrete group, and if and only if $L^1(G)$ contains a unit. In general, $L^1(G)$ becomes an algebra with a unit if we adjoin the Dirac point mass at the origin $e = \delta_0$, that is, consider the algebra $L^1(G) + \mathbb{C} \cdot e$. We call this subalgebra of $\mathcal{M}(G)$ the unital group algebra. In this subsection, for the case of a nondiscrete LCA group, we improve the upper estimate for $c_1(\delta)$ contained in Theorem 2.1.1 above. In the next Section 3 we show that this new estimate for $L^1(G) + \mathbb{C} \cdot e$ is sharp.

2.4.2. THEOREM. — *The critical constant $\delta_1(L^1(G) + \mathbb{C} \cdot e, \widehat{G})$ for the unital group algebra of a nondiscrete LCA group G is less than or equal to $\frac{1}{2}$. Moreover, for all $\frac{1}{2} < \delta \leq 1$,*

$$(2.4.1) \qquad\qquad c_1\big(\delta, L^1(G) + \mathbb{C} \cdot e, \widehat{G}\big) \leq \frac{1}{2\delta - 1}.$$

For $n \geq 2$ we still have estimates (2.1.1).

Proof. — Since $\|\lambda\delta_0 + f \, dm\| = |\lambda| + \|f\|_1$ for all $f \in L^1(G)$ and $\lambda \in \mathbb{C}$, the algebra $L^1(G) + \mathbb{C} \cdot e$ splits at the unit. Moreover, the Riemann-Lebesgue lemma implies that

$$\lambda = \lim_{\gamma \to \infty} \widehat{\mu}(\gamma)$$

for $\mu = \lambda e + f \, dm \in (L^1(G) + \mathbb{C} \cdot e)$. Hence, for an element μ with $\|\mu\| \leq 1$ and $|\widehat{\mu}(\gamma)| \geq \delta > \frac{1}{2}$, $\gamma \in \widehat{G}$, we can apply Lemma 1.4.3 which gives $\|\mu^{-1}\| \leq (2\delta - 1)^{-1}$, and (2.4.1) follows. Since the algebra $L^1(G) + \mathbb{C} \cdot e$ is $\widehat{G}$-symmetric, Lemma 1.4.2 implies that

$$c_n\big(\delta, L^1(G) + \mathbb{C} \cdot e, \widehat{G}\big) \leq c_1\big(\delta^2, L^1(G) + \mathbb{C} \cdot e, \widehat{G}\big) \leq (2\delta^2 - 1)^{-1}$$

for all $n \geq 2$ and $\frac{1}{\sqrt{2}} < \delta \leq 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.4.3. *The algebra of almost periodic functions $\mathcal{F}\mathcal{M}_d(G)$, the semigroup algebra $\mathcal{L}\mathcal{M}_a(S)$, and the algebra of Dirichlet series $\mathcal{L}\mathcal{M}_d(S)$.* Let $\mathcal{M}_d(G)$ be the algebra of discrete mesures on an LCA group G, and $\mathcal{M}_a(G)$ the algebra of measures absolutely continuous with respect to Haar measure m_G. We denote by $\mathcal{M}_d(S)$, respectively by $\mathcal{M}_a(S)$, their subalgebras of measures supported by a subsemigroup S satisfying hypotheses (i)–(iii) of Subsection 2.2.1. The case of the group algebra $\mathcal{M}_a(G) = L^1(G)$ is considered in Subsections 2.4.1 and 2.4.2 above, here we make some remarks on three other algebras.

- The unital semigroup algebra

$$\mathcal{M}_a(S) + \mathbb{C} \cdot e = L^1(S) + \mathbb{C} \cdot e$$

is an inversion stable subalgebra of $\mathcal{M}(S)$, and so Theorem 2.2.2 gives the same estimates as in $\mathcal{M}(S)$:

$$\delta_1 \leq \frac{1}{2}, \quad c_1 \leq \frac{1}{2\delta - 1} \quad \text{for} \quad \frac{1}{2} < \delta \leq 1.$$

It will be shown that these estimates are sharp; see Section 3 below. Comments 2.3.4 and 2.3.5 are still valid in this case too.

• The algebra $\mathcal{F}\mathcal{M}_d(G)$ of Fourier transforms of discrete measures is the algebra of almost periodic functions with absolutely convergent Fourier series. It is obvious and well known that $\mathcal{M}_d(G)$ is an inversion stable subalgebra of $\mathcal{M}(G)$. (Indeed, if $\mu \in \mathcal{M}_d(G) \cap \mathcal{G}(\mathcal{M}(G))$ and ν is its inverse in $\mathcal{M}(G)$, then $\delta_0 = \mu * \nu = \mu * \nu_d + \mu * \nu_c$, where ν_d, ν_c stand for the discrete and continuous parts of ν respectively; hence, $\mu * \nu_c = 0$ and, finally, $\nu_c = 0$.) Thus, Theorem 2.1.1 works for this algebra too (the visible spectrum is, of course, $X = \widehat{G}$). As for the entire algebra $\mathcal{M}(G)$, we will show in Section 3 that the critical constant $\delta_1(\mathcal{F}\mathcal{M}_d(G), \widehat{G})$ is at least $\frac{1}{2}$. It should be mentioned that here, as before, the absence of the norm-controlled inversion, that is, the fact that $c_1(\delta, \mathcal{M}_d(G), \widehat{G}) = \infty$ for $0 < \delta \le \frac{1}{2}$, is not related to the evident insufficiency of $X = \widehat{G}$ for recognition of all the spectrum $\mathfrak{M}(\mathcal{M}_d(G))$. Indeed, despite the fact that the latter is much larger than $\widehat{G}$ and, in fact, coincides with the so-called Bohr compactum $(\widehat{G})^-$, we show that even staying on the Bohr compactum we still have $\delta_1(\mathcal{M}_d(G), (\widehat{G})^-) \ge \frac{1}{2}$; see Corollary 3.3.8 below.

• Similarly, $\mathcal{F}\mathcal{M}_d(S)$ is the algebra of absolutely convergent Dirichlet series

$$f(\gamma) = \sum_{x \in S} \mu(\{x\})(x, \gamma), \quad \gamma \in \widehat{S}_b, \quad \sum_{x \in S} |\mu(\{x\})| < \infty.$$

The case of the classical Dirichlet series

$$z \longmapsto \sum_{k \ge 1} a_k \, e^{-z x_k}, \quad \operatorname{Re}(z) \ge 0,$$

corresponds to the case $S = \mathbb{R}_+$, $\widehat{S}_b = \{z \in \mathbb{C} : \operatorname{Re}(z) \ge 0\}$ with the pairing $(x, z) \mapsto e^{-zx}$. Like $\mathcal{M}_d(G)$, the algebra $\mathcal{M}_d(S)$ is an inversion stable subalgebra of $\mathcal{M}(S)$. Hence, Theorem 2.2.2 and Comments 2.3.4–2.3.5 are still valid for this algebra as well.

2.5. Other semigroups.

Probably, many other semigroups can be treated in a similar way, even if they do not satisfy our requirements (i)–(iii). As an example, we consider the *quasinilpotent semigroup* $S = [0, 1]$ endowed with the quotient operation of $\mathbb{R}_+/(1 + \mathbb{R}_+)$: $x \circ y = x + y$ for $x + y < 1$, and $x \circ y = 1$ for $x + y \ge 1$. The convolution of two Borel measures on S is well defined as the restriction to S of its convolution on $\mathbb{R}_+$; in particular, for $\mu = f \, dx$, $\nu = g \, dx$ with $f, g \in L^1(0, 1)$ we have for $0 < x < 1$

$$\mu * \nu = (f * g) \, dx, \quad \text{where } f * g(x) = \int_0^x f(y) g(x - y) \, dy.$$

In what follows we consider measures without point mass at point 1, or, equivalently, endow the space $\mathcal{M}(S)$ with the seminorm $\|\mu\|_{[0,1)} = \|\mu|_{[0,1)}\|$. On the subspace $\mathcal{M}([0,1))$, this seminorm coincides with the usual norm. It is easy to see that if $\mu|_{[0,1/n]} = 0$, then

$$\|\mu * \cdots * \mu\|_{[0,1)} = 0.$$

This implies that $\lim\limits_{n} \|\mu^n\|_{[0,1)}^{1/n} = 0$ for all measures with $\mu(\{0\}) = 0$. Hence, the space $\mathcal{M}([0,1))$ endowed with the above convolution becomes a unital Banach algebra with the only nonzero complex homomorphism being

$$\varphi: \mu \longmapsto \mu(\{0\}).$$

It is also obvious that the algebra $\mathcal{M}([0,1))$, as well as its subalgebra $L^1(0,1) + \mathbb{C} \cdot e$, splits at the unit and is X-symmetric with $X = \mathfrak{M} = \{\varphi\}$. Therefore, using Theorem 1.4.7, we obtain the following result.

2.5.1. THEOREM. — *For the convolution algebra $\mathcal{M}([0,1))$, and for $X = \mathfrak{M}(\mathcal{M}([0,1))) = \{\varphi\}$, we have $\delta_1(\mathcal{M}([0,1)), X) \leq \frac{1}{2}$. Moreover,*

$$c_1\big(\delta, \mathcal{M}([0,1))\big) \leq \frac{1}{2\delta - 1} \quad \text{for} \quad \frac{1}{2} < \delta \leq 1,$$

$$c_n\big(\delta, \mathcal{M}([0,1))\big) \leq \frac{1}{2\delta^2 - 1} \quad \text{for} \quad n \geq 2 \ \text{and} \ \frac{1}{\sqrt{2}} < \delta \leq 1.$$

In particular, if

$$J : L^1(0,1) \longrightarrow L^1(0,1)$$

is an operator of integration,

$$Jf(x) = f * \chi_{[0,1)}(x) = \int_0^x f(y)\,dy, \quad x \in (0,1),$$

and $p(z) = \sum\limits_{k=0}^{m} a_k z^k$ is a polynomial such that

$$\|p(*\chi_{[0,1)})\|_{\mathcal{M}(S)} = \|p(J)\| \leq 1 \quad \text{and} \quad \frac{1}{2} < |p(0)|,$$

then

$$\|(p(J))^{-1}\| \leq \frac{1}{2|p(0)| - 1}. \qquad \square$$

2.5.2. Remark. — It is worth mentioning that the same integration operator J behaves in a different way on other classical spaces. For example, for the norm closed algebra $A = A(J, L^2)$ generated by

$$J : L^2(0,1) \longrightarrow L^2(0,1),$$

and for $X = \mathfrak{M} = \{\varphi\}$, we have $\delta_1(A, X) = 1$, and, thus, the spectrum of A is $(\delta\text{-}1)$-invisible for all δ, $0 < \delta < 1$. See Theorem 5.3.5 below. On the other hand, for the algebra $A = A(J, H^2)$ generated in a similar way by the same operator

$$J : H^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D}),$$

but on the classical Hardy space, it can be shown that $\delta_1(A, X) = 0$; see [N2]. Whence we see that the spectrum of this algebra is $(\delta\text{-}1)$-visible for all δ, $0 < \delta \leq 1$. See [N2] for details.

3. A lower estimate for measure algebras on groups and semigroups.

Our goal in this section is to determine a common lower estimate of $c_1(\delta, A, X)$ for the measure algebras of groups and semigroups and for their subalgebras considered in Section 2 above. In particular, we show that the critical constant δ_1 is greater than or equal to $\frac{1}{2}$ for every infinite LCA group and for most semigroups. This finishes the proof of Theorems A and B of Subsection 0.3.

In fact, we have at our disposal two approaches to lower estimates. The first one is based directly on the existence and properties of measures carried by independent Cantor sets; in what follows a special kind of such measures will be called *Sreider measures*. The corresponding theory is surely one of the most subtle chapters of measure algebra theory; see [GRS], [Ru1], [GMG]. The second method is inspired by H. Shapiro's example [Sh1]. In our setting, it depends on some improvements to the technique of exponential norm behaviour $\| e^{it\mu} \|_{\mathcal{M}(G)}$ for $t > 0$. This technique is well known to be the base of the theory of functions operating on an algebra; see [Ru1], [GMG], [K]. The first way is faster, and we use it in this section. The second way is more explicit, and can be found in [ENZ] and [Sh2].

We start by recalling some classical facts about measure algebras, essentially as exposed in [Ru1].

3.1. Measure algebra prerequisites.

For an LCA group G, we denote by $\mathfrak{M} = \mathfrak{M}(\mathcal{M}(G))$ the spectrum of the algebra $\mathcal{M}(G)$.

3.1.1. *Sreider measures.* — Suppose G is not discrete. Then there exists a positive measure $\mu \in \mathcal{M}(G)$ such that

$$\widehat{\mu}(\widehat{G}) \subset [-1,1] \quad \text{and} \quad \widehat{\mu}(\mathfrak{M}) = \overline{\mathbb{D}}, \quad \|\mu\| = 1;$$

see Sreider [Sr] for $G = \mathbb{R}$, and [Ru1] Theorem 5.3.4, for the general case and history. We call such μ's *Sreider measures*. Moreover, it is known that there exist continuous Sreider measures μ such that the convolution powers and their translates, μ^k, $\delta_x * \mu^k$, are all mutually singular; see [Ru1], [GMG]. It is worth mentioning that $0 \in \widehat{\mu}(\widehat{G})$ for all known examples of such measures.

3.1.2. *The Bochner characterization of $\mathcal{F}\mathcal{M}(G)$.* — Let φ be a continuous function on $\widehat{G}$. The following are equivalent:

(i) there exists $\mu \in \mathcal{M}(G)$ such that $\varphi = \widehat{\mu}$ and $\|\mu\| \leq N$;

(ii) $\left|\sum a_k \varphi(\gamma_k)\right| \leq N\|f\|_\infty$ for every trigonometric polynomial f on G of the form $f(x) = \sum a_k(x, \gamma_k)$, $\gamma_k \in \widehat{G}$.

See [Ru1], Theorem 1.9.1. This theorem implies that $\|\mu\| \leq \varliminf_{i \in I} \|\mu_i\|$ if a net of Fourier transforms $(\widehat{\mu}_i)_{i \in I}$ of $\mu_i \in \mathcal{M}(G)$ converges to $\widehat{\mu}$ pointwise.

