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ON A PROBLEM OF WALKS

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1. Introduction.

It was such a rainy day in Aussois after the workshop on Graph Theory (May 1995), that walks in the mountains were impossible. So F. Jaeger and M.-C. Heydemann began to work together on a conjecture from [9] on binary operations which are related to homomorphisms of de Bruijn digraphs. But very quickly, they considered the following problem of walks.

PROBLEM 1. — *Characterize the class of digraphs G such that for any $k \geq 0$, G has exactly m walks of length k.*

Since the number of walks of length k of a digraph G is equal to the number of vertices of its k-iterated line digraph, Problem 1 is related to the characterization of periodic iterated line digraphs. These digraphs have been studied by several authors and this article recalls results from the survey on line digraphs [7].

We give in Section 2 definitions and explain in Section 3 the motivation for such a study. Section 4 and Section 5 contain results on Problem 1. The main common result of François Jaeger and M.-C. Heydemann was Theorem 1 and Problem 2. François never commented the first draft written at the end of 1995. Charles Delorme brought recently some results about the main conjecture.

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So this article contains more questions than answers.

2. Notation and preliminary results.

Definitions not given here can be found in [2]. Algebraic notions can be found in [8] and [10].

2.1. Sequences.

A sequence $(w_k)_{k\geq 0}$ is *periodic* if there exists a positive integer p such that, for any $k \geq 0$, $w_{k+p} = w_k$. The smallest integer p with this property is called the *period* of $(w_k)_{k\geq 0}$.

A sequence $(w_k)_{k\geq 0}$ is *ultimately periodic* if there exists an integer n such that $(w_{k-n})_{k\geq n}$ is periodic.

2.2. Digraphs.

Definitions not given here can be found in [2].

In this paper, we deal only with directed graphs without multiple arcs (digraphs). A digraph G with *vertex set* $V = V(G)$ and *arc set* $A = A(G)$ is written $G = (V, A)$. An arc from vertex u to vertex v is denoted (u, v) (or $[u, v]$ if the vertex u or v is expressed with parenthesis). If (u, v) is an arc, then u is said to be an *in-neighbour* of v and symmetrically v is an *out-neighbour* of u. The set of in-neighbours of u is $\Gamma_G^-(u) = \{v \in V(G) : (v, u) \in A(G)\}$ and $\delta_G^-(u) = |\Gamma_G^-(u)|$ is called the *in-degree* of u. Similarly, the set of vertices of out-neighbours of u is $\Gamma_G^+(u) = \{v \in V(G) : (u, v) \in A(G)\}$ and $\delta_G^+(u) = |\Gamma_G^+(u)|$ is the *out-degree* of u.

A *trivial digraph* is a subdigraph with only one vertex and no loop.

K_n^+ denotes the complete digraph on n vertices with a loop at each vertex. Any digraph G with n vertices is isomorphic to a subdigraph of K_n^+.

Let n, D be two positive integers. $\mathbb{Z}_n$ denotes the set of integers $\{0, 1, \ldots, n-1\}$. The *de Bruijn digraph* $B(n, D)$ is defined by

$$V(B(n,D)) = \mathbb{Z}_n^D$$
$$A(B(n,D)) = \{(x_1x_2\ldots x_D, x_2\ldots x_D x_{D+1}) | x_i \in \mathbb{Z}_n, 1 \leq i \leq D+1\}$$

$M(G)$ denotes the *incidence matrix* (also called adjacency matrix) of G. It is a square matrix $n \times n$, with entries given by $M(G)_{i,j} = 1$ if (i, j) is an arc of G and $M(G)_{i,j} = 0$ otherwise.

2.3. Walks.

A *walk* of length ℓ from vertex u to vertex v in digraph G is a sequence of vertices $u = u_0, u_1, \ldots, u_\ell = v$ such that (u_i, u_{i+1}) is an arc of G for any i, $0 \le i \le \ell - 1$. If $u_0 = u_\ell$ the walk is said to be *closed*.

A *directed elementary path* (for short, *dipath*) is a walk in which all vertices are distinct; if the only vertex repetition is $u_0 = u_\ell$, it is a *directed cycle*.

We denote by $W_k(G)$ the number of walks of G of length k and by $\alpha_k(G)$ the number of closed walks of length k.

