Annales de l'institut Fourier

Miguel Abánades
Wojciech Kucharz

Algebraic equivalence of real algebraic cycles

Annales de l'institut Fourier, tome 49, nº 6 (1999), p. 1797-1804
<http://www.numdam.org/item?id=AIF_1999__49_6_1797_0>

Ann. Inst. Fourier, Grenoble
49, 6 (1999), 1797–1804

ALGEBRAIC EQUIVALENCE OF REAL ALGEBRAIC CYCLES

by M. ABÁNADES & W. KUCHARZ

1. Introduction.

Let X be a nonsingular, n-dimensional, quasiprojective variety over $\mathbb{R}$ (that is, an irreducible, n-dimensional, quasiprojective scheme over $\mathbb{R}$, smooth over $\mathbb{R}$). We endow the set $X(\mathbb{R})$ of $\mathbb{R}$-rational points of X with the topology induced by the usual metric topology on $\mathbb{R}$, and assume that $X(\mathbb{R})$ is nonempty and compact. Thus $X(\mathbb{R})$ is a C^∞, closed, n-dimensional manifold. Given a nonnegative integer k, we let $Z^k(X)$ denote the group of algebraic $(n-k)$-cycles on X (that is, the free Abelian group on the set of closed, $(n-k)$-dimensional subvarieties of X). There exists a unique group homomorphism

$$\mathrm{cl}_{\mathbb{R}} : Z^k(X) \to H^k(X(\mathbb{R}), \mathbb{Z}/2)$$

such that for every closed, $(n-k)$-dimensional subvariety V of X, the cohomology class $\mathrm{cl}_{\mathbb{R}}(V)$ is Poincaré dual to the homology class in $H_{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$ determined by $V(\mathbb{R})$ (*cf.* [5] for the definition of this homology class). In the present paper we study the cohomology classes of the form $\mathrm{cl}_{\mathbb{R}}(z)$, where z is a cycle in $Z^k(X)$ algebraically equivalent to 0 (we refer to [7] for the theory of algebraic equivalence of cycles). Such cohomology classes need not be trivial, but as we shall see below they must satisfy quite restrictive conditions.

Both authors were partially supported by NSF Grant DMS-9503138.
Keywords: Real algebraic variety – Algebraic cycles – Algebraic cohomology.
Math. classification: 14P25 – 55N22.

The extreme cases, $k = 0$ and $k = n$, are easy to analyze. Obviously, a cycle z in $Z^0(X)$ is algebraically equivalent to 0 if and only if $z = 0$. On the other hand, every cycle in $Z^n(X)$ of the form $x_0 - x_1$, where x_0 and x_1 are points in $X(\mathbb{R})$, is algebraically equivalent to 0. We have $\mathrm{cl}_{\mathbb{R}}(x_0 - x_1) \neq 0$ whenever x_0 and x_1 belong to distinct connected components of $X(\mathbb{R})$. It follows that a cohomology class u in $H^n(X(\mathbb{R}), \mathbb{Z}/2)$ can be written as $u = \mathrm{cl}_{\mathbb{R}}(z)$ for some cycle z in $Z^n(X)$ algebraically equivalent to 0 if and only if the homology class in $H_0(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to u is represented by an even number of points of $X(\mathbb{R})$. In view of these facts, we concentrate our attention on the intermediate cases, $1 \leq k \leq n-1$.

Given a continuous map $f : M \to N$ between topological spaces, we denote by $H^k(f) : H^k(N, \mathbb{Z}/2) \to H^k(M, \mathbb{Z}/2)$ the homomorphism induced by f. Recall that a cohomology class u in $H^k(M, \mathbb{Z}/2)$ with $k \geq 1$ is said to be *spherical* if $u = H^k(f)(c)$, where $f : M \to S^k$ is a continuous map into the unit k-sphere S^k, and c is the generator of $H^k(S^k, \mathbb{Z}/2) \cong \mathbb{Z}/2$. We denote, as usual, by $\cup$ and $< -,- >$ the cup product of cohomology classes and the Kronecker index (pairing) of cohomology and homology classes, *cf.* [11]. If M is a C^∞, closed manifold of dimension n, we denote by $w_k(M)$ the kth Stiefel-Whitney class of M and by μ_M the fundamental homology class of M in $H_n(M, \mathbb{Z}/2)$.