3.1.3. *$\widehat{G}$-semicontinuity of inversion.* — Let $(\mu_i)_{i \in I} \subset \mathcal{M}(G)$ be a net of measures on G, and $\mu \in \mathcal{M}(G)$ such that $\widehat{\mu}(\gamma) = \lim_{i \in I} \widehat{\mu}_i(\gamma)$ for $\gamma \in \widehat{G}$. Then $\varliminf_{i \in I} \|\mu_i^{-1}\| \geq \|\mu^{-1}\|$, where we mean $\|\nu^{-1}\| = \infty$ in the case where ν is not invertible.

Indeed, let $\lim_j \|\mu_{i_j}^{-1}\| < \infty$ for a subnet. Since $|\widehat{\nu}(\gamma)| \leq \|\nu\|$ for all $\gamma \in \widehat{G}$ and $\nu \in \mathcal{M}(G)$, it is clear that

$$\lim_j \mathcal{F}(\mu_{i_j}^{-1})(\gamma) = \lim_j \left(\mathcal{F}\mu_{i_j}(\gamma)\right)^{-1} = \frac{1}{\widehat{\mu}(\gamma)}$$

is a continuous function on $\widehat{G}$. Then, passing to the limit in 3.1.2 (ii) written for $\varphi_j = \mathcal{F}\mu_{i_j}^{-1}$, we get a measure $\nu \in \mathcal{M}(G)$ such that $\widehat{\nu} = 1/\widehat{\mu}$ on $\widehat{G}$ and $\|\nu\| \leq \lim_j \|\mu_{i_j}^{-1}\|$. Hence $\mu * \nu = \delta_0$. $\qquad\square$

3.1.4. *Bohr compactification.* — For an LCA group G, we denote by $\widehat{G}_d$ the dual group $\widehat{G}$ endowed with the discrete topology, and set $\overline{G} = (\widehat{G}_d)\widehat{}$. The compact group $\overline{G}$ is called the *Bohr compactification* of G; in fact, G is a dense subset of $\overline{G}$, and $\overline{G} = \mathfrak{M}(\mathcal{M}(\widehat{G}_d)) = \mathfrak{M}(\mathcal{M}_d(\widehat{G}))$.

3.2. Lower estimates on groups.

Here we prove a general statement, Theorem 3.2.2 below, which contains the required lower estimates for all three algebras in question in the case of groups: $\mathcal{M}(G)$, $L^1(G) + \mathbb{C} \cdot \delta_0$, and $\mathcal{M}_d(G)$. We use the following simple method, which we informally refer to as to "a walk to the Bohr compact". Namely, the method consists of the following steps:

1) staying on an infinite group G, we lift ourselves up to the Bohr group $\overline{G} \supset G$;

2) being nondiscrete, $\overline{G}$ carries a Sreider measure, say ν, whose polynomials give the required lower estimate (see Lemma 3.2.1 below);

3) using almost periodic approximations of $\mathcal{F}\nu$, we get the same lower estimate on G (via the Bochner criterion 3.1.2). Of course, for a nondiscrete group G, steps 1) and 3) are not necessary.

3.2.1. LEMMA. — *Let ν be a Sreider measure on a nondiscrete LCA group G, and let*

$$\mu = \delta e + (1 - \delta)\nu^2,$$

where $\frac{1}{2} < \delta \leq 1$, and $e = \delta_0$ stands for the unit of $\mathcal{M}(G)$. Then

$$\|\mu\| = 1, \quad \widehat{\mu}(\widehat{G}) \subset [\delta, 1], \quad \|\mu^{-1}\| = \frac{1}{2\delta - 1}.$$

Proof. — It is clear from the definition and the spectral mapping theorem that $\nu^2 = \nu * \nu$ is also a Sreider measure with $(\mathcal{F}\nu^2)(\widehat{G}) \subset [0, 1]$ and $\sigma(\nu^2) = (\nu^2)\widehat{}(\mathfrak{M}) = \overline{\mathbb{D}}$. Hence, $\widehat{\mu}(\widehat{G}) \subset [\delta, 1]$ and $\sigma(\mu) = \widehat{\mu}(\mathfrak{M}) = \delta + (1 - \delta)\overline{\mathbb{D}}$. This implies that

$$\|\mu^{-1}\| \geq \frac{1}{\text{dist}(0, \sigma(\mu))} = \frac{1}{2\delta - 1}.$$

On the other hand, $\mu^{-1} = \delta^{-1} \sum_{k \geq 0} \left(\frac{1 - \delta}{\delta}\right)^k \nu^{2k}$. Therefore,

$$\|\mu^{-1}\| \leq \delta^{-1} \sum_{k \geq 0} \left(\frac{1 - \delta}{\delta}\right)^k = \frac{1}{2\delta - 1}. \qquad \square$$

To accomplish step 3) of the program described above, we need a kind of a strengthened weak topology, precisely, a version of Beurling's well-known narrow topology.

- Namely, we say that a net $(\mu_i)_{i \in I} \subset \mathcal{M}(G)$ is $\widehat{G}$-*convergent to a measure* $\mu \in \mathcal{M}(G)$ if $\varliminf_{i \in I} \|\mu_i\| \leq \|\mu\|$ and $\lim_{i \in I} \widehat{\mu}_i(\gamma) = \widehat{\mu}(\gamma)$ for $\gamma \in \widehat{G}$. (In fact, $\lim_i \|\mu_i\| = \|\mu\|$ due to 3.1.2.)

- A set $A \subset \mathcal{M}(G)$ is said to be $\widehat{G}$-*dense* in a set $B \subset \mathcal{M}(G)$ if every $\mu \in B$ is a $\widehat{G}$-limit of a net of measures belonging to A.

3.2.2. THEOREM. — *Let G be an infinite LCA group, and let A be a unital subalgebra of $\mathcal{M}(G)$ (not necessarily closed) verifying the following conditions:*

(i) *A is $\widehat{G}$-symmetric;*

(ii) *A is $\widehat{G}$-dense in $\mathcal{M}(G)$.*

Further, given a number δ, $\frac{1}{2} < \delta \leq 1$, let

$$A\big([\delta, 1]\big) = \big\{ \alpha \in A : \|\alpha\| \leq 1, \ \widehat{\alpha}(\widehat{G}) \subset [\delta, 1] \big\}.$$

Then

$$\sup_{\alpha \in A([\delta,1])} \|\alpha^{-1}\| \geq \frac{1}{2\delta - 1}.$$

In particular, $\delta_1(A, \widehat{G}) \geq \frac{1}{2}$, and $c_1(\delta, A, \widehat{G}) \geq (2\delta - 1)^{-1}$ for $\frac{1}{2} < \delta \leq 1$.

Proof. — We want to use Lemma 3.2.1 but, in the case where G is a discrete group, there are no Sreider measures on G. In accordance with the idea described at the beginning of this section, we consider a Sreider measure ν on the Bohr compactum $\overline{G}$. Let $\sigma \subset \widehat{G}$ be a finite set, and let $\epsilon > 0$. Since G is dense in $\overline{G}$, there exists a finitely supported measure $\tau \in \mathcal{M}(G)$ such that $\|\tau\| \leq 1$ and

$$\big| \widehat{\nu}(\gamma) - \widehat{\tau}(\gamma) \big| < \epsilon, \quad \gamma \in \sigma.$$

(To get such a measure, consider Riemann sum approximations for the integrals $\int_{\overline{G}}(-x, \gamma)\, d\nu(x)$, $\gamma \in \sigma$.)

Now, using (ii), we choose a measure $\beta \in A$ such that $\|\beta\| \leq 1$ and

$$\big| \widehat{\beta}(\gamma) - \widehat{\tau}(\gamma) \big| < \epsilon, \quad \gamma \in \sigma.$$

Taking

$$\alpha = \delta e + (1 - \delta)(\beta * \beta_*),$$

where β_* is the measure symmetric to β as defined in Subsection 1.4.1, we get a measure $\alpha \in A([\delta, 1])$ approximating the measure $\mu = \delta e + (1 - \delta)\nu^2$ of Lemma 3.2.1 on the set σ:

$$\left|\widehat{\mu}(\gamma) - \widehat{\alpha}(\gamma)\right| < 4\epsilon, \quad \gamma \in \sigma.$$

Letting $\epsilon \to 0$, we obtain

$$\left|\sum_{\gamma \in \sigma} a_\gamma \widehat{\mu}^{-1}(\gamma)\right| \leq \sup_{\alpha \in A([\delta,1])} \left|\sum_{\gamma \in \sigma} a_\gamma \widehat{\alpha}^{-1}(\gamma)\right| \leq \|f\|_\infty \cdot \sup_{\alpha \in A([\delta,1])} \|\alpha^{-1}\|,$$

where, as in 3.1.2, f stands for a linear combination of characters $f(x) = \sum_{\gamma \in \sigma} a_\gamma(x, \gamma)$, $x \in G$; $a_\gamma \in \mathbb{C}$. Using Bochner's characterization 3.1.2 (for $\mathcal{FM}(\overline{G})$) and Lemma 3.2.1, we finish the proof. $\qquad\square$

3.2.3. Remark. — In fact, we have proved a slightly stronger result: if $\mathcal{M}_f$ is the algebra of finitely supported measures on G, B_f is the unit ball of $\mathcal{M}_f$, and $\mathcal{M}_{f\delta}^+ = \{\rho : \rho = \delta e + (1 - \delta)(\tau * \tau_*), \tau \in B_f\}$, then

$$\sup_{\rho \in \mathcal{M}_{f\delta}^+} \|\rho^{-1}\| \geq \frac{1}{2\delta - 1}.$$

Another comment is that Lemma 3.2.1 is a statement of the same type as Theorem 1.5.1. The difference is that, due to the properties of Sreider measures, we do not need any estimate: we simply use the spectral mapping theorem.

And the last remark on the Sreider measures approach is that no functions of an independent Sreider measure containing 0 in its visible spectrum can help to improve the above estimate for the critical constant $\delta_1(A, \widehat{G})$ and, therefore, to penetrate into the gap interval $\left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$. Indeed, let μ be a Sreider measure with mutually singular convolution powers μ^k, $k \geq 0$ (see 3.1.1 above) and let

$$A = A(\mu) = \mathrm{span}_{\mathcal{M}(G)}(\mu^k : k \geq 0)$$

be the norm closed algebra generated by μ and e. Clearly, the mapping $p \mapsto p(\mu)$, defined first for complex polynomials p, has a unique continuous

extension up to an isometric isomorphism from the Wiener algebra $\mathcal{F}\ell^1(\mathbb{Z}_+)$ onto A. The visible spectrum of A, say X, can be naturally identified with $\widehat{\mu}(\widehat{G}) \subset [-1,1]$. Since $0 \in \widehat{\mu}(\widehat{G})$ and the algebra $\mathcal{F}\ell^1(\mathbb{Z}_+)$ splits at the unit, we get $\delta_1(A, X) \leq \frac{1}{2}$ and $c_1(\delta, A, X) \leq (2\delta - 1)^{-1}$; see Lemma 1.4.3 above. So, the algebras $A(\mu)$ cannot provide us with any sharper example than we already have. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

3.2.4. *Subalgebras of* $\mathcal{M}(G)$. — Now, we are in a position to finish the proof of Theorem A, Subsection 0.3. Namely, the following is an immediate consequence of Theorem 3.2.2.

3.2.5. COROLLARY. — *Let* G *be an infinite LCA group. Then the algebras* $A = \mathcal{M}(G)$, $A = L^1(G) + \mathbb{C} \cdot e$, *and* $A = \mathcal{M}_d(G)$ *satisfy the hypotheses of Theorem 3.2.2. Consequently, for each of these algebras*

$$c_1(\delta, A, \widehat{G}) \geq \frac{1}{2\delta - 1} \quad \text{for} \quad \tfrac{1}{2} < \delta \leq 1.$$

Moreover,

$$c_1(\delta, A, \widehat{G}) = \infty \quad \text{for} \quad 0 < \delta \leq \tfrac{1}{2}\,;$$

whence

$$\delta_1(A, \widehat{G}) \geq \frac{1}{2}.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

3.3. Lower estimates on semigroups.

In this subsection we prove a semigroup counterpart of Theorem 3.2.2 by using the same method of moving to the Bohr compactum as in Subsection 3.2. However, the construction of measures giving the maximum $\|\mu^{-1}\|$ when $\inf |\widehat{\mu}|$ is fixed is necessarily different. Indeed, the previous one is based on the $\widehat{G}$−symmetry of $\mathcal{M}(G)$ and the positive definiteness of the corresponding Fourier transforms; see Subsection 3.2. Both reasons fail for semigroups. Instead, we are using a more complicated, but general, construction of Theorem 1.5.1. By the way this gives another proof to Theorem 3.2.2 too; see Remark 3.3.3 below. Moreover, to realize the technique of narrow approximations on a semigroup instead of the entire group, we restrict ourselves to a class of semigroups S described in Subsection 3.3.4. This class contains all frequently used semigroups, in particular, all examples of Subsection 2.3 above.

We start with three lemmas on measure algebras on groups, then describe the class of subalgebras of $\mathcal{M}(S)$ we are working with, and finally prove Theorem 3.3.7, our main result on lower estimates on semigroups. In this subsection, we denote by $\mathcal{A}$ a subalgebra of $\mathcal{M}(G)$, and by A a subalgebra of $\mathcal{M}(S)$.