Notice that $W_k(G)$ is also the sum of entries of $M(G)^k$ and $\alpha_k(G)$ is the trace of the matrix $M(G)^k$.

2.4. Line digraphs.

Let G be a digraph. We denote $L(G)$ the *line digraph* of G. The vertices of $L(G)$ are the arcs of G, i.e. $V(L(G)) = A(G)$, and the arcs of $L(G)$ are all the couples $[(x, y), (y, z)]$, $x \in V(G)$, $y \in \Gamma_G^+(x)$, $z \in \Gamma_G^+(y)$. A survey about line digraphs can be found in [7].

The *p-iterated line digraph* is defined recursively:

$$L^p(G) = L(L^{p-1}(G)) \text{ for } p \ge 2.$$

Following [1], a digraph is said to be *periodic* if there exist some integers k, n, $n > 0$, such that $L^k(G) \cong L^{n+k}(G)$. In this case, the smallest value n for which it is true is called *the period* of G. Thus, a digraph is periodic of period n if and only if the sequence $(L^k(G))_{k \ge 0}$ is ultimately periodic of period n.

Considering G as a subdigraph of K_n^+, by the following proposition, $L^p(G)$ can also be considered as a subdigraph of the de Bruijn digraph $B(n, p + 1)$.

PROPOSITION 1 ([3]). — *For $n \ge 2$ and $p \ge 2$, $K_n^+ = B(n, 1)$ and $B(n, p) = L(B(n, p - 1)) = L^{p-1}(K_n^+)$.*

Therefore $L^p(G)$ is isomorphic to the digraph with vertices $u_0 u_1 \cdots u_p$, for $(u_i, u_{i+1}) \in E(G)$, $0 \leq i \leq p$, and arcs of the form $(u_0 u_1 \cdots u_p, u_1 \cdots u_p u_{p+1})$. In the following we will consider that vertices of $L^p(G)$ are walks of length p of G written shortly $u_0 u_1 \cdots u_p$. We will also use the following equality and property:

$$W_k(G) = |V(L^k(G))|. \tag{1}$$

LEMMA 1. — *For any k, $k \geq 0$, G contains m walks of length $k+1$ iff $L(G)$ contains m walks of length k, i.e. $W_k(L(G)) = W_{k+1}(G)$.*

Proof. — By Equation (1),

$$W_{k+1}(G) = |V(L^{k+1}(G))| = |V(L^k(L(G)))| = W_k(L(G)).$$

□

3. Motivation.

The motivation for our work is a conjecture from [9] on binary operations on $\mathbb{Z}_n$ which are related to uniform homomorphisms of de Bruijn digraphs. We first explain the original conjecture by recalling results from [9].

DEFINITION 1 ([9]). — *Let D be an integer, $D \geq 2$. Let $\diamond$ be a binary operation on $\mathbb{Z}_n$ such that for any $y_1, \ldots, y_{D-1} \in \mathbb{Z}_n$, the set of $D-1$ equations*

$$x_i \diamond x_{i+1} = y_i, \qquad 1 \leq i \leq D-1 \tag{2}$$

for unknowns $x_1, x_2, \ldots, x_D$ has exactly n distinct solutions such that $x_i \in \mathbb{Z}_n$ for all i. Then it is said that operation $\diamond$ satisfies Property (P_D).

It is easy to obtain binary operations on $\mathbb{Z}_n$ satisfying Property (P_D) for any D. For example, take $x \diamond y = x$, for any $x, y \in \mathbb{Z}_n$. More generally, if operation $\diamond$ satisfies at least one of the two properties:

(B) $\forall\, \alpha \in \mathbb{Z}_n$ the mapping $x \mapsto \alpha \diamond x$ is a bijection on $\mathbb{Z}_n$;

(B′) $\forall\, \alpha \in \mathbb{Z}_n$ the mapping $x \mapsto x \diamond \alpha$ is a bijection on $\mathbb{Z}_n$;

then $\diamond$ satisfies Property (P_D) for any D. The problem is to find all binary operations on $\mathbb{Z}_n$ satisfying Property (P_D) for any D. Are there such operations satisfying neither Property (B) nor Property (B′)?

The answer is positive if n is not prime. Before giving such examples, we translate binary operation on $\mathbb{Z}_n$ and Property (P_D) in terms of arc-colorings of K_n^+ and arc-coloring property.

