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THE MILGRAM NON-OPERAD

by Michael BRINKMEIER

Introduction.

R.J. Milgram introduced in [10] geometric models J^n for the iterated loop-space operads $\Omega^n\Sigma^n$. Later J.M. Boardman, R.M. Vogt and P. May proved that n-fold loop spaces are closely related to E_n operads in general, and thus to the little cube operads (see [5], [6], [8]). Hence the question arises if the operad structure of $\Omega^n\Sigma^n$ translates to the geometric model J^n.

C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R.M. Vogt construct in [1] an operad $\mathcal{M}_n$, which codifies n-fold monoidal categories, a categorial analog of n-fold loop spaces. They observe that an equivalent *preoperad* is embedded in $\mathcal{M}_n$, whose free space is of the same homotopy type as J^nX. For $n = 2$ the spaces are even homeomorphic (see [1], §§ 13.12–13.14]).

Due to the underlying polytopes, the permutohedra, the Milgram-construction J^n is of some importance for the examination of coherent homotopy-commutativity. Similar to the associahedra introduced by Stasheff in [11], Williams uses the permutohedra and the Milgram-construction in [12] to define his notion of C_n-spaces, which is used in several subsequent papers, and is occuring in papers of McGibbon and Hemmi (see for example [9] and [7]).

In fact there exists an operad structure with the permutohedra as underlying spaces (this was pointed out to me by Clemens Berger and Zig Fiedorowicz), simply by using the convex extension of the permutation operad. But since the permutohedra are contractible this is an E_∞ operad in the sense of Boardman and Vogt (*i.e.* the symmetric group action does

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not need to be free). Therefore an algebra of this operad is homotopy equivalent to an infinite loop space and hence its associated monad can not be of the same homotopy type as the Milgram construction J^n which is just an n-fold loop space.

C. Berger conjectured in [3] and [4] an E_n operad structure of the permutohedra, whose associated monad is the Milgram construction J^n.

I will show that Berger's construction does not work. The first observation is that the would-be operad bears a structure far too strong, namely that of strictly abelian monoids. This collapse of structure is then used to show that the suggested structure does not define an operad at all. In fact the proof shows that the multiplication defined by Berger does not respect the degeneration conditions.

Nonetheless the construction defines preoperads $J^{(n)}$, which are homotopy equivalent to the little n-cubes. In particular, the k-th space $J_k^{(n)}$ is Σ_k-equivariantly homotopy equivalent to the real configuration space $F(\mathbb{R}^n, k)$. Furthermore the "non-monad" associated to the preoperad $J^{(n)}$ is an alternative description of the Milgram construction J_n.

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1. The permutohedra and Berger's construction.

Let Λ be the category of finite sets $\mathbf{n} = \{1, \dots, n\}$ and injective maps. Each injective map $\varphi : \mathbf{n} \to \mathbf{m}$ has an unique decomposition of the form $\varphi = \varphi^{\mathrm{inc}} \circ \varphi^{\natural}$, such that $\varphi^{\mathrm{inc}} : \mathbf{n} \to \mathbf{m}$ is increasing and $\varphi^{\natural} : \mathbf{n} \to \mathbf{n}$ is a permutation.

Definition 1.1. — *The permutation preoperad $\Sigma : \Lambda^{\mathrm{op}} \to \mathbf{Top}$ is the functor with $\Sigma(\mathbf{n}) = \Sigma_n$ and $\varphi^* := \Sigma(\varphi) : \Sigma_m \to \Sigma_n$ given by $\sigma \mapsto (\sigma \circ \varphi)^{\natural}$ for $\varphi \in \Lambda(\mathbf{n}, \mathbf{m})$.*

The multiplications of the permutation operad Σ are given by

$$\begin{aligned}\gamma^{\Sigma}_{n;i_1,\dots,i_n} : \Sigma_n \times \Sigma_{i_1} \times \cdots \times \Sigma_{i_n} &\longrightarrow \Sigma_{i_1+\cdots+i_n}\\ (\sigma; \tau_1, \dots, \tau_n) &\longmapsto \sigma(i_1, \dots, i_n) \circ (\tau_1 \oplus \cdots \oplus \tau_n)\end{aligned}$$

where $\sigma(i_1, \ldots, i_n)$ permutes the blocks (see [4], 1.15 (a)) and $- \oplus -$: $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ is the canonical product of permutations.