3.3.1. LEMMA. — *Let G be a nondiscrete LCA group. Then the algebra $\mathcal{A} = \mathcal{M}(G)$ satisfies (1.5.1). Moreover, for every $\epsilon > 0$, there exists a measure $\mu \in \mathcal{A}^\epsilon$ ($\mathcal{A}^\epsilon$ is defined in Theorem 1.5.1) such that for all $p \geq 0$ and $b_k \geq 0$*

$$\left\| \sum_{k=0}^{p} b_k \mu^k \right\| = \sum_{k=0}^{p} b_k.$$

Proof. — Let $\tau \in \mathcal{M}(G)$ be a Sreider measure with mutually singular powers (see 3.1.1 for the definition), and let

$$\nu = (e - \tau * \tau)^N.$$

Then $\widehat{\nu}(\widehat{G}) \subset [0,1]$, and

$$\|\nu\| = \left\| \sum_{k=0}^{N} \binom{n}{k} (-1)^k \tau^{2k} \right\| = \sum_{k=0}^{N} \binom{n}{k} \|\tau^{2k}\| = \sum_{k=0}^{N} \binom{n}{k} = 2^N.$$

Hence, $\sup_{\widehat{G}} |\widehat{\nu}| \leq 1 < \epsilon \|\nu\|$ for N sufficiently large. Moreover,

$$\left\| \sum_{k=0}^{p} b_k \nu^k \right\| = \left\| \sum_{k=0}^{p} b_k \sum_{j=0}^{kN} \binom{kN}{j} (-\tau^2)^j \right\| = \left\| \sum_{j=0}^{pN} (-\tau^2)^j \sum_{j/N \leq k \leq p} b_k \binom{kN}{j} \right\|$$

$$= \sum_{j=0}^{pN} \|\tau^{2j}\| \sum_{j/N \leq k \leq p} b_k \binom{kN}{j} = \sum_{j=0}^{pN} \sum_{j/N \leq k \leq p} b_k \binom{kN}{j}$$

$$= \sum_{k=0}^{p} b_k 2^{kN} = \sum_{k=0}^{p} b_k \|\nu\|^k.$$

Setting $\mu = \nu/\|\nu\|$, we obtain the required measure.   □

3.3.2. LEMMA. — *Let G be an infinite LCA group, and let A be a subalgebra of $\mathcal{M}(G)$ which is $\widehat{G}$-symmetric and $\widehat{G}$-dense in $\mathcal{M}(G)$ (but not necessarily closed and/or unital). Then, A satisfies (1.5.1) with $X = \widehat{G}$.*

Proof. — If $\mathcal{A}$ does not contain the unit, we add it setting $\mathcal{A}_e = \mathcal{A} + \mathbb{C} \cdot e$. Suppose first that G is nondiscrete. Let $\epsilon > 0$, let N be such that $\epsilon 2^N > 1$, and let ν and τ be the measures participating in the proof of Lemma 3.3.1 for some numbers $b_k \geq 0$ given in advance (and for $\mathcal{A}_e$ instead of $\mathcal{A}$). Let further $(\tau_i)_{i \in I}$ be a net in $\mathcal{A}_e$ $\widehat{G}$-convergent to the measure τ. Without loss of generality, we may assume that $\|\tau_i\| = 1$ for $i \in I$. This implies that $0 \leq |\widehat{\tau}_i|^2 = (\tau_i * (\tau_i)_*)^\smallfrown \leq 1$, where $(\tau_i)_*$ stands for the measure $\widehat{G}$-symmetric to τ_i.

Now, we set

$$\nu_i = \big(\delta_0 - \tau_i * (\tau_i)_*\big)^N,$$

where, as usual, δ_0 stands for the Dirac measure at zero. We have $\nu_i \in \mathcal{A}_e$ and $\widehat{\nu}_i(\widehat{G}) \subset [0,1]$ for all $i \in I$. Moreover, $\lim_i \widehat{\nu}_i(\gamma) = \widehat{\nu}(\gamma)$ for $\gamma \in \widehat{G}$, whence we see that $\varliminf_i \|\nu_i\| \geq \|\nu\| = 2^N$. This implies that $\mu_i = \nu_i / \|\nu_i\| \in \mathcal{A}_e^\epsilon$ for indices i large enough ($\mathcal{A}^\epsilon$ is defined in Theorem 1.5.1).

Denoting $\rho = \sum\limits_{k=0}^{p} b_k \mu^k$ and $\rho_i = \sum\limits_{k=0}^{p} b_k \mu_i^k$, we get $\|\rho\| = \sum\limits_{k=0}^{p} b_k$ (Lemma 3.3.1). On the other hand, $\lim_i \widehat{\rho}_i = \widehat{\rho}$ on $\widehat{G}$, which implies $\varliminf_i \|\rho_i\| \geq \|\rho\|$. Hence,

$$\sup_{a \in \mathcal{A}_e^\epsilon} \left\| \sum_{k=0}^{p} b_k a^k \right\| \geq \sum_{k=0}^{p} b_k.$$

Since the converse inequality is obvious, the result follows in the case of a nondiscrete group G. For a general infinite G, consider the Bohr compactification $\overline{G}$. Since $(\overline{G})^\smallfrown = (\widehat{G})_d$, the algebra $\mathcal{M}(G)$ is $(\overline{G})^\smallfrown$-dense in $\mathcal{M}(\overline{G})$. Hence, so is $\mathcal{A}$. Applying the part already proved to $\mathcal{A}_e \subset \mathcal{M}(\overline{G})$, we get the result for the algebra $\mathcal{A}_e$.

For the case of $\mathcal{A}$ without the unit, to finish the proof we apply Lemma 1.5.2 to the algebra $\mathcal{A}_e$, and observe that $(\mathcal{A}_e)_0^\epsilon = \mathcal{A}^\epsilon$. The hypothesis of this lemma is fulfilled as has been explained in Remark 2.1.2 above.   □

3.3.3. *Remark.* — Lemma 3.3.2 and Theorem 1.5.1 give another proof of Theorem 3.2.2 and Corollary 3.2.5.

Now, we describe the class of semigroups we are working with to get lower estimates for $c_1(\delta)$.

3.3.4. *Absorbing semigroups.* — Let S be a semigroup embedded into an LCA group G, $S \subset G$, and satisfying hypotheses (i)–(iii) of Subsection 2.2.1. We say that G *verifies the absorption condition* if the following is true:

(iv) For every compact set $K \subset G$ there exists an element $x \in G$ such that $x + K \subset S$.

If S is absorbing, in fact, there exists an element $x \in S$ such that $x + K \subset S$ (consider $K \cup \{0\}$), and, therefore, (iv) implies the following generating property: $G = S - S$. On the other hand, if S is generating and $S \setminus \{0\}$ is open, or S contains an open generating part, then S is absorbing. For instance, this is the case for all examples of Subsection 2.3.

3.3.5. LEMMA. — *Let $S \subset G$ be an absorbing semigroup of an LCA group G, and let $\mu \in \mathcal{M}(G)$. Then, given $\epsilon > 0$, there exists $x \in S$ such that*

$$\sup_{x' \in x+S} \left\| P_{G \setminus S}(\delta_{x'} * \mu) \right\| < \epsilon,$$

where P_E stands for the projection of restriction to a subset $E \subset G$:
$P_E \mu = \mu_{|E}.$

Proof. — Let $\epsilon > 0$, and K be a compact set such that $\|\mu_{|(G \setminus K)}\| < \epsilon$. This implies $\|(\delta_x * \mu)_{|(G \setminus (x+K))}\| < \epsilon$, whence $\|(\delta_x * \mu)_{|(G \setminus S)}\| < \epsilon$ if $x \in S$ is chosen in such a way that $x + K \subset S$.   □

Now, we describe the class of subalgebras of $\mathcal{M}(S)$ we are working with.

3.3.6. *S-subalgebras of $\mathcal{M}(S)$.* — Roughly speaking, we suppose that our subalgebra $A \subset \mathcal{M}(S)$ contains the restriction to S of a "small" subalgebra of $\mathcal{M}(G)$.

Namely, let S be a semigroup, $S \subset G$. We say that A is an *S-subalgebra of $\mathcal{M}(S)$* if A is a subalgebra of $\mathcal{M}(S)$ containing a "small" subalgebra of the form $P_S \mathcal{A} + \mathbb{C} \cdot e$,

$$A \supset P_S \mathcal{A} + \mathbb{C} \cdot e,$$

where $\mathcal{A} \subset \mathcal{M}(G)$ stands for a subalgebra of $\mathcal{M}(G)$ verifying the following conditions (compare with the conditions of Theorem 3.2.2):

(i) $\mathcal{A}$ is $\widehat{G}$-symmetric;

(ii) $\mathcal{A}$ is $\widehat{G}$-dense in $\mathcal{M}(G)$;

(iii) $\mathcal{A}$ is S-invariant, that is, $\delta_x * \mathcal{A} \subset \mathcal{A}$ for $x \in S$;

(iv) $\widehat{G}$ is a boundary for $\mathfrak{M}(\mathcal{A})$, that is, $\lim\limits_{n} \|\mu^n\|^{1/n} = \sup_{\widehat{G}} |\widehat{\mu}|$ for every $\mu \in \mathcal{A}$.

Observe that $\mathcal{A}$ is not assumed to be unital. The standard subalgebras

$$A = \mathcal{M}_d(S) \quad \text{and} \quad A = L^1(S) + \mathbb{C} \cdot e,$$

with obvious group counterparts $\mathcal{A} = \mathcal{M}_d(G)$ and $\mathcal{A} = L^1(G)$, respectively, are S-subalgebras of $\mathcal{M}(S)$. Hence, so is any bigger subalgebra. For instance, $\mathcal{M}_d(S) + L^1(S)$, and $\mathcal{M}(S)$ itself, are S-subalgebras. Another example is the algebra $\mathcal{M}_f(S)$ of finitely supported measures on S. Now, we are ready to prove of the following lower estimate for semigroups.

3.3.7. THEOREM. — *Let S be a semigroup satisfying conditions (i)-(iii) of Subsection 2.2.1 and the absorption condition (iv). Let A be an S-subalgebra of $\mathcal{M}(S)$ (not necessarily closed). Then*

$$\delta_1(A, \widehat{S}_b) \geq \frac{1}{2}, \quad c_1(\delta, A, \widehat{S}_b) \geq \frac{1}{2\delta - 1} \quad \text{for all } \delta, \ \tfrac{1}{2} < \delta < 1.$$

Proof. — We check (1.5.1) with $X = \widehat{S}_b$. Namely, we prove that

$$\sup_{a \in A^\epsilon} \left\| \sum_{k=0}^{p} b_k a^k \right\| = \sum_{k=0}^{p} b_k$$

for every $\epsilon > 0$ and for all $p \geq 0$ and $b_k \geq 0$. Here, as in Theorem 1.5.1,

$$A^\epsilon = \left\{ a \in A : \sup_{\widehat{S}_b} |\widehat{a}| < \epsilon, \ \|a\| = 1 \right\}.$$

It is clear that it suffices to check this property for S-subalgebras of the form

$$A = P_S \mathcal{A} + \mathbb{C} \cdot e,$$

where $\mathcal{A}$ stands for an algebra on G satisfying conditions (i)–(iv) of Definition 3.3.6.

Now, we fix an integer $p \geq 0$ and numbers $b_k \geq 0$ for $0 \leq k \leq p$. By Lemma 3.3.2 applied to $\mathcal{A}$, given positive numbers ϵ, ϵ', there exists an element $\alpha \in \mathcal{A}$ such that $\sup_{\widehat{G}} |\widehat{\alpha}| < \epsilon$, $\|\alpha\| = 1$, and

$$\left\| \sum_{k=0}^{p} b_k \alpha^k \right\| > (1 - \epsilon') \sum_{k=0}^{p} b_k.$$

First of all, we show that whatever is x in S, there exists $n \in \mathbb{N}$ such that the latter inequality still holds true if we replace α by the shifted measure $\delta_{x'} * \alpha$ with $x' = nx$. To this end, let

$$\beta = \sum_{k=0}^{p} b_k \alpha^k, \quad \beta_y = \sum_{k=0}^{p} b_k (\delta_y * \alpha)^k,$$

where $y \in G$, and let $f(t) = \sum_{\gamma \in \sigma} c_\gamma (t, \gamma)$, $t \in G$, be a trigonometric polynomial satisfying $\sup_{G} |f| \leq 1$, σ being a finite subset of $\widehat{G}$. Then $\beta_y \in \mathcal{A}$ for every $y \in S$, and

$$\widehat{\beta}_{nx}(\gamma) = \sum_{k=0}^{p} b_k (x, \gamma)^{nk} \widehat{\alpha}(\gamma)^k$$

for all $\gamma \in \widehat{G}$. Since $|(x, \gamma)| = 1$, we can use the classical Kronecker theorem and choose a sequence $n = n_j$, $j = 1, 2, \ldots$, such that $\lim_j (x, \gamma)^{n_j} = 1$ for every $\gamma \in \sigma$. Hence, $\lim_j \widehat{\beta}_{n_j x}(\gamma) = \widehat{\beta}(\gamma)$ for $\gamma \in \sigma$, and, therefore,

$$\left| \sum_{\gamma \in \sigma} c_\gamma \widehat{\beta}(\gamma) \right| \leq \overline{\lim_n} \left| \sum_{\gamma \in \sigma} c_\gamma \widehat{\beta}_{nx}(\gamma) \right| \leq \overline{\lim_n} \|\beta_{nx}\|.$$

Bochner's Criterion 3.1.2 implies that $\|\beta\| \leq \overline{\lim_n} \|\beta_{nx}\|$. Consequently, there exists infinitely many indices n such that

$$\|\beta_{nx}\| > (1 - \epsilon') \sum_{k=0}^{p} b_k.$$

Now, we observe that the latter inequality is stable under sufficiently small (say, of size ϵ'') perturbations of the element $\delta_{nx} * \alpha$. Applying Lemma 3.3.5 for $\mu = \alpha$ and $\epsilon = \epsilon''$, we find a point $x \in S$ such that

$$\left\| \delta_{nx} * \alpha - P_S(\delta_{nx} * \alpha) \right\| < \epsilon''$$

for all $n \in \mathbb{N}$. Consequently, there exist infinitely many indices n such that

$$\left\| \sum_{k=0}^{p} b_k \frac{a^k}{\|a\|^k} \right\| > (1 - \epsilon') \sum_{k=0}^{p} b_k,$$

where $a = P_S(\delta_{nx} * \alpha)$. Clearly, $\delta_{nx} * \alpha \in \mathcal{A}$.

By Definition 3.3.6, $a \in A$. In fact, $a/\|a\| \in A^\epsilon$. To prove that, we recall that

$$\sup_{\widehat{G}} |\mathcal{F}(\delta_{nx} * \alpha)| < \epsilon \|\delta_{nx} * \alpha\|,$$

and the written sup is the spectral radius of $\delta_{nx} * \alpha$ (condition (iv) of 3.3.6). The spectral radius is a norm continuous function on $\mathcal{M}(G)$, which implies that $\lim_{m} \|a^m\|^{1/m} < \epsilon \|a\|$ for ϵ'' small enough. It is also clear that $|\hat{a}(\gamma)| \leq \lim_{m} \|a^m\|^{1/m}$ for any bounded character of S. Hence $a/\|a\| \in A^\epsilon$ (with $X = \widehat{S}_b$). Passing to the limit as $\epsilon' \to 0$, we finish the verification of (1.5.1).

