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A WEIGHTED GRAPH POLYNOMIAL FROM CHROMATIC INVARIANTS OF KNOTS

by S.D. NOBLE and D.J.A. WELSH (1)

1. Introduction.

In a series of three papers [2], [3], [4], Chmutov, Duzhin and Lando examine the combinatorial aspects of the relationship between chord diagrams and Vassiliev invariants of knots.

A key concept in the treatment in [4] is the homomorphism associating a chord diagram with its intersection graph, and this turns out to be just a map on weighted graphs modulo a weighted chromatic relation. This is essentially the same as the study of a chromatic type polynomial on weighted graphs. However, as the authors point out, the treatment given in [4] does not allow an extension to more general Tutte Grothendieck invariants.

This paper originated in an attempt to clarify the reasons for this and we show that a fairly simple modification of the definition of [4] does allow such an extension. In a sense this is not too surprising in view of the established relation between Tutte polynomials and knot polynomials, see for example Jaeger [8] or Jaeger, Vertigan and Welsh [9]. Here we define a polynomial of weighted graphs which contains the invariants of [4] as a specialisation but moreover has a wide range of other specialisations in combinatorics. These include the Tutte polynomial, stability polynomial and matching polynomial of ordinary graphs and a polymatroid polynomial studied by Oxley and Whittle [16].

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The plan of this paper is as follows. In Section 2 we introduce weighted graphs and our weighted polynomial W. We then give a brief overview of the relationship between Vassiliev invariants, chord diagrams and weight functions, and explain how they are linked with our polynomial W. In Section 4 we study the basic properties of W. Then in Section 5 we study it as an invariant of ordinary graphs and in Section 6 relate this specialisation to a symmetric function generalisation of the chromatic polynomial introduced by Stanley [18]. We close with a discussion of complexity issues.

2. A new polynomial on weighted graphs.

Our notation is fairly standard except that our graphs are allowed to have loops and parallel edges. $V(G)$ and $E(G)$, or just V and E, are used to denote the vertex set and edge set of a graph G. We assume that V and E are both finite. If $A \subseteq E$ then $G|A$ is the *restriction* of G to A formed by deleting all edges except those contained in A. Given a subset U of V, we define $G{:}U$ to be the subgraph of G with vertex set U and edge set consisting of those edges in E with both endpoints in U. The number of connected components of G is denoted by $k(G)$. The *rank* of a set, A, of edges is denoted by $r(A)$ and is given by

$$r(A) = |V(G)| - k(G|A).$$

A graph is said to be *simple* if it has no loops or parallel edges. A *parallel class* is a maximal set of parallel edges.

A *weighted graph* consists of a finite graph G with vertex set $V = \{v_1, \dots, v_n\}$ and edge set E, together with a weight function $\omega: V \to \mathbb{Z}^+$. We call $\omega(v_i)$ the *weight* of vertex v_i. More generally, if $U \subseteq V$, we define the *weight* of U, $\omega(U)$, to be $\sum_{v \in U} \omega(v)$.

We need to introduce the notion of deletion and contraction in weighted graphs.

If e is an edge of (G, ω) then let (G'_e, ω) denote the graph obtained from G by deleting e and leaving ω unchanged.

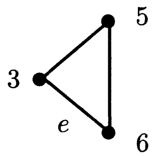
If e is not a loop of (G, ω) then let (G''_e, ω) be the graph obtained from G by contracting e, that is deleting e, identifying its endpoints v, v' into a single vertex v'' and setting

$$\omega(v'') = \omega(v) + \omega(v'),$$

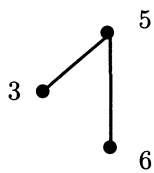
the other vertices of (G''_e, ω) having the same weight as in (G, ω). If e is a loop then we regard (G''_e, ω) as (G'_e, ω).

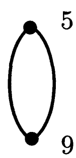
If e is an edge of a simple weighted graph, (G, ω), then (G^c_e, ω) is the graph formed from (G''_e, ω) by replacing every parallel class by a single edge. This is the notion of contraction for weighted graphs used in [4] where they denote it by G''_e.

Example. — Suppose

$(G, \omega) =$ 

then

$(G'_e, \omega) =$ 

$(G''_e, \omega) =$ 

$(G^c_e, \omega) =$ 

From here onwards, we will often write just G instead of (G, ω).

We associate with any weighted graph (G, ω) a multivariate polynomial $W_G(\boldsymbol{x}, y)$ which is defined as follows. Let $y, x_1, x_2, \ldots$ be commuting indeterminates.

Now let $W_G(\boldsymbol{x}, y)$ be defined recursively by the following rules which are reminiscent of the Tutte-Grothendieck decomposition well-known in combinatorics:

• If G consists of m isolated vertices with weights $a_1, \ldots a_m$ then $W_G(\boldsymbol{x}, y) = x_{a_1} \cdots x_{a_m}$.

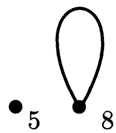
• If G has a loop e then

$$W_G(\boldsymbol{x}, y) = y W_{G'_e}(\boldsymbol{x}, y).$$

• If G has an edge e which has distinct endpoints then W satisfies the deletion/contraction rulé

$$W_G(\boldsymbol{x}, y) = W_{G'_e}(\boldsymbol{x}, y) + W_{G''_e}(\boldsymbol{x}, y).$$

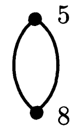
Thus, as a trivial example,

if $(G, \omega) =$ 

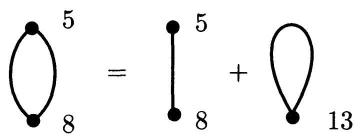
then $W_G(\boldsymbol{x}, y) = x_5 x_8 y.$

For a slightly less simple example consider the following.

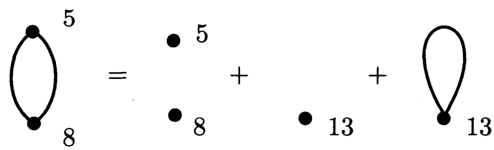
Example. — Let

$(G, \omega) =$ 

Now



therefore



and so $W_G(\boldsymbol{x}, y) = x_5 x_8 + x_{13} + x_{13} y.$

In the next section we explain the relationship between W and Vassiliev invariants.

3. Chord diagrams and intersection graphs.

We assume familiarity with the basics of classical combinatorial knot theory, at a level equivalent to say [21]. A key notion in the Vassiliev theory is that of a *singular knot* and its associated diagram. In a singular knot certain crossings are called *double points.* Geometrically they are points where the knot intersects itself and the tangent lines cross. An example of a singular knot with three double points is shown in Figure 1.

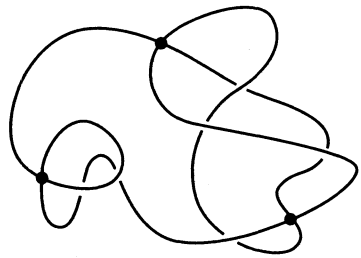


Figure 1. A knot with three double points

Given an orientation of a singular knot K a double point can be eliminated or unfolded to create either a positive or negative crossing, K_+ or K_-, as shown in Figure 2.