Let G be a digraph and c an arc-coloring of G with n colors, i.e. a mapping $c : A(G) \to \mathbb{Z}_n$. We say that c satisfies the *uniform path coloring property* if, for any $\ell \geq 1$, and for any ℓ-sequence of colors $c_1, c_2, \ldots, c_\ell$, $c_i \in \mathbb{Z}_n, 1 \leq i \leq \ell$, there exist n distinct walks in G, $(u_0^j, u_1^j, \ldots, u_\ell^j)$, $0 \leq j \leq n-1$, colored with colors $c_1, c_2, \ldots, c_\ell$ in good order, i.e. $c(u_i^j, u_{i+1}^j) = c_{i+1}$, for $0 \leq i \leq \ell - 1$.

Notice that an arc-coloring c of K_n^+ satisfies the *uniform path coloring property* on K_n^+ if and only if the binary operation $\diamond$ defined by $i \diamond j = c(i,j)$ satisfies Property (P_D) for any integer $D \geq 2$.

Therefore we consider the problem of finding arc-colorings c of K_n^+ with the uniform path coloring property. Notice that the translation of Property (B) gives examples of colorings such that each color appears exactly once on the out-going arcs of each vertex (in other words by taking $\{c(u,v); v \in \Gamma^+(u)\} = \mathbb{Z}_n$). Similarly, a binary operation with Property (B$'$) corresponds to a coloring such that each color appears exactly once on the in-going arcs of each vertex ($\{c(v,u); v \in \Gamma^-(u)\} = \mathbb{Z}_n$).

If n is not prime, it is possible to construct less obvious examples. Before giving them, we introduce another way of considering the P_D property.

Let c be an arc-coloring of K_n^+. We may use the matrices $M^{[k]}$, $0 \leq k \leq d-1$, where $M^{[k]}$ is the adjacency matrix of the digraph defined by the arcs of color k (in the $\diamond$ setting, the entries are $(M^{[k]})_{i,j} = \delta(k, i \diamond j)$ with δ the Kronecker symbol). Hence counting the solutions in a system of equations $x_i \diamond x_{i+1} = y_i$, $1 \leq i \leq D-1$, or counting the paths of length $D-1$ with prescribed colors, amounts to summing the entries of the matrix $M_y = M^{[y_1]} M^{[y_2]} \ldots M^{[y_{D-1}]}$. Thus finding an arc-coloring c of K_n^+ which satisfies the uniform arc-coloring property is equivalent to find a set S of $n \times n$ matrix filled with 1's and 0's such that

- for any ordered pair i,j, exactly one matrix $M \in S$ satisfies $M_{j,i} = 1$,
- for any product p of elements of S, $\mathbf{1}_n p \mathbf{1}_n^* = n$ (where $\mathbf{1}_n$ is the $1 \times n$ matrix filled with 1's).

LEMMA 2. — *If c satisfies the uniform path coloring property on K_n^+, then all the products of matrices of S have entries in $\{0, 1\}$.*

If a is the maximum number of 1's in a row and b is the maximum number of 1's in a column, then $ab \leq n$.

Proof. — The entries of the products are obviously nonnegative integers, and one cannot have a product of the adjacency matrices with an entry larger than 1. If such a product has entry $p_{i,j} > 1$, then some adjacency matrix corresponding to a color has an entry $m_{j,i} = 1$, and since all matrices m have entries 0 or 1 only, all entries of $q = mp$ are nonnegative integers, and thus $q_{j,j} > 1$, and $(q^k)_{j,j} \geq (q_{j,j})^k$ grows too fast and prevents $\mathbf{1}_n q^k \mathbf{1}_n^*$ to be n.

