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ANTISYMMETRIC FLOWS AND STRONG COLOURINGS OF ORIENTED GRAPHS

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Dedicated to the memory of François Jaeger

1. Introduction.

Cycle Double Cover type conjectures and Nowhere-Zero Flows Problems lie in the core of graph theory [30]. They present a complex web of results, reductions and transformations and conjectures which unify algebraical and topological approach to graphs. The area became classic in the work of Tutte, Szekeres, Jaeger, Seymour, Sabidussi and others, see [12], and still a very nice authoritative survey [13], another more recent survey [26].

This central area of graph theory led to important modifications (such as Bouchet theory of "bidirected flows", see *e.g.* [8], [17], [32]), see also [4]. Here we present yet another approach which is motivated by the recently defined version of oriented chromatic numbers (of directed graphs), see *e.g.* [19], [20], [23]. Among others, we define notions of Antisymmetric-Flow (ASF) (in the analogy to Jaeger B-flow) [14], strong-oriented chromatic number $\overrightarrow{\chi_s}$ (as the modulo version of oriented chromatic number) and strong colourable Cycle Double Cover. All these notions depend on the particular orientation of the graph.

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We shall give several proofs of the fact that an ASF exists for any oriented graph without oriented 2-cut. This allows us to define upper and lower ASF number of any 3-edge connected undirected graph. While the lower ASF number is closely connected to the NZF number, to investigate upper ASF number is much harder and, in fact, presently it is unknown whether such upper number is bounded by a constant. This is another contribution to the well known phenomenon that minor-based theories are still undeveloped and even difficult to create for oriented graphs.

In the positive direction we relate ASF to oriented chromatic number [5], [11], [23] and in particular we show that any planar graph has a $(\mathbb{Z}_6)^5$-ASF. The strong-oriented chromatic number of planar graphs is bounded by 7776. This is indeed a large number and perhaps largest number presently involved in colourings of planar graphs, compare [2], [10], [16], [18], [23], [24], [31].

This paper is organized as follows:

In Section 2 we introduce all the necessary notions.

In Section 3 we state some elementary properties and we prove that any oriented graph has an ASF.

In Section 4 we reveal the original motivation of our study: we relate all the previous results to the oriented chromatic number and prove that the class of all planar graphs has a $(\mathbb{Z}_6)^5$-strong-oriented colouring.

Section 5 contains several problems and remarks which illuminate what we believe is an interesting niche of classical problems.

2. Definitions.

Let $G = (V, E)$ be an oriented graph; if $S \subset V$ we denote by $\omega^+(S)$ the set of the edges which begin in S and terminate not in S. We write

$$\omega^-(S) = \omega^+(V \setminus S).$$

If M is an abelian group (with additive notation), then a M-flow is a mapping ϕ from E to M such that: for all $S \subset V$,

$$\sum_{e\in\omega^+(S)} \phi(e) - \sum_{e\in\omega^-(S)} \phi(e) = 0.$$

Now if $B \subset M$, such that $0 \notin B$ and $B = -B$ then a B-nowhere-zero flow (shorter B-NZF) of a graph G is a flow of G where the values are taken in B (see [14]).

As is customary a M-NZF is defined as a B-NZF for $B = M \setminus \{0\}$.

When $M = \mathbb{Z}$ and $B \subseteq [1-k, -1] \cup [1, k-1]$ $(k \geq 2)$, we will speak about k-NZF. A classical result of Tutte [28] is that for a graph a $\mathbb{Z}_k$-NZF exists if and only if a k-NZF exists.

In this context we propose the following:

DEFINITION OF ANTISYMMETRIC-FLOW. — *If $B \subset M$ is such that $B \cap -B = \emptyset$, then a flow with values taken in B will be called an antisymmetric-flow. In this case we will say that G has a M-antisymmetric-flow (shortly M-ASF).*

One could also say that an antisymmetric-flow is a nowhere-inverse flow.

No graph has a $(\mathbb{Z}_2)^k$-ASF and any Eulerian graph with an Eulerian orientation (*i.e.* balanced: $\forall\, x \in V$, $d^+(x) = d^-(x)$) has a $\mathbb{Z}_3$-ASF.

If $A = \mathbb{Z}$ and all the values of the antisymmetric-flow are belonging to $[1-k, k-1]$ $(k \geq 2)$ then we will say that G has a k-ASF.

If G is an undirected graph we will say that G has a M-ASF, if it has a M-ASF for some orientation.

It is a non trivial fact that 3-edge connected graphs have ASF (this will be proved below).

Let G be a graph and D an orientation of G, we will denote by

$$\operatorname{asf}(G, D)$$

the smallest order k of a group M such that G has a M-ASF, under the orientation D. The *upper antisymmetric-flow number* of G is by definition:

$$\operatorname{asf}_U(G) = \max_D \operatorname{asf}(G, D),$$

the maximum is taken over all the possible orientations of G. We also call

$$\operatorname{asf}_L(G) = \min_D \operatorname{asf}(G, D),$$

the *lower antisymmetric-flow number* of graph G.