The result follows from Theorem 1.5.1.   □

3.3.8. COROLLARY. — *Let S be a semigroup satisfying conditions (i)–(iii) of Subsection 2.2.1 and the absorption condition (iv). Let A be one of the following subalgebras of $\mathcal{M}(S) : \mathcal{M}(S)$ itself; $\mathcal{M}_d(S)$; $L^1(S) + \mathbb{C} \cdot e$; $\mathcal{M}_d(S) + L^1(S)$; or $\mathcal{M}_f(S)$, the algebra of finitely supported measures on S. Then the conclusion of Theorem 3.3.7 holds true for A.*   □

4. Remarks on finite groups and semigroups.

In this section we briefly consider the measure algebras $\mathcal{M}$ on finite groups G, $\mathcal{M} = \mathcal{M}(G) = L^1(G)$, and on finite semigroups, $\mathcal{M} = \mathcal{M}(S) = L^1(S)$. In general, exact computations of the majorants $c_n(\delta, \mathcal{M}, X)$ for finite groups and semigroups are, probably, even more complicated than in the infinite case. This is why we mainly restrict ourselves to two examples: namely, to cyclic groups $C_d = \mathbb{Z}/d\mathbb{Z}$ of order $d \geq 1$, and to nilpotent semigroups $Z_d = \mathbb{Z}_+/(d + \mathbb{Z}_+)$ of order d. In a sense, the groups C_d "exhaust" the group $\mathbb{Z}$, and the semigroups Z_d "exhaust" $\mathbb{Z}_+$. Essentially, we study the asymptotic behaviour of $c_1(\delta, \mathcal{M}(C_d), \widehat{C}_d)$ and $c_1(\delta, \mathcal{M}(Z_d), \widehat{Z}_d)$ as $d \to \infty$.

4.1. Preliminaries on finite groups.

Let G be a finite group, written additively, and m_G the invariant (Haar) measure normalized by $m_G(\{x\}) = 1$ for all $x \in G$. Clearly, the space $\mathcal{M}(G) = L^1(G) = \ell^1(G)$ is a convolution Banach algebra with the unit $e = \delta_0$. The Fourier (Gelfand) transformation is

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \sum_{x \in G} f(x)(-x, \gamma), \quad \gamma \in \widehat{G},$$

where $\widehat{G}$ stands for the dual group of unimodular characters written multiplicatively. The Haar measure $m = m_{\widehat{G}}$ is normalized to have total mass 1, so that the Fourier transformation $\mathcal{F}$ is a unitary operator from $L^2(G)$ to $L^2(\widehat{G})$. For the spectrum of $\mathcal{M}(G)$ we have

$$\mathfrak{M}(\mathcal{M}(G)) = \widehat{G}.$$

Moreover, it is easy to see that, as a subset of $\mathcal{M}(G)$, every convolution algebra A on G is of the form

$$A = \{\mu \in \mathcal{M}(G) : \hat{\mu}_{|\sigma_i} = \text{const}, \ i \in I\},$$

where $\{\sigma_i\}_{i \in I}$ stands for a partition of $\widehat{G}$. Hence, the spectrum $\mathfrak{M}(A)$ can be identified with a subset of $\widehat{G}$. Therefore, we can regard the dual group $\widehat{G}$ as the visible spectrum for any convolution algebra on G. That is, in our previous notation, we set $X = \widehat{G}$. Moreover, the description mentioned of the convolution algebras on G obviously implies that every such algebra A is $\mathfrak{M}(A)$-symmetric, see Definition 1.4.1.

The following easy lemma shows that on finite groups the majorants $c_1(\delta, A, \widehat{G})$ always have linear growth rate as $\delta \to 0$.

4.1.1. LEMMA. — *Let G be a finite abelian group, and A a unital convolution Banach algebra on G. Then $\delta_n(A, \widehat{G}) = 0$ for all $n \geq 1$, and for all $0 < \delta \leq 1$*

$$c_n(\delta, A, \widehat{G}) \leq k(A) \min\left(\frac{1}{\delta^2}, \frac{n^{\frac{1}{2}}}{\delta}\right).$$

Moreover, $\|\mu^{-1}\| \leq k(A)/\delta$ for every $\mu \in A$ satisfying $|\hat{\mu}(\gamma)| \geq \delta$, $\gamma \in \widehat{G}$. Here $k(A)$ is a constant depending on A only, and

$$k(\mathcal{M}(G)) \leq (\text{card}(G))^{\frac{1}{2}}.$$

Proof. — The norm $\|\cdot\|_A$ is equivalent to $\|\widehat{\mu}\|_\infty = \max\limits_{\gamma \in \widehat{G}} |\widehat{\mu}(\gamma)|$, $\mu \in A$, and we see that there exists a constant $k(A)$ such that

(4.1.1) $$\|\mu\|_A \leq k(A)\|\widehat{\mu}\|_\infty$$

for $\mu \in A$. In particular, for every μ with $|\widehat{\mu}(\gamma)| \geq \delta$, $\gamma \in \widehat{G}$,

$$\|\mu^{-1}\|_A \leq k(A)\left\|\frac{1}{\widehat{\mu}}\right\|_\infty \leq \frac{k(A)}{\delta}.$$

This proves the desired estimate for $c_1(\delta, A, \widehat{G})$.

For $n > 1$ we use Lemma 1.4.2 and the symmetry of A mentioned above. This yields

$$c_n(\delta, A, \widehat{G}) \leq \frac{k(A)}{\delta^2}.$$

Another way to bound a solution $\nu = (\nu_1, \dots, \nu_n)$ of the equation

$$\sum_{i=1}^n \mu_i * \nu_i = e$$

is to set $\widehat{\nu}_i = \overline{f}_i|f|^{-2}$, where $f_i = \widehat{\mu}_i$ and $|f|^2 = \sum\limits_{i=1}^n |f_i|^2$, and apply inequality (4.1.1) n times:

$$\|\nu\|_{A^n} = \Big(\sum_{i=1}^n \|\nu_i\|_A^2\Big)^{\frac{1}{2}} \leq k(A)\Big(\sum_{i=1}^n \|\widehat{\nu}_i\|_\infty^2\Big)^{\frac{1}{2}}$$

$$\leq k(A)\Big(\sum_{i=1}^n \Big\|\frac{1}{|f|}\Big\|_\infty^2\Big)^{\frac{1}{2}} \leq k(A)\frac{n^{\frac{1}{2}}}{\delta}.$$

Hence, $c_n(\delta, A, \widehat{G}) \leq k(A)n^{\frac{1}{2}}/\delta^2$.

In the case where $A = \mathcal{M}(G) = \ell^1(G)$, we can write

$$\|\mu^{-1}\|_{L^1(G)} \leq \big(\mathrm{card}(G)\big)^{\frac{1}{2}}\|\mu^{-1}\|_{L^2(G)} = \big(\mathrm{card}(G)\big)^{\frac{1}{2}}\|1/\widehat{\mu}\|_{L^2(\widehat{G},m)}$$

$$\leq \big(\mathrm{card}(G)\big)^{\frac{1}{2}}\|1/\widehat{\mu}\|_{L^\infty(\widehat{G},m)} \leq \big(\mathrm{card}(G)\big)^{\frac{1}{2}}\delta^{-1}. \qquad \square$$

4.2. The cyclic groups C_d.

Now, we take $G = C_d = \mathbb{Z}/d\mathbb{Z} = \{0, 1, \ldots, d-1\}$ with the quotient convolution on C_d. The dual group $\widehat{C}_d$ is the group of d-th roots of unity

$$\widehat{C}_d = \{\zeta_k = \zeta^k : 0 \leq k \leq d-1\}, \quad \text{where } \zeta = \zeta(d) = e^{2\pi i/d}.$$

The Fourier transform of an element $f \in \mathcal{M}(C_d)$ is

$$\mathcal{F}f(\zeta_k) = \hat{f}(\zeta_k) = \sum_{0 \leq s < d} f(s)\zeta_k^s, \quad \zeta_k \in \widehat{C}_d,$$

and the norm is $\|f\| = \sum_{0 \leq s < d} |f(s)|$. Let $e_k = \chi_{\{k\}}$ be the basic functions on C_d. The convolution on C_d, which we denote by $\circ$, follows the rule

$$e_r \circ e_s = e_t,$$

where $t \in C_d$, $t \equiv (r + s) \bmod (d)$.

Note that, instead of $\{0, 1, \ldots, d-1\}$, we can choose an arbitrary interval $\{k, \ldots, k+d-1\}$, $k \in \mathbb{Z}$, of length d as a model for $\mathbb{Z}/d\mathbb{Z}$, endowing it with the d-periodic convolution $e_r \circ e_s = e_t$, where $t \in \{k, \ldots, k+d-1\}$, $t \equiv (r + s) \bmod (d)$.

Now, we consider the behaviour of the upper bound $c_1(\delta, \{C_d\})$ defined by

$$(4.2.1) \qquad c_1(\delta, \{C_d\}) = \sup\{c_1(\delta, \mathcal{M}(C_d), \widehat{C}_d) : d \geq 1\}, \quad 0 < \delta \leq 1.$$

First, we observe a consequence of our general upper estimate for symmetric algebras.

4.2.1. LEMMA. — *For all δ, $\dfrac{1}{\sqrt{2}} < \delta \leq 1$, we have*

$$c_1(\delta, \{C_d\}) \leq \frac{1}{2\delta^2 - 1}.$$

Proof. — This is immediate from Theorem 2.1.1. □

The following lemma shows that, in a sense, the algebras $\mathcal{M}(C_d)$ approximate the algebra $\mathcal{M}(\mathbb{Z}) = \ell^1(\mathbb{Z})$.

4.2.2. LEMMA. — *For all* $0 < \delta \leq 1$

$$c_1\left(\delta, \{C_d\}\right) \geq c_1\left(\delta, \mathcal{M}(\mathbb{Z}), \mathbb{T}\right).$$

Proof. — Let $\delta_1(\mathcal{M}(\mathbb{Z}), \mathbb{T}) < \delta \leq 1$, and let $\epsilon > 0$. We take a finitely supported measure $p \in \mathcal{M}(\mathbb{Z})$ such that $\delta \leq |\widehat{p}(\zeta)| \leq \|p\|_{\mathcal{M}(\mathbb{Z})} \leq 1$ for $\zeta \in \mathbb{T}$, and

$$\|p^{-1}\|_{\mathcal{M}(\mathbb{Z})} > c_1\left(\delta, \mathcal{M}(\mathbb{Z}), \mathbb{T}\right) - \epsilon.$$

Here p^{-1} means the inverse in $\mathcal{M}(\mathbb{Z})$: $p * p^{-1} = e = \delta_0$, where $e = e_0$. Let

$$N = \deg(\widehat{p}) = \sup\{|k| : p(k) \neq 0, \ k \in \mathbb{Z}\},$$

and let $C_d = \{-n, \dots, 0, \dots, n\}$ be the cyclic group of order $d = 2n + 1$, where $n \gg N$. For $a, b \in \mathbb{Z}$, we denote by $[a, b]$ an interval of $\mathbb{Z}$, $[a, b] = \{k \in \mathbb{Z} : a \leq k \leq b\}$, so that $C_d = [-n, n]$. We write $p^{-1} = q + r$, where

$$q = \sum_{|s| \leq n - N} (p^{-1})(s) e_s,$$

and regard p and q as elements of C_d identifying them with the restrictions to $C_d = [-n, n]$. Let us check that

(4.2.2) $q \circ p = e + \epsilon_n,$

where $\epsilon_n \in \mathcal{M}(C_d)$ with

$$\lim_n \|\epsilon_n\|_{\mathcal{M}(C_d)} = 0.$$

Indeed, by the definition of the convolution on C_d, we have

$$q \circ p = (q * p)_{|[-n,n]},$$

and $(q * p)_{|(\mathbb{Z} \setminus [-n,n])} = 0$. Hence,

$$q \circ p = ((p^{-1} - r) * p)_{|[-n,n]} = e - (r * p)_{|[-n,n]} = e + \epsilon_n,$$

where $\epsilon_n = -(r * p)_{|[-n,n]}$. Clearly,

$$\|\epsilon_n\|_{\mathcal{M}(C_d)} = \|\epsilon_n\|_{\mathcal{M}(\mathbb{Z})} \leq \|r * p\|_{\mathcal{M}(\mathbb{Z})} \leq \|r\|_{\mathcal{M}(\mathbb{Z})},$$

and the latter norm tends to zero as $n \to \infty$. It follows from the representation (4.2.2) that $p^{(-1)} = q \circ (e + \epsilon_n)^{(-1)}$, where $p^{(-1)}$ stands for the inverse in $\mathcal{M}(C_d)$. This implies that

$$\|q\|_{\mathcal{M}(C_d)} = \|p^{(-1)} \circ (e + \epsilon_n)\|_{\mathcal{M}(C_d)} \leq \|p^{(-1)}\|_{\mathcal{M}(C_d)} \left(1 + \|\epsilon_n\|_{\mathcal{M}(C_d)}\right).$$

Since $\lim\limits_{n} \|\epsilon_n\|_{\mathcal{M}(C_d)} = 0$, we get

$$\|p^{(-1)}\|_{\mathcal{M}(C_d)} \geq \|q\|_{\mathcal{M}(C_d)} - \epsilon > \|p^{-1}\|_{\mathcal{M}(\mathbb{Z})} - 2\epsilon > c_1(\delta, \mathcal{M}(\mathbb{Z}), \mathbb{T}) - 3\epsilon$$

for n sufficiently large. Since the element $p \in \mathcal{M}(C_d)$ satisfies

$$\delta \leq |\widehat{p}(\zeta_k)| = |\mathcal{F}p(\zeta_k)| \leq \|p\|_{\mathcal{M}(C_d)} \leq 1$$

for $\zeta_k \in \widehat{C}_d$, we obtain

$$c_1\left(\delta, \mathcal{M}(C_d), \widehat{C}_d\right) \geq c_1\left(\delta, \mathcal{M}(\mathbb{Z}), \mathbb{T}\right) - 3\epsilon.$$

The lemma follows.   □

As in (4.2.1), we can define the upper bound $c_n(\delta, \{C_d\})$ for an arbitrary $n \geq 1$, and then prove an analogue of Lemma 4.2.2: $c_n(\delta, \{C_d\}) \geq c_n(\delta, \mathcal{M}(\mathbb{Z}), \mathbb{T})$.

We summarize the preceding lemmas as follows.

4.2.3. THEOREM.

(i) $c_n(\delta, \mathcal{M}(C_d), \widehat{C}_d) \leq d^{\frac{1}{2}} \min(1/\delta^2, n^{\frac{1}{2}}/\delta)$ for all $0 < \delta \leq 1$ and $n \geq 1$.