The product $\Sigma_{i_1} \oplus \cdots \oplus \Sigma_{i_n} \subset \Sigma_{i_1+\cdots+i_n}$ will be denoted with $\Sigma(i_1, \ldots, i_n)$.

For more details about (pre)operads in general the reader is referred to [3], [4] or [8].

In contrary to Clemens Berger I will use the left action of the symmetric group on $\mathbb{R}^n$, which seems to be the more common description.

DEFINITION 1.2. — *For $n \geq 1$ the symmetric group Σ_n acts on $\mathbb{R}^n$ from the left by permuting the coordinates in the following way:*

$$\sigma(x_1, \ldots, x_n) = \big(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\big).$$

The n-th permutohedron $P_n \subset \mathbb{R}^n$ is the convex hull of the orbit of $(1, 2, \ldots, n) \in \mathbb{R}^n$ under this operation.

The convex hull of the orbit of $\Sigma(i_1, \ldots, i_r) \subset \Sigma_n$ will be denoted with $P(i_1, \ldots, i_r) \subset P_n$. The point $\sigma(1, \ldots, n) \in P_n$ will be denoted with σ.

Remark 1.3. — The notation of σ for the point $\sigma(1, \ldots, n) = \big(\sigma^{-1}(1), \ldots, \sigma^{-1}(n)\big)$ seems somewhat confusing. But since we will extend the permutation operad to the permutohedra, we can calculate the vertices right from the permutations, without applying it to the $\mathbb{R}^n$.

The geometric and simplicial properties of these polygons were examined in [10], [12] and [2]. Here I will give only a rough sketch of the few details I will use.

P_1 consists of only one point, P_2 is homeomorphic to the unit interval in $\mathbb{R}$ and P_3 to the hexagon in $\mathbb{R}^2$. In general P_n is a $(n-1)$-dimensional polytope.

Obviously there exists a left Σ_n-action on P_n. The vertices are mapped to vertices, and for each $\sigma \in \Sigma_n$ the map $P_n \to P_n$ with $x \mapsto \sigma x$ is a homeomorphism. But unfortunately this action is not free, since the barycenter of each permutohedron is a fixed point.

An arbitrary point $x \in P_n$ will be denoted by a linear combination of permutations

$$x = \sum_{\sigma \in \Sigma_n} t_\sigma \sigma \quad \text{with} \quad \sum_{\sigma \in \Sigma_n} t_\sigma = 1.$$

If $x = \sum_{\sigma\in\Sigma_n} s_\sigma \sigma$ and $y = \sum_{\tau\in\Sigma_m} t_\tau \tau$ are points of P_n and P_m respectively, the point $x \oplus y \in P_{n+m}$ is given by

$$x \oplus y = \sum_{\sigma\in\Sigma_n} \sum_{\tau\in\Sigma_m} s_\sigma t_\tau \sigma \oplus \tau.$$

Milgram defined in [10] maps $I_k : P_k \times P_{n-k} \to P_n$ given by $(x, y) \mapsto x \oplus y$ mapping the product of two permutohedra into certain faces of a higher dimensional permutohedron. More general the codimension $(r-1)$-faces of P_n are in one-to-one correspondence with the ordered partitions of $\{1, \ldots, n\}$ of type $(i_1, \ldots, i_r)$ with $i_1 + \cdots + i_r = n, i_k \geq 1$.

Remark 1.4. — Each partition of type $(i_1, \ldots, i_r)$ can be interpreted as a permutation of $i_1 + \cdots + i_r$ elements. If the classes are given by

$$\{j_1, \ldots, j_{i_1}\}, \{j_{i_1+1}, \ldots, j_{i_1+i_2}\}, \ldots, \{j_{i_1+\cdots+i_{r-1}+1}, \ldots, j_{i_1+\cdots+i_r}\}$$

with $j_k < j_\ell$ for $i_m \leq k < \ell < i_{m+1}$, then the corresponding permutation is given by $k \mapsto j_k$ for $1 \leq k \leq i_1 + \cdots + i_r$.

The converse does not hold, since the same permutation can be associated to different partitions. For example the identity in Σ_3 corresponds to the partion $\{1\}, \{2, 3\}$ and to $\{1, 2\}, \{3\}$.