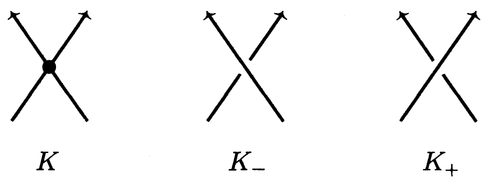


Figure 2. Unfolding a double point

If $\mathcal{K}^n$ denotes the collection of singular knots with n double points, and $\mathcal{K} = \bigcup_{n\geq 0} \mathcal{K}^n$, then a *Vassiliev invariant* is any function V on $\mathcal{K}$ which takes values in some abelian group A, which is invariant under isotopy and satisfies

$$V(K) = V(K_+) - V(K_-).$$

The invariant V is said to be of *order* n if V vanishes on any knot of more than n double points.

A *chord diagram* of *degree* m is a circle, oriented anti-clockwise, together with m chords marked on it. Thus what we are really considering is arrangements of pairs of distinct points on the circle.

A *weight function of degree* n is a function ω, mapping all chord diagrams with n chords into an abelian group A and satisfying the two relations (1T) and (4T) listed below.

(1T) The *one term relation* says that if chord diagram D has a chord which does not meet another then $\omega(D) = 0$.

(4T) The *four term relation* is more complicated and says that if D_N,D_S,D_E,D_W differ only as shown in Figure 3 then

$$\omega(D_N) - \omega(D_S) = \omega(D_W) - \omega(D_E).$$

Also it is assumed that no chord, other than those shown, has an endpoint in the shaded areas.

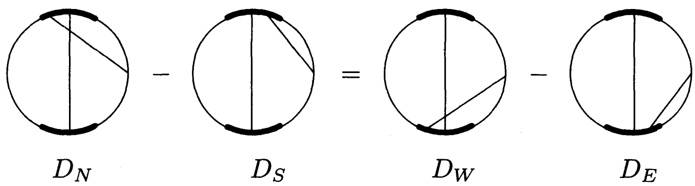


Figure 3. Four term relation

A key result is the following

THEOREM 3.1. — *If V is a Vassiliev invariant of order n with values in the abelian group A then there is a naturally defined weight function $\omega(V)$ on chord diagrams of degree n and taking values in A. Conversely given a weight function f of degree n and taking values in a field of characteristic zero there is an associated naturally defined Vassiliev invariant $V(f)$ of order n such that $\omega(V(f)) = f$ and this ω is unique, modulo invariants of smaller order.*

For good accounts of the relationship between Vassiliev invariants and chord diagrams see [2] and [23].

The *intersection graph* of a chord diagram is obtained by taking a vertex for each chord of the diagram and an edge between two vertices if the corresponding (straight line) chords intersect.

The *Intersection Graph Conjecture* made in [4] is that weight functions depend only on the intersection graph of diagrams. The Intersection Graph Conjecture has been verified in [2] for weight functions up to degree 8. It has also [3] been shown to be true when the intersection graph is a tree and hence a forest. In other words, if two chord diagrams have isomorphic intersection graphs and this graph is a tree then the diagrams are equivalent modulo the (4T) and (1T) relations. It is important to recognise that the intersection graph is a much rougher concept than a chord diagram and it has recently been reported in [22] that a counterexample to the Intersection

Graph Conjecture has been constructed by Le Tu using a pair of mutant knots.

As it stands the mapping from chord diagrams to graphs has not taken into account the additional structure due to the 4-term relations. In [4] the authors move towards this as follows. A function f defined on the set $\mathcal{W}_n$ of weighted graphs with total weight n is called a *weighted chromatic invariant of order* n if it satisfies the weighted chromatic relation

$$f(G) - f(G') - f(G^c) = 0. \tag{1}$$

Given a chord diagram D, its intersection graph can be regarded as a weighted graph with *all weights set to* 1.

A main result of [4] is to show that if f is a weighted chromatic invariant on intersection graphs then regarded as an invariant on chord diagrams, it will satisfy the more complicated condition demanded by the (4T) relation, and this is straightforward combinatorics (graph theory).

However, it is important to draw attention to a fairly recent paper of Lieberum [12] which shows that the Vassiliev invariants of knots corresponding to the chromatic weight systems distinguish exactly the same set of knots as a particular one variable specialisation of the Homfly and 2-variable Kauffman polynomials. They are in fact related in a quite straightforward way to well known classical invariants coming from simple Lie groups. We refer to Lieberum [12] for details.

4. Properties of W.

We begin with some basic properties of W and move on to consider various expansions and a Recipe Theorem. We will often write $W(G)$ or just W instead of $W_G(\boldsymbol{x}, y)$.

PROPOSITION 4.1. — *The polynomial W is well defined in the sense that the resulting multivariate polynomial is independent of the order in which the edges are deleted/contracted.*

Proof. — First note that if we have a weighted graph G and edges e and f then $G/e \setminus f = G \setminus f/e$ so the order in which we contract or delete edges does not affect the graph which we obtain. We prove the result by induction on the number of edges. If G has at most one edge then there is nothing to be proved. So suppose G has at least two edges. Choose any two edges e and f which we will assume are not loops and not in parallel. We will show that computing W in any way that starts with deleting and contracting e leads to the same answer as starting with f. Starting with e gives us $W(G'_e) + W(G''_e)$ which by induction is not dependent on the order in which we delete or contract the rest of the edges but

$$\begin{aligned} W(G'_e) + W(G''_e) &= W\big((G'_e)'_f\big) + W\big((G'_e)''_f\big) + W\big((G''_e)'_f\big) + W\big((G''_e)''_f\big) \\ &= W\big((G'_f)'_e\big) + W\big((G'_f)''_e\big) + W\big((G''_f)'_e\big) + W\big((G''_f)''_e\big) \\ &= W(G'_f) + W(G''_f) \end{aligned}$$

which is what we get if we delete and contract f first. It is easy to modify this if either e or f or both is a loop, or if they are in parallel. □

The following properties of W are easily verified:

- If G consists of connected components $G_1, \ldots, G_k$ then

$$W(G) = \prod_{i=1}^{k} W(G_i).$$

- For any G, each monomial in the polynomial $W_G(\boldsymbol{x}, y)$ is of the form $x_{a_1} x_{a_2} \cdots x_{a_m} y^t$ where $a_1 + \cdots + a_m$ equals the sum of the weights of G.

- $W_G(\boldsymbol{x}, y)$ has no term in y if and only if G is a forest.

We first show that W contains as a specialisation the weighted chromatic invariants of [4] as an evaluation of the form $W_G(\boldsymbol{x}, 0)$.

THEOREM 4.2. — *Let f be a weighted chromatic invariant with the property that on any graph consisting just of k isolated vertices with weights $a_1, \ldots, a_k$, f takes the value $x_{a_1} \cdots x_{a_k}$, then for any simple weighted graph G, $f(G) = W_G(\boldsymbol{x}, 0)$.*

Proof. — Let G be a graph with no loops. Suppose e_1, e_2 are parallel edges. We claim $W_G(\boldsymbol{x}, 0) = W_{G'_{e_1}}(\boldsymbol{x}, 0)$. This follows from the recursive definition of W, $W_G(\boldsymbol{x}, 0) = W_{G'_{e_1}}(\boldsymbol{x}, 0) + W_{G''_{e_1}}(\boldsymbol{x}, 0)$ but $W_{G''_{e_1}}(\boldsymbol{x}, 0) = 0$ because G''_{e_1} contains a loop e_2. Hence, for any e, $W_{G''_e}(\boldsymbol{x}, 0) = W_{G^c_e}(\boldsymbol{x}, 0)$. Now the result follows by induction on the number of edges of G using the recursive definition of W and Equation 1. □

THEOREM 4.3. — *$W_G(\boldsymbol{x},y)$ has a states model representation of the form*

$$W_G(\boldsymbol{x},y) = \sum_{A \subseteq E} x_{c_1} x_{c_2} \cdots x_{c_k} (y-1)^{|A|-r(A)}$$

where c_i, $1 \leq i \leq k$ is the total weight of the ith component of the weighted subgraph $(G|A,\omega)$.