THEOREM 1.1. — *Let X be a nonsingular, n-dimensional, quasiprojective variety over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty and compact. Let z be a cycle in $Z^k(X)$ that is algebraically equivalent to 0. Then the cohomology class $\mathrm{cl}_{\mathbb{R}}(z)$ in $H^k(X(\mathbb{R}), \mathbb{Z}/2)$ satisfies $\mathrm{cl}_{\mathbb{R}}(z) \cup \mathrm{cl}_{\mathbb{R}}(z) = 0$ in $H^{2k}(X(\mathbb{R}), \mathbb{Z}/2)$ and*

$$< \mathrm{cl}_{\mathbb{R}}(z) \cup w_{i_1}(X(\mathbb{R})) \cup \ldots \cup w_{i_r}(X(\mathbb{R})),\ \mu_{X(\mathbb{R})} > = 0$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = n-k$. Furthermore, if $k = 1$ or if $k = n-1 \geq 1$ with $X(\mathbb{R})$ connected, then the cohomology class $\mathrm{cl}_{\mathbb{R}}(z)$ is spherical.

Let us note that, in general, the cohomology class $\mathrm{cl}_{\mathbb{R}}(z)$ of Theorem 1.1 need not be spherical. Indeed, suppose $X = X' \times X''$ (product over $\mathrm{Spec}\mathbb{R}$), where X' and X'' are nonsingular, projective varieties over $\mathbb{R}$ such that $X'(\mathbb{R})$ is nonempty and $X''(\mathbb{R})$ is disconnected. Let z' be any algebraic cycle on X'. Choose two points p_0 and p_1 in $X''(\mathbb{R})$ that belong to distinct connected components. Since the 0-cycle $z'' = p_0 - p_1$ on X'' is algebraically equivalent to 0, the cycle $z' \times z''$ on X is algebraically equivalent to 0 as well. Furthermore, the cohomology class $\mathrm{cl}_{\mathbb{R}}(z' \times z'') = \mathrm{cl}_{\mathbb{R}}(z') \times \mathrm{cl}_{\mathbb{R}}(z'')$ is spherical if and only if the cohomology class $\mathrm{cl}_{\mathbb{R}}(z')$ is spherical (for p_0 and p_1 belong to distinct connected components of $X''(\mathbb{R})$). Taking $X' = \mathbb{P}^m_{\mathbb{R}}$, we have $\mathrm{cl}_{\mathbb{R}}(Z^k(X')) = H^k(X'(\mathbb{R}), \mathbb{Z}/2)$, and the unique nontrivial cohomology class in $H^k(X'(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2$ is not spherical, provided that $1 \le k \le m-1$ and m is even. In particular, "connected" cannot be omitted in the last part of Theorem 1.1.

If $\mathrm{cl}_{\mathbb{R}}(z)$ is spherical, then $\mathrm{cl}_{\mathbb{R}}(z) \cup \mathrm{cl}_{\mathbb{R}}(z) = 0$ is automatically satisfied, and Theorem 1.1 is in some sense the best possible result. More precisely, we have the following.

THEOREM 1.2. — *Let M be a C^∞, closed, n-dimensional manifold and let u be a spherical cohomology class in $H^k(M, \mathbb{Z}/2)$ with $1 \le k \le n-1$. Then the following conditions are equivalent :*

(a) *There exist a nonsingular, projective algebraic variety X over $\mathbb{R}$ and a C^∞ diffeomorphism $\varphi : X(\mathbb{R}) \to M$ such that $H^k(\varphi)(u) = \mathrm{cl}_{\mathbb{R}}(z)$ for some cycle z in $Z^k(X)$ algebraically equivalent to 0;*

(b) *$< u \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M),\ \mu_M >= 0$ for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = n - k$.*

Let us mention that Theorem 1.2 is an improvement upon inefficient [10], Theorem 2.4.