(ii) $c_n(\delta, \mathcal{M}(C_d), \widehat{C}_d) \leq 1/(2\delta^2 - 1)$ for all $\frac{1}{\sqrt{2}} < \delta \leq 1$ and $n \geq 1$.

(iii) $c_1(\delta, \{C_d\}) \geq c_1(\delta, \mathcal{M}(\mathbb{Z}), \mathbb{T})$ for all $0 < \delta \leq 1$, where $c_1(\delta, \{C_d\})$ is defined in (4.2.1). In particular, $c_1(\delta, \{C_d\}) = \infty$ for $0 < \delta \leq \frac{1}{2}$, and $c_1(\delta, \{C_d\}) \geq (2\delta - 1)^{-1}$ for all $\frac{1}{2} < \delta \leq 1$.   □

4.3. The nilpotent semigroups Z_d.

The semigroup $Z_d = \mathbb{Z}_+/(d + \mathbb{Z}_+)$ is defined as the set $Z_d = \{0, 1, \ldots, d - 1, d\}$ endowed with the operation $(s, t) \mapsto \min(s + t, d)$, and with the measure $m(\{s\}) = 1$ for $0 \leq s < d$ and $m(\{d\}) = 0$. Therefore, on

the basic functions e_s, $e_s(t) = \delta_{s,t}$ (the Kronecker delta), the convolution is defined by the formula

$$e_s \circ e_t = e_{s+t}$$

for $s + t < d$, and $e_s \circ e_t = 0$ for $s + t \geq d$.

The space $\mathcal{M}(Z_d) = L^1(Z_d, m)$ of all measures (functions) on Z_d endowed with this convolution and with the usual L^1 norm is a unital d-nilpotent Banach algebra. Namely, the algebra $\mathcal{M}(Z_d)$ has a generator e_1 such that $e_1^d = 0$. Hence, the only character on Z_d is the trivial one: $0 \mapsto 1$ and $s \mapsto 0$ for $s > 0$. We write $\widehat{Z}_d = \{0\}$, and $\mathfrak{M}(\mathcal{M}(Z_d)) = \{0\}$ with the only homomorphism on $\mathcal{M}(Z_d)$,

$$\mu = \big(\mu(s)\big)_{0 \leq s < d} \longmapsto \mu(0).$$

The following theorem gives the exact value of $c_1(\delta, \mathcal{M}(Z_d), \{0\})$.

4.3.1. THEOREM. — *For all $0 < \delta \leq 1$,*

$$c_1\big(\delta, \mathcal{M}(Z_d), \{0\}\big) = \delta^{-1} \sum_{0 \leq k < d} \left(\frac{1-\delta}{\delta}\right)^k,$$

and, therefore, $c_1(\delta, \mathcal{M}(Z_d), \{0\}) \sim \delta^{-d}$ as $\delta \to 0$. Moreover, for all $0 < \delta \leq 1$

$$c_1(\delta, \{Z_d\}) =: \sup\{c_1(\delta, \mathcal{M}(Z_d), \{0\}) : d \geq 1\} = c_1\big(\delta, \mathcal{M}(\mathbb{Z}_+), \overline{\mathbb{D}}\big);$$

that is,

$$c_1\big(\delta, \{Z_d\}\big) = \begin{cases} \infty & \text{for } 0 < \delta \leq \frac{1}{2}, \\ \dfrac{1}{2\delta - 1} & \text{for } \frac{1}{2} < \delta \leq 1. \end{cases}$$

Proof. — If $\mu \in \mathcal{M}(Z_d)$ and $\delta \leq |\mu(0)| \leq \|\mu\| \leq 1$, then $\mu = \mu(0)e_0 + e_1 \circ \nu$, where $\|\nu\| = \|\mu\| - |\mu(0)| \leq 1 - \delta$. Therefore, since $e_1^d = 0$, we have

$$\mu^{-1} = \mu(0)^{-1} \sum_{k \geq 0} \frac{e_1^k \circ \nu^k}{\mu(0)^k} = \mu(0)^{-1} \sum_{0 \leq k < d} \frac{e_1^k \circ \nu^k}{\mu(0)^k},$$

which implies

$$\|\mu^{-1}\| \leq |\mu(0)|^{-1} \sum_{0 \leq k < d} \frac{\|\nu\|^k}{|\mu(0)|^k} \leq \delta^{-1} \sum_{0 \leq k < d} \left(\frac{1-\delta}{\delta}\right)^k.$$

To check the converse inequality, we simply take $\mu = \delta e_0 + (1-\delta)e_1$, for which choice

$$\|\mu^{-1}\| = \left\|\mu(0)^{-1} \sum_{0 \leq k < d} \frac{(1-\delta)^k e_1^k}{\mu(0)^k}\right\| = \delta^{-1} \sum_{0 \leq k < d} \left(\frac{1-\delta}{\delta}\right)^k.$$

The theorem is proved. □

5. Spectral hulls and norm-controlled functional calculi.

In the four preceding sections, we considered the problem of uniform upper bounds for inverses when staying on a given subset $X \subset \mathfrak{M}$ of the maximal ideal space $\mathfrak{M}$. Here we treat a more general form of the same problem: the so-called full X-spectral hull $h(\sigma, X)$ of a given set $\sigma \subset \mathbb{C}$ is defined and studied. In this language, the uniform boundedness of inverses is equivalent to the property $0 \notin h(\sigma_\delta, X)$, where σ_δ stands for the annulus $\{z \in \mathbb{C} : \delta \leq |z| \leq 1\}$. It is proved that $h(\sigma, X)$ is the minimal set satisfying the following uniform calculus property: for every open set $\Omega \subset \mathbb{C}$ containing $h(\sigma, X)$ there exists a constant $k = k(\Omega, X)$ such that $\|f(a)\| \leq k\|f\|_\Omega$ for all $f \in \mathrm{Hol}(\Omega)$ and for all $a \in A$ with $\sigma(a) \subset \sigma$ and $\|a\| \leq 1$. Precisely this latter requirement distinguishes our approach from classical studies of functions operating on a Banach algebra; see [HKKR], [Ru1].

5.1. Full spectral hulls and resolvent majorants.

Let A be a unital Banach algebra, and let X be a subset of the maximal ideal space $\mathfrak{M}(A)$ endowed with the induced topology, so that $A \subset C(X)$.

5.1.1. DEFINITION. — *Let $\sigma \subset \overline{\mathbb{D}}$. We set*

$$A(\sigma; X) = \{a \in A : \|a\| \leq 1, \hat{a}(X) \subset \sigma\},$$

$$C(\lambda, \sigma; X) = C(\lambda, \sigma; A, X)$$
$$= \sup\{\|(\lambda e - a)^{-1}\| : a \in A(\sigma; X)\}, \quad \lambda \in \mathbb{C},$$

where we take $\|(\lambda e - a)^{-1}\| = \infty$ *for* $\lambda \in \sigma(a)$. *Let also*

$$h(\sigma; X) = h(\sigma; A, X) = \{\lambda \in \mathbb{C} : C(\lambda, \sigma; X) = \infty\}.$$

The set $h(\sigma; X)$ *is called the* X-*spectral, or* (A, X)-*spectral, hull of* σ; *the full spectral hull of* σ *is*

$$h(\sigma) = h\big(\sigma; A, \mathfrak{M}(A)\big).$$

The complement $\rho(\sigma; X) = \mathbb{C} \setminus h(\sigma; X)$ *is called the* norm-controlled $(X$-$)$ *resolvent complement of* σ.

For a positive constant $k > 0$, *we introduce*

$$h(\sigma, k; X) = \{\lambda \in \mathbb{C} : C(\lambda, \sigma) > k\}$$

and $\rho(\sigma, k; X) = \mathbb{C} \setminus h(\sigma, k; X)$, *and call them the* k-*level* $(X$-$)$ *spectral hull of* σ, *and the* k-*level resolvent complement of* σ, *respectively. For* $X = \mathfrak{M}(A)$ *we simplify the notation as follows:*

$$A(\sigma) = A\big(\sigma; \mathfrak{M}(A)\big), \ h(\sigma, k) = h\big(\sigma, k; \mathfrak{M}(A)\big), \ \rho(\sigma, k) = \rho\big(\sigma, k; \mathfrak{M}(A)\big).$$

It is clear that the definition of the $(\delta$-$1)$-*visibility is a special case of the definition of the norm-controlled resolvent complement. Indeed, let* σ_δ *be an annulus,*

$$\sigma_\delta = \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}.$$

The spectrum of A *is* $(\delta$-$1)$-*visible if and only if* $0 \in \rho(\sigma_\delta; X)$. *Moreover,*

$$c_1(\delta, A, X) = C(\sigma_\delta, 0; A, X), \quad 0 < \delta \leq 1.$$

Note that n-*variable counterparts of* $A(\sigma; X), C(\lambda, \sigma; X)$, *etc., could be considered as well, but in this paper we restrict ourselves to the case* $n = 1$.

The next subsection contains some general properties of $C(\lambda, \sigma; A, X)$ *and* $h(\sigma, k; A, X)$ *we need for applications.*

5.1.2. *General properties.* — *Let* A *be a unital Banach algebra, let* $X \subset \mathfrak{M}(A)$, *and let* σ *be a closed subset of the unit disc* $\overline{\mathbb{D}}$.

(i) *The* X-*resolvent complement* $\rho(\sigma; X)$ *is an open subset of* $\mathbb{C}$, $\rho(\sigma; X) \supset \mathbb{C} \setminus \overline{\mathbb{D}}$, *and*

$$\frac{1}{\operatorname{dist}(\lambda, h(\sigma; X))} \leq C(\lambda, \sigma; X) \leq \frac{C(z, \sigma; X)}{1 - |z - \lambda| C(z, \sigma; X)}$$

for all $z \in \rho(\sigma; X)$ *and* $|z - \lambda| < 1/C(z, \sigma; X)$.

Indeed, let $C(z, \sigma; X) < \infty$, and let $a \in A(\sigma; X)$. Then

$$\|(\lambda e - a)^{-1}\| = \|(ze - a)^{-1} \cdot (e - (z - \lambda)(ze - a)^{-1})^{-1}\|$$

$$\leq \sum_{k \geq 0} |z - \lambda|^k \cdot \|(ze - a)^{-1}\|^{k+1} \leq \frac{C(z, \sigma; X)}{1 - |z - \lambda| C(z, \sigma; X)}$$

for $|z - \lambda| < 1/C(z, \sigma; X)$. Hence $\lambda \in \rho(\sigma; X)$, and

$$C(\lambda, \sigma; X) \leq \frac{C(z, \sigma; X)}{1 - |z - \lambda| C(z, \sigma; X)}.$$

This implies the left hand side inequality too.   □

(ii) $C(\,.\,, \sigma; X)$ *is a positive continuous function from* $\mathbb{C}$ *to* $\overline{\mathbb{R}}_+ = (0, \infty]$, *and* $\log(C(\,.\,, \sigma; X))$ *is subharmonic on* $\rho(\sigma; X)$. *Moreover,* $h(\sigma, k; X)$ *is an open set, and*

$$h(\sigma; X) = \bigcap_{k < \infty} h(\sigma, k)$$

is a closed subset of the unit disc $\overline{\mathbb{D}}$ *such that* $\sigma \subset h(\sigma; X)$.

Indeed, it is clear that $\lambda e \in A(\sigma)$ for $\lambda \in \sigma$, and so $C(\lambda, \sigma; X) = \infty$. This means that $\sigma \subset h(\sigma; X)$.

The right hand side inequality of (i) implies that $C(\cdot, \sigma; X)$ is a continuous function on $\rho(\sigma; X)$. The left hand side inequality gives $\lim_{\lambda \to z} C(\lambda, \sigma; X) = \infty$ for every $z \in h(\sigma; X)$. Hence, $C(\cdot, \sigma; X)$ is continuous on $\mathbb{C}$.

In particular, the set $h(\sigma, k; X)$ is open. The set $h(\sigma; X) = \mathbb{C} \setminus \rho(\sigma; X)$ is closed as the complement of an open set.

The function $\log(C(\cdot, \sigma; X))$ is subharmonic as an upper bound of a family of subharmonic functions $\lambda \mapsto \log |\langle (\lambda e - a)^{-1}, \varphi \rangle|$, where $a \in A(\sigma; X)$ and $\varphi \in A^*$, $\|\varphi\| \leq 1$.   □

(iii) *The sequence* $(h(\sigma, k; X))_{k \geq 1}$ *decreases and uniformly converges to* $h(\sigma; X)$.

The inclusions $h(\sigma, k; X) \supset h(\sigma, k + 1; X)$ are obvious. It follows from the continuity of $C(\,.\,, \sigma; X)$ that

$$\text{clos}\big(h(\sigma, k; X)\big) =: \bar{h}(\sigma, k; X) =: \big\{z \in \mathbb{C} : C(z, \sigma; X) \geq k\big\}$$

and we see that

$$h(\sigma;X) = \bigcap_{k \geq 1} \bar{h}(\sigma, k;X).$$

Since $\|(\lambda e - a)^{-1}\| \leq (|\lambda| - 1)^{-1}$ for all $a \in A(\sigma;X)$ and $|\lambda| > 1$, the sets $\bar{h}(\sigma, k;X)$ are compact, and, therefore, the previous equality implies that the convergence of $\bar{h}(\sigma, k;X)$ to $h(\sigma;X)$ is uniform: for every open neighbourhood $\Omega \supset h(\sigma;X)$, there exists k such that $\Omega \supset h(\sigma, k;X)$.   □

(iv) $h(\sigma;X) = \sigma$ for every closed $\sigma \subset \bar{\mathbb{D}}$ if and only if $\delta_1(A, X) = 0$, where $\delta_1(A, X)$ is the critical constant for the pair $(A, , X)$ (see also (vi) below).

Necessity: taking $\sigma_\delta = \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}$ for a given $\delta > 0$, we get $\sigma_\delta = h(\sigma_\delta;X)$, and so $0 \notin h(\sigma_\delta;X)$. As mentioned before, the latter is equivalent to $c_1(\delta, A, X) < \infty$. Hence, $\delta_1(A, X) = 0$.