Note that in this and some later theorems, the number of components k is not constant and depends on A. Setting $y = 0$ gives a corollary concerning the form of any weighted chromatic invariant.

COROLLARY 4.4. — *Every weighted chromatic invariant is of the form*

$$\sum_{A \subseteq E} x_{c_1} \cdots x_{c_k} (-1)^{|A|-r(A)}.$$

Proof of Theorem. — We let $c_i(A)$ be the total weight of the ith component of $G|A$. Let

$$S(G) = \sum_{A \subseteq E} x_{c_1(A)} x_{c_2(A)} \cdots x_{c_k(A)} (y-1)^{|A|-r(A)}.$$

We show that $S(G) = W(G)$ by induction on the number of edges of G. If G has no edges then $S(G) = W(G)$ so suppose e is an edge of G that is not a loop:

$$\begin{aligned}
S(G) &= \sum_{A \subseteq E} x_{c_1(A)} x_{c_2(A)} \cdots x_{c_k(A)} (y-1)^{|A|-r(A)} \\
&= \sum_{A \subseteq E \setminus e} x_{c_1(A)} x_{c_2(A)} \cdots x_{c_k(A)} (y-1)^{|A|-r(A)} \\
&\qquad + \sum_{A:\, e \in A \subseteq E} x_{c_1(A)} x_{c_2(A)} \cdots x_{c_k(A)} (y-1)^{|A|-r(A)} \\
&= S(G'_e) + \sum_{A \subseteq E \setminus e} x_{c_1(A \cup e)} x_{c_2(A \cup e)} \cdots x_{c_k(A \cup e)} (y-1)^{|A \cup e|-r(A \cup e)} \\
&= S(G'_e) + S(G''_e),
\end{aligned}$$

which by induction is equal to $W(G'_e)+W(G''_e) = W(G)$. If e is a loop then

$$\begin{aligned} S(G) &= \sum_{A\subseteq E} x_{c_1(A)}x_{c_2(A)}\cdots x_{c_k(A)}(y-1)^{|A|-r(A)} \\ &= \sum_{A\subseteq E\backslash e} x_{c_1(A)}x_{c_2(A)}\cdots x_{c_k(A)}\big((y-1)^{|A|-r(A)}+(y-1)^{|A\cup e|-r(A\cup e)}\big) \\ &= \sum_{A\subseteq E\backslash e} x_{c_1(A)}x_{c_2(A)}\cdots x_{c_k(A)}(y-1)^{|A|-r(A)}y \\ &= yS(G'_e) \end{aligned}$$

which by induction equals $yW(G'_e) = W(G)$. □

As an immediate consequence, we have by putting $y = 1$:

Corollary 4.5. — *The polynomial $W_G(\boldsymbol{x},1)$ is a homogeneous polynomial in $(x_1,x_2,\ldots)$ and is a weighted sum over all spanning forests of G, namely*

$$W_G(\boldsymbol{x},1) = \sum_{F\in\mathcal{F}(G)} x(F),$$

where if F is a spanning forest with connected components having vertex sets $V_1,\ldots,V_k$, $x(F)$ is the monomial $x_{\omega(V_1)}x_{\omega(V_2)}\cdots x_{\omega(V_k)}$ and $\mathcal{F}(G)$ is the set of all spanning forests of G.

The *Tutte polynomial* of a graph is the two variable polynomial given by

$$T(G;x,y) = \sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

Corollary 4.6. — *If $x_i = \theta$ for each i, then the resulting 2-variable polynomial $W_G(x_i = \theta, y)$ is independent of its weights and is given by*

$$W(x_i = \theta, y) = \theta^{k(G)}T(G; 1+\theta, y),$$

where $k(G)$ is the number of connected components of G.

Note. — Here and throughout this paper we slightly abuse notation by letting $W(x_i = \theta, y)$ denote the polynomial (function) obtained by substituting θ for each x_i.

Proof. — Putting $x_i = \theta$ for all i in Theorem 4.3 gives

$$\begin{aligned} W_G(x_i = \theta, y) &= \sum_{A \subseteq E} \theta^{k(G|A)} (y-1)^{|A|-r(A)} \\ &= \sum_{A \subseteq E} \theta^{|V(G)|-r(A)} (y-1)^{|A|-r(A)} \\ &= \theta^{k(G)} T(G; 1+\theta, y). \end{aligned}$$ □

This instantly gives a host of specialisations of W, see for example [21].

A second interpretation of W in terms of the Tutte polynomial is contained in the following theorem.

If $\pi = (V_1, \dots, V_k)$ denotes a partition of the vertex set V it determines subgraphs $G_1, \dots, G_k$ where G_i denotes the subgraph of G induced by V_i. We say a partition is *connected* if each of $G_1, \dots, G_k$ is connected. The partition π of V also determines the monomial defined by

$$x(\pi) = x_{\omega(V_1)} \cdots x_{\omega(V_k)}.$$

Then we have

THEOREM 4.7. — *A states model for W is given by*

$$W_G(\boldsymbol{x}, y) = \sum_{\pi} x(\pi) T(G_1; 1, y) \cdots T(G_k; 1, y) \tag{2}$$

where the sum is over all connected partitions π of $V(G)$.

Again, by putting $y = 0$ we obtain a corollary concerning chromatic invariants.

COROLLARY 4.8. — *Any weighted chromatic invariant has the form*

$$\sum_{\pi} x(\pi) T(G_1; 1, 0) \dots T(G_k; 1, 0)$$

where the summation is over all connected partitions of $V(G)$.

Proof of theorem. — Given a set A of edges, we define $\Pi(A)$ to be the partition with ground set V whose blocks correspond to the connected components of $G|A$. We have from Theorem 4.3 that

$$W_G(\boldsymbol{x}, y) = \sum_{A \subseteq E} x_{c_1} x_{c_2} \cdots x_{c_k} (y-1)^{|A|-r(A)}$$

where c_i, $1 \leq i \leq k$ is the total weight of the ith component of the weighted subgraph $(G|A, \omega)$. Now

$$W_G(\boldsymbol{x}, y) = \sum_{\pi} \sum_{A:\Pi(A)=\pi} x_{\omega(V_1)} x_{\omega(V_2)} \cdots x_{\omega(V_k)} (y-1)^{|A|-r(A)}$$

where π is the partition $(V_1, \ldots, V_k)$. If $\Pi(A) = \pi$ then we can write

$$A = A_1 \cup \ldots \cup A_k$$

where for each i, the edges in A_i have both endpoints in V_i and $G{:}V_i$ is connected. Hence