2. Proofs.

Let X be a nonsingular, n-dimensional, quasiprojective algebraic variety over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty and compact. Recall that if an algebraic cycle z in $Z^k(X)$ is rationally equivalent to 0, then $\mathrm{cl}_{\mathbb{R}}(z) = 0$ (*cf.* [5], 5.13) and hence $\mathrm{cl}_{\mathbb{R}}$ induces a homomorphism, also denoted by $\mathrm{cl}_{\mathbb{R}}$, from the Chow group $A^k(X)$ of X into $H^k(X(\mathbb{R}), \mathbb{Z}/2)$. It is known that $\mathrm{cl}_{\mathbb{R}} : A^*(X) \to H^*(X(\mathbb{R}), \mathbb{Z}/2)$ is a homomorphism of graded rings [5], p. 495. Thus

$$H^*_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = \mathrm{cl}_{\mathbb{R}}(Z^*(X)) = \mathrm{cl}_{\mathbb{R}}(A^*(X))$$

is a graded subring of $H^*(X(\mathbb{R}), \mathbb{Z}/2)$. We shall need the following result [10], Theorem 2.1 :

$$< \mathrm{cl}_{\mathbb{R}}(z) \cup v,\ \mu_{X(\mathbb{R})} >= 0 \tag{1}$$

for all cycles z in $Z^k(X)$ algebraically equivalent to 0 and all v in $H^{n-k}_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$.

Assume now that X is projective. Then the set $X(\mathbb{C})$ of $\mathbb{C}$-rational points of X is a compact complex manifold of complex dimension n. There exists a unique group homomorphism

$$\mathrm{cl}_{\mathbb{C}} : Z^k(X) \to H^{2k}(X(\mathbb{C}), \mathbb{Z})$$

such that for every closed, $(n-k)$-dimensional subvariety V of X, the cohomology class $\mathrm{cl}_{\mathbb{C}}(V)$ is Poincaré dual to the homology class in $H_{2n-2k}(X(\mathbb{C}), \mathbb{Z})$ determined by $V(\mathbb{C})$ (*cf.* [5] for the definition of this homology class). In other words, if $\pi : X_{\mathbb{C}} = X \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C} \to X$ is the canonical projection, then $\mathrm{cl}_{\mathbb{C}}(z)$ is the cohomology class corresponding to the pullback algebraic cycle $\pi^*(z)$ on $X_{\mathbb{C}}$, *cf.* [5], 4.2 or [7], Chapter 19. In particular,

$$\mathrm{cl}_{\mathbb{C}}(z) = 0 \tag{2}$$

for all cycles z in $Z^k(X)$ algebraically equivalent to 0, *cf.* [5], 4.14 or [7], Proposition 19.1.1. Furthermore, it follows from the proof of [2], Theorem A that

$$\mathrm{cl}_{\mathbb{R}}(z) \cup \mathrm{cl}_{\mathbb{R}}(z) = \text{ the reduction modulo 2 of } r(\mathrm{cl}_{\mathbb{C}}(z)) \tag{3}$$

for all z in $Z^k(X)$, where $r : H^{2k}(X(\mathbb{C}), \mathbb{Z}) \to H^{2k}(X(\mathbb{R}), \mathbb{Z})$ is the homomorphism induced by the inclusion map $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$.

Proof of Theorem 1.1. — By Hironaka's resolution of singularities theorem [8], 3, we may assume that X is projective.

We obtain $\mathrm{cl}_{\mathbb{R}}(z) \cup \mathrm{cl}_{\mathbb{R}}(z) = 0$ directly from (2) and (3).

It follows from [5], p. 498 that $w_i(X(\mathbb{R}))$ is in $H^i_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$, and hence if $i_1, \dots, i_r$ are nonnegative integers with $i_1 + \cdots + i_r = n - k$, then the cohomology class

$$v = w_{i_1}(X(\mathbb{R})) \cup \ldots \cup w_{i_r}(X(\mathbb{R}))$$

belongs to $H^{n-k}_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$. In view of (1), we have $< \mathrm{cl}_{\mathbb{R}}(z) \cup v, \mu_{X(\mathbb{R})} >= 0$, which completes the proof of the first part of the theorem.