Sufficiency: if $\delta_1(A, X) = 0$, the spectrum of A is 1-visible (Lemma 1.1.2), which gives $\delta_1(A, \mathfrak{M}(A)) = \delta_1(A, X) = 0$ and $h(\sigma;X) = h(\sigma;\mathfrak{M}(A))$. By Lemma 1.3.2, the algebra A obeys the distance controlled resolvent growth, which immediately yields $h(\sigma;\mathfrak{M}(A)) = \sigma$; see estimate (1.3.1).   □

(v) If $d_\lambda = \text{dist}(\lambda, \sigma)$ and $|\lambda| \leq 1$, one has

$$C(\lambda, \sigma;X) \leq c_1\Big(\frac{d_\lambda}{1 + |\lambda|}, A, X\Big);$$

for $|\lambda| > 1$, we simply have $C(\lambda, \sigma;X) \leq (|\lambda| - 1)^{-1}$.

The second inequality is obvious. To check the first one, we take $f \in A(\sigma;X)$ and set $g = (\lambda e - f)/\|\lambda e - f\|$ for $\|\lambda e - f\| > 1$, and $g = \lambda e - f$ for $\|\lambda e - f\| \leq 1$. Clearly, $\|\lambda e - f\| \leq 1 + |\lambda|$, which implies $|\hat{g}| \geq d_\lambda/(1 + |\lambda|)$ on the set X. Hence

$$\|(\lambda e - f)^{-1}\| \leq \|g^{-1}\| \leq c_1\Big(\frac{d_\lambda}{1 + |\lambda|}, A, X\Big)$$

and the assertion follows.   □

(vi) If $\delta_1(A, X)$ stands for the critical constant for (A, X), one has

$$h(\sigma;X) \subset \{\lambda \in \bar{\mathbb{D}} : \text{dist}(\lambda, \sigma) \leq (1 + |\lambda|)\delta_1(A, X)\}$$

$$\subset \{\lambda \in \bar{\mathbb{D}} : \text{dist}(\lambda, \sigma) \leq 2\delta_1(A, X)\}.$$

This is immediate from (v).   □

Note that the latter inclusion is informative for $\delta_1(A, X) < \frac{1}{2}$ only. For pairs (A, X) satisfying some extra conditions we prove another inclusion and an estimate in the next subsection.

5.1.3. Horodisc expansions and an upper estimate. — Denote

$$D(z,r) = \left\{ \zeta \in \mathbb{C} : |\zeta - z| < r \right\}, \quad \overline{D}(z,r) = \left\{ \zeta \in \mathbb{C} : |\zeta - z| \leq r \right\}.$$

In the hyperbolic geometry of the unit disc $\mathbb{D}$, the discs

$$\overline{D}(z, 1 - |z|), \quad z \in \overline{\mathbb{D}}$$

are called *horodiscs*. Given a closed set $\sigma \subset \overline{\mathbb{D}}$, we call the set

$$\mathrm{hor}(\sigma) = \bigcup_{z \in \sigma} \left(\overline{D}(z, 1 - |z|) \right)$$

the *horodisc expansion* of σ. See Subsection 5.1.5 below for some pictures related to $\mathrm{hor}(\sigma)$.

The following theorem will be useful for measure algebras $A \subset \mathcal{M}(S)$ on semigroups; see 5.3.2 below.

5.1.4. THEOREM. — *Let A be a unital Banach algebra satisfying the splitting hypothesis of Subsection 1.4, and let the visible part of the spectrum $X \subset \mathfrak{M}(A)$ be such that $\varphi_e \in \mathrm{clos}(\delta_x : x \in X)$ (see 1.4.8 for details).*

(i) *One has $h(\sigma; X) \subset \mathrm{hor}(\sigma)$ and for $\lambda \in \mathbb{C}$*

$$C(\lambda, \sigma; X) \leq \frac{1}{\mathrm{dist}(\lambda, \mathrm{hor}(\sigma))} \cdot$$

(ii) *One has*

$$C(\lambda, \sigma; X) \leq c_1(d_\lambda R_\lambda^{-1}, A, X)$$

where

$$d_\lambda = \mathrm{dist}(\lambda, \sigma), \quad R_\lambda = \max(r_\lambda, 1)$$

$$r_\lambda = \sup_{z \in \sigma} \left(|\lambda - z| + (1 - |z|) \right)$$

and $c_1(.)$ stands for the standard visibility constant defined by formula $(0.2.5)$. Note that $R_0 = 1$ and

$$h(\sigma; X) \subset \left\{ \lambda \in \overline{\mathbb{D}} : d_\lambda \leq R_\lambda \delta_1(A, X) \right\}.$$

Proof. — (i) First, we observe that

$$\operatorname{dist}(\lambda, \operatorname{hor}(\sigma)) = \inf_{z \in \sigma} \left(|\lambda - z| - (1 - |z|) \right)^+$$

for all $\lambda \in \mathbb{C}$, where $x^+ = \max(0, x)$, $x \in \mathbb{R}$.

Let $\lambda \in \mathbb{C} \setminus \operatorname{hor}(\sigma)$, and let $f \in A(\sigma; X)$, which means $\|f\| \leq 1$ and $\hat{f}(X) \subset \sigma$. By the hypothesis, $z =: \varphi_e(f) \in \sigma$. Moreover, since $\lambda e - f = (\lambda - z)(e - (\lambda - z)^{-1}(f - ze))$ and $|\lambda - z| - (1 - |z|) > 0$, we have

$$\left\| (\lambda e - f)^{-1} \right\| \leq |\lambda - z|^{-1} \frac{1}{1 - |\lambda - z|^{-1} \cdot \|f - ze\|}$$

$$\leq \frac{1}{|\lambda - z| - (1 - |z|)} \leq \frac{1}{\operatorname{dist}(\lambda, \operatorname{hor}(\sigma))}.$$

Therefore, $C(\lambda, \sigma; X) \leq 1/\operatorname{dist}(\lambda, \operatorname{hor}(\sigma)) < \infty$, and, in particular, $\lambda \notin h(\sigma, X)$.

(ii) To prove this property, we slightly amplify the reasoning in 5.1.2, part (v). As in part (i) above, we take $f \in A(\sigma; X)$ and set $z =: \varphi_e(f) \in \sigma$. This implies

$$\|\lambda e - f\| = |\lambda - z| + \|f - ze\| \leq |\lambda - z| + (1 - |z|) = r.$$

Setting $g = (\lambda e - f)/\|\lambda e - f\|$ for $\|\lambda e - f\| > 1$, we get

$$|\hat{g}| \geq \frac{d_\lambda}{\|\lambda e - f\|} \geq \frac{d_\lambda}{r} \geq \frac{d_\lambda}{R_\lambda}$$

on the set X. In the case where $\|\lambda e - f\| \leq 1$, we set $g = \lambda e - f$ and obtain

$$|\hat{g}| \geq d_\lambda \geq \frac{d_\lambda}{R_\lambda}.$$

Hence $\left\| (\lambda e - f)^{-1} \right\| \leq \|g^{-1}\| \leq c_1(d_\lambda/R_\lambda, A, X)$. The assertion follows. $\square$

5.1.5. COROLLARY. — *The following inclusions and inequalities are special cases of Theorem 5.1.4 (for pairs (A, X) satisfying the hypotheses of the theorem, in particular, for the analytic Wiener algebra $A = \mathcal{F}\mathcal{M}(\mathbb{Z}_+)$ with $X = \overline{\mathbb{D}}$, and for all algebras A from Theorem 3.3.7 with $X = \widehat{S}_b$).*

(i) $h(\sigma_\delta; X) \subset \operatorname{hor}(\sigma_\delta) = \sigma_{2\delta-1}$ *for* $\frac{1}{2} < \delta < 1$ *(see Figures 2 and 3), where* $\sigma_\delta = \{ z \in \mathbb{C} : \delta \leq |z| \leq 1 \}$. *Moreover,*

$$C(\lambda, \sigma_\delta; X) \leq \frac{1}{(2\delta - 1) - |\lambda|}$$

$$h(\sigma_\delta) = \sigma_{2\delta-1} \qquad\qquad h(\sigma_\delta) = \overline{\mathbb{D}}$$
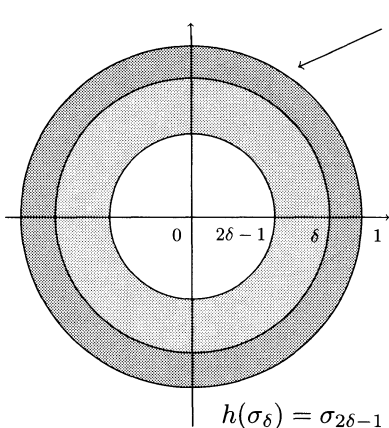
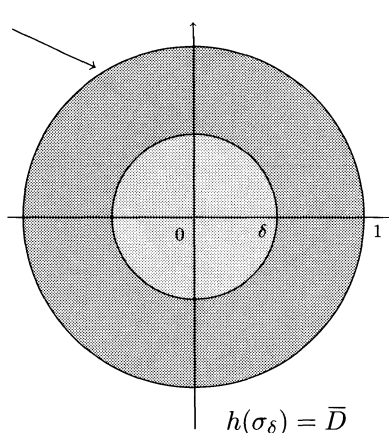Figure 2: $\delta > \frac{1}{2}$ Figure 3: $\delta \leq \frac{1}{2}$

for $|\lambda| < 2\delta - 1$. Recall that $c_1(\delta, A, X) = C(\sigma_\delta, 0; X)$.

(ii) For a disc segment $s_\delta = \{z \in \overline{\mathbb{D}} : \mathrm{Re}(z) \geq \delta\}$, we have

$$h(s_\delta; X) \subset \mathrm{hor}(s_\delta) = s_\delta \cup s_\delta^*$$

where s_δ^* is the mirror reflection of s_δ in the line $\mathrm{Re}(z) = \delta$ (see Figure 4). In particular, $h(s_\delta; X) \neq \overline{\mathbb{D}}$ for all $\delta > 0$.

(iii) For a disc $\sigma = \overline{D}(\zeta, \epsilon)$, where $0 < \epsilon < \min(\delta, 1 - \delta)$ and $\delta = |\zeta|$, the spectral hull $h(\overline{D}(\zeta, \epsilon); X)$ is contained in

$$\mathrm{hor}\big(\overline{D}(\zeta, \epsilon)\big);$$

the latter is a subset of $\overline{\mathbb{D}}$ which is symmetric with respect to the diameter passing by ζ, contains the shadow arc $\gamma \subset \mathbb{T}$ of $\overline{D}(\zeta, \epsilon)$ (of length $|\gamma| = 2\arctg(\epsilon/\delta)$), and satisfies (see Figure 5)

$$(\zeta\mathbb{R}) \cap h(\overline{D}(\delta, \epsilon); X) = [2(\delta - \epsilon) - 1, 1]\frac{\zeta}{|\zeta|}. \qquad\qquad \square$$

Now, we pass to converse inclusions for $h(\sigma; X)$ and to lower estimates for $C(\lambda, \sigma; X)$.

5.1.6. *Microlocalization and a lower estimate.* — It is clear that lower estimates for spectral hulls and resolvent majorants heavily depend on the
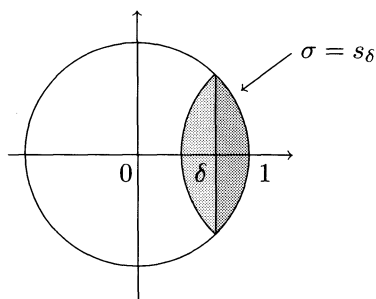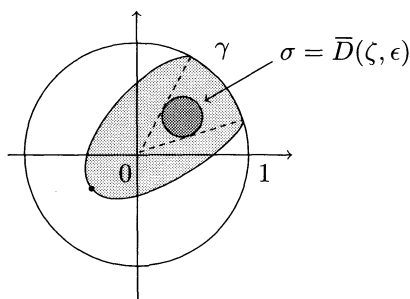
Figure 4: $h(s_\delta) = s_\delta \cup s_\delta^*$ Figure 5: $\sigma = \overline{D}(\zeta, \epsilon)$

geometry of a given subset $\sigma \subset \overline{\mathbb{D}}$. For instance, $A(\sigma; X) = \{\text{const}\}$ for every $X \subset \mathfrak{M}(A)$ equipped with an analytic structure and for every σ with $\text{int}(\sigma) = \emptyset$; see 5.3.1 for examples. Having in mind this last constraint, we restrict ourselves to the case where

$$\sigma = \text{clos}\left(\text{int}(\sigma)\right).$$

In this case, the behaviour of $C(\lambda, \sigma; X)$ depends on the following microlocal version of the critical constants and the inversion majorants.

Let A be a unital Banach algebra, let $X \subset \mathfrak{M}(A)$, and let $0 < \delta \leq 1$. A *microlocal upper bound* (majorant) for inverses is defined by

$$c_1^0(\delta, A, X) = \inf_{\epsilon > 0}\left(\sup\{\|f^{-1}\| : \|f\| \leq 1, \ \hat{f}(X) \subset \sigma_\delta, \ \text{diam } \hat{f}(X) < \epsilon\}\right)$$

and the *microlocal critical constant* is defined by

$$\delta_1^0(A, X) = \inf\{\delta : c_1^0(\delta, A, X) < \infty\};$$

here, as before, $\sigma_\delta = \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}$.

General properties of these microlocal constants are similar to those of the general ones, that is, of $\delta_1(A, X)$ and $c_1(\delta, A, X)$ defined in Section 1. Namely, the following is true.

(i) $c_1^0(\delta, A, X)$ *is a decreasing function of* δ.

(ii) $c_1^0(\delta, A, X) \leq c_1(\delta, A, X)$, *and* $\delta_1^0(A, X) \leq \delta_1(A, X)$.

(iii) *If a pair* (A, X) *satisfies* (1.5.1) *of Theorem 1.5.1, then* $\delta_1^0(A, X) \geq \frac{1}{2}$ *and* $c_1^0(\delta, A, X) \geq (2\delta - 1)^{-1}$.

If, in addition, A splits at the unit and φ_e is X-dominated (see 1.4.4–1.4.6 about that), we have

$$\delta_1^0(A, X) = \delta_1(A, X) = \frac{1}{2}$$

and for $\frac{1}{2} < \delta \le 1$

$$c_1^0(\delta, A, X) = c_1(\delta, A, X) = (2\delta - 1)^{-1}.$$

The claimed inequalities are easy consequences of the proof of Theorem 1.5.1, since this proof is based on a microlocal construction with $\mathrm{diam}(\hat{f}(X)) < \epsilon$ for a test function.

