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A GENERALIZATION OF JAEGER-NOMURA'S BOSE MESNER ALGEBRA ASSOCIATED TO TYPE II MATRICES

by Makoto MATSUMOTO

1. Spin model and its Bose-Mesner algebra.

Let X be a finite set with cardinality n, and $\mathbf{C}[X]$ be the vector space with basis X over the complex number field $\mathbf{C}$. We consider the set of matrices $M_X := \mathrm{Hom}(\mathbf{C}(X), \mathbf{C}(X))$, which is identified with the algebra on $n \times n$ matrices, by identifying $X = \{1, 2, \dots, n\}$. An $n \times n$ matrix $W \in M_X$ is said to be a (two-weight[1]) *spin model*, if it satisfies three algebraic conditions called type I, type II, and type III conditions. These names came from the corresponding Reidemeister moves in the Knot theory, see Jones [6] and Bannai-Bannai [1]. Spin models were introduced to describe a link invariant. For a matrix $M \in M_X$, we denote by $M(a,b)$ its (a,b)-component for every $a, b \in X$. The usual product of matrices, denoted by $M'M$ or $M' \cdot M$, is defined by $M'M(a,b) = \sum_{x \in X} M'(a,x)M(x,b)$, while the *Hadamard product* $M' \circ M$ is defined by $(M' \circ M)(a,b) := M'(a,b)M(a,b)$, which is commutative. We denote by J a matrix whose components are all 1, which is the unit for Hadamard product. If M is invertible with respect to the Hadamard product, its inverse is denoted by M°. Then, the type II condition is described by

$$M \cdot {}^tM^\circ = nI,$$

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[1] Here a spin model means two weight, unless otherwise stated.

where I denote the unit matrix of size n. The type I condition is automatically satisfied if M has both type II and type III properties. Type III condition will be explained later.

1.1. A Bose-Mesner algebra associated with a type II matrix.

A matrix B which satisfies the type II condition above is called a *type II matrix*. For a type II matrix, we introduced [4] after Jaeger [5] and Nomura [7] an algebra $N(W)$ as follows. For each $b, c \in X$, define a vector

$$\mathbf{y}_{b,c} := \left(\frac{W(b,x)}{W(c,x)} \right)_{x \in X} \in \mathbf{C}[X].$$

Then we define an algebra (possibly called Jaeger-Nomura algebra of W)

$$N(W) := \{ M \in M_X \mid \mathbf{y}_{b,c} \text{ is } M\text{'s eigen vector for all } b, c \in X \}.$$

It can be proved that $N(W)$ is a Bose-Mesner (BM) algebra, *i.e.* a commutative subalgebra of M_X closed under Hadamard product and transpose with identity $I \in M_X$ for usual product and identity $J \in M_X$ for Hadamard product. (Thus, it corresponds to an association scheme.)

2. Spin models of index two.

Jaeger-Nomura [5] introduced the notion of index of a spin model. I don't recall the definition, but according to their result we may define a spin model W of index two to be a spin model with the following form:

$$W = \begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ -B & B & C & C \\ B & -B & C & C \end{pmatrix}, \tag{1}$$

which was fully used in [5] to construct a spin model with *chiral* link invariant, *i.e.* one which distinguishes some link from its mirror image.

PROPOSITION 1 ([5], Proposition 8). — *Let A, C be a symmetric matrix of the same size. Then, the matrix W in* (1) *is a spin model if and only if the following conditions are satisfied:*

(A) *A, C are spin models and B is a type II matrix.*

(B) *Let n be the size of W, and D be one of $\pm\sqrt{n}$. Then the following four equalities hold:*

$$\sum_y A(a,y)\frac{B(y,b)}{B(y,c)} = DC(b,c)^{-1}\frac{B(a,b)}{B(a,c)} \tag{2}$$

$$\sum_y C(a,y)\frac{B(b,y)}{B(c,y)} = DA(b,c)^{-1}\frac{B(b,a)}{B(c,a)} \tag{3}$$

$$\sum_y A(a,y)^{-1}B(y,b)B(y,c) = -DC(b,c)^{-1}B(a,b)^{-1}B(a,c)^{-1} \tag{4}$$

$$\sum_y C(a,y)^{-1}B(b,y)B(c,y) = -DA(b,c)^{-1}B(b,a)^{-1}B(c,a)^{-1}. \tag{5}$$

The concrete result of this article is just the following.

THEOREM 2. — *Let A, B, C, D be type II matrices. Then the conditions* (2) *and* (3) *are equivalent, and the conditions* (4) *and* (5) *are equivalent.*

The first equivalence is a direct consequence of [4], Theorem 1, as shown below, but the second equivalence is more delicate and I prove it by introducing something like a duality on something like BM algebra.