The vertices of the codimension $(r-1)$-face, corresponding to a partition of type $(i_1, \ldots, i_r)$, with associated permutation $\sigma \in \Sigma_n$, are given by the coset $\sigma\Sigma(i_1, \ldots, i_r)$. In addition there is a homeomorphism $I_\sigma : P_{i_1} \times \cdots \times P_{i_r} \to \sigma P(i_1, \ldots, i_r) \subset P_n$ with

$$(x_1, \ldots, x_r) \longmapsto \sigma(x_1 \oplus \cdots \oplus x_r).$$

Using the *(right) weak Bruhat order* on the symmetric groups, the 1-skeleton of these faces can be oriented.

Definition 1.5. — *The inversion index* $\mathrm{inv}(\sigma)$ *of a permutation* $\sigma \in \Sigma_n$ *is the number of ordered pairs* (i, j), $1 \leq i < j \leq n$, *whose orders are inverted by* σ, *i.e.* $\sigma(i) > \sigma(j)$.

The (*right*) *weak Bruhat order of* Σ_n *is the partial order generated by* $\sigma < \tau$, *if* τ *is the composition of* σ *and a transposition of two subsequent numbers, that is* $\tau = \sigma \circ (i, i+1)$, *and* $\mathrm{inv}(\sigma) < \mathrm{inv}(\tau)$.

Remark 1.6. — Since we use the left action of Σ_n on $\mathbb{R}^n$, we have to use the right weak Bruhat order, instead of the left weak Bruhat order, as Clemens Berger did.

Example 1.7. — Σ_3 is given by the poset

$$\begin{tikzcd}
 & (123) \arrow[dl] \arrow[dr] & \\
(213) \arrow[d] & & (132) \arrow[d] \\
(231) \arrow[dr] & & (312) \arrow[dl] \\
 & (321) &
\end{tikzcd}$$

Here and in the following the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$$

will be denoted with $(\sigma(1), \sigma(2), \dots, \sigma(n))$ (the commas will be left, if unnecessary).

Remark 1.8. — If the vertices of the poset (*i.e.* the permutations) are interpreted as points in $\mathbb{R}^3$, it does not seem to describe the border of P_3 – the points $(2,3,1)$ and $(3,1,2)$ have to be exchanged. But since we use the left action of Σ_n on $\mathbb{R}^n$, the permutation (231) corresponds to the point $(3,1,2)$ and the permutation (312) to the point $(2,3,1)$. In fact the correspondence holds for all vertices of P_3 and the poset Σ_3.

Applying this partial order to the permutohedra the edges will be oriented. The face corresponding to a certain partition of type $(i_1, \dots, i_r)$ and associated permutation σ has exactly one initial vertex, given by σ and a unique terminal vertex, given by the permutation that turns the classes of the partition "upside-down", *i.e.*

$$\ell \longmapsto k_{\ell'} \quad \text{with} \quad \ell' = 2(i_1 + \dots + i_\ell) + i_{\ell+1} - \ell + 1 \quad \text{if} \quad i_\ell < \ell \leq i_{\ell+1}.$$

DEFINITION 1.9. — *An interval $[\sigma_1, \sigma_2]$ in Σ_n with the weak Bruhat order is called admissible, if it is the vertex set of some coset $\sigma\Sigma(i_1, \dots, i_r)$. Hence σ_1 is the initial and σ_2 the terminal vertex of a face of P_n.*

The geodesic of an arbitrary interval $[\sigma_1, \sigma_2]$ of the weak Bruhat order in Σ_n, is the average of all oriented edge-paths between σ_1 and σ_2 in P_n.

The barycenter of $[\sigma_1, \sigma_2]$ is the barycenter of its geodesic.

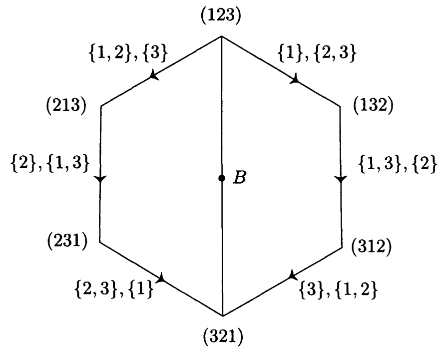

Figure 1. The geodesic and barycenter of P_3

The barycenter of an admissible interval coincides with the barycenter of the corresponding cell. For example the geodesic of $[(123),(213)] \subset \Sigma_3$ is the corresponding edge of P_3, and its barycenter the barycenter of the interval. The geodesic of $[(123),(321)] \subset \Sigma_3$ is the line between (123) and (321) and its barycenter is the point B. The geodesic of the non-admissible intervall $[(123),(231)] \subset \Sigma_3$ consists of the two edges $[(123),(213)]$ and $[(213),(231)]$ of P_3. Hence its barycenter is the point $(213) \in P_3$ (see Fig. 1).

