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Algebraic equivalence of real algebraic cycles

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ALGEBRAIC EQUIVALENCE
OF REAL ALGEBRAIC CYCLES

by M. ABÁNADES & W. KUCHARZ

1. Introduction.

Let $X$ be a nonsingular, $n$-dimensional, quasiprojective variety over $\mathbb{R}$ (that is, an irreducible, $n$-dimensional, quasiprojective scheme over $\mathbb{R}$, smooth over $\mathbb{R}$). We endow the set $X(\mathbb{R})$ of $\mathbb{R}$-rational points of $X$ with the topology induced by the usual metric topology on $\mathbb{R}$, and assume that $X(\mathbb{R})$ is nonempty and compact. Thus $X(\mathbb{R})$ is a $C^\infty$, closed, $n$-dimensional manifold. Given a nonnegative integer $k$, we let $Z^k(X)$ denote the group of algebraic $(n-k)$-cycles on $X$ (that is, the free Abelian group on the set of closed, $(n-k)$-dimensional subvarieties of $X$). There exists a unique group homomorphism

$$\text{cl}_R : Z^k(X) \to H^k(X(\mathbb{R}), \mathbb{Z}/2)$$

such that for every closed, $(n-k)$-dimensional subvariety $V$ of $X$, the cohomology class $\text{cl}_R(V)$ is Poincaré dual to the homology class in $H_{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$ determined by $V(\mathbb{R})$ (cf. [5] for the definition of this homology class). In the present paper we study the cohomology classes of the form $\text{cl}_R(z)$, where $z$ is a cycle in $Z^k(X)$ algebraically equivalent to 0 (we refer to [7] for the theory of algebraic equivalence of cycles). Such cohomology classes need not be trivial, but as we shall see below they must satisfy quite restrictive conditions.

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The extreme cases, $k = 0$ and $k = n$, are easy to analyze. Obviously, a cycle $z$ in $Z^0(X)$ is algebraically equivalent to 0 if and only if $z = 0$. On the other hand, every cycle in $Z^n(X)$ of the form $x_0 - x_1$, where $x_0$ and $x_1$ are points in $X(\mathbb{R})$, is algebraically equivalent to 0. We have $\text{cl}_R(x_0 - x_1) \neq 0$ whenever $x_0$ and $x_1$ belong to distinct connected components of $X(\mathbb{R})$.

It follows that a cohomology class $u$ in $H^n(X(\mathbb{R}), \mathbb{Z}/2)$ can be written as $u = \text{cl}_R(z)$ for some cycle $z$ in $Z^n(X)$ algebraically equivalent to 0 if and only if the homology class in $H_0(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to $u$ is represented by an even number of points of $X(\mathbb{R})$. In view of these facts, we concentrate our attention on the intermediate cases, $1 \leq k \leq n - 1$.

Given a continuous map $f : M \rightarrow N$ between topological spaces, we denote by $H^k(f) : H^k(N, \mathbb{Z}/2) \rightarrow H^k(M, \mathbb{Z}/2)$ the homomorphism induced by $f$. Recall that a cohomology class $u$ in $H^k(M, \mathbb{Z}/2)$ with $k \geq 1$ is said to be spherical if $u = H^k(f)(c)$, where $f : M \rightarrow S^k$ is a continuous map into the unit $k$-sphere $S^k$, and $c$ is the generator of $H^k(S^k, \mathbb{Z}/2) \cong \mathbb{Z}/2$.

We denote, as usual, by $\cup$ and $\langle \cdot, \cdot \rangle$ the cup product of cohomology classes and the Kronecker index (pairing) of cohomology and homology classes, cf. [11]. If $M$ is a $C^\infty$, closed manifold of dimension $n$, we denote by $w_k(M)$ the $k$th Stiefel-Whitney class of $M$ and by $\mu_M$ the fundamental homology class of $M$ in $H_n(M, \mathbb{Z}/2)$.

**Theorem 1.1.** — Let $X$ be a nonsingular, $n$-dimensional, quasiprojective variety over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty and compact. Let $z$ be a cycle in $Z^k(X)$ that is algebraically equivalent to 0. Then the cohomology class $\text{cl}_R(z)$ in $H^k(X(\mathbb{R}), \mathbb{Z}/2)$ satisfies $\text{cl}_R(z) \cup \text{cl}_R(z) = 0$ in $H^{2k}(X(\mathbb{R}), \mathbb{Z}/2)$ and

$$< \text{cl}_R(z) \cup w_{i_1}(X(\mathbb{R})) \cup \ldots \cup w_{i_r}(X(\mathbb{R})), \mu_{X(\mathbb{R})} > = 0$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = n - k$. Furthermore, if $k = 1$ or if $k = n - 1 \geq 1$ with $X(\mathbb{R})$ connected, then the cohomology class $\text{cl}_R(z)$ is spherical.

