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HULLS OF SUBSETS OF THE TORUS IN $\mathbb{C}^2$

by Herbert ALEXANDER

Introduction.

We consider in $\mathbb{C}^2$ the polynomial convex hull $\hat{X}$ of a compact subset X of the unit torus $\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$. If a point p in the open unit polydisk Δ^2 is contained in a 1-dimensional analytic subvariety V of the polydisk and if $bV \subseteq X$, then the maximum principle implies that $p \in \hat{X}$. One can ask if the hull of $X \subseteq \mathbb{T}^2$ can be larger than X without the existence of such a variety V with $bV \subseteq X$. It is known that in general a polynomial hull need not contain such analytic structure: this was first demonstrated by G. Stolzenberg [S]. Subsequently John Wermer [W] gave an example of a set X in $\mathbb{C}^2$ lying over the unit circle (*i.e.* $X \subseteq \{(z,w) \in \mathbb{C}^2 : |z| = 1\}$) such that $\hat{X}$ contains no analytic structure. Our main result here is that such a set can be found in $\mathbb{T}^2$.

THEOREM. — *There exists a compact subset X of $\mathbb{T}^2$ such that $\hat{X} \setminus X$ is a non-empty subset of Δ^2 and $\hat{X} \setminus X$ contains no analytic subset of positive dimension. Moreover, if V is any pure 1-dimensional analytic subvariety of Δ^2 with $bV \subseteq \mathbb{T}^2$ and Ω is any neighborhood of $\bar{V}$ in $\bar{\Delta}^2$, then we can choose $\hat{X}$ to be contained in Ω.*

Our proof is parallel to Wermer's [W]–however the details differ because we need to construct varieties with boundaries in $\mathbb{T}^2$ and consequently, the linear structure in the w-variable, which underlies Wermer's

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construction, cannot be used here. Instead, in order to keep the boundaries in $\mathbb{T}^2$, we iterate the composition of proper holomorphic correspondences. The first step is to obtain a good explicit approximation to the identity. The "identity" correspondence in this case being the diagonal variety $\{z = w\}$ in Δ^2. Our approximation is by a subvariety V^ϵ that "doubles" the identity and that satisfies $bV^\epsilon \subseteq \mathbb{T}^2$; V^ϵ approaches the identity in the Hausdorff metric as $\epsilon \to 0$. The set X is then obtained essentially as a limit of iterates of the V^ϵ for a sequence of ϵ's approaching 0.

The method of Stolzenberg mentioned above is based on a different type of construction. This has been further developed in recent work of Fornaess and Levenberg [FL] and Duval and Levenberg [DL]. Davidson and Salinas [DS] have applied the theory of hulls of subsets X of $\mathbb{T}^2$ to study operator theoretical variants of $\mathrm{Ext}(X)$.

1. Preliminary remarks and notations.

(a) Notations. — We will denote a point of $\mathbb{C}^2$ by (z, w) and denote the two coordinate functions by z and w. We put $\Delta(\beta, r) = \{z \in \mathbb{C} : |z-\beta| < r\}$ and write $\Delta'(\beta, r) = \{z \in \mathbb{C} : 0 < |z-\beta| < r\}$ for the punctured disk. Δ denotes the open unit disk and Δ^2 the unit polydisk in $\mathbb{C}^2$ with the unit torus $\mathbb{T}^2 = \{(z, w) : |z| = 1, |w| = 1\}$, as its distinguished boundary. Recall that the polynomially convex hull of a compact set $X \subseteq \mathbb{C}^n$ is the set

$$\hat{X} = \{z \in \mathbb{C}^n : |P(z)| \leq ||P||_X \text{ for all polynomials } P \text{ in } \mathbb{C}^n\}$$

where $||P||_X$ is the supremum of $|P|$ over X. For a subset Z of $\mathbb{C}^2$ we denote the fiber of Z by the map z over the point $\lambda \in \mathbb{C}$ by Z_λ , this is defined as the set $\{w \in \mathbb{C} : (\lambda, w) \in Z\}$.

(b) Semicontinuity of the hull. — We recall that the operation of taking the polynomially hull of X is "semi-continuous" in the sense that for all open sets $\mathcal{W} \supseteq \hat{X}$ there exists an open set $\mathcal{V} \supseteq X$ such that $\hat{K} \subseteq \mathcal{W}$ provided that $K \subseteq \mathcal{V}$. We shall often use this fact below without an explicit reference.

