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## EDGE-DISJOINT ODD CYCLES IN GRAPHS WITH SMALL CHROMATIC NUMBER

by C. BERGE and B. REED

*Dedicated to the memory of François Jaeger*

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### 1. A positional game.

An interesting positional game can be described as follows: two players, called respectively Red and Blue, play alternately by coloring with their color an edge in a graph  $G$ . The first one who achieves a monochromatic odd cycle wins and his opponent loses; if no monochromatic odd cycle is obtained, it is a draw. It would be interesting to know for which graphs one of the players can win.

A non-constructive proof for the existence of a winning strategy is:

**THEOREM 1.** — *If the players Red and Blue play the game with a graph  $G$  of chromatic number  $> 4$ , there exists a winning strategy for the first player.*

*Proof.* — Assume that Red is the first player and has no winning strategy with a graph  $G$ ; we shall show that this leads to a contradiction.

Clearly, for his opponent, no strategy  $\sigma$  is winning, because otherwise Red could win by using the same strategy  $\sigma$  (determined for a fictitious game where Blue starts, his first move being an arbitrary chosen edge),

which is a contradiction. Thus both Red and Blue can guarantee a draw. At the end of the game, all the edges of  $G$  have been colored; the red edges define a partial graph  $G_R$  with no odd cycles and the blue edges define a partial graph  $G_B$  with no odd cycles. Since the graph  $G_R$  is bipartite, its vertices can be colored with only two colors, say  $\alpha$  and  $\alpha'$ , so that no two adjacent vertices have the same color; also, the vertices of  $G_B$  can be colored with two colors  $\beta$  and  $\beta'$  so that no two adjacent vertices of  $G_B$  have the same color. If we assign to each vertex a label  $(\alpha, \beta)$ ,  $(\alpha, \beta')$ ,  $(\alpha', \beta)$  or  $(\alpha', \beta')$ , corresponding to the pair of colors used for this vertex in  $G_R$  and in  $G_B$ , no two adjacent vertices of  $G$  have the same label. This implies that the graph  $G$  is 4-colorable. A contradiction.

For 4-colorable graphs, and in particular for planar graphs, no simple structural property has been found so far to see that the game is unfair.

## 2. The König property.

Given a simple graph  $G = (X, E)$ , the *edge-transversal number*  $\tau_e(G)$  is the least number of edges to remove from  $G$  in order to destroy all the odd cycles. The *edge-packing number*  $\nu_e(G)$  is the maximum number of pairwise edge-disjoint odd cycles; if  $G$  has no odd cycles, we put  $\nu_e(G) = 0$ .

Clearly, we have always  $\tau_e(G) \geq \nu_e(G)$ , and if for some graph  $G$  this inequality holds with equality, it is easier to check that some edge-transversal is minimum (or that some edge-packing is maximum). It would be interesting to determine the graphs for which the values of these two coefficients can be obtained in polynomial time (see [2], [9] for some related results where the words "edge" and "vertex" are interchanged).

When these two coefficients are equal, we say that we have the *König property (for the odd cycles)*.

We do not always have this König property, even for planar graphs; for instance, the complete graph  $K_4$  is planar, but  $\tau_e(K_4) = 2$  and  $\nu_e(K_4) = 1$ . However we have proved in [3] that for all the planar graphs having a fixed edge-packing number, their edge-transversal numbers are bounded.

Graphs for which each pair of odd cycles has at least one vertex in common and yet the number of vertices needed to hit them all is large, have been considered by Lovász [7]. Similarly, we have considered in [3] a family of graphs  $G$  for which  $\nu_e(G) = 1$  and their edge-transversal numbers can be arbitrarily large.

We do not know any simple characterization for the graphs whose odd cycles satisfy the König property, but we have:

**THEOREM 2.** — *Let  $G$  be a graph such that  $\tau_e(G') = \nu_e(G')$  for every partial graph  $G'$  of  $G$ . Then its chromatic number is  $< 4$ .*

*Proof.* — Catlin [4] proved that every graph with chromatic number  $\geq 4$  contains a subdivision of the complete graph  $K_4$  such that each triangle of  $K_4$  is subdivided to form an odd cycle (see [10] and [11]). If the graph  $G$  was not 3-colorable, then a partial graph  $G'$  would be a subdivision of  $K_4$  of the type described above. However  $\tau_e(G') = 2$  is different from  $\nu_e(G') = 1$ . A contradiction.

Similar results can also be obtained by using some basic results of Hypergraph Theory. Let  $H_G$  be the hypergraph whose vertices are the edges of  $G$  and whose edges are the elementary odd cycles of  $G$ . So, using usual notations  $\tau$  and  $\nu$  for the two main coefficients of a hypergraph (see for instance [1]), we can write  $\tau_e(G) = \tau(H_G)$  and  $\nu_e(G) = \nu(H_G)$ .

Every hypergraph  $H$  satisfies  $\tau(H) \geq \nu(H)$ . If  $\tau(H) = \nu(H)$ , the hypergraph  $H$  is said to have the *König property*. A hypergraph  $H$  has the *Helly property* if every family of pairwise intersecting edges has a non-empty intersection. For a hypergraph  $H$  with the Helly property, the most usual way to prove that  $H$  satisfies the König property is to show that the line-graph (or “intersection graph”)  $L(H)$  is a perfect graph; but unfortunately, as we have seen above, the hypergraph  $H_G$  does not necessarily satisfy the Helly property.

The property of  $G$  described in the statement of Theorem 2 implies that  $H_G$  is a *normal* hypergraph (Lovász [6]). Fournier and Las Vergnas [5] have shown that the vertex-set of a normal hypergraph can be split into two classes, the “red” vertices and the “blue” vertices, so that no edge of the hypergraph is monochromatic. This shows that *this graph  $G$  can be split into two partial graphs which are bipartite*. This result is implied by Theorem 2. However, perhaps this alternative approach, via Hypergraph Theory, can shed some light on the problem.

## BIBLIOGRAPHY

- [1] C. BERGE, *Hypergraphs, Combinatorics of finite sets*, North-Holland, Amsterdam, New York, 1989.
- [2] C. BERGE, J.-L. FOUQUET, On the optimal transversals of the odd cycles, *Discrete Math.*, 169 (1997), 169–176.
- [3] C. BERGE, B. REED, Optimal packings of edge disjoint odd cycles, to appear.
- [4] P.C. CATLIN, Hajós' Graph Coloring Conjecture, *J. of Combinat. Theory*, B26 (1979), 268–274.
- [5] J.-C. FOURNIER, M. LAS VERGNAS, Une classe d'hypergraphes bichromatiques, *Discrete Math.*, 2 (1979), 407–410 (see also [1], p. 156).
- [6] L. LOVÁSZ, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.*, 2 (1972), 253–267.
- [7] L. LOVÁSZ, Private communication.
- [8] B. REED, Mango and Blueberries (to appear).
- [9] B. REED, Tree width and tangles, a new measure of connectivity and some applications, *Survey in Combinatorics*, R. Bailey editor, London Math. Soc. Lecture Notes Series 241, Cambridge University Press, Cambridge 1997, 87–162.
- [10] C. THOMASSEN, On the presence of disjoint subgraphs of a specified type, *J. of Graph Theory*, 12 (1988), 101–111.
- [11] B. TOFT, T.R. JENSEN, *Graph coloring problems*, Wiley Interscience, 1995.

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