Let us note that, in general, the cohomology class $\text{cl}_R(z)$ of Theorem 1.1 need not be spherical. Indeed, suppose $X = X' \times X''$ (product over $\text{Spec} \mathbb{R}$), where $X'$ and $X''$ are nonsingular, projective varieties over $\mathbb{R}$ such that $X'(\mathbb{R})$ is nonempty and $X''(\mathbb{R})$ is disconnected. Let $z'$ be any algebraic cycle on $X'$. Choose two points $p_0$ and $p_1$ in $X''(\mathbb{R})$ that belong to distinct connected components. Since the 0-cycle $z'' = p_0 - p_1$ on $X''$ is algebraically equivalent to 0, the cycle $z' \times z''$ on $X$ is algebraically equivalent to 0 as well. Furthermore, the cohomology class $\text{cl}_R(z' \times z'') = \text{cl}_R(z') \times \text{cl}_R(z'')$ is
spherical if and only if the cohomology class $cl_\mathbb{R}(z')$ is spherical (for $p_0$ and $p_1$ belong to distinct connected components of $X''(\mathbb{R})$). Taking $X' = \mathbb{P}^m_\mathbb{R}$, we have $cl_\mathbb{R}(Z^k(X')) = H^k(X'(\mathbb{R}), \mathbb{Z}/2)$, and the unique nontrivial cohomology class in $H^k(X'(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2$ is not spherical, provided that $1 \leq k \leq m-1$ and $m$ is even. In particular, "connected" cannot be omitted in the last part of Theorem 1.1.

If $cl_\mathbb{R}(z)$ is spherical, then $cl_\mathbb{R}(z) \cup cl_\mathbb{R}(z) = 0$ is automatically satisfied, and Theorem 1.1 is in some sense the best possible result. More precisely, we have the following.

**Theorem 1.2.** — Let $M$ be a $C^\infty$, closed, $n$-dimensional manifold and let $u$ be a spherical cohomology class in $H^k(M, \mathbb{Z}/2)$ with $1 \leq k \leq n-1$. Then the following conditions are equivalent:

(a) There exist a nonsingular, projective algebraic variety $X$ over $\mathbb{R}$ and a $C^\infty$ diffeomorphism $\varphi : X(\mathbb{R}) \to M$ such that $H^k(\varphi)(u) = cl_\mathbb{R}(z)$ for some cycle $z$ in $Z^k(X)$ algebraically equivalent to 0;

(b) $u \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), \mu_M \geq 0$ for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \ldots + i_r = n - k$.

Let us mention that Theorem 1.2 is an improvement upon inefficient [10], Theorem 2.4.

2. Proofs.

Let $X$ be a nonsingular, $n$-dimensional, quasiprojective algebraic variety over $\mathbb{R}$ with $X(\mathbb{R})$ nonempty and compact. Recall that if an algebraic cycle $z$ in $Z^k(X)$ is rationally equivalent to 0, then $cl_\mathbb{R}(z) = 0$ (cf. [5], 5.13) and hence $cl_\mathbb{R}$ induces a homomorphism, also denoted by $cl_\mathbb{R}$, from the Chow group $A^k(X)$ of $X$ into $H^k(X(\mathbb{R}), \mathbb{Z}/2)$. It is known that $cl_\mathbb{R} : A^*(X) \to H^*(X(\mathbb{R}), \mathbb{Z}/2)$ is a homomorphism of graded rings [5], p. 495. Thus

$$H^*_\text{alg}(X(\mathbb{R}), \mathbb{Z}/2) = cl_\mathbb{R}(Z^*(X)) = cl_\mathbb{R}(A^*(X))$$

is a graded subring of $H^*(X(\mathbb{R}), \mathbb{Z}/2)$. We shall need the following result [10], Theorem 2.1:

(1) $< cl_\mathbb{R}(z) \cup v, \mu_X(\mathbb{R}) > = 0$

for all cycles $z$ in $Z^k(X)$ algebraically equivalent to 0 and all $v$ in $H^{n-k}_\text{alg}(X(\mathbb{R}), \mathbb{Z}/2)$.
Assume now that $X$ is projective. Then the set $X(\mathbb{C})$ of $\mathbb{C}$-rational points of $X$ is a compact complex manifold of complex dimension $n$. There exists a unique group homomorphism

$$\text{cl}_C : Z^k(X) \to H^{2k}(X(\mathbb{C}), \mathbb{Z})$$

such that for every closed, $(n - k)$-dimensional subvariety $V$ of $X$, the cohomology class $\text{cl}_C(V)$ is Poincaré dual to the homology class in $H_{2n-2k}(X(\mathbb{C}), \mathbb{Z})$ determined by $V(\mathbb{C})$ (cf. [5] for the definition of this homology class). In other words, if $\pi : X_C = X \times_{\text{Spec} \mathbb{R}} \text{Spec} \mathbb{C} \to X$ is the canonical projection, then $\text{cl}_C(z)$ is the cohomology class corresponding to the pullback algebraic cycle $\pi^*(z)$ on $X_C$, cf. [5], 4.2 or [7], Chapter 19. In particular,

$$(2) \quad \text{cl}_C(z) = 0$$

for all cycles $z$ in $Z^k(X)$ algebraically equivalent to 0, cf. [5], 4.14 or [7], Proposition 19.1.1. Furthermore, it follows from the proof of [2], Theorem A that

$$(3) \quad \text{cl}_R(z) \cup \text{cl}_R(z) = \text{the reduction modulo 2 of } r(\text{cl}_C(z))$$

for all $z$ in $Z^k(X)$, where $r : H^{2k}(X(\mathbb{C}), \mathbb{Z}) \to H^{2k}(X(\mathbb{R}), \mathbb{Z})$ is the homomorphism induced by the inclusion map $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$.

**Proof of Theorem 1.1.** — By Hironaka’s resolution of singularities theorem [8], 3, we may assume that $X$ is projective.

We obtain $\text{cl}_R(z) \cup \text{cl}_R(z) = 0$ directly from (2) and (3).

It follows from [5], p. 498 that $w_i(X(\mathbb{R}))$ is in $H_{\text{alg}}^i(X(\mathbb{R}), \mathbb{Z}/2)$, and hence if $i_1, \ldots, i_r$ are nonnegative integers with $i_1 + \cdots + i_r = n - k$, then the cohomology class

$$v = w_{i_1}(X(\mathbb{R})) \cup \ldots \cup w_{i_r}(X(\mathbb{R}))$$

belongs to $H_{\text{alg}}^{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$. In view of (1), we have $\langle \text{cl}_R(z) \cup v, \mu_{X(\mathbb{R})} \rangle = 0$, which completes the proof of the first part of the theorem.

Given an invertible sheaf $\mathcal{L}$ on $X$, we denote by $\mathcal{L}_R$ (resp. $\mathcal{L}_C$) the topological real (resp. complex) line bundle on $X(\mathbb{R})$ (resp. $X(\mathbb{C})$) determined by $\mathcal{L}$ in the usual way. If $\mathcal{L}$ corresponds to a Weil divisor $D$ on $X$, then

$$w_1(\mathcal{L}_R) = \text{cl}_R(D) \quad \text{and} \quad c_1(\mathcal{L}_C) = \text{cl}_C(D),$$

where $w_1(-)$ and $c_1(-)$ stand for the first Stiefel-Whitney class and the first Chern class, respectively, cf. [5], p. 498, p. 489. Note that the restriction $\mathcal{L}_C|X(\mathbb{R})$ of $\mathcal{L}_C$ to $X(\mathbb{R})$ is the complexification of $\mathcal{L}_R$, and hence

$$c_1(\mathcal{L}_C|X(\mathbb{R})) = \beta(w_1(\mathcal{L}_R)), $$
where $\beta : H^1(X(\mathbb{R}), \mathbb{Z}/2) \to H^2(X(\mathbb{R}), \mathbb{Z})$ is the Bockstein homomorphism that appears in the long exact sequence

$$\ldots \to H^1(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^1(X(\mathbb{R}), \mathbb{Z}) \to H^1(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^2(X(\mathbb{R}), \mathbb{Z}) \to \ldots,$$

cf. [11], Problems 15-C and D. The last equality can be written in an equivalent form

$$r(cl_{C}(D)) = \beta(cl_{R}(D)), \quad (4)$$

where $r : H^2(X(\mathbb{C}), \mathbb{Z}) \to H^2(X(\mathbb{R}), \mathbb{Z})$ is the homomorphism induced by the inclusion map $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$.