$$r(A) = r(A_1) + \cdots + r(A_k).$$

This gives

$$\begin{aligned} W_G(\boldsymbol{x}, y) &= \sum_{\pi} x_{\omega(V_1)} x_{\omega(V_2)} \cdots x_{\omega(V_k)} \sum_{A_1} (y-1)^{|A_1|-r(A_1)} \\ &\qquad \cdots \sum_{A_k} (y-1)^{|A_k|-r(A_k)} \\ &= \sum_{\pi} x_{\omega(V_1)} x_{\omega(V_2)} \cdots x_{\omega(V_k)} T(G_1; 1, y) \cdots T(G_k; 1, y), \end{aligned}$$

where the final summation is over all connected partitions of $V(G)$. □

Suppose we are given a connected graph G and a total ordering on its edges. Consider a spanning tree T of G. An edge e in $G \setminus T$ is *externally active* with respect to T if it is the largest edge in the unique cycle contained in $T \cup e$. Let $\operatorname{ext}(T)$ be the number of externally active edges with respect to T. A theorem of Tutte [20] implies that

$$T(G; 1, y) = \sum_{T} y^{\operatorname{ext}(T)}$$

where the summation is over all spanning trees. This implies that the number of spanning trees with a given external activity is a graph invariant and not dependent on the choice of ordering. We extend the definition of external activity to forests in the obvious way: given a graph G and a spanning forest F of G, an edge e of G is *externally active* with respect to F if $F \cup e$ contains a unique cycle and e is the largest edge in that cycle. An edge is externally active with respect to F if both its endpoints are incident with vertices in the same component T of F and e is externally active with respect to the spanning tree T of $G: V(T)$. Given a weighted graph (G,ω) and a spanning forest F with connected components having vertex sets $V_1,\dots V_k$ the term $x(F)$ is given by

$$x(F) = x_{\omega(V_1)}x_{\omega(V_2)}\cdots x_{\omega(V_k)}.$$

This leads to yet another states model for W over the set of all spanning forests.

THEOREM 4.9. — *For any weighted graph (G,ω)*

$$W_G(\boldsymbol{x},y) = \sum_{F\in\mathcal{F}(G)} x(F)\, y^{\operatorname{ext}(F)}$$

where the summation is over all spanning forests of G and $\operatorname{ext}(F)$ is the number of edges externally active with respect to F.

COROLLARY 4.10. — *Any weighted chromatic invariant can be written in the form*

$$\sum_F x(F)$$

where the summation is over all spanning forests of external activity zero.

Proof of theorem. — For a spanning forest F we let $\Pi(F)$ be the partition of V induced by the connected components of F. Using Theorem 4.7 we have

$$W_G(\boldsymbol{x},y) = \sum_\pi x(\pi)T(G_1;1,y)\cdots T(G_k;1,y)$$

where the summation is over all connected partitions $\pi = (V_1,\dots,V_k)$ of the vertex set and for each i, G_i is the subgraph induced by V_i. This gives

$$W_G(\boldsymbol{x},y) = \sum_\pi x(\pi)\sum_{T_1} y^{\operatorname{ext}(T_1)}\cdots\sum_{T_k} y^{\operatorname{ext}(T_k)}$$

where for each i the summation over T_i is over all spanning trees of G_i. The union of these trees is just a spanning forest and so

$$W_G(\boldsymbol{x},y) = \sum_\pi x(\pi)\sum_{F:\Pi(F)=\pi} y^{\operatorname{ext}(F)} = \sum_{F\in\mathcal{F}(G)} x(F)\, y^{\operatorname{ext}(F)}$$

where the summation is over all spanning forests of G. □

PROPOSITION 4.11. — *If G has n vertices each with weight one or weight two and A is the set of vertices of weight two then the coefficient of*

$$x_2^{k+|A|}\,x_1^{n-2k-|A|}$$

in $W_G(\boldsymbol{x},1)$ gives the number of matchings of size k in which no edge has an endpoint in A.

Proof. — Theorem 4.9 shows that

$$W_G(\boldsymbol{x},1)=\sum_F x(F)$$

where the summation is over all spanning forests of G. The spanning forests such that $x(F)=x_2^{k+|A|}x_1^{n-2k-|A|}$ are precisely the matchings with k edges in which no edge has an endpoint in A. □

We now move to a version of the Recipe Theorem of [15].

THEOREM 4.12. — *Let f be a function on weighted graphs defined recursively by the following conditions:*

1) *For any non-loop edge e,*

$$f(G)=af(G'_e)+bf(G''_e)$$

where a, b are any two non-zero scalars.

2) *If e is a loop then $f(G)=yf(G'_e)$.*

3) *If G consists of isolated vertices with weights $a_1,\dots,a_m$ then $f(G)=x_{a_1}\cdots x_{a_m}$.*

Then

$$f(G)=a^{|E|-|V|}\,b^{|V|}\,W_G\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right).$$

Proof. — We prove the result by induction on the number of edges in G. If G consists of isolated vertices with weights $a_1,\dots,a_m$ then

$$f(G)=x_{a_1}\cdots x_{a_m}=a^{|E|-|V|}\,b^{|V|}\,W\left(G;\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right)$$

so the result is true if G has no edges. Let e be an edge of G and suppose first that e is a loop. Using induction we have that

$$\begin{aligned} f(G) &= yf(G'_e) \\ &= y\,a^{|E(G'_e)|-|V(G'_e)|}\,b^{|V(G'_e)|}\,W_{G'_e}\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right) \\ &= \frac{y}{a}\,a^{|E(G)|-|V(G)|}\,b^{|V(G)|}\,W_{G'_e}\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right) \\ &= a^{|E(G)|-|V(G)|}\,b^{|V(G)|}\,W_G\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right) \end{aligned}$$

so the result is true in this case. Now let e be any non-loop edge of G. Using induction we have that

$$\begin{aligned} f(G) &= af(G'_e) + bf(G''_e) \\ &= a^{1+|E(G'_e)|-|V(G'_e)|}\,b^{|V(G'_e)|}\,W_{G'_e}\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right) \\ &+ a^{|E(G''_e)|-|V(G''_e)|}\,b^{1+|V(G''_e)|}\,W_{G''_e}\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right) \\ &= a^{|E(G)|-|V(G)|}\,b^{|V(G)|}\,W_{G'_e}\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right) \\ &\qquad + a^{|E(G)|-|V(G)|}\,b^{|V(G)|}\,W_{G''_e}\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right) \\ &= a^{|E(G)|-|V(G)|}\,b^{|V(G)|}\,W_G\left(\frac{a\boldsymbol{x}}{b},\frac{y}{a}\right) \end{aligned}$$

hence the result is true in general. □

5. An invariant of ordinary graphs.

We now consider the natural invariant of graphs obtained from W by treating an ordinary, unweighted graph G as a weighted graph in which each vertex has weight equal to unity. In this case we write W as $U_G(\boldsymbol{x}, y)$.

Apart from a certain intrinsic interest as a combinatorial invariant it is also the case that in the original motivating situation (weighted chord diagrams) the weights are initially all one but the recursive definition does not permit remaining in this "unweighted category". First we summarise the basic properties of U which follow from what we know about W.

PROPOSITION 5.1. — *If G is a graph on n vertices then $U_G(\boldsymbol{x},y)$ is a polynomial in $x_1, \dots, x_n, y$ such that*

1) *U_G is independent of y if and only if G is a forest.*

2) *Each monomial in* U_G *is of the form* $x_{a_1} \cdots x_{a_m} y^t$ *where* $\sum a_i = n$.