Conversely, assuming that A splits at the unit and φ_e is X-dominated, we take $\delta > \frac{1}{2}$, $\epsilon' > 0$ and an element $f \in A$ satisfying $\hat{f}(X) \subset \sigma_\delta$ and $\mathrm{diam}\, \hat{f}(X) < \epsilon$. Lemma 1.4.6 implies

$$\varphi_e(f) \in \mathrm{conv}\big(\hat{f}(X)\big).$$

Since $\mathrm{conv}(\hat{f}(X)) \subset \sigma_\delta(1 - \epsilon')$ for $\epsilon > 0$ small enough, we infer from Lemma 1.4.3 that

$$\|f^{-1}\| \le \big(2\delta(1 - \epsilon') - 1\big)^{-1}.$$

By the above definition of $c_1^0(\delta, A, X)$, we get $c_1^0(\delta, A, X) \le (2\delta - 1)^{-1}$. The result follows. $\qquad\square$

The following is a special case of (iii).

(iv) *For the same measure algebras $A \subset \mathcal{M}(S)$ on semigroups as in Subsections 2.2, 2.4 and 3.3.6, one has*

$$c_1^0(\delta, A, \widehat{S}_b) = c_1(\delta, A, \widehat{S}_b) = (2\delta - 1)^{-1} \text{ and } \delta_1^0(A, \widehat{S}_b) = \delta_1(A, \widehat{S}_b) = \frac{1}{2}.$$

In particular, this is the case for $A = \mathcal{M}(S), \mathcal{M}_a(S), L^1(S) + \mathbb{C} \cdot e$ on semigroups S satisfying (i)–(iii) of 2.2.1 and (iv) of 3.3.4. $\qquad\square$

Now we apply microlocal constants to spectral hulls.

5.1.7. Theorem. — *Let A be a unital Banach algebra splitting at the unit, and let φ_e be X-dominated (see 1.4.4–1.4.6 for details). Let $\sigma = \operatorname{clos}(\operatorname{int}(\sigma)) \subset \overline{\mathbb{D}}$, and $\delta_1^0 = \delta_1^0(A, X)$. Then, $\delta_1^0 \leq \frac{1}{2}$, and*

$$\operatorname{hor}(\sigma, \delta_1^0) \subset h(\sigma; A, X),$$

where $\operatorname{hor}(\sigma, \delta) = \bigcup_{z \in \sigma} \left(\overline{D}\left(z, \frac{\delta}{1-\delta}(1 - |z|)\right) \right)$ *and moreover,*

$$C(\lambda, \sigma; A, X) \geq \sup_{z \in \sigma}\left(r_z^{-1} c_1^0(\delta_z, A, X)\right),$$

where $r_z = |\lambda - z| + (1 - |z|)$ and $\delta_z = |\lambda - z|/r_z$, that is, $\dfrac{\delta_z}{1 - \delta_z} = \dfrac{|\lambda - z|}{1 - |z|}.$

Proof. — Let $\lambda \in \mathbb{D}$, $z \in \operatorname{int}(\sigma)$, and let $\delta, \epsilon > 0$ be such that $\delta > \delta_z$ and $D(z, \epsilon) \subset \sigma$.

Let $f \in A$ be an "almost extremal" element from the definition of $c_1^0(\delta, A, X)$, that is, $\|f\| \leq 1$, $\hat{f}(X) \subset \sigma_\delta$, $\operatorname{diam} \hat{f}(X) < \frac{1}{2}\epsilon\delta$, and

$$\|f^{-1}\| > (1 - \epsilon')c_1^0(\delta, A, X)$$

(with an obvious modification in the case where $c_1^0(\delta, A, X) = \infty$). There exists a point $\zeta \in \sigma_\delta$ such that $\hat{f}(X) \subset D(\zeta, \frac{1}{2}\epsilon\delta)$. Now, we infer from Lemma 1.4.6 that $\varphi_e(f) \in \overline{D}(\zeta, \frac{1}{2}\epsilon\delta)$, that is, $\varphi_e(f) = \zeta + t$ where $|t| < \frac{1}{2}\epsilon\delta$. Let

$$g = \lambda e - \frac{\lambda - z}{\zeta}f.$$

By the definition,

$$\hat{g}(X) = \lambda - \frac{\lambda - z}{\zeta}\hat{f}(X) \subset \lambda - \frac{\lambda - z}{\zeta}D\left(\zeta, \frac{1}{2}\epsilon\delta\right) \subset D(z, \epsilon).$$

Moreover, for $\epsilon > 0$ sufficiently small

$$\|g\| = \left|\lambda - \frac{\lambda - z}{\zeta}\varphi_e(f)\right| + \left|\frac{\lambda - z}{\zeta}\right|(\|f\| - |\varphi_e(f)|)$$

$$= \left|\lambda - \frac{\lambda - z}{\zeta}(\zeta + t)\right| + \left|\frac{\lambda - z}{\zeta}\right|(\|f\| - |\zeta + t|)$$

$$= \left|z - \frac{\lambda - z}{\zeta}t\right| + \left|\frac{\lambda - z}{\zeta}\right|(\|f\| - |\zeta + t|)$$

$$\leq |z| + \epsilon + \left|\frac{\lambda - z}{\zeta}\right|(1 - |\zeta|) + \epsilon$$

$$\leq |z| + \frac{|\lambda - z|}{\delta}(1 - \delta) + 2\epsilon < |z| + \frac{|\lambda - z|}{\delta_z}(1 - \delta_z).$$

Since $|z| + \dfrac{|\lambda - z|}{\delta_z}(1 - \delta_z) = 1$, we obtain $\|g\| \leq 1$. Therefore,

$$C(\lambda, \sigma; X) \geq \|(\lambda e - g)^{-1}\| = \left|\dfrac{\zeta}{\lambda - z}\right| \cdot \|f^{-1}\| > \dfrac{\delta_z}{|\lambda - z|}(1 - \epsilon')c_1^0(\delta, A, X).$$

Passing to the limit as $\epsilon' \to 0$ and $\delta \to \delta_z$, we get

$$C(\lambda, \sigma; A, X) \geq r_z^{-1}c_1^0(\delta_z, A, X).$$

Since $z \in \text{int}(\sigma)$ is arbitrary, and $\sigma = \text{clos}(\text{int}(\sigma))$, the stated inequality follows.

Moreover, since $1 - |\lambda| \leq r_z \leq 2$ and $c_1^0(\delta_z, A, X) = \infty$ for $\delta_z \leq \delta_1^0(A, X)$, we get $C(\lambda, \sigma; X) = \infty$ if $\min\limits_{z \in \sigma} \delta_z \leq \delta_1^0(A, X)$. We conclude that $\lambda \in h(\sigma; X)$ if there exists $z \in \sigma$ such that $\delta_z \leq \delta_1^0(A, X)$, that is, if $\lambda \in \overline{D}\left(z, \dfrac{\delta_1^0}{1 - \delta_1^0}(1 - |z|)\right)$. The inclusion $\text{hor}(\sigma, \delta_1^0) \subset h(\sigma; A, X)$ follows. $\square$

5.1.8. Comments. — Observe that under the splitting hypothesis we always have

$$\delta_1^0(A, X) \leq \delta_1(A, X) \leq \dfrac{1}{2}$$

(and, therefore, $\delta_1^0/(1 - \delta_1^0) \leq 1$), and

$$\text{hor}(\sigma, \delta_1^0) \subset \text{hor}\left(\sigma, \dfrac{1}{2}\right) = \text{hor}(\sigma).$$

The equality $\text{hor}(\sigma, \delta_1^0) = \text{hor}(\sigma)$ holds for all $\sigma \subset \overline{\mathbb{D}}$ if and only if $\delta_1^0(A, X) = \frac{1}{2}$.

It should be mentioned that the converse inclusion $h(\sigma; A, X) \subset \text{hor}(\sigma)$ is proved in Theorem 5.1.4 (i) under a stronger hypothesis,

$$\varphi_e \in \text{clos}(\delta_x : x \in X),$$

than those of Theorem 5.1.7 (namely, $\varphi_e \in \text{conv}(\delta_x : x \in X)$). In particular, Theorem 5.1.7 is available for both pairs $A = \mathcal{M}(G), X = \widehat{G}$ and $A = \mathcal{M}(S)$, $X = \widehat{S}_b$, whereas Theorem 5.1.4 (i) does it for the latter one only.

We combine both theorems in the following statement.

5.1.9. COROLLARY. — *Let A be a unital splitting Banach algebra, $X \subset \mathfrak{M}(A)$ be a subset dominating φ_e, and let $\sigma = \mathrm{clos}(\mathrm{int}(\sigma)) \subset \overline{\mathbb{D}}$.*

(i) *One has*

$$\mathrm{hor}(\sigma, \delta_1^0) \subset h(\sigma; A, X) \subset \mathrm{hor}\left(\sigma, \frac{1}{2}\right) = \mathrm{hor}(\sigma),$$

and

$$\sup_{z \in \sigma}\left(r_z^{-1} c_1^0(\delta_z, A, X)\right) \leq C(\lambda, \sigma; A, X) \leq \frac{1}{\mathrm{dist}(\lambda, \mathrm{hor}(\sigma))},$$

where $\delta_1^0 = \delta_1^0(A, X) \leq \frac{1}{2}$ and the left hand sides of these statements are defined in Theorem 5.1.7.

(ii) *If $\delta_1^0(A, X) = \frac{1}{2}$, we have*

$$h(\sigma; A, X) = \mathrm{hor}(\sigma).$$

(iii) *If $\delta_1^0(A, X) = \frac{1}{2}$ and $c_1^0(\delta, A, X) = (2\delta - 1)^{-1}$, we have $h(\sigma; A, X) = \mathrm{hor}(\sigma)$ and*

$$C(\lambda, \sigma; X) = \frac{1}{\mathrm{dist}(\lambda, \mathrm{hor}(\sigma))}. \qquad \square$$

5.2. Norm-controlled calculi.

The Gelfand theory guarantees the existence of a holomorphic calculus on the spectrum of every element of a Banach algebra A: if $a \in A$, the function of a

$$f(a) = \frac{1}{2\pi i}\int_{\partial\Omega} f(\lambda)(\lambda e - a)^{-1}\, d\lambda$$

is well defined for every $f \in \mathrm{Hol}(\overline{\Omega})$ and for every open neighbourhood of the spectrum $\Omega \supset \sigma(a)$ (the Riesz-Dunford calculus). As we know from Sections 1 through 4, this does not guarantee any estimate of the norm $\|f(a)\|$, even for the simplest functions such as $f(z) = 1/z$, and even if we add the normalizing condition $\|a\| \leq 1$ to the spectral inclusion $\sigma(a) \subset \Omega$. A real question is the following: What are the relations between the spectrum $\sigma(a)$ (or the visible spectrum $\hat{a}(X)$) and a domain Ω that guarantee the uniform continuity of the Ω-calculus? That is, does there exist a compact set $K \subset \Omega$ and a constant $c > 0$ such that $\|f(a)\| \leq c\sup_K |f|$ for every $f \in \mathrm{Hol}(\Omega)$ and every $a \in A(\sigma)$? We formalize this by the following definition.

5.2.1. DEFINITION. — *Let A and X be as above, and let $\sigma = \overline{\sigma} \subset \overline{\mathbb{D}}$. We say that an open set $\Omega \subset \mathbb{C}$ X-dominates the set σ, or is a norm-controlled calculus domain for σ, if $\sigma \subset \Omega$ and the calculi*

$$f \longmapsto f(a), \quad f \in \mathrm{Hol}(\Omega)$$

are well defined and uniformly continuous for

$$a \in A(\sigma; X) = \{a \in A : \hat{a}(X) \subset \sigma, \ \|a\| \le 1\}.$$

This is equivalent to the existence of a compact set $K \subset \Omega$ and a constant $c(K) = c(K, X) > 0$ such that

$$\|f(a)\| \le c(K)\|f\|_K$$

for every function $f \in \mathrm{Hol}(\Omega)$ and every $a \in A(\sigma; X)$. Here $\|f\|_K = \max\{|f(z)| : z \in K\}$.

Obviously, the definition of $(\delta$-$1)$-visibility, as well as the definition of the distance-controlled resolvent growth (see Subsection 1.3.1) are special cases of the latter concept. The following theorem shows that the domains that X-dominate a given set σ can be described in terms of the spectral hulls $h(\sigma; X)$.

5.2.2. THEOREM. — *Let A be a unital Banach algebra, let $X \subset \mathfrak{M}(A)$, and let $\sigma \subset \overline{\mathbb{D}}$ and $\Omega \subset \mathbb{C}$ be a closed and an open set, respectively. The following assertions are equivalent:*

   (i) *Ω is an X-dominating domain for σ;*

   (ii) *$\Omega \supset h(\sigma; X)$;*

   (iii) *there exists $k > 0$ such that $\Omega \supset h(\sigma, k; X)$.*

Proof. — (i) $\Rightarrow$ (ii). If $\lambda \notin \Omega$ and $f_\lambda(z) = (\lambda - z)^{-1}$, then $f_\lambda \in \mathrm{Hol}(\Omega)$, and $\|(\lambda e - a)^{-1}\| \le c(K)\|f_\lambda\|_K$ for all $a \in A(\sigma; X)$. By the definition of $h(\sigma; X)$ (see 5.1.1), this implies $\lambda \notin h(\sigma; X)$.

(ii) $\Rightarrow$ (iii). This follows from Theorem 5.1.2 (iv).

(iii) $\Rightarrow$ (i). The function $C(\,.\,, \sigma; X)$ is continuous by Theorem 5.1.2 (ii). Hence, the $(k+1)$-level spectral hull $h(\sigma, k+1; X)$ is compactly contained in the k-level spectral hull $h(\sigma, k; X)$, that is, $\mathrm{clos}(h(\sigma, k+1; X)) \subset h(\sigma, k; X)$. Therefore, there exists a domain Ω with rectifiable boundary $\partial\Omega$ such that

$\text{clos}(h(\sigma, k + 1; X)) \subset \Omega \subset \overline{\Omega} \subset h(\sigma, k; X)$ (in fact, there exists a polygon with these properties). The standard Riesz-Dunford functional calculus gives

$$f(a) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\lambda)(\lambda e - a)^{-1} \, d\lambda,$$

and

$$\|f(a)\| \leq c \, \frac{k + 1}{2\pi} \|f\|_{\partial\Omega},$$

where c stands for the length of $\partial\Omega$. The result follows. $\square$

5.3. Examples of spectral hulls.

Here we exhibit examples of three different types.