It follows from Theorem 2 below that $\operatorname{asf}_L(G)$ is defined for every bridgeless graph and from Theorem 4 that $\operatorname{asf}_U(G)$ is defined for every 3-edge connected graph.

We review our approach somewhat in the reverse order. The notion of antisymmetric-flow and antisymmetric-flow numbers was inspired by a variant of the oriented chromatic number which we are going to explain now.

Given two oriented graphs $G = (V, A)$ and $G' = (V', A')$ a homomorphism from G to G' is any mapping $f: V \to V'$ satisfying

$$(x, y) \in A \implies \big(f(x), f(y)\big) \in A'.$$

The existence of a homomorphism from G to G' will be denoted by $G \to G'$. In [20] we consider the following problem:

> Given an oriented graph $G = (V, A)$, find the smallest number of vertices of an oriented graph $G' = (V', A')$ for which $G \to G'$. This number is denoted by $\vec{\chi}(G)$ and called the *oriented chromatic number* of G. For an undirected graph G the *oriented chromatic number* of G is denoted by $\vec{\chi}(G)$ and it denotes the maximal oriented chromatic number of an orientation of G.

In this paper we consider the following more restrictive definition:

Given an oriented graph $G = (V, A)$ find the smallest q such that there exists an oriented graph $G' = (V', A')$ with q vertices labelled by the q elements of an abelian additive group M for which there exists a homomorphism f: $G \to G'$ such that for any pair of arcs of G (x, y) and (x', y') (not necessarily distinct), we have $f(y) - f(x) \neq -(f(y') - f(x'))$ (this condition will be called the no-opposite value condition). Then we will say that f is a *M-strong-oriented colouring.*

The smallest number q will be called the strong-oriented chromatic number, and denoted by $\vec{\chi_s}(G)$. If the graph is undirected $\vec{\chi_s}(G)$ will be the maximum of $\vec{\chi_s}(G)$ taken over all the possible orientations.

We recall now the definition of the oriented dual graph. Given an oriented plane graph G corresponding to each face f of G there is a vertex f^* of the dual graph G^*, and corresponding to each arc e of G there is an arc e^* of G^*. Two vertices f^* and g^* are joined by the arc e^* if and only if the corresponding faces are separated by the arc e in G. The arc e^* is oriented so that if we go from the initial vertex of e to the terminal vertex then e is crossed by the arc e^* from right to left.

It is then easy to see by comparing the corresponding definitions that if G is a oriented planar graph then G has a $\mathbb{Z}_k$-strong-oriented colouring if and only if the oriented dual graph G^* of G has a $\mathbb{Z}_k$-ASF.

3. Elementary properties.

We start the following easy lemma:

LEMMA 1. — *If an undirected graph G has a M-flow ϕ such that $\phi(e) \neq -\phi(e)$ for any $e \in E$, then it has a M-ASF for some orientation.*

Proof. — Let a be an element of M, we change the orientation of all the arcs having a flow's value equal to $-a$. We do the same to all the values of the flows, then we obtain an M-ASF. □

COROLLARY 2. — *If an undirected graph G has a M-flow, M being a group of odd order then there is an orientation of G with an M-ASF.*

Remark. — If a graph has a p-ASF, then it has a p'-ASF for any $p' \geq p$. Moreover if a graph has a $\mathbb{Z}_p$-ASF then it has $\mathbb{Z}_{p'}$-ASF for any $p' \geq p$ (p and p' odd).

We will use the following result of Tutte [28]:

THEOREM 1. — *If $G = (V,E)$ has a $\mathbb{Z}_k$-NZF f, then it has a k-NZF g such that $f(e) \equiv g(e) \bmod k$ for any $e \in E$ and the flows f and g are defined under the same orientation.*

OBSERVATION 1. — *If an undirected graph G has a p-NZF (p odd) then it has a $\mathbb{Z}_p$-ASF.*

Proof. — If G has a p-NZF for some orientation, then it admits a $\mathbb{Z}_p$-NZF under the same orientation. By reversing the direction of some arcs, we find an orientation of G for which the NZF uses only the values $\left\{\overline{1,\ldots,\tfrac{1}{2}(p-1)}\right\}$. It is obvious that this is a $\mathbb{Z}_p$-ASF. □

Note that if G has a p-NZF (p even) then it has a $\mathbb{Z}_{p+1}$-ASF.

In the other direction we have

OBSERVATION 2. — *If an undirected graph G has a $\mathbb{Z}_p$-ASF then it has a p-ASF.*

Proof. — We choose an orientation for which the $\mathbb{Z}_p$-ASF f is such that

- $f(E) = \{\overline{1}, \dots, \overline{\frac{1}{2}p - 1}\}$ if p is even or
- $f(E) = \{\overline{1}, \dots, \overline{\frac{1}{2}(p - 1)}\}$ if p is odd.

By the theorem of Tutte there exists a p-NZF g of G under the same orientation, such that

$f(e) \equiv g(e) \bmod p$. Hence for p even, if $f(e) = i$ then g(e) can be i or $i - p$. We have then

$$g(E) \subseteq \{1, 1 - p, 2, 2 - p, \dots, \tfrac{1}{2}p - 1, \tfrac{1}{2}p - 1 - p\}.$$

It is easy to see that $g(E)$ contains no opposite values. The same verification holds for p odd. □

Note that the converse of Observation 2 holds only for p odd.