If the product A has a row, say row α with a 1's and B has column β with b 1's, the product BCA, where $C_{\beta,\alpha} = 1$ has a submatrix of size $b \times a$ filled with 1's. □

COROLLARY 1. — *If n is 2 or 3, Property (B) or (B′) is satisfied.*

Example 2. — Let $n = pq$ with integers $p > 1$ and $q > 1$. We choose $M^{[k]}$ (the matrix of the color k) in the following way. Let c_u be the matrix of size $p \times p$ made from null entries, but for the column u, $0 \leq u < p$ filled with 1's. Let ℓ_v the matrix of size $q \times q$, made from null entries, but for the row $v, 0 \leq v < q$ filled with 1's. Then, the set of matrices $M^{[qu+v]} = c_u \otimes \ell_v$ (tensor product of matrices), $0 \leq u < p, 0 \leq v < q$, satisfies: $\sum_{k=0}^{d-1} M^{[k]}$ is the $n \times n$ matrix filled with 1's and $M^{[qu+v]}M^{[qu'+v']} = M^{[qu'+v]}$. Since each matrix $M^{[qu+v]}$ has p non null rows, each of which has q non null entries that all fall in the same q columns, the cardinality of 1's in the matrix is equal to $d = pq$. Thus, any product $M^{[y_1]}M^{[y_2]} \ldots M^{[y_{D-1}]}$ has exactly n 1's and $n^2 - n$ 0's. The associated arc-coloring of K_n^+ satisfies the *uniform path coloring property*, but neither Property (B) nor Property (B′). The corresponding binary operation $\diamond$ is of course given by $(qu + v) \diamond (qu' + v') = qu' + v$.

If two couples of set and binary operation $(A, \diamond)$ and $(C, \circledast)$ with property P_D for any D are given, it is possible to build a new one by a cartesian product with set $A \times C$ and operation $(a, c) \diamondsuit (b, d) \stackrel{\text{def}}{=} (a \diamond b, c \circledast d)$.

If $\diamond$ on A has not property (B) and $\circledast$ on C has not property (B′) then $\diamondsuit$ has neither.

In particular, when $a \diamond b = a$ and $c \circledast d = d$ for all $a, b \in A$ and $c, d \in C$, we have an operation isomorphic to the one of Example 1.

Example 2. — Another generalization of Example 1 is given in [9]. An illustration of this construction for $n = 6$ is the following:

$\diamond$	0	1	2	3	4	5
0	5	5	5	1	1	1
1	4	4	4	2	2	2
2	0	0	0	3	3	3
3	3	3	3	2	2	2
4	4	4	4	0	0	0
5	1	1	1	5	5	5

We now translate the construction of [9] in terms of colored paths.

Let $n = pq$ with integers $p > 1$ and $q > 1$. Consider a partition of the vertex set V of K_n^+ into subsets $V_i, 1 \leq i \leq p$, each one of cardinality q.

For each color c, $0 \leq c \leq n-1$, for each subset $V_i, 1 \leq i \leq p$, successively

- choose one vertex $b_{i,c} \in V_i$, such that not all its outgoing arcs are already colored,
- choose one subset $V_{j(i,c)}$ such that all the arcs $(b_{i,c}, x)$, $x \in V_{j(i,c)}$, are still not colored,
- color all the arcs $(b_{i,c}, x), x \in V_{j(i,c)}$ with the color c.

Then c is an arc-coloring such that

- for each color c, $0 \leq c \leq n-1$, and for each subset $V_i, 1 \leq i \leq p$, there exists exactly one vertex $b_{i,c}$ with out-going arcs colored c,
- if $c(i,j) = c$ and $j \in V_m$, then for any arc (i,j') with $c(i,j') = c$, $j' \in V_m$ and for any vertex $j' \in V_m$, $c(i,j') = c$.

For the above example: $p = 2, q = 3$, $V_1 = \{0,1,2\}$, $V_2 = \{3,4,5\}$, $b_{1,0} = 2$, $b_{1,1} = 0$, $b_{1,2} = 1, \ldots$, $V_{j(1,0)} = V_1, V_{j(1,1)} = V_{j(1,2)} = V_2, \ldots$ In other words, for each subset V_l, each color colors the out-going arcs of exactly one vertex and these arcs cover exactly one set V_m. Since $q > 1$, it is clear that the arc-coloring does not satisfy Property (B) (the out-going arcs of each vertex are colored with at most $p < d$ colors). It is sufficient to choose one color c, two integers $i \neq i'$ and $j(i,c) = j(i',c)$ in order that Property (B$'$) is also not satisfied.