Suppose now that $k = 1$, that is, $z$ is a Weil divisor on $X$. By (2) and (4), we have $\beta(cl_{R}(z)) = 0$, which means that $cl_{R}(z)$ is the reduction modulo 2 of a cohomology class in $H^1(X(\mathbb{R}), \mathbb{Z})$. This last fact implies that the cohomology class $cl_{R}(z)$ is spherical, cf. [9], p. 49.

Let us now assume that $k = n - 1 \geq 1$ and $X(\mathbb{R})$ is connected. We already know that $< cl_{R}(z) \cup w_{1}(X(\mathbb{R})), \mu_{X(\mathbb{R})} > = 0$, which in view of the connectedness of $X(\mathbb{R})$ is equivalent to $cl_{R}(z) \cup w_{1}(X(\mathbb{R})) = 0$. The last condition implies that the homology class in $H_{1}(X(\mathbb{R}), \mathbb{Z}/2)$ Poincaré dual to $cl_{R}(z)$ can be represented by a $C^\infty$, closed curve in $X(\mathbb{R})$, with trivial normal vector bundle, cf. for example [4], p. 599. This in turn implies that $cl_{R}(z)$ is spherical, cf. [12], Théorème II.1. Thus the proof is complete. \Box

Proof of Theorem 1.2. — By Theorem 1.1, (a) implies (b), and we show below that (b) implies (a).

Choose a nonsingular, irreducible algebraic subset $W$ of $\mathbb{R}^{k+1}$, which has precisely two connected components $W_{0}$ and $W_{1}$, each diffeomorphic to the unit $k$-sphere $S^{k}$ (for example, $W = \{(x_{1}, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_{1}^{4} - 4x_{1}^{3} + 1 + x_{2}^{2} + \cdots + x_{k+1}^{2} = 0\}$). Let $c$ be the unique generator of the group $H^{k}(W_{0}, \mathbb{Z}/2) \cong \mathbb{Z}/2$, viewed as a subgroup of $H^{k}(W, \mathbb{Z}/2)$. Since the cohomology class $u$ is spherical, there exists a $C^{\infty}$ map $h : M \to W$ such that $h(M) \subseteq W_{0}$ and $u = H^{k}(h)(c)$. Choose a regular value $y_{0}$ of $h$ in $W_{0}$. Then $u$ is Poincaré dual to the homology class in $H_{n-k}(M, \mathbb{Z}/2)$ represented by the $C^{\infty}$ submanifold $h^{-1}(y_{0})$ of $M$, cf. [5], 2.15. Clearly, there exists a unique $C^{\infty}$ map $f : M \to W$ such that for every connected component $S$ of $M$ and every point $x$ in $S$, we have $f(x) = h(x)$ if $S \cap h^{-1}(y_{0}) \neq \emptyset$ and $f(x) = y_{0}$ if $S \cap h^{-1}(y_{0}) = \emptyset$. The map $f$ satisfies

$$f(M) \subseteq W_{0} \text{ and } u = H^{k}(f)(c). \quad (5)$$
Furthermore, each connected component of $f^{-1}(y_0)$ is a $C^\infty$ submanifold of $M$ of dimension either $n - k$ or $n$. Also, each connected component of $M$ contains a connected component of $f^{-1}(y_0)$. Since $n - k \geq 1$, we can find a $C^\infty$ closed curve $C$ in $M$ such that

$$f(C) = \{y_0\},$$

the normal vector bundle of $C$ in $M$ is trivial, and each connected component of $M$ contains a connected component of $C$. Choose an integer $d$ with $2n + 1 \leq d$ and let $D$ be a compact, nonsingular, irreducible, 1-dimensional algebraic subset of $\mathbb{R}^d$ that has the same number of connected components as $C$. Replacing $M$ by its image under a suitable $C^\infty$ embedding into $\mathbb{R}^d$, we may assume that

$$D = C \subseteq M \subseteq \mathbb{R}^d.$$

By Tognoli's theorem [13] or [1], Corollary 2.8.6, there exists a nonsingular real algebraic subset $A$ of $\mathbb{R}^p$, for some $p$, diffeomorphic to $M$. Consider the disjoint union $\tilde{N} = M \amalg A$ and the $C^\infty$ map $F : \tilde{N} \to W$ defined by $F(x) = f(x)$ for $x$ in $M$ and $F(x) = y_0$ for $x$ in $A$. We assert that if $w$ is a cohomology class in $H^\ell(W,\mathbb{Z}/2)$ and if $j_1, \ldots, j_s$ are nonnegative integers with $j_1 + \cdots + j_s = n - \ell$, then