For k even, it may happen that for a graph G there is an k-ASF while no $\mathbb{Z}_k$-ASF exists. This is due to the fact that an k-ASF f needs not satisfy the condition $f(e) + f(e') \neq 0 \bmod k$ for any pair of edges e, e' of G. A k-ASF satisfying this additional property (which is obviously necessary) implies the existence of $\mathbb{Z}_k$-ASF. In this case we have the equivalence between the existence of a k-ASF and a $\mathbb{Z}_k$-ASF for any integer $k > 2$.

THEOREM 2. — *Every bridgeless graph G has an orientation for wich G has a $\mathbb{Z}_7$-ASF (i.e.* $\mathrm{asf}_L(G) \leq 7$ *for every bridgeless graph G).*

Proof. — By Seymour's theorem [25], any bridgeless graph has a $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF. By the theorem of Tutte it has a 6-NZF wich is also a 7-NZF, by Observation 1 it admits a $\mathbb{Z}_7$-ASF. □

Also the Jaeger's theorem [15] giving the existence of a 8-NZF for any bridgless graph, implies that any bridgless graph has a $\mathbb{Z}_9$-ASF.

THEOREM 3. — *Let p be an odd number, G an undirected graph. If for an orientation G has a $\mathbb{Z}_{2p}$-ASF then for some other orientation it has a $\mathbb{Z}_p$-ASF. On the other hand every $\mathbb{Z}_p$-ASF may be thought as $\mathbb{Z}_{2p}$-ASF and thus the existence of orientations with $\mathbb{Z}_{2p}$-ASF and $\mathbb{Z}_p$-ASF are equivalent.*

Proof. — First, suppose that G has a $\mathbb{Z}_{2p}$-ASF, denoted by f. Clearly we may assume that the values of the antisymmetric-flow f are taken in $\{\overline{1},\dots,\overline{p-1}\}$. By the theorem of Tutte there exists a $2p$-NZF g under the same orientation, such that: $f(e) \equiv g(e) \bmod 2p$. If $f(e) = i$ then $g(e)$ can be i or $i - 2p$. We have then

$$g(E) \subseteq \{1, 1-2p, 2, 2-2p, \dots, p-1, p-1-2p\}.$$

Let $g'(e)$ be the residue of $g(e)$ modulo p. Since $g(e) \neq 0, \pm p$, it follows that $g'(e) \in \mathbb{Z}_p$ is non-zero. Hence G has a $\mathbb{Z}_p$-NZF. And by reversing the orientation of certain arcs, we will have a $\mathbb{Z}_p$-ASF in G.

The reverse implication follows by Observation 1: if G has a $\mathbb{Z}_p$-ASF then it has a $\mathbb{Z}_{2p}$-ASF. □

Let us remark that for p even we can only prove that if a graph has a $\mathbb{Z}_{2p}$-ASF then it has a $\mathbb{Z}_p$-NZF.

COROLLARY 2. — *If G has a $\mathbb{Z}_4$-ASF then it is an Eulerian graph.*

Proof. — By using the same argument as in theorem 3 it is easy to see that G has a $\mathbb{Z}_2$-NZF. It follows that G is an Eulerian graph. □

For a given orientation of a graph G, the existence of a $\mathbb{Z}_p$-ASF is stronger that the existence of a p-ASF. This may be seen as follows:

Consider a cubic bipartite graph $G = (V, E)$ with the bipartition $V = X \cup Y$ (where $|X| = |Y|$). The edges of a matching are oriented from X to Y and the remaining edges from Y to X. Let f be the valuation of the edges of G such that $f(e) = 2$ if e is an arc belonging to the matching and $f(e) = 1$ for the arcs not belonging to the matching. It is easy to see that f is a 3-ASF, but neither does G admit a $\mathbb{Z}_3$-ASF nor a $\mathbb{Z}_4$-ASF under this orientation. It admits a $\mathbb{Z}_5$-ASF under the same orientation.

4. Antisymmetric-flows.

Let G be an undirected graph. If G contains an isthmus or a 2-cut then G does not admit a nowhere-zero flow. If G contains an edge-cut made of two edges e_1,e_2 between components K_1 and K_2 and if we take an orientation of the graph such that the edges e_1 and e_2 are both oriented from K_1 to K_2 then clearly G with this orientation has no antisymmetric-flow. A 2-edge-cut will be also called a 2-cut.

Now we prove that every graph without a 2-cut has an antisymmetric-flow. We shall use the following concept and an easy lemma:

LEMMA 2. — *A connected bridgeless graph G has no 2-cut if and only if for every pair $\{e,e'\}$ of distinct edges of G there exits a separating circuit C (i.e. a circuit which contains exactly one of the edges e,e').*

Proof. — Suppose that $G \setminus \{e, e'\}$ is connected, let $e = \{x, y\}$. Then a path which joins x to y in $G \setminus \{e, e'\}$ together with the edge e forms a cycle which does not contain e' in G.