Given an invertible sheaf $\mathcal{L}$ on X, we denote by $\mathcal{L}_{\mathbb{R}}$ (resp. $\mathcal{L}_{\mathbb{C}}$) the topological real (resp. complex) line bundle on $X(\mathbb{R})$ (resp. $X(\mathbb{C})$) determined by $\mathcal{L}$ in the usual way. If $\mathcal{L}$ corresponds to a Weil divisor D on X, then

$$w_1(\mathcal{L}_{\mathbb{R}}) = \mathrm{cl}_{\mathbb{R}}(D) \quad \text{and} \quad c_1(\mathcal{L}_{\mathbb{C}}) = \mathrm{cl}_{\mathbb{C}}(D),$$

where $w_1(-)$ and $c_1(-)$ stand for the first Stiefel-Whitney class and the first Chern class, respectively, *cf.* [5], p. 498, p. 489. Note that the restriction $\mathcal{L}_{\mathbb{C}}|X(\mathbb{R})$ of $\mathcal{L}_{\mathbb{C}}$ to $X(\mathbb{R})$ is the complexification of $\mathcal{L}_{\mathbb{R}}$, and hence

$$c_1(\mathcal{L}_{\mathbb{C}}|X(\mathbb{R})) = \beta(w_1(\mathcal{L}_{\mathbb{R}})),$$

where $\beta : H^1(X(\mathbb{R}), \mathbb{Z}/2) \to H^2(X(\mathbb{R}), \mathbb{Z})$ is the Bockstein homomorphism that appears in the long exact sequence

$$\ldots \to H^1(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^1(X(\mathbb{R}), \mathbb{Z}) \to H^1(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^2(X(\mathbb{R}), \mathbb{Z}) \to \ldots,$$

cf. [11], Problems 15-C and D. The last equality can be written in an equivalent form

$$r(\mathrm{cl}_{\mathbb{C}}(D)) = \beta(\mathrm{cl}_{\mathbb{R}}(D)), \tag{4}$$

where $r : H^2(X(\mathbb{C}), \mathbb{Z}) \to H^2(X(\mathbb{R}), \mathbb{Z})$ is the homomorphism induced by the inclusion map $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$.

Suppose now that $k = 1$, that is, z is a Weil divisor on X. By (2) and (4), we have $\beta(\mathrm{cl}_{\mathbb{R}}(z)) = 0$, which means that $\mathrm{cl}_{\mathbb{R}}(z)$ is the reduction modulo 2 of a cohomology class in $H^1(X(\mathbb{R}), \mathbb{Z})$. This last fact implies that the cohomology class $\mathrm{cl}_{\mathbb{R}}(z)$ is spherical, *cf.* [9], p. 49.

Let us now assume that $k = n - 1 \geq 1$ and $X(\mathbb{R})$ is connected. We already know that $< \mathrm{cl}_{\mathbb{R}}(z) \cup w_1(X(\mathbb{R})),\ \mu_{X(\mathbb{R})} >= 0$, which in view of the connectedness of $X(\mathbb{R})$ is equivalent to $\mathrm{cl}_{\mathbb{R}}(z) \cup w_1(X(\mathbb{R})) = 0$. The last condition implies that the homology class in $H_1(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to $\mathrm{cl}_{\mathbb{R}}(z)$ can be represented by a C^∞, closed curve in $X(\mathbb{R})$, with trivial normal vector bundle, *cf.* for example [4], p. 599. This in turn implies that $\mathrm{cl}_{\mathbb{R}}(z)$ is spherical, *cf.* [12], Théorème II.1. Thus the proof is complete. □

Proof of Theorem 1.2. — By Theorem 1.1, (a) implies (b), and we show below that (b) implies (a).

Choose a nonsingular, irreducible algebraic subset W of $\mathbb{R}^{k+1}$, which has precisely two connected components W_0 and W_1, each diffeomorphic to the unit k-sphere S^k (for example, $W = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1^4 - 4x_1^2 + 1 + x_2^2 + \cdots + x_{k+1}^2 = 0\}$). Let c be the unique generator of the group $H^k(W_0, \mathbb{Z}/2) \cong \mathbb{Z}/2$, viewed as a subgroup of $H^k(W, \mathbb{Z}/2)$. Since the cohomology class u is spherical, there exists a C^∞ map $h : M \to W$ such that $h(M) \subseteq W_0$ and $u = H^k(h)(c)$. Choose a regular value y_0 of h in W_0. Then u is Poincaré dual to the homology class in $H_{n-k}(M, \mathbb{Z}/2)$ represented by the C^∞ submanifold $h^{-1}(y_0)$ of M, *cf.* [5], 2.15. Clearly, there exists a unique C^∞ map $f : M \to W$ such that for every connected component S of M and every point x in S, we have $f(x) = h(x)$ if $S \cap h^{-1}(y_0) \neq \emptyset$ and $f(x) = y_0$ if $S \cap h^{-1}(y_0) = \emptyset$. The map f satisfies