3) *If* $n_1, \ldots, n_k$ *are the number of vertices of the different components of* $G|A$ *then*

$$U_G(\boldsymbol{x},y) = \sum_{A \subseteq E} x_{n_1} \cdots x_{n_k} (y-1)^{|A|-r(A)}.$$

We now turn to specific evaluations of U in terms of known combinatorial polynomials. From Corollary 4.6 we know that the Tutte polynomial is given by

$$T(G;x,y) = (x-1)^{-k(G)}\, U_G(x_i = x-1, y).$$

In particular, the chromatic polynomial $P_G(\lambda)$ is given by

$$P_G(\lambda) = (-1)^{|V|}\, U_G(x_i = -\lambda, y = 0).$$

Despite its apparent similarity to the Tutte polynomial, U is a much stronger invariant. Any two graphs with the same matroid share a common Tutte polynomial whereas

$$U(P_4) = x_1^4 + 3x_1^2 x_2 + 2x_1 x_3 + x_2^2 + x_4,$$
$$U(St_4) = x_1^4 + 3x_1^2 x_2 + 3x_1 x_3 + x_4$$

where P_4 and St_4 are respectively the path and star with four vertices.

Another specialisation of U gives the *stability polynomial*, $A(G;p)$, of a loopless graph. This was introduced by Farr in [5] and is the one-variable polynomial given by

$$A(G;p) = \sum_{\substack{U \subseteq V(G) \\ U \in \mathcal{S}(G)}} p^{|U|} (1-p)^{|V(G) \setminus U|},$$

where $\mathcal{S}(G)$ is the set of all stable sets of G.

THEOREM 5.2. — *If* G *is loopless then* $A(G;p)$ *is given by*

$$A(G;p) = U_G\Big(x_1 = 1, x_j = -(-p)^j \text{ if } j \geq 2, y = 0\Big).$$

Proof. — Theorem 13 in [5] shows that

$$A(G;p) = \sum_{A \subseteq E(G)} (-1)^{|A|} p^{f(A)}$$

where $f(A)$ is the number of vertices incident with an edge of A. In the case when all weights are one and $y = 0$, Theorem 4.3 reduces to

$$W_G(\boldsymbol{x}, 0) = \sum_{A \subseteq E} x_{|V(G_1)|} x_{|V(G_2)|} \cdots x_{|V(G_k)|} (-1)^{|A|-r(A)}$$

where G_i, $1 \leq i \leq k$ is the i-th component of the weighted subgraph $(G|A, \omega)$. Now $\sum_{i:|V(G_i)>1|} |V(G_i)|$ is equal to $f(A)$ and $\sum_i (|V(G_i)| - 1)$ is equal to $r(A)$. Hence

$$(-1)^{r(A)} = \prod_i (-1)^{(|V(G_i)|-1)}.$$

Thus when we set $x_1 = 1$ and $x_j = -(-p)^j$ for $j \geq 2$ we have

$$U_G(\boldsymbol{x}, 0) = W_G(\boldsymbol{x}, 0) = \sum_{A \subseteq E} p^{f(A)} (-1)^{r(A)} (-1)^{|A|-r(A)} = A(G;p). \quad \square$$

In [16] Oxley and Whittle study a polynomial S that is the analogue of the Tutte polynomial for 2-polymatroids. In the case that the polymatroid has as its ground set the edge set of a graph G and rank function, f, where $f(A)$ is the number of vertices incident with an edge in A, the polynomial is the two variable polynomial given by

$$S(G;u,v) = \sum_{A \subseteq E} u^{f(E)-f(A)} v^{2|A|-f(A)}.$$

This polynomial, which clearly contains the stability polynomial A as a specialisation, turns out also to be a specialisation of U.

THEOREM 5.3. — *Let G be a loopless graph with no isolated vertices. Then $S(G;u,v)$ is given by*

$$S(G;u,v) = U_G\Big(x_1 = u,\ x_2 = 1,\ x_j = v^{j-2} \text{ for } j > 2,\ y = v^2 + 1\Big).$$

The restriction on isolated vertices is not really important because the polynomial S is unaffected by the addition of isolated vertices.

Proof. — The proof is similar to that of the preceding theorem. In the case when all weights are one and $y = v^2 + 1$, Theorem 4.3 reduces to

$$W_G(\boldsymbol{x}, y) = \sum_{A \subseteq E} x_{|V(G_1)|}\, x_{|V(G_2)|} \cdots x_{|V(G_k)|}\, v^{2(|A|-r(A))}$$

where G_i, $1 \le i \le k$ is the i-th component of the weighted subgraph $(G|A, \omega)$. Now the number of components of $G|A$ consisting of just a single vertex is $f(E) - f(A)$. Also

$$r(A) = \sum_{i:|V(G_i)|\ge 2} (|V(G_i)| - 1)$$

and hence

$$2r(A) - f(A) = \sum_{i:|V(G_i)|\ge 2} (|V(G_i)| - 2).$$

Thus, setting $x_1 = u$, $x_2 = 1$, $x_j = v^{j-2}$ for $j \ge 2$ and $y = v^2 + 1$ gives that

$$\begin{aligned} U_G(\boldsymbol{x}, y) = W_G(\boldsymbol{x}, y) &= \sum_{A \subseteq E} u^{|f(E)-f(A)|}\, v^{2r(A)-f(A)}\, v^{2(|A|-r(A))} \\ &= S(G; u, v). \end{aligned}$$

□

We now show that the number of cliques of given size appear as coefficients of monomials in U.

Proposition 5.4. — *If G is simple and has n vertices then the coefficient of $x_k x_1^{n-k} y^{\binom{k}{2}-k+1}$ in $U_G(\boldsymbol{x},y)$ gives the number of cliques of size k in G.*

The proposition follows from Theorem 4.7 and the following easy lemma.

Lemma 5.5. — *Suppose G is a simple connected graph on n vertices, then G is a clique if and only if the coefficient of $y^{\binom{n}{2}-n+1}$ in $T(G; 1, y)$ is positive.*

Proof. — The evaluation of T at $(1, y)$ is given by

$$T(G;1,y) = \sum_A (y-1)^{|A|-n+1}$$

where the summation is over subsets of edges spanning G and so if G is a clique, taking $A = E(G)$ will give a term in $y^{\binom{n}{2}-n+1}$. Conversely if the coefficient of $y^{\binom{n}{2}-n+1}$ in $T(G;1,y)$ is non-zero then G must have $\binom{n}{2}$ edges and hence be a clique. □

Proof of proposition. — Suppose H is a set of vertices of G corresponding to a k-clique. Equation (2) and Lemma 5.5 show that the partition of V into $n-k+1$ blocks, one of which is H and the others are singletons, will lead to a term in $x_k x_1^{n-k} y^{\binom{k}{2}-k+1}$. Such terms can arise only if the partition in the sum in Equation (2) consists of one block with k vertices and singleton blocks otherwise. Then for some i, $T(G_i;1,y)$ contains a term of the form $y^{\binom{k}{2}-k+1}$ and hence by Lemma 5.5, G_i is a clique. □

The *matching polynomial* of G is the polynomial $m(G;t)$ given by

$$m(G;t) = \sum_{k\geq 0} m_k t^k,$$

where m_k is the number of matchings of G with k edges. A consequence of Proposition 4.11 is that $m(G;t)$ is an evaluation of U.