To define an operad-multiplication on the permutohedra we have to define maps $P_r \times P_{i_1} \times \cdots \times P_{i_r} \to P_{i_1+\cdots+i_r}$ which satisfy the associativity conditions. But the intention to formulate an operad, whose associated monad is the Milgram construction gives certain additional restrictions to the choice of the multiplication.

The operad structure has to extend the permutation operad Σ. This can be done very easily by mapping the vertices of the product $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ to the corresponding vertices of $P_{i_1+\cdots+i_r}$, given by Σ. But the extension of this map can be done in two different ways.

The first possibility is the *convex* extension of the permutation operad, by mapping $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ to the convex hull of the image vertices (see Fig. 2). This construction does in fact define an operad whose n-th space is P_n. It is E_∞ in the sense of Boardman and Vogt, *i.e.* its underlying spaces are contractible but the actions of the symmetric groups does not need to be free. Thus its algebras are homotopy equivalent to infinte loop spaces (cmp. [6], § VI.3). But since the Milgram construction is only a model for n-fold loop spaces the associated monad of the convex extension is of the wrong homotopy type.

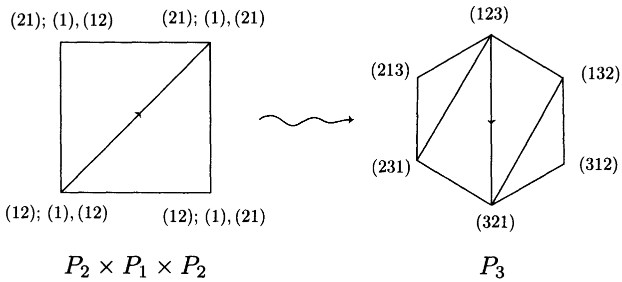

Figure 2. The convex multiplication

Berger tried to get the correct operad by application of two changes to the convex extension of the permutation operad. First he deformed the multiplication such that it respects the relations on the borders of the permutohedra, given in Milgrams construction. But this does not affect the homotopy type of the spaces. In a second step he made the action of the symmetric group free, which would give the correct homotopy type.

In the Milgram construction the border of $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ has to be mapped to the border of $P_{i_1+\cdots+i_r}$. Berger did this by using a *cubical* extension of the symmetric operad instead of the convex one. He defined the operad-multiplication $\gamma^P_{n;i_1,\ldots,i_n} : P_n \times P_{i_1} \times \cdots \times P_{i_n} \to P_{i_1+\cdots+P_{i_n}}$ as the affine extension of the permutation operad, such that the barycenter of any interval in $\Sigma_{i_1+\cdots+i_n}$ is mapped to the barycenter of its geodesics in $P_{i_1+\cdots+i_n}$ (see Fig. 3). Hence for $n = 2$ and $i_1 + i_2 = 3$ the squares A, B, C and D of $P_2 \times P_1 \times P_2$, resp. $P_2 \times P_2 \times P_1$ are mapped to the corresponding segments of P_3.

The second step, in which the symmetric group action is made free, is done in the definition of the would-be operad. For each permutation in Σ_n a copy of P^k_n is added and an appropriate quotient of the space $P^k_n \times \Sigma_n$ is taken to be the n-th space of the new E_k operad.

Here I will only describe the suggested construction for the operad $J^{(2)}$ corresponding to J_2, *i.e.* 2-fold loop spaces.

Definition 1.10 (cmp. [4], § 2.12). — *Let $J^{(2)}_n$ be the quotient space of $P_n \times \Sigma_n$ under the relation*

$$(\tau x; \sigma) \sim (x; \tau\sigma)$$

for any partition $\tau \in \Sigma_n$ of type $(i_1, \ldots, i_r), i_1 + \cdots + i_r = n$, where $x \in P\,(i_1, \ldots, i_r)$ and $\sigma \in \Sigma_n$.