Proof of (2) $\Leftrightarrow$ (3). — Let us define $\mathbf{y} = b, c$ as in §1.1. Then (2) for all $a \in X$ is equivalent to

$$A\mathbf{y}_{b,c} = DC(b,c)\mathbf{y}_{b,c},$$

and thus $A \in N(B)$. Also this shows that $\Phi_W(A) = D \cdot C^\circ$, where Φ_W is defined as follows.

THEOREM 3 ([4], Theorem 1). — *Let W be a type II matrix of size n. Let $M \in N(W)$. Write $\Phi_W(M)(a,b)$ for the eigenvalue of M with respect to $\mathbf{y}_{a,b}$. Thus, $\Phi_W(M) \in M_X$. Then, Φ_W induces a dual pair*

$$\Phi_W : N(W) \to N({}^tW),$$

i.e. a linear isomorphism satisfying $\Phi_W(M'M) = \Phi_W(M') \circ \Phi_W(M)$, $\Phi_W(M' \circ M) = \frac{1}{n}\Phi_W(M')\Phi_W(M)$, and $\Phi_{{}^tW}(\Phi_W(M)) = n\cdot {}^tM$.

By this, the condition (2) is equivalently transformed to $\Phi_W(A^{-1}) = D^{-1}C$ (since inverse is mapped to Hadamard inverse), then by taking $\Phi_{{}^tW}$ we have $\Phi_{{}^tW}(\Phi_W(A^{-1})) = D^{-1}\Phi_{{}^tW}(C)$. The left hand side is $n^t(A^{-1})$, which is A°, since by type II condition on A we have ${}^tA^\circ = n \cdot A^{-1}$. Thus it is equivalent to $\Phi_{{}^tW}(C) = DA^\circ$, *i.e.* (3).

For the rest of Theorem 2, we use a similar generalization according to a categorical concept.

3. A category N.

We shall define a category N (possibly called Jaeger-Nomura category), which is a generalization of Jaeger-Nomura algebra $N(W)$. N has a beautiful duality Φ, which is not a functor unfortunately.

3.1. An abstract nonsense.

Although it is a bit confusing, by $\mathbf{y}$ we denote a function $X \times X \to \mathbf{C}[X] - \{0\}$, or equivalently $n \times n$ nonzero vectors in $\mathbf{C}[X]$ indexed by $X \times X$, *i.e.*

$$\mathbf{y} = (\mathbf{y}_{a,b} \in \mathbf{C}[X], \neq 0 \mid a, b \in X).$$

(*I.e.* $\mathbf{y}$ is not a single vector but an ordered set of vectors, from now on.) Let $\mathbf{y}' := (\mathbf{y}'_{a,b} \mid a, b \in X)$ be another set of vectors. We define the set of *morphisms* from $\mathbf{y}$ to $\mathbf{y}'$ by

$$\mathrm{Hom}_N(\mathbf{y}, \mathbf{y}') := \{M \in M_X \mid M\mathbf{y}_{a,b} = c_{a,b}\mathbf{y}'_{a,b} \text{ for some } c_{a,b} \in \mathbf{C} \quad \text{for all } \ a, b \in X\},$$

which is a subvector space of $M_X = \mathrm{Hom}(\mathbf{C}[X], \mathbf{C}[X])$. It is clear that a morphism from $\mathbf{y}$ to $\mathbf{y}'$ can be composed with one from $\mathbf{y}'$ to $\mathbf{y}''$ to obtain one from $\mathbf{y}$ to $\mathbf{y}''$ (*i.e.* we defined a cateory C with object $\mathbf{y}$'s and morphisms $\mathrm{Hom}_N(\mathbf{y}, \mathbf{y}')$). We define a linear homomorphism

$$\underset{\mathbf{y},\mathbf{y}'}{\Phi} : \mathrm{Hom}_N(\mathbf{y}, \mathbf{y}') \to M_X$$

by $\Phi(M) := (c(a,b))_{a,b}$. It follows that

$$\Phi(M'M) = \Phi(M') \circ \Phi(M)$$

for $M' \in \mathrm{Hom}_N(\mathbf{y}', \mathbf{y}'')$. Thus, Φ is a functor from C to the category consisting of one object $*$, with homomorphisms M_X, with composition by Hadamard product and identity J. We denote this category by M_X°.

If there is no fear of confusion, we omit the suffix of Φ.