In the reverse direction, suppose that for any pair of distinct edges there exist a separating cycles. Thus given a pair $\{e, e'\}$ of distinct edges find separating cycles with $e \in C_e$, $e' \notin C_e$. The end vertices of e belong to the same component thus $G \setminus \{e, e'\}$ is connected. □

Let G be a graph. A family $C_1, \ldots, C_t$ of circuits of G is said to be *edge-distinguishing* if any pair $\{e, e'\}$ of distinct edges of G there exists C_i which is containing exactly the edge e and not containing the edge e'. The above lemma can be formulated as follows: every 3-edge connected graph G has a family of distinguishing cycles.

Note that for 3-edge connected planar graphs the set of facial cycles is distinguishing.

More generally we have

LEMMA 3. — *For a 3-edge connected graph any cycle basis is edge distinguishing.*

Proof. — Assume the contrary: let $\mathcal{C}$ be a cycle basis and let e, e' be distinct edges such that any cycle $C \in \mathcal{C}$ contains either both edges e, e' or none of them. As $\mathcal{C}$ is a basis there is no circuit which contains exactly one of the edges. This contradics lemma 2. □

THEOREM 4. — *Let G be a graph, $C_1, \ldots, C_t$ a family of edge-distinguishing cycles. Then then for any orientation G has a $\mathbb{Z}_3^t$-ASF.*

Proof. — Let $G = (V, E)$ with a fixed orientation. Let C_i denote also the set of arcs. For each C_i by doing a walk on C_i we define the flow

$$f_i : C_i \longrightarrow \mathbb{Z}_3$$

by $$f_i(xy) = \begin{cases} +1 & \text{for forwarding arcs of } C_i, \\ 2 & \text{for the backwarding arcs of } C_i, \\ 0 & \text{for the arcs not belonging to } C_i. \end{cases}$$

Put

$$f(xy) = \big(f_i(xy);\ i = 1, \ldots, t\big).$$

We prove that f is an antisymmetric-flow.

Obviously $f(x, y) \neq (0, \ldots, 0)$. Let a and b two arcs of G. Let C_i be a cycle containing exactly one of the arcs a,b; let say $a \in C_i$ and $b \notin C_i$. Then $f_i(a) \neq 0$ and $f_i(b) = 0$. Thus $f(a) \neq -f(b)$ (in $\mathbb{Z}_3^t$) (note that we must use $\mathbb{Z}_3^t$ as in $\mathbb{Z}_2^t$ for every $x = -x$). □

Denote by $\mathcal{C}(G)$ the cycle space of G.

COROLLARY 3 (*Existence of ASF*). — *Any orientation of a 3-edge connected graph G has a $\mathbb{Z}_3^{\dim \mathcal{C}(G)}$-ASF.*

Remark. — Another, although less explicite proof of the fact that any orientation of any 3-edge connected graph has an antisymmetric-flow may be sketched as follows:

Existence of ASF (second proof). — Let $G = (V, E)$ be an orientation of a 3-edge connected graph with m edges and n vertices. The set of all $\mathbb{R}$-flows on G ($\mathbb{R}$-flows are usually called circulations) is the set of all solutions $(x_e\ ;\ e \in E)$ defined by equations:

$$\forall u \in V\colon \quad \sum_{v \in V} x_{uv} - \sum_{v \in V} x_{vu} = 0 \tag{1}$$

the variable x_{uv} is defined if and only if $(uv) \in E(G)$.

The set of all solutions form a $(m-n)$-dimensional sub-space $\mathcal{V}$ of $\mathbb{R}^E$. The projection of $\mathcal{V}$ on any coordinate (*i.e.* edge e of G) is 1-dimensional as otherwise e would be a cut. Similarly the projection of $\mathcal{V}$ to any pair of coodinates e, e' has dimension 2 (for otherwise there would be no circuit which would distinguish e and e' and thus $\{e, e'\}$ would be a 2-cut). Thus there exists a solution of the system (1) which misses any of the hyperplanes

$$x_e = 0, \quad x_{uv} + x_{u'v'} = 0.$$

And there exists also an integer solution. □

Let us remark that this algebraical argument implies a slight improvement of the existence of ASF:

COROLLARY 4. — *Let G be a 2-edge connected oriented graph without an oriented 2-cut. Then G has an ASF.*

The following is an easy corollary of Theorem 4:

COROLLARY 5. — *Let $G_i = (V_i, E_i)$ (where $i \in \{1,2\}$) be graphs. Let G_i has a family of t_i edge-distinguishing cycles. Then $G_1 \cup G_2$ has a $\mathbb{Z}_3^{t_1+t_2}$-ASF.*

(Observe that the union of edge-distinguishing cycle families in G_i is edge-distinguishing.)

Continuing in the same direction we get the following:

Let k be a fixed number. We say that a class $\mathcal{G}$ of graph is k-edge degenerated if any graph from $\mathcal{G}$ can be obtained as subsequent strict amalgamation of 3-edge connected graphs with at most k edges.