$$f(M) \subseteq W_0 \ \text{ and } \ u = H^k(f)(c). \tag{5}$$

Furthermore, each connected component of $f^{-1}(y_0)$ is a C^∞ submanifold of M of dimension either $n-k$ or n. Also, each connected component of M contains a connected component of $f^{-1}(y_0)$. Since $n-k \geq 1$, we can find a C^∞ closed curve C in M such that

$$f(C) = \{y_0\}, \tag{6}$$

the normal vector bundle of C in M is trivial, and each connected component of M contains a connected component of C. Choose an integer d with $2n+1 \leq d$ and let D be a compact, nonsingular, irreducible, 1-dimensional algebraic subset of $\mathbb{R}^d$ that has the same number of connected components as C. Replacing M by its image under a suitable C^∞ embedding into $\mathbb{R}^d$, we may assume that

$$D = C \subseteq M \subseteq \mathbb{R}^d. \tag{7}$$

By Tognoli's theorem [13] or [1], Corollary 2.8.6, there exists a nonsingular real algebraic subset A of $\mathbb{R}^p$, for some p, diffeomorphic to M. Consider the disjoint union $N = M \amalg A$ and the C^∞ map $F : N \to W$ defined by $F(x) = f(x)$ for x in M and $F(x) = y_0$ for x in A. We assert that if w is a cohomology class in $H^\ell(W, \mathbb{Z}/2)$ and if $j_1, \ldots, j_s$ are nonnegative integers with $j_1 + \cdots + j_s = n - \ell$, then

$$< H^\ell(F)(w) \cup w_{j_1}(N) \cup \ldots \cup w_{j_s}(N),\ \mu_N >= 0.$$

Indeed, first note that $w = 0$, unless $\ell = 0$ or $\ell = k$. If $\ell = 0$, then either $H^0(F)(w) = 0$ or $H^0(F)(w) = 1$. In the latter case the assertion holds since M and A are diffeomorphic. If $\ell = k$, then either $H^k(F)(w) = 0$ or $H^k(F)(w) = u$ (we view $H^k(M, \mathbb{Z}/2)$ as a subgroup of $H^k(N, \mathbb{Z}/2)$). In the latter case the assertion follows from condition (b). Thus the assertion is proved. It implies that there exist a C^∞ compact manifold $\tilde{N}$ with boundary $\partial\tilde{N} = N$ and a C^∞ map $\tilde{F} : \tilde{N} \to W$ satisfying $\tilde{F}|N = F$, cf. [6], 17.3. In other words, the map $f : M \to W$ and the constant map $A \to W$, which sends A to y_0, represent the same class in the unoriented bordism group of W. By construction, the normal bundle of D in M is trivial, so the restriction $\nu|D$ to D of the normal bundle ν of M in $\mathbb{R}^d$ admits an algebraic structure. Therefore, by [1], Theorem 2.8.4 and in view of (6) and (7), one can find a nonnegative integer e, a nonsingular algebraic subset V of $\mathbb{R}^d \times \mathbb{R}^e$, a C^∞ diffeomorphism $\varphi : V \to M$, and a regular map $g : V \to W$ (the latter designates the restriction to V of a rational map from $\mathbb{R}^d \times \mathbb{R}^e$ into $\mathbb{R}^{k+1}$ which has no poles on V and maps V into W) such that g is homotopic to $f \circ \varphi$ and $D \times \{0\} \subseteq V$. Since D is irreducible and each connected component of V contains a connected component of

$D \times \{0\}$, it follows that V is irreducible as well (this is the only place where D is needed).