(c) Composition of holomorphic correspondences. — Let V be a pure 1-dimensional subvariety of the polydisk Δ^2 with $bV \subseteq \mathbb{T}^2$. This class of varieties can also be described as the set of proper holomorphic

correspondences of the disk Δ with itself. See K. Stein [St] for a general discussion. For these correspondences there is an operation of composition that can be described as follows: if V_1 is given locally by functions $w = W_1^k(z)$, $1 \leq k \leq m_1$ and and V_2 is given locally by functions $w = W_2^j(z)$, $1 \leq j \leq m_2$, then the composition $V_1 \circ V_2$ is given locally by the $m_1 m_2$ functions $W_1^k \circ W_2^j$. In particular, identifying a function with its graph, if these correspondences are functions (i.e., $m_1 = 1$, $m_2 = 1$), then this is just the usual composition of functions. For us the main point is that the family of pure 1-dimensional subvarieties V of the polydisk Δ^2 with $bV \subseteq \mathbb{T}^2$ (*i.e.* the class of *proper* holomorphic correspondences) is closed under composition. The varieties that we construct below will be of the form $V_1 \circ V_2 \circ \cdots \circ V_n$. We remark that Slodkowski [Sl] has proved more generally that the composition of analytic multifunctions is (when defined) also an analytic multifunction. We shall not need this here.

2. Approximation of the identity.

We want to approximate the diagonal $\{w = z\}$ in Δ^2 by a subvariety of Δ^2 with boundary in $\mathbb{T}^2$. More precisely we approximate the diagonal with multiplicity two, $\{(w - z)^2 = 0\}$, by an irreducible subvariety of Δ^2 with boundary in $\mathbb{T}^2$. To do this we shall modify the coefficients of lower order powers of w in the defining equation

$$w^2 - 2zw + z^2 = 0. \tag{1}$$

We construct a family $\{V^\epsilon\}$ of such subvarieties depending on a positive parameter ϵ, $0 < \epsilon < 1$. We define V^ϵ as the set of all $(z, w) \in \mathbb{C}^2$ with $|z| < 1$ and satisfying the equation

$$w^2 - 2A_\epsilon(z)w + B(z) = 0 \tag{2}$$

where

$$A_\epsilon(z) = (1 - \epsilon)(z - \epsilon^2) \tag{3}$$

and

$$B_\epsilon(z) = z\frac{z - \epsilon^2}{1 - \epsilon^2 z}. \tag{4}$$

We shall see below that $V^\epsilon \subseteq \Delta^2$. Note that as $\epsilon \to 0$, $A_\epsilon(z) \to z$ and $B_\epsilon(z) \to z^2$ and so the coefficients of the powers of w on the left hand side of the equation (2) approach the corresponding coefficients of the powers of w on the left hand side of the equation (1). The main point here, which requires some computations, is that $bV^\epsilon = \overline{V^\epsilon} \setminus V^\epsilon \subseteq \mathbb{T}^2$.

Note that for $|z| = 1$,

$$|A_\epsilon(z)| \leq (1-\epsilon)(1+\epsilon^2) < 1 \tag{5}$$

and that

$$|B_\epsilon(z)| = 1, \tag{6}$$

since B_ϵ is a finite Blaschke product. Define a rational function g_ϵ of z by

$$g_\epsilon(z) = \frac{B_\epsilon}{A_\epsilon^2} = \frac{z}{(1-\epsilon)^2(1-\epsilon^2 z)(z-\epsilon^2)}. \tag{7}$$

LEMMA 1. — *On the unit circle $\mathbb{T}$, g_ϵ is real-valued, positive and satisfies*

$$g_\epsilon = |g_\epsilon| > 1. \tag{8}$$

Proof. — By a direct computation one shows that $\overline{g_\epsilon(z)} = g_\epsilon(z)$ for $|z| = 1$ (multiply on the left by z^2 in the numerator and denominator). And so g_ϵ is real-valued on $\mathbb{T}$. Also

$$|g_\epsilon| = \frac{|B_\epsilon|}{|A_\epsilon|^2} = \frac{1}{|A_\epsilon|^2} > 1 \tag{9}$$

on $\mathbb{T}$ by (5), (6) and (7). Hence $g_\epsilon(z) > 1$ or $g_\epsilon(z) < -1$ at each $z \in \mathbb{T}$. Finally since $g_\epsilon(1) > 1$ we conclude, by the connectedness of $\mathbb{T}$, that (8) holds on $\mathbb{T}$. $\square$