Here a strict amalgamation of graphs G_1 and G_2, $G_i = (V_i, E_i)$ $(i \in \{1, 2\})$ is a graph $G = (V, E)$ such that

$$V = V_1 \cup V_2, \quad E = E_1 \cup E_2, \quad \text{with} \quad E_1 \cap E_2 = \emptyset.$$

Then we have

COROLLARY 6. — *Any class of k-edge degenerate graphs has a bounded antisymmetric-flow for any orientation.*

Proof. — A k-degenerated graph G is a disjoint edge union of 3-edge connected graphs G_i each having at most k edges. Thus we can define an antisymmetric-flow on G by defining it on each of the graphs G_i. It follows that k-degenerated graphs have a bounded ASF. □

We have also

THEOREM 5. — *Let G be any graph. Let G' be the multigraph which arises by multiplying every edge of G a least 3 times. Then any orientation of G' has a $\mathbb{Z}_5$-ASF.*

Proof. — Let K_2^m $(m \geq 2)$ the graph having 2 vertices linked by m edges. It is easy to check that any orientation of the graph K_2^3 has a $\mathbb{Z}_5$-ASF using the values $\overline{1}$ and $\overline{2}$, the same can be verified for any orientation of the graphs K_2^4 and K_2^5. For $m \geq 3$ We have the following:

- if $m \equiv 0 \bmod 3$, K_2^m is an union of graphs K_2^3;
- if $m \equiv 1 \bmod 3$, K_2^m is an union of graphs K_2^3 and a K_2^4;
- if $m \equiv 2 \bmod 3$ K_2^m is an union of graphs K_2^3 and a K_2^4.

Hence for any $m \geq 3$ K_2^m has a $\mathbb{Z}_5$-ASF using the values $\overline{1}$ and $\overline{2}$. We contruct any oriented graph G' edge by edge from G by replacing the edges of G by the desired orientations of K_2^m with the $\mathbb{Z}_5$-ASF values. Then it is clear that G' has a $\mathbb{Z}_5$-ASF. This completes the proof. □

It is interesting to note that this theorem is not true for graphs with multiplicities at most 2:

Consider the graph $\overrightarrow{K}_{2,4}$ depicted in Figure 1. It is easy to see that the strong-oriented chromatic number of this graph is at least 6. The oriented dual graph $\overrightarrow{K}^*_{2,4}$ of the graph $\overrightarrow{K}_{2,4}$ is a cycle of length 4 with double arcs. It is then clear that $\mathrm{asf}_U(\overrightarrow{K}^*_{2,4}) \geq 6$.

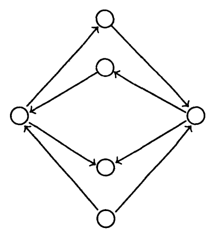

Figure 1. The set $\overrightarrow{K}_{2,4}$

5. Strong-oriented colouring.

A *k-colouring* of the vertices of an undirected graph is said to be *acyclic* if the subgraph induced by the vertices with any two colours has no cycle. The acyclic chromatic number $a(G)$ of a graph G is the smallest m such that G as an acyclic m-colouring.

We are going to prove now that every oriented graph G has a strong-oriented colouring which depends only on the acyclic chromatic number $a(G)$ of the graph G.

THEOREM 6. — *Let G be an undirected graph with acyclic chromatic number at most k. Then any orientation of G admits a $(\mathbb{Z}_6)^k$-strong-oriented colouring. Thus $\overrightarrow{\chi_s}(G) \leq 6^k$ for any graph G with $a(G) \leq k$.*

Before proving Theorem 6 let us list some corollaries and consequences.

The result for the planar graphs is obtained as a consequence of the following theorem [5]:

THEOREM 7 (Borodin, 1979). — *If G is a planar graph then $a(G) \leq 5$.*

Theorem 7 combined with Theorem 6 gives the corollary:

COROLLARY 7. — *Every oriented planar graph is strongly oriented 6^5-colourable.*

By duality we have

COROLLARY 8. — *If G is a 3-edge connected planar graph, then* $\mathrm{asf}_U(G) \leq 6^5$.

A cycle double cover (CDC) of an undirected graph G is a family $C_1, \dots, C_t$ of cycles (considered as edge sets) such that each edge of G belongs to exactly two cycles of the family.

A cycle double cover is said to be p-colourable if all the edges of G can be covered by p Eulerian subgraphs $\mathcal{E}_1, \dots, \mathcal{E}_p$ such that each edge is covered twice. This is the same as to colour the cycles $C_1, \dots, C_t$ by p colours such that any two cycles which share the same colour are edge disjoint. It has been conjectured by Celmins and Preissmann [9], [22] that any 2-edge connected graph has a 5-colourable CDC. We shall present a stronger (oriented) variant of these concepts.

Let G be an undirected graph. We say that a cycle double cover $C_1, \dots, C_t$ is p-strongly colourable if for every tournament R on $\{1, \dots, t\}$, there exists a mapping $f\colon \{C_1, \dots, C_t\} \to \mathbb{Z}_p$ with the following properties:

1) $f(C_i) \neq f(C_j)$ whenever $C_i \cap C_j \neq \emptyset$.