With this formulation, it is easier than in [9] to prove that the constructed digraph satisfies the *uniform path coloring property*. Considering one sequence of colors $y_1, \dots, y_{D-1} \in \mathbb{Z}_n$, once the first vertex is chosen among the p possible b_{i,y_1}, $1 \le i \le p$, the following vertices of the path are unique except the last one for which there exist q possibilities. Therefore there exist exactly pq paths colored with the sequence of colors $y_1, \dots, y_{D-1}$. Notice that the same construction applies with in-going arcs instead of out-going arcs.

Problem 18 of [9] asks whether other types of constructions than Example 2 exist if n is not prime. The answer is yes:

Example 3. — Let S be a set of matrices P_i, with entries in $\{0,1\}$, of size $(n \times n)$, with some subsets such that the sum of the matrices in them is the matrix J (filled with 1's), such that any product M of these matrices P_i satisfies $\mathbf{1}_n M \mathbf{1}_n^* = n$, and a set of m matrices Q_j with entries in $\{0,1\}$ of size $m \times m$ summing to J_m, such that any product N of Q_j's satisfies $\mathbf{1}_m N \mathbf{1}_m^* = m$, we can build new sets of matrices of size $mn \times mn$ by building from each Q_i a set of n new matrices, say $R_{i,j}$ by replacing each null entry in the Q_i by a null block of size $n \times n$ and each other entry by some matrix from S, in such a way that $\sum\limits_j R_{i,j}$ is $J \otimes Q_i$ (each null entry of Q_i is replaced by the null block and the other entries by J).

For example, with $m = n = 2$, let M be the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$; let S be the set $\{I, J-I, M, J-M\}$, $Q_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $Q_1 = J - Q_0$, and the description by blocks of the R_i's (other choices are of course possible):

$$R_{0,0} = \begin{bmatrix} I, & 0 \\ M, & 0 \end{bmatrix}, \; R_{0,1} = \begin{bmatrix} J-I, & 0 \\ J-M, & 0 \end{bmatrix}, \; R_{1,0} = \begin{bmatrix} 0, & M \\ 0, & I \end{bmatrix}, \; R_{1,1} = \begin{bmatrix} 0, & J-M \\ 0, & J-I \end{bmatrix},$$

In this example the set of (non void) products of matrices R has 16 elements.

Note that the operation obtained that way is neither of the kind of example 2, nor the reverse kind.

For n prime, the constructions given in the preceding examples do not work. We now formulate the conjecture which motivated our work.

CONJECTURE 1 ([9]). — *Let n prime and $\diamond$ be a binary operation on $\mathbb{Z}_n$ which satisfies (P_D), for any integer $D \ge 2$. Then operation $\diamond$ satisfies at least one of the two properties:*

– *$\forall\, \alpha \in \mathbb{Z}_n$, the mapping $x \mapsto \alpha \diamond x$ is a bijection on $\mathbb{Z}_n$.*

– *$\forall \alpha \in \mathbb{Z}_n$, the mapping $x \mapsto x \diamond \alpha$ is a bijection on $\mathbb{Z}_n$.*

Conjecture 1 can be formulated in terms of colored walks in the digraph K_n^+ as follows.

CONJECTURE 2. — *Let n prime and c an arc-coloring of K_n^+, such that c satisfies the uniform path coloring property on K_n^+. Then coloring c satisfies at least one of the two properties:*

– *For any vertex u of K_n^+, the n in-going arcs are colored with all the n colors of $\mathbb{Z}_n$, i.e. $\{c(v,u); v \in \Gamma^-(u)\} = \mathbb{Z}_n$.*

– *For any vertex u of K_n^+, the n out-going arcs are colored with all the n colors of $\mathbb{Z}_n$, i.e. $\{c(u,v); v \in \Gamma^+(u)\} = \mathbb{Z}_n$.*

If c satisfies the uniform path coloring property on K_n^+, then each color induces a subdigraph of K_n^+ with exactly n walks of each length. Our hope was to find necessary conditions on digraphs with constant number of walks in order to obtain necessary conditions on c and solve Conjecture 2. In a first approach, we consider digraphs with ultimately periodic number of walks.

4. Digraphs with ultimately periodic number of walks.

Notice first that it is easy to construct digraphs with ultimately periodic number of walks.