Define a function h_ϵ on $\mathbb{T}$ by

$$h_\epsilon = \sqrt{g_\epsilon - 1}.$$

By Lemma 1, we can choose the square root so that h_ϵ is real and positive on $\mathbb{T}$. Then clearly h_ϵ extends to be holomorphic in a neighborhood of $\mathbb{T}$. Now we solve the equation (2) for w and get

$$w = A_\epsilon \pm \sqrt{A_\epsilon^2 - B_\epsilon}. \tag{10}$$

By (7) $B_\epsilon = A_\epsilon^2 g_\epsilon$ and we have $A_\epsilon^2 - B_\epsilon = A_\epsilon^2(1-g_\epsilon) = -A_\epsilon^2 h_\epsilon^2$. We get

$$w = A_\epsilon \pm i A_\epsilon h_\epsilon,$$

for z in some neighborhood of $\mathbb{T}$. For $z \in \mathbb{T}$ we get

$$w = A_\epsilon(1 \pm i h_\epsilon)$$

and so, since h_ϵ is positive on $\mathbb{T}$, on $\mathbb{T}$ we have:

$$|w| = |A_\epsilon||1 \pm ih_\epsilon| = |A_\epsilon|\sqrt{1+h_\epsilon^2} = |A_\epsilon|\sqrt{|g_\epsilon|} = |A_\epsilon|\frac{1}{|A_\epsilon|} = 1. \tag{11}$$

From (11) we conclude that $bV^\epsilon \subseteq \mathbb{T}^2$.

Consider the discriminant $D_\epsilon = A_\epsilon^2 - B_\epsilon$ of (2). By (5) and (6) D_ϵ has no zeros on $\mathbb{T}$. We claim that D_ϵ has two distinct zeros in the unit disk. Since B_ϵ has two zeros in the unit disk it follows from Rouché's theorem that D_ϵ also has two zeros in the unit disk, provided that $|D_\epsilon + B_\epsilon| < |B_\epsilon|$ on $\mathbb{T}$. This last inequality follows from (5) and (6), since $|D_\epsilon + B_\epsilon| = |A_\epsilon|^2 < 1 = |B_\epsilon|$ on $\mathbb{T}$. We want we locate the two zeros of D_ϵ in Δ more precisely. One of these zeros is ϵ^2. We claim that the other zero in the unit disk is in the real interval $(-\epsilon, 0)$. Write $D_\epsilon = (z - \epsilon^2)H_\epsilon(z)$ where

$$H_\epsilon(z) = (1-\epsilon)^2(z-\epsilon^2) - \frac{z}{1-\epsilon^2 z}.$$

In fact clearly $H_\epsilon(0) < 0$ and so we need only show that $H_\epsilon(-\epsilon) > 0$. By a short calculation, $((1+\epsilon^3)/\epsilon)H_\epsilon(-\epsilon) = 1 - (1-\epsilon)^2(1+\epsilon)(1+\epsilon^3) > 0$.

Our constructions below will be based on the varieties V^ϵ. The next result collects the facts that we shall need.

PROPOSITION 2. — *The equation (2) defines subvarieties V^ϵ of Δ^2 with the following properties:*

(a) *The boundary bV^ϵ of V^ϵ is contained in the torus $\mathbb{T}^2$ and consists of two disjoint simple closed real curves each of which is mapped by the coordinate function z diffeomorphically to $\mathbb{T}$.*

(b) *The map $z : V^\epsilon \to \Delta$ is a branched analytic cover of order 2. There are precisely two points in Δ over which the mapping branches: ϵ^2 is one of these points and the second point lies on the negative real axis in the interval $(-\epsilon, 0)$.*

(c) *The sets $\overline{V^\epsilon}$ converge in the Hausdorff metric to the diagonal set $\{(z,w) \in \overline{\Delta^2} : z = w\}$ as $\epsilon \to 0$.*

Moreover let V be a pure 1-dimensional subvariety of Δ^2 with $bV \subseteq \mathbb{T}^2$ and let $\mathcal{U}$ be a neighborhood of bV in $\mathbb{T}^2$. If ϵ is sufficiently small, then $b(V \circ V^\epsilon) \subseteq \mathcal{U}$.

Remark. — As we have noted above, $V \circ V^\epsilon$ is a subvariety of Δ^2 with $b(V \circ V^\epsilon) \subseteq \mathbb{T}^2$.