PROPOSITION 5.6. — *The matching polynomial is given by*

$$m(G;t) = U_G\Big(x_1 = 1,\, x_2 = t,\, x_j = 0 \text{ for } j > 2,\, y = 1\Big).$$

Proof. — Proposition 4.11 implies that if G has n vertices then the coefficient of $x_2^k x_1^{n-2k}$ in $U_G(\boldsymbol{x},1)$ is the number of matchings of size k in G. The result follows from this observation. □

6. A symmetric function application.

A more interesting and less apparent specialisation of U_G is that it gives the symmetric function generalisation X_G of the chromatic polynomial developed in [18].

For any graph G, X_G is a homogeneous symmetric function in $\boldsymbol{x} = (x_1, x_2, \ldots)$ of degree $n = |V|$ defined by

$$X_G = X_G(\boldsymbol{x}) = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)}\, x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

where the sum ranges over all proper colourings $\kappa : V \to \mathbb{Z}^+$. Note that the sum is infinite since we allow any proper colouring using positive integers.

In [18] Stanley develops X_G and its properties in terms of the standard 'natural' bases for the space of symmetric functions. The basis which is of most interest here is the *power sum basis* $\{p_r\}$ given by (see Macdonald [14])

$$p_0 = 1 \quad \text{and} \quad p_r(x_1, x_2, \ldots) = \sum_i x_i^r, \quad \text{for} \quad r \geq 1.$$

THEOREM 6.1. — *For any graph G*

$$X_G = (-1)^{|V|}\, U_G\big(x_j = -p_j, y = 0\big).$$

Proof. — For $S \subseteq E$, let $\lambda(S)$ be the partition of $n = |V|$, whose parts are equal to the vertex sizes of the connected components of $G|S$. For a partition π with blocks of size $s_1, \ldots, s_k$ we let

$$p_\pi = p_{s_1} \cdots p_{s_k},$$

(see [14]). Then from Theorem 2.5 of [18] we know

$$X_G = \sum_{S \subseteq E} (-1)^{|S|}\, p_{\lambda(S)}.$$

Now use the representation we have given of W_G in Theorem 4.3 and noting that the number of parts (or length) of $\lambda(S)$ is exactly $k(G)$, it is easy to see that the substitution $x_j = -p_j$ gives X_G up to $(-1)^{|V|}$. □

Many of the results in [18] now follow from our earlier interpretations of U_G.

In a later paper Stanley [19] suggests the following symmetric function generalisation of the Tutte polynomial of a graph. He does this by using the equivalent representation in terms of the *bad colouring* polynomial of [21] which gives T as the generating function for the number of 'bad' or monochromatic edges over *all* colourings $\kappa: V(G) \to \{1, 2, \ldots\}$.

DEFINITION. — *Let $\boldsymbol{x} = (x_1, x_2, \ldots)$ and t be indeterminates and define*

$$X_G(\boldsymbol{x}, t) = \sum_{\kappa: V \to \mathbb{Z}^+} (1+t)^{b(\kappa)} x_{\kappa(v_1)}\, x_{\kappa(v_2)} \cdots x_{\kappa(v_n)},$$

where $b(\kappa)$ denotes the number of bad edges in the colouring κ and the sum is over all colourings.

Note that the sum is infinite since we allow all possible colourings using positive integers. If $f(1^n)$ denotes the substitution $x_1 = x_2 = \cdots = x_n = 1$, $x_{n+1} = x_{n+2} = \cdots = 0$ then

$$X_G(1^n, t) = n^{k(G)}\, t^{r(G)}\, T\Big(G; \frac{t+n}{t}, t+1\Big)$$

and

$$X_G(\boldsymbol{x}, -1) = X_G(\boldsymbol{x}).$$

THEOREM 6.2. — *For any G, U_G and $X_G(\boldsymbol{x},t)$ determine each other. In particular X_G is easily obtained from U_G by the substitution*

$$X_G(\boldsymbol{x},t) = t^{|V|}\, U_G\left(x_j = \frac{p_j}{t},\, y = t+1\right).$$

Conversely, if we can expand X_G in terms of the power basis of symmetric functions then we can recover U_G.

Proof. — A result attributed in [19] to T. Chow gives the representation

$$X_G(\boldsymbol{x}, t) = \sum_{S \subseteq E} t^{|S|}\, p_{\lambda(S)}$$

where we are using the same notation as in the previous theorem. Now put $x_j = p_j/t$ and $y = t+1$ in the representation for U_G given in Theorem 4.3 and the result follows. □

One consequence of Theorem 6.2 and our original definition of W and which does not seem obvious from their definitions in [19] is:

COROLLARY 6.3. — *The function $X_G(\boldsymbol{x},t)$ and hence the symmetric chromatic polynomial X_G have recursive delete/contract derivations analogous to those of the normal chromatic polynomial.*

7. Complexity issues.

Consider now the problem of computing W. This is clearly going to be a hard problem for general G and $\boldsymbol{x}$ because the Tutte polynomial is a specialisation of W and this is shown in [9] to be $\#P$-hard to compute at most points. In a sense it appears that W should be no harder to compute than the Tutte polynomial as both have a similar recursive formula [15]. Here however, we show that unlike the Tutte polynomial, this polynomial is hard to compute even for the special cases where the input graph is a tree or a complete graph.

First we clarify the problems.

PROBLEM 7.1: W-coefficient.

- *Input*: *Weighted graph G, finite ordered (multi)-set $P=\{p_1, \dots, p_k\}$ of positive integers and a positive integer q.*
- *Output*: *Coefficient of $x_{p_1}, \dots, x_{p_k} y^q$ in $W_G(\boldsymbol{x}, y)$.*

PROBLEM 7.2: W-evaluation.

- *Input*: *Weighted graph G, finite set $S = \{(i, a_i) : i \in I\}$ where $I \subseteq \mathbb{N}$ and each a_i is an integer together with an integer y_0.*
- *Output*: *The value which W takes on substituting*

$$x_i = \begin{cases} a_i & i \in I, \\ 0 & i \notin I \end{cases} \quad \text{and } y = y_0.$$

Note. — We have made the slightly artificial restriction that the a_i be integral to avoid unnecessary complications. As in [9] we could have allowed the a_i to take values in any algebraic field but since we are going to prove hardness results for most graphs and most integer points it does not seem worthwhile.

THEOREM 7.3. — *Computing W-coefficient is #P-hard even if G is restricted to being a tree.*

To prove the theorem we need to consider the following four problems, the first of which is a counting version of one of Karp's original twelve *NP*-complete problems. Note that in the following we take $\mathbb{Z}^+$ to be the set of strictly positive integers so that we do not allow elements to have size 0.

PROBLEM 7.4: #partition.

- *Input*: *Finite set A and a size $s(a) \in \mathbb{Z}^+$ for each $a \in A$.*
- *Output*: *The number of subsets $A' \subseteq A$ with $\sum_{a \in A'} s(a) = \sum_{a \in A \setminus A'} s(a)$.*

PROBLEM 7.5: $\#\frac{1}{2}$partition.