PROPOSITION 2. — *For any periodic sequence $(w_k)_{k\geq 0}$ of period p, there exists a digraph G such that $W_k(G) = w_k + p + 1 + \sum_{i=1}^{p} w_i$ for $k \geq 2$.*

Proof. — Consider the digraph G with vertices $a, c_i, d_i^j, 0 \leq i \leq p-1, 1 \leq j \leq w_{i+2}$, consisting of a directed cycle $c_0, c_1, \ldots, c_{p-1}, c_0$, arcs $(c_i, d_i^j), 0 \leq i \leq p-1, 1 \leq j \leq w_{i+2}$, and another arc (a, c_0).

For any $k > 0$, the number of walks of length k in G which begin at a vertex $c_i, 0 \leq i \leq p-1$, is independent of k and equal to $p + \sum_{i=1}^{p} w_k$. The number of walks of length k which begin at vertex a is equal to $w_k + 1$ for $k \geq 2$. Thus, for any $2 \leq k$, $W_k(G) = w_k + p + 1 + \sum_{i=1}^{p} w_k$. □

Notice that, for the graph G constructed in the proof of Proposition 2, $L^{p+1}(G)$ is isomorphic to $L(G)$ plus isolated vertices and $L^{p+2}(G) = L^2(G)$. Thus G is periodic and the period of the sequences $(w_k)_{k\geq 0}$ and $(L^p(G))_{k\geq 0}$ are the same.

More generally, digraphs with ultimately periodic number of walks are exactly periodic digraphs according to Theorem 1 below.

THEOREM 1. — *For a digraph G the following properties are equivalent:*

(i) *The sequence $(W_k(G))_{k\geq 0}$ is bounded;*

(ii) *The sequence $(W_k(G))_{k\geq 0}$ is ultimately periodic;*

(iii) *G is periodic, i.e. the sequence $(L^k(G))_{k\geq 0}$ is ultimately periodic.*

(iv) *The strongly connected components of G are directed cycles or trivial subgraphs. Moreover there exists no directed path joining two distinct directed cycles.*

Characterizations of periodic digraphs are given in [7]. The equivalence of (iii) and (iv) is exactly Theorem 9.1 (iii) of [7], but we give proofs which are not available in [7]. In order to prove Theorem 1, we begin with two lemmas.

LEMMA 3. — *Let G be a digraph such that the sequence of the number of its closed walks of length k is bounded. Then the strongly connected components of G are directed cycles or trivial subgraphs.*

Proof. — We will prove that if G is strongly connected with at least one arc and if the sequence $(\alpha_k(G))_{k\geq 0}$ of the number of its closed walks of length k is bounded, then G is a directed cycle.

We apply Perron-Frobenius theorem to $A(G)$ (see for example [4] or [8]), which is an irreducible matrix (owing to the strong connectivity) with nonnegative entries. The maximum eigenvalue of G, λ_1, is real positive.

We first prove by contradiction that $\lambda_1 = 1$.

$\lambda_1 < 1$ would imply $\lambda_1^k \to 0$ with $k \to \infty$ and imply also the trace $\mathrm{tr}(A(G)^k) \to 0$, which is impossible since the number of closed walks of length k is at least 1 for k multiple of ℓ.

$\lambda_1 > 1$ would imply λ_1^k tends to infinity with k and $\mathrm{tr}(A(G)^k)$ is not bounded, which is impossible since the number of closed walks of length k

is bounded.

Thus $\lambda_1 = 1$.

Now, if G is a non-trivial, strongly connected digraph that is not a cycle, then each vertex belongs to at least one elementary cycle. Then if L is the least common multiple of the lengths of these cycles, each diagonal entry of $A(G)^L$ is at least 1, and since some vertex x has $\delta^+_G(x)\delta^-_G(x) > 1$, and thus lies on more than one cycle, we see that the trace satisfies $n\lambda_1^L \geq \operatorname{tr} A(G)^L > n$, hence $\lambda_1 > 1$. □

LEMMA 4. — *Let G a digraph such that the sequence $(W_k(G))_{k\geq 0}$ is bounded. Then there exists no directed path joining two distinct directed cycles of G.*

Proof. — By contradiction. Assume G contains two directed cycles C (of length l) and C', joined by a path beginning at vertex u on C and ending at a vertex of C'. The number of walks of length $k \geq l$ beginning at u is at least $\lfloor k/l \rfloor + 1$, and therefore is not bounded. □

We now prove Theorem 1.

Proof.