2) If $(i, j) \in R$, $(i', j') \in R$, and $C_i \cap C_j \neq \emptyset$, $C_{i'} \cap C_{j'} \neq \emptyset$ then

$$f(C_j) - f(C_i) \neq -\big(f(C_{j'}) - f(C_{i'})\big).$$

(The tournament R may be viewed as an arbitrary orientation of the abstract dual defined by the CDC $C_1, \dots, C_t$. In case of a planar graph G such an orientation may result from orientation of edges of G). Then we have the following:

COROLLARY 9. — *Every 2-connected planar graph has a 7776-strongly colourable CDC.*

Proof. — Consider facial cycles and a strong colouring of arbitrary orientation of the dual. Then apply Corollary 8. □

As the oriented chromatic number, the strong-oriented chromatic number is linked to acyclic chromatic number:

COROLLARY 10. — *For any class $\mathcal{K}$ of graphs the following statements are equivalent:*

1) *The acyclic chromatic number is bounded for all graphs from $\mathcal{K}$.*

2) *The oriented chromatic number is bounded for all graphs from $\mathcal{K}$.*

3) *The strong-oriented chromatic number is bounded for all graphs from $\mathcal{K}$.*

Proof. — The equivalence $1 \Leftrightarrow 2$ is proved in [19]. The implication $1 \Rightarrow 3$ follows from Theorem 6. The implication $3 \Rightarrow 2$ is trivial. □

Before proving Theorem 6 let us state the following: Let $\vec{C_6}$ be the directed cycle with 6 vertices labelled by elements of $\mathbb{Z}_6$ in the order $\overline{1}, \overline{3}, \overline{5}, \overline{1}, \overline{3}, \overline{5}$. It is a bipartite graph and the two stable sets, denoted by A and B, of the bipartition have the vertices labelled by $\overline{1}, \overline{3}, \overline{5}$.

We have the following easy proposition:

PROPOSITION 1. — *Let T be a tree and V_1 and V_2 be any bipartition of its set of vertices according to the fact that a tree is a bipartite graph. Then there exists a homomorphism of T into $\overrightarrow{C_6}$ which maps V_1 into A and V_2 into B.*

Proof. — It easy to see that there is a homomorphism between T and $\overrightarrow{C_6}$. For example we can use a depth first search of the tree T from any root of V_1 and map the root vertex into any vertex of $\overrightarrow{C_6}$ according to the fact that we want to map this vertex into A. Then each new vertex x encountered will be mapped into a vertex of $\overrightarrow{C_6}$ according to the orientation of the edge linking the vertex x to another already visited vertex. It is then clear that V_1 is mapped into A and V_2 into B. □

Proof of Theorem 6. — In the proof we proceed analogously as in [23], but we use a different target graph H:

The set of vertices of H consists from all k-tuples of form

$$(a_1, a_2, \dots, a_{i-1}, \overline{0}, a_{i+1}, \dots, a_k)$$

where $i \in [1,k]$ and $a_\ell \in \{\overline{1}, \overline{3}, \overline{5}\} \subset \mathbb{Z}_6$ for $\ell \in [1, i-1] \cup [i+1, k]$.

Thus the cardinality of $V(H)$ is then $k \cdot 3^{k-1}$ and the vertices are labelled by elements of $(\mathbb{Z}_6)^k$.

The set of arcs of H is defined as follows:

Let $i < j$, and let

$$u = (a_1, a_2, \dots, a_{i-1}, \overline{0}, a_{i+1}, \dots, a_j, \dots, a_k),$$
$$v = (b_1, b_2, \dots, b_i, \dots, b_{j-1}, \overline{0}, b_{j+1}, \dots, b_k)$$

be vertices of H. Then u and v are linked by an arc of the same direction of the arc linking $a_j \in A$ and $b_i \in B$ in $\overrightarrow{C_6}$.

We prove now that the graph H satisfies the no-opposite value condition.

Let uv an arc of H with $u = (a_1, a_2, \dots, a_{i-1}, \overline{0}, a_{i+1}, \dots, a_j, \dots, a_k)$ and $v = (b_1, b_2, \dots b_i, \dots, b_{j-1}, \overline{0}, b_{j+1}, \dots, b_k)$ with $i < j$ (for $j < i$ the reasoning is the same). Consider the difference $v - u$, we have

$$v - u = (b_1 - a_1, \dots, b_{i-1} - a_{i-1}, b_i, b_{i+1} - a_{i+1}, \dots,$$
$$b_{j-1} - a_{j-1}, -a_j, b_{j+1} - a_{j+1}, \dots, b_k - a_k).$$

$b_\ell - a_\ell \in \{\overline{0}, \overline{2}, \overline{4},\}$ for $\ell \neq i, j$ and $b_i, -a_j \in \{\overline{1}, \overline{3}, \overline{5}\}$. If by contradiction we assume that there is an other arc tw of H such that $v - u = -(w - t)$, then the zero components of t and w must be at the same place as the zero components of u and v. Since uv is an arc, it follows from the definition of H that $a_j = x$ and that $b_i = x + \overline{2}$, $x \in \{\overline{1}, \overline{3}, \overline{5}\}$.