• One of these examples pertains to algebras A whose hull operation is trivial in the sense that $h(\sigma; \mathfrak{M}(A)) = \sigma$ for every $\sigma = \overline{\sigma} \subset \overline{\mathbb{D}}$. This is Example 5.3.6 below.

• For algebras A of the second type, the same operation $\sigma \mapsto h(\sigma; \mathfrak{M}(A))$ is also trivial, but in a different way, namely, $h(\sigma; \mathfrak{M}(A)) = \overline{\mathbb{D}}$ for every nonempty $\sigma = \overline{\sigma} \subset \overline{\mathbb{D}}$, even for singletons. This is Example 5.3.4 below.

• For the middle type algebras the full spectral hull $h(\sigma; \mathfrak{M}(A))$ essentially depends on σ but is different from it. For instance, this is the case for the algebras $A = \mathcal{M}(S)$ on semigroups of the same class as before. In this case we can completely compute the full spectral hulls $h(\sigma; \widehat{S}_b)$ and the resolvent majorants $C(\cdot, \sigma; \widehat{S}_b)$. This is Theorem 5.3.2 below. These computations represent a refinement of our results on the visibility constants $c_n(\delta, \mathcal{M}(S), \widehat{S}_b)$; see Theorem B, Section 0.3.

5.3.1. *Spectral hulls for measure algebras on semigroups.* — We start with a remark about the analytic Wiener algebra $A = \mathcal{M}(\mathbb{Z}_+) = \ell^1(\mathbb{Z}_+)$, or, equivalently, $A = \mathcal{F}\ell^1(\mathbb{Z}_+) = W_+$, in which case $\mathfrak{M}(A) = \overline{\mathbb{D}}$. Since a nonconstant holomorphic function is an open mapping, the sets

$$A(\sigma) = A(\sigma, \overline{\mathbb{D}}) = \{f \in W_+ : \|f\| \leq 1, f(\overline{\mathbb{D}}) \subset \sigma\}$$

and $A(\sigma')$, where

$$\sigma' = \text{clos}(\text{int}(\sigma)),$$

differ from each other by constant functions taking values in $\sigma \setminus \sigma'$. Hence, $h(\sigma, \overline{\mathbb{D}}) = h(\sigma', \overline{\mathbb{D}}) \cup (\sigma \setminus \sigma')$, and we can restrict ourselves to the case where $\sigma = \sigma'$.

In fact, supposing $\sigma = \sigma'$, we easily derive from Corollary 5.1.9 a description of $h(\sigma; \widehat{S}_b)$ for all subalgebras of $\mathcal{M}(S)$ considered in Sections 2 and 3.

5.3.2. THEOREM. — *Let* $A = \mathcal{F}\mathcal{M}(S)$, *or let* A *be any algebra* $A \subset \mathcal{M}(S)$ *satisfying the hypotheses of Theorem 3.3.7. Let* σ *be a closed subset of* $\mathbb{D}$ *such that* $\sigma = \mathrm{clos}(\mathrm{int}(\sigma))$. *The following properties hold:*

(i) $h(\sigma; \widehat{S}_b) = \mathrm{hor}(\sigma)$.

(ii) $C(\lambda, \sigma; \widehat{S}_b) = 1/\mathrm{dist}(\lambda, \mathrm{hor}(\sigma))$ *for* $\lambda \in \mathbb{C}$.

(iii) *An open set* Ω *is a norm-controlled calculus domain for* σ *if and only if* $\Omega \supset \mathrm{hor}(\sigma)$.

Proof. — Assertion (iii) is a straightforward consequence of (i) and Theorem 5.2.2 (see also Definition 5.2.1 of norm-controlled calculi).

Assertions (i) and (ii) are special cases of Corollary 5.1.9, because $\delta_1^0(A, \widehat{S}_b) = \frac{1}{2}$ by Theorem 3.3.7, and $\varphi_e(f) = f(0) \in \hat{f}(\widehat{S}_b)$ for every $f \in \mathcal{M}(S)$. □

5.3.3. COROLLARY. — *Let* $A = \mathcal{F}\mathcal{M}(S)$ *be an algebra satisfying the hypotheses of Theorem 3.3.7 (in particular, the analytic Wiener algebra* $A = \mathcal{F}\mathcal{M}(\mathbb{Z}_+)$ *is allowed). The following values of full spectral hulls and resolvent majorants are special cases of the theorem:*

(i) $h(\sigma_\delta) = \sigma_{2\delta-1}$ *for* $\frac{1}{2} < \delta < 1$ *and* $h(\sigma_\delta) = \overline{\mathbb{D}}$ *for* $0 < \delta \leq \frac{1}{2}$. *Moreover,*

$$C(\lambda, \sigma_\delta; \widehat{S}_b) = \frac{1}{(2\delta - 1) - |\lambda|}$$

for $|\lambda| < 2\delta - 1$ *and* $C(\lambda, \sigma_\delta; \widehat{S}_b) = \infty$ *for* $2\delta - 1 \leq |\lambda| \leq 1$. *Here, as above,* $\sigma_\delta = \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}$. *Recall that* $c_1(\delta, A, \widehat{S}_b) = C(\sigma_\delta, 0; \widehat{S}_b)$.

(ii) $h(\sigma) = \overline{\mathbb{D}}$ *if* $0 \in \mathrm{clos}(\mathrm{int}(\sigma))$, *whence* $C(\lambda, \sigma; \widehat{S}_b) = \infty$ *for all* $|\lambda| \leq 1$.

(iii) *For the spectral hulls of a disc segment* $s_\delta = \{z \in \overline{\mathbb{D}} : \mathrm{Re}(z) \geq \delta\}$, $h(s_\delta; \widehat{S}_b) = s_\delta \cup s_\delta^*$, *and a small disc* $\overline{D}(\delta, \epsilon)$ *with* $0 < \epsilon < \min(\delta, 1 - \delta)$, *see Corollary 5.1.5 above.* □

5.3.4. *Full spectral hulls equal to* $\overline{\mathbb{D}}$. — Here we give examples of bad Banach algebras for which the full spectral hull of every spectrum is equal to $\overline{\mathbb{D}}$. To this end, we need a bit of model operators. All properties claimed below can be found in [N1].

Let Θ be a singular inner function in $\mathbb{D}$, and let

$$\Theta = \exp\left(\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, d\mu(\zeta)\right), \quad z \in \mathbb{D}$$

be its canonical integral representation, where μ is a positive measure on $\mathbb{T}$ singular with respect to the Lebesgue measure dm. Sometimes, we write $\Theta = \Theta_\mu$. The simplest example is

$$\Theta_{\delta_\zeta} = \exp\left(\frac{z+\zeta}{z-\zeta}\right), \quad z \in \mathbb{D},$$

where $\zeta \in \mathbb{T}$. Further, let

$$K_\Theta = H^2 \ominus \Theta H^2$$

be the orthogonal complement of the corresponding z-invariant subspace ΘH^2 of the Hardy space H^2. The compression

$$f \longmapsto M_\Theta f = P_\Theta z f, \quad f \in K_\Theta$$

of the multiplication operator is called a *model operator*; P_Θ stands for the orthogonal projection on K_Θ. An H^∞ function of M_Θ is well defined by

$$\varphi(M_\Theta)f = P_\Theta \varphi f, \quad f \in K_\Theta,$$

and the mapping $\varphi \mapsto \varphi(M_\Theta)$ is an algebra homomorphism from H^∞ to the algebra $L(K_\Theta)$ of all bounded linear operators on K_Θ. Moreover, $\varphi(M_\Theta) = 0$ if and only if $\varphi \in \Theta H^\infty$, and

$$\left\|\varphi(M_\Theta)\right\| = \inf\left\{\|\varphi + \Theta h\|_{\mathbb{D}} : h \in H^\infty\right\}.$$

Therefore, an operator $\varphi(M_\Theta)$ is invertible if and only if there exist two functions $\psi, h \in H^\infty$ such that $\varphi\psi + \Theta h = 1$. The Carleson corona Theorem (with subsequent improvements, see Subsection 1.2.2) shows that there exists a constant $c > 0$ such that the norm of the inverse $\varphi(M_\Theta)^{-1} = \psi(M_\Theta)$ satisfies

$$\frac{\delta^{-2}}{c} \leq \left\|\varphi(M_\Theta)^{-1}\right\| \leq c\,\delta^{-2}\log\left(\frac{1}{\delta}\right)$$

where

$$\delta = \inf_{z \in \mathbb{D}}\left(|\varphi(z)| + |\Theta(z)|\right).$$

It is an exercise to show that $\sigma(M_\Theta) = \mathrm{supp}(\mu) \subset \mathbb{T}$ and $\|M_\Theta\| = 1$.

Now, we define our algebra $A = A_\Theta$ as the norm closure of polynomials in M_Θ.

It is worth mentioning that, for $\Theta = \Theta_{\delta_1}$, there exists a unitary operator $U : K_\Theta \to L^2(0,1)$ such that the algebra $U A_\Theta U^{-1}$ is the norm closure of polynomials of the integration operator

$$Jf(x) = \int_0^x f(t)\,dt, \quad x \in (0,1)$$

on $L^2(0,1)$. It can be proved that

$$\mathfrak{M}(A_\Theta) = \sigma(M_\Theta) = \mathrm{supp}(\mu)$$

and, thus, $\sigma(\varphi(M_\Theta)) = \varphi(\mathrm{supp}(\mu))$ for every $\varphi \in C(\overline{\mathbb{D}}) \cap H^\infty$.

5.3.5. THEOREM. — *Let $A = A_\Theta$ be the above Banach algebra corresponding to a singular inner function $\Theta = \Theta_\mu$ with the representing measure μ supported on a subset of zero Lebesgue measure : $m(\mathrm{supp}(\mu)) = 0$. Then for every closed subset $\sigma \subset \overline{\mathbb{D}}$, $\sigma \neq \emptyset$, we have*

$$h\big(\sigma; \mathfrak{M}(A)\big) = \overline{\mathbb{D}}.$$

Proof. — Clearly, it suffices to prove the theorem for a singleton $\sigma = \{s\}$, $s \in \overline{\mathbb{D}}$. Fixing points $t, \lambda \in \mathbb{D}$, we denote by ζ the point satisfying $\zeta \in \overline{\mathbb{D}}$ and

$$b_\lambda(\zeta) = s,$$

where $b_\lambda(z) = (\lambda - z)/(1 - \bar{\lambda}z)$ stands for a conformal automorphism of the unit disc.

By the classical Rudin interpolation theorem, [Ru2], see also [Gar], there exists a disc algebra function $h \in C(\overline{\mathbb{D}}) \cap H^\infty$ such that

$$h(z) = \frac{1}{b_t(z)} \quad \text{for } z \in \mathrm{supp}(\mu) \text{ and } \|h\|_{\mathbb{D}} = 1.$$

Setting $g = \zeta\, b_t h$, we get a function in $C(\overline{\mathbb{D}}) \cap H^\infty$ such that

$$g(z) = \zeta \quad \text{for } z \in \mathrm{supp}(\mu), \ g(t) = 0, \ \|g\|_{\mathbb{D}} \leq 1.$$

Finally, the function f defined by $f = b_\lambda \circ g$ satisfies

$$f(z) = s \quad \text{for} \quad z \in \text{supp}(\mu), \ f(t) = \lambda, \ \|f\|_\mathbb{D} \leq 1.$$

We conclude that $f(M_\Theta) \in A_\Theta$, $\sigma(f(M_\Theta)) = f(\text{supp}(\mu)) = \{s\}$, and $\|f(M_\Theta)\| \leq 1$, that is, $f(M_\Theta) \in A_\Theta(\{s\}; \mathfrak{M}(A_\Theta))$.

On the other hand,

$$\left\| (f(M_\Theta) - \lambda I)^{-1} \right\| \geq c\,\delta^{-2},$$

where $\delta = \inf\limits_{z \in \mathbb{D}} (|f(z) - \lambda| + |\Theta(z)|)$. Taking $z = t$, we get

$$\left\| (f(M_\Theta) - \lambda I)^{-1} \right\| \geq c|\Theta(t)|^{-2}.$$

Therefore, $C(\lambda, \{s\}; \mathfrak{M}(A_\Theta)) \geq c|\Theta(t)|^{-2}$ for every $t \in \mathbb{D}$, and hence

$$C\big(\lambda, \{s\}; \mathfrak{M}(A_\Theta)\big) = \infty.$$

The result follows. $\square$

5.3.6. Spectrally closed sets. — It is natural to call subsets $\sigma \subset \mathbb{C}$ (A, X)-*spectrally closed* (or A-*spectrally closed* in the case where $X = \mathfrak{M}(A)$) if

$$h(\sigma; A, X) = \sigma.$$

Theorem 5.1.2 (v) says that all closed subsets $\sigma \subset \mathbb{C}$ are (A, X)-closed if and only if $\delta_1(A, X) = 0$. The paper [ENZ] is especially devoted to this case, and many algebras with this property are exhibited there. For instance, all weighted Beurling-Sobolev algebras with regularly varying weights tending to infinity have this property. Actually we mean the following pairs (A, X) of convolution algebras and their maximal ideal spaces $X = \mathfrak{M}(A)$:

   - $A = \ell^p(\mathbb{Z}, w)$, $X = \{z \in \mathbb{C} : r_- \leq |z| \leq r_+\}$, where $r_\pm = \lim\limits_{n \to \pm\infty} (w(n))^{1/n}$;

   - $A = \ell^p(\mathbb{Z}_+, w)$, $X = \{z \in \mathbb{C} : |z| \leq r_+\}$, where $r_+ = \lim\limits_{n \to \infty} (w(n))^{1/n} > 0$.

Hence, we can say that among the weighted Beurling algebras $\ell^1(\mathbb{Z}, w)$, $\ell^1(\mathbb{Z}_+, w)$ with regularly varying weights and positive spectral radius ($r_+ > 0$) the only exception with nontrivial spectral hulls are the "flat" Wiener algebras $\ell^1(\mathbb{Z}), \ell^1(\mathbb{Z}_+)$. Reasons for such a behaviour are discussed in [ENZ].

Radical weighted algebras of formal power series, that is, $\ell^1(\mathbb{Z}_+, w)$ with $r_+ = 0$ and $X = \mathfrak{M}(A) = \{0\}$, are different: both possibilities $\delta_1(A, X) = 0$ and $\delta_1(A, X) > 0$ can occur even for sufficiently regular weights w. See [N2] for details.

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