Proof. — The V^ϵ are defined as subvarieties of $\Delta \times \mathbb{C}$ and are clearly bounded sets. We have seen above that for $(z,w) \in V^\epsilon$, $(z,w) \to \mathbb{T}^2$ as $|z| \to 1$. Thus we can apply the maximum principle to the function w on V^ϵ to conclude that $|w| < 1$ on V^ϵ; *i.e.*, $V^\epsilon \subseteq \Delta^2$.

By the discussion above, $bV^\epsilon \subseteq \mathbb{T}^2$ is the union of the two curves $\{(z, A_\epsilon(z)(1 + ih_\epsilon(z))) : z \in \mathbb{T}\}$ and $\{(z, A_\epsilon(z)(1 - ih_\epsilon(z))) : z \in \mathbb{T}\}$. Since $h_\epsilon \neq 0$ on $\mathbb{T}$, the two curves are disjoint. This gives part (a). We have shown above that D_ϵ has precisely the two zeros in Δ that are given in (b). Since $A_\epsilon(z) \to z$ and $B_\epsilon(z) \to z^2$ uniformly on $\bar{\Delta}$ as $\epsilon \to 0$, (c) follows from the explicit formula (10) for V^ϵ.

Finally the fact that $b(V \circ V^\epsilon) \subseteq \mathcal{U}$ for small enough ϵ follows directly from (c). □

Remark. — The varieties V^ϵ that we have used to approximate the diagonal are annuli such that bV^ϵ is a union of two disjoint simple closed curves in $\mathbb{T}^2$. The referee has pointed out a different approximation parameterized by the unit disk by the map $\lambda \mapsto (\lambda^2, \lambda(\lambda - \epsilon)/(1 - \epsilon\lambda))$. The boundaries of these disks are single curves in $\mathbb{T}^2$ with one self-intersection. (Any sufficiently good approximation to the diagonal by a disk will have such a self- intersection at the boundary.) These disks could be used in place of the V^ϵ in an appropriate version of Proposition 2 and then, without further changes, in our proof of the theorem.

3. The doubling lemma.

The next lemma gives an approximation of a given variety by one with twice the number of sheets and introduces branching over a given point.

LEMMA 3. — *Let V be a pure one-dimensional analytic subvariety of Δ^2 with $bV \subseteq \mathbb{T}^2$ so that $z : V \to \Delta$ is a branched cover of order m. Let $\mathcal{U} \subseteq \mathbb{T}^2$ be an open neighborhood of bV in $\mathbb{T}^2$. Let $\lambda \in \Delta$ be a point over which V is not branched. We can thus choose s so that $V \cap z^{-1}(\Delta(\lambda, s))$ is the union of m components each of which is mapped biholomorphically by z to $\Delta(\lambda, s)$. Assume further that : (*) w maps these m components biholomorphically to mutually disjoint open subsets in $\mathbb{C}$. Then for all sufficiently small $\epsilon > 0$ there exists a pure one-dimensional analytic subvariety W of Δ^2 with $bW \subseteq \mathbb{T}^2$ such that*

(a) $bW \subseteq \mathcal{U}$

and, setting $\beta = (\lambda + \epsilon^2)/(1 + \epsilon^2\lambda)$,

(b) *there exists* $r > 0$ *with* $\overline{\Delta(\beta, r)} \subseteq \Delta(\lambda, s)$ *and such that*

$$W \cap z^{-1}(b\Delta(\beta, r)) = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_m$$

with each γ_j *a (connected) Jordan curve such that* $z : \gamma_j \to b\Delta(\beta, r)$ *is a 2-to-1 covering projection.*

Proof. — We consider first the case $\lambda = 0$. We define the variety W to be the composition $V \circ V^\epsilon$. If ϵ is sufficiently small then (a) holds.

By hypothesis there are m single-valued analytic functions $w_1, w_2, \cdots, w_m$ in $\Delta(0, s)$ so that $V \cap z^{-1}(\Delta(0, s))$ is the union of the graphs of these m functions.