Irreducibility of V and W allows us to choose nonsingular, quasiprojective varieties T and Y over $\mathbb{R}$ with $T(\mathbb{R}) = V$ and $Y(\mathbb{R}) = W$. By Hironaka's resolution of singularities theorem [8], 3, we may assume that T and Y are projective (and still nonsingular). Let $\tilde{g} : U \to Y$ be an algebraic morphism over $\mathbb{R}$, defined on a Zariski open neighborhood of $T(\mathbb{R}) = V$ in T, such that $\tilde{g}|T(\mathbb{R}) = g$. By applying Hironaka's theorem on removing points of indeterminacy [8], 3, we can find a nonsingular, projective algebraic variety X over $\mathbb{R}$ and an algebraic morphism $G : X \to Y$ over $\mathbb{R}$ satisfying $X(\mathbb{R}) = T(\mathbb{R})$ and $G|X(\mathbb{R}) = g$.

Let y_1 be a point in W_1 and let β be the class in $A^k(Y)$ of the 0-cycle $y_0 - y_1$ on Y. By (5), $u = H^k(f)(\mathrm{cl}_{\mathbb{R}}(\beta))$ (although, of course, $\mathrm{cl}_{\mathbb{R}}(\beta) \neq c$). Since $G|X(\mathbb{R}) = g$ is homotopic to $f \circ \varphi$, we obtain

$$\begin{aligned} H^k(\varphi)(u) &= H^k(\varphi)(H^k(f)(\mathrm{cl}_{\mathbb{R}}(\beta))) = H^k(f \circ \varphi)(\mathrm{cl}_{\mathbb{R}}(\beta)) \\ &= H^k(G|X(\mathbb{R}))(\mathrm{cl}_{\mathbb{R}}(\beta)) = \mathrm{cl}_{\mathbb{R}}(G^*(\beta)), \end{aligned}$$

where the last equality is a consequence of the functorial property of $\mathrm{cl}_{\mathbb{R}} : A^* \to H^*$, *cf.* [5], 5.12. Let z be a cycle in $Z^k(X)$ that represents in the Chow group $A^k(X)$ the pullback class $G^*(\beta)$. Then $\mathrm{cl}_{\mathbb{R}}(z) = \mathrm{cl}_{\mathbb{R}}(G^*(\beta)) = H^k(\varphi)(u)$. The proof is now complete since the cycle $y_0 - y_1$ is algebraically equivalent to 0 on Y and hence the cycle z is algebraically equivalent to 0 on X. □

BIBLIOGRAPHY

[1] S. AKBULUT and H. KING, Topology of Real Algebraic Sets, Mathematical Sciences Research Institute Publications, Springer, 1992.

[2] S. AKBULUT and H. KING, Transcendental submanifolds of $\mathbb{R}^n$, Comm. Math. Helv., 68 (1993), 308–318.

[3] E. BIERSTONE and P. MILMAN, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math., 128 (1997), 207–302.

[4] J. BOCHNAK and W. KUCHARZ, Algebraic models of smooth manifolds, Invent. Math., 97 (1989), 585–611.

[5] A. BOREL et A. HAEFLIGER, La classe d'homologie fondamentale d'un espace analytique, Bull. Soc. Math. France, 89 (1961), 461–513.

[6] P.E. CONNER, Differentiable Periodic Maps, Lecture Notes in Math., Vol. 738, Berlin-Heidelberg-New York, Springer, 1979.

[7] W. FULTON, Intersection Theory, Ergebnisse der Math., Vol. 2, Berlin-Heidelberg-New York, Springer, 1984.

[8] H. HIRONAKA, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math., 79 (1964), 109–326.

[9] S.T. HU, Homotopy Theory, New York, Academic Press, 1959.

[10] W. KUCHARZ, Algebraic equivalence and homology classes of real algebraic cycles, Math. Nachr., 180 (1996), 135–140.

[11] J. MILNOR and J. STASHEFF, Characteristic Classes, Ann. of Math. Studies, Vol. 76, Princeton Univ. Press, 1974.

[12] R. THOM, Quelques propriétés globales de variétés différentiables, Comm. Math. Helv., 28 (1954), 17–86.

[13] A. TOGNOLI, Su una congettura di Nash, Ann. Scuola Norm. Sup. Pisa, 27 (1973), 167–185..

Manuscrit reçu le 16 avril 1999,
accepté le 26 mai 1999.

M. ABÁNADES & W. KUCHARZ,
University of New Mexico
Department of Mathematics and Statistics
Albuquerque, NM 87131-1141 (USA).
abanades@math.unm.edu
kucharz@math.unm.edu