Suppose that

$$t = (c_1, c_2, \dots, c_{i-1}, \overline{0}, c_{i+1}, \dots, c_j, \dots, c_k),$$
$$w = (d_1, d_2, \dots d_i, \dots, d_{j-1}, \overline{0}, d_{j+1}, \dots, d_k).$$

Similarly, as tw is an arc, we have by definition of H that $c_j = y$ and that $d_i = y + \overline{2}$. Then $u - v = -(w - t)$ implies $x + \overline{2} = -(y + \overline{2})$ and $x = -y$, which implies $\overline{4} = \overline{0}$, a contradiction.

The other possibility is the case where the zero component of t is at the place j and the zero component of w is at the place i. Since tw is an arc, it follows from the definition of H that $c_i = y$ and that $d_j = y + \overline{2}$. Since $v - u = -(w - t)$ we have $x + \overline{2} = y$ and $y + \overline{2} = x$, which implies again $\overline{4} = \overline{0}$, a contradiction. Hence the no-opposite value condition is satisfied.

We define the homomorphism $f\colon G \to H$:

Let $G = (V, E)$ be a graph with acyclic colouring

$$V = V_0 \cup V_2 \cup \ldots \cup V_{k-1},$$

and $\overrightarrow{G}$ an orientation of G. Denoted by $G_{i,j}$ the graph induced by $V_i \cup V_j$ $(0 \leq i < j \leq k)$ and $\overrightarrow{G}_{i,j}$ the orientation inherited from $\overrightarrow{G}$. For every i, j, $G_{i,j}$ is a forest.

Hence by Proposition 1 there exists a homorphism $c_{i,j}$ of $\overrightarrow{G}_{i,j}$ in $\overrightarrow{C}_6$ which maps V_i (resp. V_j) into A (resp. B).

Let u be a vertex of G; then u belongs to one V_i for some $i \in [1, k]$. The image of u, will be

$$f(u) = \big(c_{1,i}(u), \ldots, c_{i-1,i}(u), \overline{0}, c_{i,i+1}(u), \ldots, c_{i,k}(u)\big).$$

We have to prove now that if $u \in V_i$ and $v \in V_j$ define an arc uv then $f(u)f(v)$ is an arc of H, assume that $i < j$. Hence

$$f(u) = \big(c_{1,i}(u), \ldots, c_{i-1,i}(u), \overline{0}, c_{i,i+1}(u), \ldots c_{i,j}(u), \ldots, c_{i,k}(u)\big)$$

and

$$f(v) = \big(c_{1,j}(v), \ldots, c_{i,j}(v) \ldots, c_{j-1,j}(v), \overline{0}, c_{j,j+1}(v), \ldots, c_{j,k}(v)\big)$$

are, by the definition of the graph H, linked by an arc according to the arc existing between $c_{i,j}(u)$ in A and $c_{i,j}(v)$ in B. Since $c_{i,j}$ is a homomorphism $f(u)f(v)$ is an arc of H. This completes the proof of Theorem 6. □

An alternative approach to prove Theorem 6 is via embedding of oriented graphs into oriented Cayley graphs. For undirected graphs such embeddings were investigated in several papers (see *e.g.* [3]) and this can be generalized (with somewhat weaker bounds) for directed graphs as follows:

For a given group $\mathcal{M}$ and a set A of its elements, we define the graph $X(\mathcal{M}, A) = (V, E)$:

$$V = \mathcal{M} \quad \text{and} \quad E = \{(g, ga)\,;\ g \in \mathcal{M},\ a \in A\}.$$

This is a Cayley digraph. We assume that A does not contain an element a together with its inverse a^{-1}. Then $X(\mathcal{M}, A)$ is an oriented graph (no symmetric arcs). Moreover if A is a set of generators then the graph is connected. Our definition of strong-oriented chromatic number may be replaced as follows:

Given an oriented graph G find a group $\mathcal{M}$ of the smallest possible order such that there is a homomorphism $G \to X(\mathcal{M}, A)$ for a suitable subset A of $\mathcal{M}$ satisfying $A \cap A_{-1} = \emptyset$. It is clear that in this case the no-opposite value condition is verified.

We prove

THEOREM 8. — *For every oriented graph $\vec{G}$, there exists a group $\mathcal{M}$ and a subset A satisfying $A \cap A^{-1} = \emptyset$ such that $\vec{G}$ is a subgraph of $X(\mathcal{M}, A)$.*

Proof. — Let G be a fixed oriented graph with n vertices, choose p such that $\frac{1}{2}p > 2^{n+2}$. Let $\mathcal{M} = \mathbb{Z}_p$. Assume that there exists a set K with the following properties:

1) $|K| = n$;

2) $1 \notin K$;

3) $x^{-1}y \neq y^{-1}x$ for any $x, y \in K$;

4) $x^{-1}y \neq t^{-1}z$ for any $x, y, z, t \in K$.

Now identify the set K with $V(G)$ and put

$$A = \{x^{-1}y\,;\ (xy) \in E(G)\}.$$

Then G is obviously a subgraph of $X(\mathcal{M}, A)$ and $X(\mathcal{M}, A)$ is an oriented graph by the above properties of K.