3.2. An object associated with a pair of type II matrices.

Let K, L be two type-II matrices on the same set X. Define a column vector by

$$\mathbf{y}_{a,b}^{K,L} := (K(a,x)L(b,x))_{x\in X}.$$

This gives a set of ordered $n \times n$ vectors indexed by $X \times X$, *i.e.* an object of C. We denoted it by $\mathbf{y}^{K,L}$.

Remark. — In this terminology, $\mathbf{y}_{b,c}$ defined in §1.1 is $\mathbf{y}_{b,c}^{W,W^\circ}$. Moreover,

$$N(W) = \mathrm{Hom}_N(\mathbf{y}^{W,W^\circ}, \mathbf{y}^{W,W^\circ}) = \mathrm{End}_N(\mathbf{y}^{W,W^\circ}),$$

which becomes an $\mathbf{C}$-algebra by the general fact on additive category. Non trivial part is that it is closed also under Hadamard product.

3.3. A quasi duality.

The following is the main result of this article.

THEOREM 4. — *Let K, L, K', L' be arbitrary type II matrices of same size. We write $\Phi_{(K,L;K',L')}$ for $\Phi_{\mathbf{y}^{K,L},\mathbf{y}^{K',L'}}$.*

(1) *$\Phi_{(K,L;K',L')}$ defines a linear map*

$$\mathrm{Hom}_N(\mathbf{y}^{K,L}, \mathbf{y}^{K',L'}) \to \mathrm{Hom}_N(\mathbf{y}^{{}^tL',{}^tL^\circ}, \mathbf{y}^{{}^tK'^\circ,{}^tK}).$$

(2) *$\Phi_{({}^tL',{}^tL^\circ;{}^tK'^\circ,{}^tK)}$ is a mapping of converse direction, and*

$$\Phi_{({}^tL',{}^tL^\circ;{}^tK'^\circ,{}^tK)}(\Phi_{(K,L;K',L')}(M)) = nM$$

holds. Thus, Φ is a linear bijection.

Proof. — All we need is to prove

$$\Phi(M)\mathbf{y}_{(g,h)}^{{}^tL',{}^tL^\circ} = nM(g,h)\mathbf{y}_{(g,h)}^{{}^tK'^\circ,{}^tK},$$

or equivalently

$$(6) \qquad \sum_b \Phi(M)(a,b)L'(b,g)L(b,h)^{-1} = nM(g,h){K'}^{-1}(a,g)K(a,h).$$

The definition of $\Phi(M)$ is given by

$$\sum_z M(c,z)K(a,z)L(b,z) = \Phi(M)(a,b)K'(a,c)L'(b,c)$$

holds for all $a, b, c \in X$. We specialize c to g, then we have

$$\Phi(M)(a,b)L'(b,g) = K'(a,g)^{-1}\sum_z M(c,z)K(a,z)L(b,z).$$

Plug this into the left hand side of (6) to have

$$\begin{aligned}\sum_b \Phi(M)(a,b)L'(b,g)L(b,h)^{-1} \\ &= K'(a,g)^{-1}\sum_b\sum_z M(c,z)K(a,z)L(b,z)L(b,h)^{-1} \\ &= K'(a,g)^{-1}\sum_z M(c,z)K(a,z)\sum_b L(b,z)L(b,h)^{-1} \\ &= nM(g,h){K'}^{-1}(a,g)K(a,h) \quad (\text{ since } L \text{ is type II}),\end{aligned}$$

as desired. □

From now on, we shall denote

$$(K,L) := \mathbf{y}^{K,L},$$
$$N(K,L;K',L') := \mathrm{Hom}_N(\mathbf{y}^{K,L}, \mathbf{y}^{K',L'}),$$

and consider only the objects of the form (K,L) in the category C. Let N be the full subcategory of C induced by these objects. In other words, we consider pairs of type II matrices (K,L), and a morphism from (K,L) to (K',L'), denoted by $N(K,L;K',L') := \mathrm{Hom}_N(\mathbf{y}^{K,L}, \mathbf{y}^{K',L'})$.

Remark. — Φ is a generalization of duality between $N(W)$ and $N({}^tW)$. We have $N(W) = N(W, W^\circ; W, W^\circ)$ by definition. If we specialize to $K = K'$ and $L = L'$, we have

$$N(K,L;K,L) \xrightarrow{\Phi} N({}^tL, {}^tL^\circ; {}^tK^\circ, {}^tK).$$

This shows that Φ is *not* a functor. By the way, the left hand side is a subalgebra of M_X with respect to the ordinary product. From 2 of Theorem 4, it follows that the right hand side is closed under the Hadamard product, and these two algebras are isomorphic as an *abstract* algebra. Imposing the right hand side to be an algebra with usual product, we may require $({}^tL, {}^tL^\circ) = ({}^tK^\circ, {}^tK)$, *i.e.* $L = K^\circ$. Then the right hand side is