- *Input*: *Finite set* A *and a size* $s(a) \in \mathbb{Z}^+$ *for each* $a \in A$.
- *Output*: *The number of subsets* $A' \subseteq A$ *with*

$$|A'| = \tfrac{1}{2}A| \quad \text{and} \quad \sum_{a\in A'} s(a) = \sum_{a\in A\setminus A'} s(a).$$

PROBLEM 7.6: #paired-partition.

- *Input*: *Finite set* $A = \{a_1, \dots a_{2n}\}$ *and a size* $s(a) \in \mathbb{Z}^+$ *for each* $a \in A$.

- *Output*: *The number of subsets* $A' \subseteq A$ *such that* A' *contains precisely one of* a_i *and* a_{2n-i} *for each* i *with* $1 \le i \le n$ *and*

$$\sum_{a\in A'} s(a) = \sum_{a\in A\setminus A'} s(a).$$

PROBLEM 7.7: #distinct-partition.

- *Input*: *Finite set* A *and a size* $s(a) \in \mathbb{Z}^+$ *for each* $a \in A$ *such that* $s(a) \neq s(a')$ *if* $a \neq a'$.

- *Output*: *The number of subsets* $A' \subseteq A$ *such that*

$$\sum_{a\in A'} s(a) = \sum_{a\in A\setminus A'} s(a).$$

We need the following lemma.

LEMMA 7.8. — *Each of problems 7.4–7.7 is* $\#P$*-complete.*

Proof. — We show

$$\#\text{partition } \alpha_T \ \#\tfrac{1}{2}\text{partition } \alpha_T \ \#\text{paired-partition} \ \alpha_T \ \#\text{distinct-partition}.$$

As far as we know there is nothing in the literature showing explicitly that #partition is $\#P$-hard. The reductions in [6], [11] showing that partition is NP-hard do not immediately give proofs of $\#P$-hardness because they do not seem to preserve the number of solutions. However using the following two problems it is possible to obtain a reduction.

PROBLEM 7.9: #3-col.

- *Input*: *A graph* G.

- *Output*: *The number of proper 3-colourings of the vertices of* G.

PROBLEM 7.10: #exact cover.

- *Input*: *Finite set* A *and a collection* C *of subsets of* A.

- *Output*: *The number of subcollections* $C' \subseteq C$ *such that every element of* A *is contained in precisely one member of* C'.

The problem #3-col is shown to be $\#P$-hard in [13] and a reduction is given from the decision problem 3-col to the decision problem exact cover in [11]. It is easy to check that this reduction preserves the number of solutions and so #exact cover is $\#P$-hard. There is a reduction to partition from a problem very similar to exact cover, namely 3-dimensional matching in [6] and this can be easily modified to give a reduction from exact cover which doubles the number of solutions. Hence #partition is $\#P$-hard.

Next we show #partition α_T $\#\frac{1}{2}$partition by giving an algorithm which constitutes a Turing reduction from #partition to $\#\frac{1}{2}$partition. We assume we are given an instance (A, s) of #partition and an oracle for $\#\frac{1}{2}$partition. The command $\frac{1}{2}$part$(A;\ s(a_1), \ldots, s(a_{2n}))$ calls the oracle with the instance (A, s) of $\#\frac{1}{2}$partition where $A = \{a_1, \ldots, a_{2n}\}$. This algorithm is shown as Algorithm 1.

To show $\#\frac{1}{2}$partition reduces to #paired-partition we suppose we have an instance (A, s) of $\#\frac{1}{2}$partition. Since the number of solutions to $\#\frac{1}{2}$partition is zero if $|A|$ is odd we may assume that $|A|$ is even. Let $A = \{a_1, \ldots, a_{2n}\}$ and let $M = \sum\limits_{a \in A} s(a)$. We construct an instance (A', s') of #paired-partition by setting $A' = \{a_1, \ldots, a_{4n}\}$ and defining s' by

$$s'(a_i) = \begin{cases} s(a_i) & \text{if } 1 \leq i \leq 2n, \\ M & \text{otherwise.} \end{cases}$$

It is clear that the number of solutions to the original instance of $\#\frac{1}{2}$partition equals the number of solutions to this instance of #paired-partition.

Finally we show #paired-partition α_T #distinct-partition. Suppose we have an instance (A, s) of #paired-partition where $A = \{a_1, \ldots, a_{2n}\}$. Any pair (a_i, a_{2n-i}) with $s(a_i) = s(a_{2n-i})$ is essentially irrelevant because

Algorithm 1: Turing reduction from partition to $\#\frac{1}{2}$partition

Requires: $A = \{a_1, \ldots, a_n\}$, $s(a_i)$, $1 \leq i \leq n$
  $M \leftarrow 2 \sum_{a \in A} s(a)$
  $count \leftarrow 0$
  for $i = 1$ **to** n
    $s'(a_i) = s(a_i)$
  end for
  for $i = 1$ **to** $\lfloor \frac{1}{2} n \rfloor$
    $A' \leftarrow \{a_1, \ldots, a_{2n-2i+2}\}$
    for $j = n+1$ **to** $2n - 2i + 1$
      $s'(a_j) \leftarrow M$
    end for
    $s'(a_{2n-2i+2}) \leftarrow (n - 2i + 1)M$
    $count \leftarrow count + \frac{1}{2} part(s(a_1), \ldots, s(a_{2n-2i+2}))$
  end for
Ensure: $count$

deleting such a pair of elements just halves the number of solutions to #paired-partition, so assume that there are no pairs of this form. Let $M = \sum_{a \in A} s(a)$. We construct an instance (A, s') of #distinct-partition where s' is given by

$$s'(a_i) = \begin{cases} s(a_i) + M^i & \text{if } 1 \leq i \leq n, \\ s(a_i) + M^{2n-i} & \text{otherwise.} \end{cases}$$

This means that if $i \neq j$ then $s'(a_i) \neq s'(a_j)$ and again it is easy to see that this instance of #distinct-partition has the same number of solutions as the original instance of #paired-partition.

We can now prove the theorem.

Proof of theorem. — We suppose we are given an instance (A, s) of #distinct-partition where $A = \{a_1, \ldots, a_n\}$. Let $M = \sum_{a \in A} s(a)$ and consider the weighted tree T, shown in Figure 4 with vertices $v_1, \ldots, v_n$, $v'_1, \ldots, v'_n$, v, v' where

$$\omega(v_i) = \omega(v'_i) = s(a_i) \quad \text{and} \quad \omega(v) = \omega(v') = M.$$

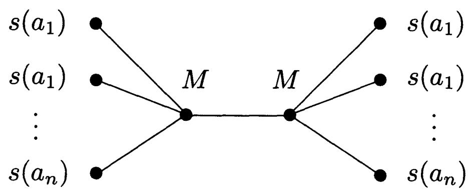


Figure 4. The weighted tree T

For each i, T has an edge joining v_i with v, and joining v_i' with v'. Finally there is an edge between v and v'. Theorem 4.7 shows that the coefficient of $x_{s(a_1)} \cdots x_{s(a_n)} x^2_{\lfloor \frac{3}{2}M \rfloor}$ in $W(T)$ is equal to the number of partitions of $V(T)$ into $n+2$ blocks such that two blocks have total weight $\lfloor \frac{3}{2}M \rfloor$ and the other have weights $\{s(a_1), \ldots, s(a_n)\}$. The number of such partitions is equal to the number of sets $A' \subseteq A$ such that $\sum\limits_{a \in A'} s(a) = \sum\limits_{a \in A \setminus A'} s(a)$. □