$$< H^\ell(F)(w) \cup w_{j_1}(N) \cup \ldots \cup w_{j_s}(N), \mu_N > = 0.$$

Indeed, first note that $w = 0$, unless $\ell = 0$ or $\ell = k$. If $\ell = 0$, then either $H^0(F)(w) = 0$ or $H^0(F)(w) = 1$. In the latter case the assertion holds since $M$ and $A$ are diffeomorphic. If $\ell = k$, then either $H^k(F)(w) = 0$ or $H^k(F)(w) = u$ (we view $H^k(M,\mathbb{Z}/2)$ as a subgroup of $H^k(N,\mathbb{Z}/2)$). In the latter case the assertion follows from condition (b). Thus the assertion is proved. It implies that there exist a $C^\infty$ compact manifold $\tilde{N}$ with boundary $\partial \tilde{N} = N$ and a $C^\infty$ map $\tilde{F} : \tilde{N} \to W$ satisfying $\tilde{F}|N = F$, cf. [6], 17.3.

In other words, the map $f : M \to W$ and the constant map $A \to y_0$, which sends $A$ to $y_0$, represent the same class in the unoriented bordism group of $W$. By construction, the normal bundle of $D$ in $M$ is trivial, so the restriction $\nu|D$ to $D$ of the normal bundle $\nu$ of $M$ in $\mathbb{R}^d$ admits an algebraic structure. Therefore, by [1], Theorem 2.8.4 and in view of (6) and (7), one can find a nonnegative integer $e$, a nonsingular algebraic subset $V$ of $\mathbb{R}^d \times \mathbb{R}^c$, a $C^\infty$ diffeomorphism $\varphi : V \to M$, and a regular map $g : V \to W$ (the latter designates the restriction to $V$ of a rational map from $\mathbb{R}^d \times \mathbb{R}^c$ into $\mathbb{R}^{k+1}$ which has no poles on $V$ and maps $V$ into $W$) such that $g$ is homotopic to $f \circ \varphi$ and $D \times \{0\} \subseteq V$. Since $D$ is irreducible and each connected component of $V$ contains a connected component of
Irreducibility of $V$ and $W$ allows us to choose nonsingular, quasiprojective varieties $T$ and $Y$ over $\mathbb{R}$ with $T(\mathbb{R}) = V$ and $Y(\mathbb{R}) = W$. By Hironaka's resolution of singularities theorem [8], 3, we may assume that $T$ and $Y$ are projective (and still nonsingular). Let $\tilde{g} : U \to Y$ be an algebraic morphism over $\mathbb{R}$, defined on a Zariski open neighborhood of $T(\mathbb{R}) = V$ in $T$, such that $\tilde{g}|T(\mathbb{R}) = g$. By applying Hironaka's theorem on removing points of indeterminacy [8], 3, we can find a nonsingular, projective algebraic variety $X$ over $\mathbb{R}$ and an algebraic morphism $G : X \to Y$ over $\mathbb{R}$ satisfying $X(\mathbb{R}) = T(\mathbb{R})$ and $G|X(\mathbb{R}) = g$.

Let $y_1$ be a point in $W_1$ and let $\beta$ be the class in $A^k(Y)$ of the 0-cycle $y_0 - y_1$ on $Y$. By (5), $u = H^k(f)(cl_{\mathbb{R}}(\beta))$ (although, of course, $cl_{\mathbb{R}}(\beta) \neq c$).

Since $G|X(\mathbb{R}) = g$ is homotopic to $f \circ \varphi$, we obtain

\[
H^k(\varphi)(u) = H^k(\varphi)(H^k(f)(cl_{\mathbb{R}}(\beta))) = H^k(f \circ \varphi)(cl_{\mathbb{R}}(\beta)) = H^k(G|X(\mathbb{R}))(cl_{\mathbb{R}}(\beta)) = cl_{\mathbb{R}}(G^*(\beta)),
\]

where the last equality is a consequence of the functorial property of $cl_{\mathbb{R}} : A^* \to H^*$, cf. [5], 5.12. Let $z$ be a cycle in $Z^k(X)$ that represents in the Chow group $A^k(X)$ the pullback class $G^*(\beta)$. Then $cl_{\mathbb{R}}(z) = cl_{\mathbb{R}}(G^*(\beta)) = H^k(\varphi)(u)$. The proof is now complete since the cycle $y_0 - y_1$ is algebraically equivalent to 0 on $Y$ and hence the cycle $z$ is algebraically equivalent to 0 on $X$.

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