$$N({}^tK^\circ, {}^tK; {}^tK^\circ, {}^tK),$$

which is closed also by the usual product, thus we recover the Jaeger-Nomura algebra. Note that the transpose induces a linear bijection

$$N(K,L;K',L') \xrightarrow{t} N(L,K;L',K'). \tag{7}$$

By composing with this transpose we have

$$N(K, K^\circ; K, K^\circ) \xrightarrow{\Phi} N({}^tK^\circ, {}^tK; {}^tK^\circ, {}^tK) \xrightarrow{t} N({}^tK, {}^tK^\circ; {}^tK, {}^tK^\circ),$$

i.e. the duality

$$N(K) \xrightarrow{\Psi} N({}^tK)$$

defined in [4], §3.1 (note that $N({}^tK)$ is $N'(K)$ there).

4. Application of the category N.

4.1. Proof of (4) $\Leftrightarrow$ (5).

(4) is equivalent to

$$A^\circ \in N({}^tB, {}^tB; {}^tB^\circ, {}^tB^\circ) \text{ and } \Phi(A^\circ) = -DC.$$

We calculate

$$\Phi(A^\circ \cdot A) = \Phi(A^\circ) \circ \Phi(A).$$

The left hand side is $\Phi(nI)$ by type II property of A, then by definition of Φ, it is nJ. The right hand side is, by condition (4), $-DC \circ \Phi(A)$. By moving DC to the left hand side, we have

$$-DC^\circ = \Phi(A).$$

Applying Φ, we have

$$\Phi(C^\circ) = -D^{-1}\Phi(\Phi(A)) = DA,$$

which is nothing but the condition (5).

4.2. Four-weight spin model.

Four-weight spin model was introduced in [1] as a generalizaion of two-weight model, which was fully used in determination of the link invariant in [5]. We can paraphrase their definition in terms on N. For the original definition, see [1].

PROPOSITION 5. — *A pair of $n \times n$ matrix (W_1, W_2) is a four weight spin model if and only if the following conditions hold:*

1. *W_1, W_2 are type II matrices.*

2. $W_1 \in N({}^tW_2{}^\circ, {}^tW_1; {}^tW_2{}^\circ, {}^tW_1)$ *and* $\Phi_{{}^tW_2{}^\circ, {}^tW_1; {}^tW_2{}^\circ, {}^tW_1}(W_1) = D \cdot {}^tW_2{}^\circ$ *(which is automatically in* $N(W_1, W_1{}^\circ; W_2, W_2{}^\circ)$*).*

3. $W_1 \in N(W_2, W_1{}^\circ; W_2, W_1{}^\circ)$ *and* $\Phi_{W_2, W_1{}^\circ; W_2, W_1{}^\circ}(W_1) = DW_2{}^\circ$ *(which is automatically in* $N({}^tW_1{}^\circ, {}^tW_1; {}^tW_2{}^\circ, {}^tW_2)$.

Proof. — In the terminology in [1], $Y^{i,j} = \mathbf{y}^{W_i, {}^tW_j}$. Their condition III_1 is equivalent to 2, and their condition III_9 is equivalent to 3. □

Remark. — It is shown that III_1 to III_8 are equivalent, and III_9 to III_{16} are equivalent in [1], where these 16 conditions naturally arize from Reidemeister move of type III with various orientations. These equivalences are easily proved by N and Φ. For example, note that III_1 is equivalent to $W_4 = {}^tW_2{}^\circ = \frac{1}{D}\Phi(W_1)$, and by taking Φ again, we have the condition III_2. Then, ${}^tW_3 = W_1^\circ$ and $\Phi(A^\circ) = n \cdot (\Phi(A)^\circ)$ implies III_3, *etc.*

Remark. — Huang-Guo [2] proved that

$$ {}^tW_1^\circ \cdot W_1^\circ \in N({}^tW_2). $$

This can be interpreted in N by

$$ W_1^\circ \in N({}^tW_2, {}^tW_2^\circ; {}^tW_1, {}^tW_1^\circ) \quad (\mathrm{III}_8) $$

and

$$ {}^tW_1{}^\circ \in N({}^tW_1, {}^tW_1^\circ; {}^tW_2, {}^tW_2^\circ) \quad (\mathrm{III}_2) $$

and thus

$$ {}^tW_1^\circ \cdot W_1^\circ \in N({}^tW_2, {}^tW_2^\circ; {}^tW_2, {}^tW_2^\circ) = N({}^tW_2). $$

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