(i) ⇒ (iii): By Equation (1), for any $k \geq 0$,

$$|V(L^k(G))| = W_k(G).$$

Hence digraphs $L^k(G)$, $k \geq 0$, belong to a finite set of (isomorphism types of) digraphs and there exist integers $k \geq 0$, $p \geq 0$ such that $L^{k+p}(G) = L^k(G)$.

(iii) ⇒ (ii): Trivial by $(W_k(G))_{k\geq 0} = (|V(L^k(G))|)_{k\geq 0}$.

(ii) ⇒ (i): Trivial.

(i) ⇒ (iv): By Lemmas 3 and 4.

(iv) ⇒ (i): For k large enough, every walk of length k meets exactly one directed cycle. Let W_k^C the number of walks of length k of G which meet C. Such a walk is of the form $P_1\ P_2\ P_3$ (concatenation of arcs), with P_1 and P_3 walks of $G - E(C)$ ending and beginning on C respectively, and P_2 a walk on C. Moreover, given P_1 and P_3 there exists at most one walk P_2 on C such that the length of the walk $P_1\ P_2\ P_3$ is a given integer k. Since the number of such walks P_1, P_3, is independant of k, $(W_k^C))_{k\geq 0}$ is bounded, and therefore $(W_k(G))_{k\geq 0}$ is bounded. □

Notice that, if G is periodic, by property (iv), the connected components of $L^k(G)$ contain only one directed cycle for k sufficiently large. The structure of a periodic digraph with a single directed cycle is described in [7] as follows.

An *eddy digraph* is defined as a digraph which consists of a directed cycle C together with trees rooted in the vertices $v_i, 0 \le i \le \ell - 1$, of C and oriented either from the root to the leaves (out-tree A_i with root $v_i \in V(C)$), or from the leaves to the root (in-tree B_j with root $v_j \in V(C)$). The *out-tree index* (resp. *in-tree index*) of an eddy digraph is the minimum positive integer r such that $A_{i+r} = A_i$ (resp. $B_{j+r} = B_j$).

If G is a periodic digraph, for k sufficiently large each connected component of $L^k(G)$ consists of an eddy digraph plus paths obtained by the Heuchenne condition (whenever there are vertex-disjoint paths of length n from u to w, v to w, u to x, then there is one path —disjoint from the others— from v to x). Then $L^{k+1}(G)$ has the same sequence of in- and out-trees as $L^k(G)$, but with the in-trees shifted one step forward on C with respect to the out-trees.

By Equation (1), the period of sequence $(W_k(G))_{k\geq 0}$ divides the period of sequence $(L^k(G))_{k\geq 0}$. This last one can be precised in the case of digraph with a single directed cycle.

PROPOSITION 3 ([6]). — *Let G be a periodic digraph with a single directed cycle. Then the period of $(L^k(G))_{k\geq 0}$ is the greatest common divisor of its out-tree and in-tree indices.*

Now a remaining problem in the study of digraphs with ultimately periodic number of walks is the following:

PROBLEM 2. — *Are the (ultimate) periods of the two sequences $(W_k(G))_{k\geq 0}$ and $(L^k(G))_{k\geq 0}$ always the same?*

5. Digraphs with ultimately constant number of walks.

We now come back to our first study: digraphs which have the same number of walks of length k for any k. We first introduce some notation. Let m, ℓ be integers, $m \geq 2, \ell \geq 0$. Let us denote by $\mathcal{G}_{m,\ell}$ the class of digraphs G such for any $k \geq \ell$, G has exactly m walks of length k: $W_k(G) = m$.

Thus we are looking for graphs of $\mathcal{G}_{m,0}$. Lemma 1 has the following easy corollary which shows how to obtain digraphs of $\mathcal{G}_{m,0}$ from digraphs of $\mathcal{G}_{m,p}$.

COROLLARY 2. — *If G belongs to $\mathcal{G}_{m,p}$, then $L^p(G)$ belongs to $\mathcal{G}_{m,0}$.*

Proof. — By induction using Lemma 1, $W_k(L^p(G)) = W_{k+p}(G)$. □

We now give some examples of digraphs of $\mathcal{G}_{m,0}$.