Hence it suffices to construct a set K. Let K the following subset of $\mathbb{Z}_p$:

$$K = \Big\{2, \ldots, \sum_{i=1}^{\ell} 2^i, \ldots, \sum_{i=1}^{n} 2^i = 2(2^n - 1)\Big\}.$$

Let x and y be two elements of K:

$$x = \sum_{i=1}^{\ell} 2^i, \quad y = \sum_{i=1}^{\ell'} 2^i.$$

Without loss of generality we assume that $\ell' > \ell$, then the element $x^{-1}y$ has the form

$$\sum_{i=\ell}^{\ell'} 2^i \quad (\text{in } \mathbb{Z}_p).$$

This element belongs to the set $\{\overline{1}, \dots, \overline{\lfloor \frac{1}{2}p \rfloor}\}$, while $y^{-1}x$ belongs to $\{\overline{\lfloor \frac{1}{2}p \rfloor} + \overline{1}, \dots, \overline{p-1}\}$. Hence K satisfies the above requirements. □

Now Theorem 8 may be used for an alternative proof of Theorem 6.

COROLLARY 11. — *Let G be an undirected graph. Then $\vec{\chi_s}(G)$ is bounded by a function of $a(G)$.*

Proof. — By [23] there exists an oriented graph Z_k with $k \cdot 2^{k-1}$ vertices such that any orientation $\vec{G}$ of a graph G with $a(G) \leq k$ satisfies $\vec{G} \to Z_k$. Applying Theorem 8 for Z_k we get a group $\mathcal{M}$ and a sub-set A satisfying $A \cap A^{-1} = \emptyset$ such that Z_k is a subgraph of $X(\mathcal{M}, A)$. □

Remark. — Let us compare the bound just obtained by Theorem 8 with the the one obtained by Theorem 6 for planar graphs:

Using Theorem 8 and the theorem of [23], we get that every planar graph is strongly oriented colourable by 2^{83} colours. However $2^{83} > 6^5$ and this is why we gave a direct proof not using embeddings of universal oriented graph with 80 vertices.

However, let us observe that the graph introduced in the proof of Theorem 6 may be thought as a subgragh of an oriented Cayley graph.

Similarly as in [6], [20] we can get better results for colourings of planar graphs of a given girth. They can be summarize as follows:

THEOREM 9. — *Let G be a planar graph. Then the following holds*:

1) *If G has girth at least 14 then* $\vec{\chi_s}(G) \leq 5$,

2) *If G has girth at least 8 then* $\vec{\chi_s}(G) \leq 7$,

3) *If G has girth at least 6 then* $\vec{\chi_s}(G) \leq 11$,

4) *If G has girth at least 5 then* $\vec{\chi_s}(G) \leq 19$.

Proof. — Recall that the circulant (directed) graph $G(n;a_1,a_2,\dots,a_k)$ is the graph whose vertex set is $\mathbb{Z}_n$ and whose arcs are those pairs (x,y) such that $\exists\ i,\ 1 \le i \le k,\ y-x \equiv a_i \pmod n$.

In the proof of Theorem 1 in [6] the used target graphs are the following circulant graphs:

$$T_5 = G(5;1,2), \quad T_7 = G(7;1,2,3), \quad T_{11} = G(11;1,3,4,5,9),$$
$$T_{19} = G(19;1,4,5,6,7,9,11,16,17).$$

It is clear that those graphs are Cayley graphs. Hence we obtain strong-oriented colouring. The result follows. □

By duality we have corresponding antisymmetric-flows. The condition on the girth becomes a condition on the edge-conectivity. We have

COROLLARY 12. — *Let G be a planar multigraph, then the following holds*:

1) *If G is 14-edge connected then* $\mathrm{asf}_U(G) \le 5$,

2) *If G is 8-edge connected then* $\mathrm{asf}_U(G) \le 7$,

3) *If G is 6-edge connected then* $\mathrm{asf}_U(G) \le 11$,

4) *If G is 5-edge connected then* $\mathrm{asf}_U(G) \le 19$. □

6. Concluding remarks.

In this new niche of classical graph theory there are a few results and many more open problems. Let us list two of them:

1) The main problem seems to be the absence of good bounds for Theorem 6 and Corollary 3. It is barely surprising as even in the much less restricted case of oriented chromatic number the gap for planar graphs is very large (16 vrs 80; see [23], [27]). But what is perhaps the central problem is the question whether the upper antisymmetric-flow number of any 3-edge connected graph is bounded by an (universal) constant.

Planar graphs show that this constant cannot be less than 16 but perhaps one should conjecture the negative answer. This is presently open even for special classes of graphs such as (general) orientations of eulerian graphs and bipartite graphs. Also asf_U of the Petersen graph P_{10} is unknown, we now that $\mathrm{asf}_U(P_{10}) \ge 11$.

2) In the spirit of the [21], we could generalize antisymmetric-flows to coloured and mixed flows. One can give a necessary and sufficient condition for the existence of such mixed-flows, it is perhaps interesting that appart from trivial obstacles such finite flows always exist.

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