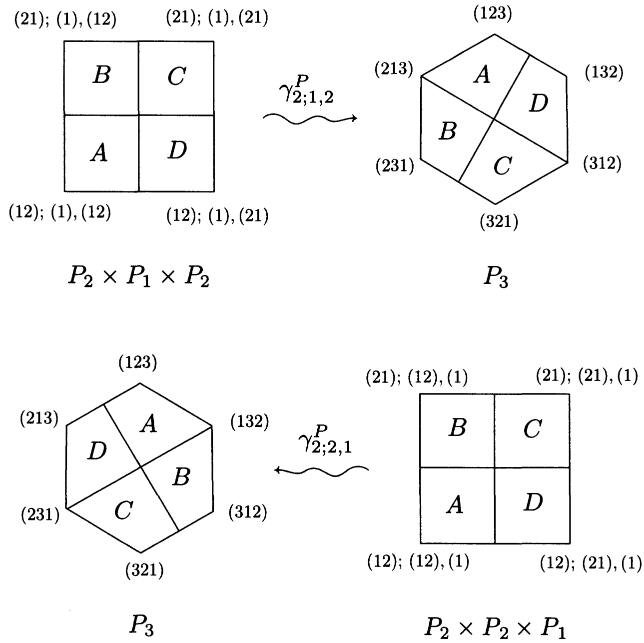

Figure 3. The cubical multiplication

The action of $\varphi \in \Lambda(\mathbf{n}, \mathbf{n})$ is induced by

$$\begin{aligned} \varphi^* : P_m \times \Sigma_m &\longrightarrow P_n \times \Sigma_n \\ (x;\sigma) &\longmapsto \big(x(\sigma\varphi)^{\mathrm{inc}};(\sigma\varphi)^{\natural}\big). \end{aligned}$$

Remark 1.11. — Since I use the left instead of the right action, the relation given by Berger has to be changed slightly.

Following [4], § 2.14, the Λ structure and the maps $\gamma^P_{i_1,\dots,i_r} \times \gamma^\Sigma_{i_1,\dots,i_r}$ induce an E_2-operad structure on $J_n^{(2)}$.

2. The commutativity of Berger's construction.

In the following we assume that Berger's construction defines an operad. If $(X, *)$ is a $J^{(2)}$-algebra, there exist maps $F_n : P_n \times \Sigma_n \times X^n \to X$.

These fulfil certain conditions, induced by the operad structure and the relations on $J_n^{(2)}$. Used here are

1) the associativity condition

$$F_n\big(s;F_{i_1}=(t_1;x_1^1,\dots,x_{i_1}^1),\dots,F_{i_n}(t_n;x_1^n,\dots,x_{i_n}^n)\big)$$
$$=F_{i_1+\dots+i_n}\big(\gamma_{n;i_1,\dots,i_n}(s;t_1,\dots,t_n);x_1^1,\dots x_{i_n}^n\big);$$

2) the degeneration relations on the associated monad, induced by the two maps $\mathbf{1}\longrightarrow\mathbf{2}$, $F_2(s;x,*)=F_1(*;x)=x=F_2(s;*,x)$;

3) and the Σ_n-equivariance

$$F_n(s\sigma;x_1,\dots,x_n)=F_n\big(s;x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n)}\big).$$

Remark 2.1. — Since only the identities of Σ_n are used, in the future this coordinate will be dropped.

There is a multiplication on X, given by $xy=F_2\big((12);x,y\big)$. The associativity of the permutation operad shows the associativity of this multiplication. Since the degeneration relations hold $*$ is a unit. Therefore X is a associative monoid with strict unit.

Obviously the map $\phi\colon I\longrightarrow P_2$ with $t\mapsto(12)(1-t)+(21)t$ is a homeomorphism. Thus there exists a homotopy $h_t\colon X\times X\longrightarrow X$ with

$$h_t(x,y)=F_2\big((12)(t-1)+(21)t;x,y\big),$$

running from $h_0(x,y)=xy$ to

$$h_1(x,y)=F_2\big((21);x,y)\big)=F_2\big((12);y,x\big)=yx.$$

Via F_3 and the degenerations one gets even stricter conditions for the commuting homotopy h_t. The maps $\gamma^P_{2;1,2}$ and $\gamma^P_{2;2,1}$ are homeomorphisms. Therefore there exist $s_i,t_i\in P_2, i=1,2$ for each $r\in P_3$ such that

$$\gamma^P_{2:2,1}\big(s_2;t_2,(1)\big)=r=\gamma^P_{2;1,2}\big(s_1;(1),t_1\big).$$