Example 4. — A digraph G such that $\delta^+_G(u) = 1$ for any vertex u belongs to $\mathcal{G}_{m,0}$ since the number of walks of length k beginning at any vertex is equal to 1. Similarly, a digraph G such that $\delta^-_G(u) = 1$ for any vertex u belongs to $\mathcal{G}_{m,0}$.

The digraphs of Example 4 satisfy furthermore $G \cong L(G)$. It is known that a connected digraph satisfies $G \cong L(G)$ if and only if G is satisfies $\delta^+_G(u) = 1$ for any vertex u or $\delta^-_G(u) = 1$ for any vertex u [5].

COROLLARY 3. — *If $G \cong L(G)$, then for any integer k, $W_k(G) = W_0(G)$. Thus G belongs to $\mathcal{G}_{m,0}$ for $m = |V(G)|$.*

Proof. — By induction on k, $G \cong L^k(G)$. By Equation (1), $W_k(G) = |V(L^k(G))| = W_0(G)$. □

A particular case is obtained by taking for G the vertex-disjoint union of a periodic digraph H such that $H \cong L^p(H)$ and its iterated line-digraphs $L(H), L^2(H), \ldots, L^{p-1}(H)$.

Notice that the converse of Corollary 3 is false since there exist digraphs G in $\mathcal{G}_{m,0}$ with $G \not\cong L(G)$:

Example 5. — Take a directed cycle C with, for any vertex $c \in V(C)$, an out-going arc (c, c'), $c \in V(C), c' \notin V(C)$. Add two new vertices a, b and two arcs $(a, c_0), (a, b)$, with $c_0 \in V(C)$. Denote by G the constructed digraph. For each vertex u of $V(C) \cup \{a\}$, the number of walks of length k beginning at u is exactly 2. Thus $G \in \mathcal{G}_{m,0}$, but $G \not\cong L(G)$ (but $L^2(G) \cong L(G)$).

Let us denote by $\mathcal{G}_m$ the union of the classes $\mathcal{G}_{m,\ell}, \ell \geq 0$, i. e. the class of digraphs with ultimately constant number m of walks (digraphs G such that there exists ℓ and for any $k \geq \ell$, G has exactly m walks of length k). Notice that this class is strictly contained in the class of digraphs with ultimately periodic number of walks since there exist digraphs not belonging to $\mathcal{G}_m$ and with ultimately periodic number of walks (see Proposition 2).

Notice that the class $\mathcal{G}_m$ is large, since it is stable by line-digraph operation and vertex-disjoint union. Furthermore, Proposition 2 has the following corollary.

COROLLARY 4. — *Any digraph with ultimately periodic number of walks is an induced subdigraph of a digraph with ultimately constant number of walks.*

Proof. — Let H be a given digraph with ultimately periodic number of walks. Let G be the digraph constructed in Proposition 2 with the sequence given by $w_k = (\max_{i \geq 0}(W_i(H))) - W_k(H)$. Then the union of the two digraphs H and G is a digraph of $\mathcal{G}_m$ containing H. □

Results of Section 4 applies to the class $\mathcal{G}_m$. In particular,

COROLLARY 5 ([7]). — *If G is strongly connected with at least one arc and belongs to the class $\mathcal{G}_m$, then G is a directed cycle.*

By Proposition 3, we get

COROLLARY 6 ([7]). — *A digraph with a single directed cycle has period 1 and therefore belongs to $\mathcal{G}_m$, if and only if its out-tree and in-tree indices are relatively prime.*

In [7] it is said that

> the only other way in which digraphs of period 1 can occur is for there to exist a collection of its connected components forming all iterated line digraphs of a unicyclic digraph with period greater than 1.

But we do not know if a digraph of $\mathcal{G}_m$ or even of $\mathcal{G}_{m,0}$ has necessarily a period equal to 1. This particular case of Problem 2 can be formulated as follows.

PROBLEM 3. — *If G belongs to the class $\mathcal{G}_{m,0}$, does there exist an integer p such that $L^p(G) \cong L^{p+1}(G)$?*

Conclusion.

In this article, we have formulated in terms of colored walks in digraphs a conjecture of [9] on some binary operations and given some related already known results on the class of digraphs with ultimate periodic iterated line-digraphs.

Besides the conjecture, a remaining problem is either to prove that, for a digraph which has the same number of walks of length k for any k, the sequence of its iterated line digraphs is ultimately constant or to find a counterexample.

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