THEOREM 7.11. — *Computing W-coefficient is #P-hard if G is restricted to being a complete graph and in the notation of Problem* 7.1,

$$P = \left\{ \left\lfloor \sum_{v \in V} \frac{1}{2} \omega(v) \right\rfloor, \sum_{v \in V} \left\lfloor \frac{1}{2} \omega(v) \right\rfloor \right\} \quad \textit{and} \quad q = 0.$$

Proof. — We use a reduction from $\# \frac{1}{2}$partition. Suppose (A, s) is an instance of $\# \frac{1}{2}$partition where $A = \{a_1, \ldots, a_{2n}\}$. Let $M = \sum\limits_{a \in A} s(a)$. Now let G be a complete weighted graph with vertices $\{v_1, \ldots, v_{2n}\}$ and $\omega(v_i) = s(a_i) + M$. Clearly

$$\sum_{v \in V} \omega(v) = (2n+1)M.$$

Adding M to $s(a_i)$ to give $\omega(v_i)$ ensures that if we partition $V(G)$ into two sets of equal weight then the two sets must each contain n vertices. Then Theorem 4.7 shows that the coefficient of $x_{\lfloor \frac{1}{2}(2n+1)M \rfloor} x_{\lfloor \frac{1}{2}(2n+1)M \rfloor}$ is equal to

$$\big(T(K_n; 1, 0)\big)^2 \frac{1}{2} \alpha(A, s)$$

where $\alpha(A, s)$ is the number of solutions to the instance of $\# \frac{1}{2}$partition. The factor of a half is because the $\# \frac{1}{2}$partition problem counts twice each partition of A into parts with the same total size. A recurrence relation to evaluate $T(K_n; 1, 0)$ is given in [1] and it is easy to see that this leads to a polynomial time algorithm for evaluating $T(K_n; 1, 0)$. □

We now turn to the problem W-evaluation.

THEOREM 7.12. — *Computing W-evaluation is #P-hard even if G is a star.*

Proof. — The proof is by a reduction from partition. Let (A, s) be an instance of partition. Let $M = \sum_{a \in A} s(a)$. Now let G be the weighted star shown in Figure 5 with $|A| + 1$ vertices where the central vertex has weight M and the other vertices have weights corresponding to sizes of distinct elements of A. The evaluation of W with $x_{\lfloor \frac{3}{2} M \rfloor} = 1$ and $x_{s(a)} = 1$ for all $a \in A$ but $x_j = 0$ otherwise, gives the number of partitions of A into two parts with equal size. □

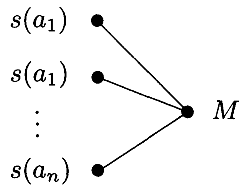


Figure 5. The graph G

It seems worthwhile to note the following obvious corollary:

COROLLARY 7.13. — *Computing W-evaluation is #P-hard for trees.*

THEOREM 7.14. — *Computing W-evaluation is #P-hard even if G is a complete graph and in the notation of Problem 7.2,*

$$S = \left\{ \left(\left\lfloor \sum_{v \in V} \tfrac{1}{2} \omega(v) \right\rfloor , 1 \right) \right\} \quad \textit{and} \quad y_0 = 0.$$

Proof. — Let $M = \sum_{v \in V} \omega(v)$. The theorem follows from the corresponding theorem concerning W-coefficient. Theorem 7.11 shows that it is #P-hard to compute the coefficient of $x^2_{\lfloor \frac{1}{2} M \rfloor}$, where $M = \sum_{v \in V} \omega(v)$. If $M \geq 4$ then the evaluation of W with $x_{\lfloor \frac{1}{2} M \rfloor} = 1$ and $x_j = 0$ otherwise, is equal to the coefficient of $x^2_{\lfloor \frac{1}{2} M \rfloor}$. □

This highlights the fact that it is not really the graph, but the weights which make W-evaluation hard to compute. Of more interest is to consider U and its complexity. Clearly computing U is hard in general. The complexity of the polynomial $U_G(\boldsymbol{x}, 0)$ when G is restricted to being an intersection graph of a chord diagram is of particular interest because these evaluations correspond to weight functions on chord diagrams. In [7], computing the chromatic number of such a graph, commonly known as a circle graph, is shown to be NP-hard. This implies the following theorem:

THEOREM 7.15. — *Computing $U_G(\boldsymbol{x},0)$ is NP-hard even if G is restricted to being the intersection graph of a chord diagram.*

Proof. — Equation (1) shows that the chromatic polynomial can be computed from $U_G(\boldsymbol{x}, 0)$ and this easily gives the chromatic number. □

The reader will notice that, unlike our earlier results, we are only claiming in Theorem 7.15 that U_G is NP-hard. We strongly believe that it is possible to replace this by $\#P$-hard, however the only proof we can see is by following through the reductions of [7] and [10] and showing that the reductions are at least weakly parsimonious. To do this rigorously would take a great deal of time and space and at this stage hardly seems justified.

8. Conclusion.

Although the functions W_G, U_G are formally very similar there does seem to be a significant difference in their computational complexity. For example, while W is hard even for the star or the complete graph, it is easy to see that

$$U(St_n; \boldsymbol{x}, y) = \sum_{k=0}^{n-1} \binom{n-1}{k} x_{k+1}\, x_1^{n-k-1}$$

if St_n denotes the star with n vertices. Similarly for the path P_n, if U_n denotes $U(P_n)$ then it is easy to prove

$$U_n = x_1 U_{n-1} + x_2 U_{n-2} + \cdots + x_{n-1} U_1 + x_n$$

so it too is easy to compute and in fact has a 'nice' generating function (see a corresponding formula for $X(P_n)$ in [18], Proposition 5.3).

Finally, for the complete graph K_n, [19] gives a generating function completely analogous to that of the Tutte polynomial, namely

$$\sum_{d\geq 0} X_{K_d}(\boldsymbol{x}, t)\frac{u^d}{d!} = \exp\Big(\sum_{m\geq 0} C_m(t)\, p_m(\boldsymbol{x})\frac{u^m}{m!}\Big)$$

where

$$C_m(t) = \sum_{i=m-1}^{\binom{m}{2}} c_{mi}\, t^i$$

and where c_{mi} is the number of connected simple graphs with m vertices and i edges. From our correspondence given by Theorem 6.2, we can read off the corresponding formula

$$\sum_{d\geq 0} U_{K_d}(\boldsymbol{x}, y)\frac{u^d}{d!} = \exp\Big((y-1)^{-(m-1)} \sum_{m\geq 0} \frac{C_m(y-1)\, x_m\, u^m}{m!}\Big).$$

It now follows in exactly the same way as in the proof of the corresponding result for ordinary Tutte polynomials by Annan [1] that there exists a polynomial time algorithm for computing $U(K_n; \boldsymbol{x}, y)$. This contrasts with Theorem 7.14 and together these results suggest that computationally U is closer to T than to W. However although it is easy to find non-isomorphic graphs with the same Tutte polynomial, until recently we knew of no pair of non-isomorphic graphs with the same polynomial U. I. Sarmiento [17] has recently found an infinite family of such pairs. We still do not know whether there exist non-isomorphic trees with the same U.

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