Hence for $x,y\in X$ one gets

$$F_2(s_2;x,y)=F_3(r;*,x,y)=F_2(t_1;x,y).$$

In the first case the homotopy h_t is mapped to the edges $[(123),(132)]$ and $[(231),(321)]$ (cmp. Fig. 4). In the second case h_t is mapped to the geodesics of the intervals $[(123),(312)]$ and $[(213),(321)]$, such that the center point of the homotopy $h_{\frac{1}{2}}(x,y)$ is mapped to (132) and (231). Thus the homotopy needs to be equal to xy on its first half and equal to yx on the second half (in the first case the edge $[(213),(231)]$ is mapped to xy and the edge $[(132),(312)]$ to yx). Thus $(X,*)$ must be an abelian monoid.

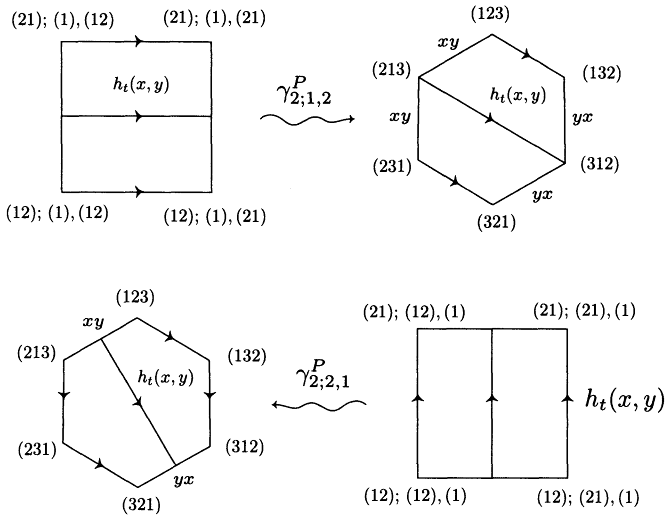

Figure 4. The commuting homotopy

Remark 2.2. — Since the permutation coordinate wasn't used, the freeness of the symmetric group action is not involved in the failure of the suggested construction. In fact the cubical extension of the multiplication does not fulfil the needed degeneration properties.

Now let X be a 2-connected CW-complex with non-degenerate base point $*$ (*i.e.* the inclusion $* \hookrightarrow X$ is a closed cofibration). Then the two-fold Moore loop space $Y\colon = \Omega^2_M X$ of X is a connected CW-complex. The canonical evaluation map $e\colon \Sigma^2\Omega^2_M X \to X$ induces a homomorphism of monoids $\Omega^2_M e\colon \Omega^2_M\Sigma^2 Y \to Y$.

If $J^{(2)}$ is an E_2 operad, whose associated monad is the Milgram construction, there exists a homomorphism of monoids $\psi\colon J^{(2)}Y \to \Omega^2_M\Sigma^2 Y$ (the map is given in [10], § 5.2). Therefore we get a homomorphism $\varphi\colon = \Omega^2_M e \circ \psi\colon J^{(2)}Y \to Y$. From the construction of ψ in [10] one can see that the diagram

$$\begin{tikzcd} J^{(2)}Y \arrow[r, "\psi"] & \Omega^2_M\Sigma^2 Y \arrow[r, "\Omega^2_M e"] & Y \\ & Y \arrow[ul, hook] \arrow[u, hook] \arrow[ur, equal] & \end{tikzcd}$$

with the inclusion $i: Y \hookrightarrow \Omega^2_M \Sigma^2 Y$ given by

$$\big(i(y)\big)(s,t) = y \wedge s \wedge t,$$

commutes. Therefore the homomorphism $\varphi = \Omega^2_M \psi$ is an extension of the identity and hence surjective.

Since $J^{(2)}Y$ is the free $J^{(2)}$-algebra of the space Y, it is an abelian monoid. The surjectivity of φ now shows that $Y = \Omega^2_M X$ is strictly commutative, too. But obviously this is wrong. Therefore $J^{(2)}$ can not be an operad whose associated monad is the Milgram construction.

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