Recall that V^ϵ is given by the two locally defined function $w_1^\epsilon, w_2^\epsilon$. Let Δ' be the punctured disk $\Delta'(\epsilon^2, \epsilon^2/2)$. A germ of w_1^ϵ at any point of Δ' can be analytically continued around every path in Δ'. Moreover, by the construction of V^ϵ, such analytic continuation of w_1^ϵ once around a circle in Δ' about 0 yields the different germ w_2^ϵ. Thus, over Δ', V^ϵ is a connected double cover without branching locus; *i.e.*, $z : V^\epsilon \cap z^{-1}(\Delta') \to \Delta'$ a covering projection of order two and $V^\epsilon \cap z^{-1}(\Delta')$ is connected. If ϵ is sufficiently small, the range of (all continuations in Δ' of) w_1^ϵ lies in $\Delta(0, s)$.

Consider the (multiple-valued) functions $w_j \circ w_1^\epsilon$ in Δ'. Each can be analytically continued on all paths in Δ'. We *claim* that such analytic continuation of $w_j \circ w_1^\epsilon$ once around a circle in Δ' about 0 yields a different germ (i.e., gives rise to a two-valued function). Otherwise, continuation would lead back to the same germ (since by (*) the m sets $w_k(\Delta(0, s))$ are mutually disjoint for $1 \leq k \leq m$). But this implies, by applying the inverse of w_j (which exists by (*)), that analytic continuation of w_1^ϵ once around a circle in Δ' about 0 yields the same germ—a contradiction. This gives the *claim*. Since W is $2m$-sheeted, we conclude from the claim that $W \cap z^{-1}(\Delta')$ is the union of m connected components, each of which is an unbranched double covering of Δ'. Hence the lemma holds in the case $\lambda = 0$ with $\beta = \epsilon^2$ and for any r with $0 < r < \epsilon^2/2$.

Next we consider the general case $\lambda \in \Delta$. For $\alpha \in \Delta$, set

$$\phi_\alpha(z) = \frac{z + \alpha}{1 + \bar{\alpha} z}$$

for $z \in \mathbb{C}$, and let

$$L_\alpha(z, w) = (\phi_\alpha(z), w).$$

L_α is a biholomorphism of Δ^2. We apply the previous case, taking $V_1 = L_{-\lambda}(V)$ for V and $\mathcal{U}_1 = L_{-\lambda}(\mathcal{U})$ for $\mathcal{U}$. Since $\phi_{-\lambda}(\lambda) = 0$, V_1 is unbranched over 0. The previous case gives, for ϵ sufficiently small, that $W_1 = L_{-\lambda}(V) \circ V^\epsilon$ satisfies $bW_1 \subseteq \mathcal{U}_1$ and that W_1 also has the appropriate branching behavior near $z = \epsilon^2$. Finally we let $W = L_\lambda(W_1)$. Since $L_\lambda \circ L_{-\lambda} =$ identity, we have $bW \subseteq \mathcal{U}$ and (b) holds with $\beta = \phi_\lambda(\epsilon^2) = (\lambda+\epsilon^2)/(1+\epsilon^2\lambda)$. Indeed for ϵ sufficiently small, $\phi_\lambda(\epsilon^2) \in \Delta(\lambda, s)$ and from the previous case we conclude that over a small deleted neighborhood of β, W is the union of m connected components, each of which is an unbranched double covering. This gives the lemma. □

4. Proof of the theorem.

LEMMA 4. — *Let W be a pure 1-dimensional analytic subvariety of Δ^2 with $bW \subseteq \mathbb{T}^2$and let $\Delta(\beta, r)$ be a disk with closure contained in Δ. Suppose that $W \cap z^{-1}(b\Delta(\beta, r))$ is the disjoint union of N smooth Jordan (connected!) curves $\gamma_1, \gamma_2, \cdots, \gamma_N$ such that $z : \gamma_j \to b\Delta(\beta, r)$ is a covering projection of order $n_j > 1$ for each $j = 1, 2, \cdots, N$. Then there exists a neighborhood $\mathcal{U}$ of bW in $\mathbb{T}^2$ with the following property: if X is compact with $X \subseteq \mathcal{U} \subseteq \mathbb{T}^2$, then $\hat{X}$ has no continuous sections over $b\Delta(\beta, r)$; i.e., there does not exist a continuous complex valued function f defined on $b\Delta(\beta, r)$ such that $Gr(f) \equiv \{(\lambda, f(\lambda)) : \lambda \in b\Delta(\beta, r)\} \subseteq \hat{X}$.*

Proof. — We can view each γ_j as a submanifold of $b\Delta(\beta, r) \times \mathbb{C}$. Let $\mathcal{N}_j$ be a small tubular neighborhood of γ_j in $b\Delta(\beta, r) \times \mathbb{C}$ with the z-coordinate constant on the fibers of this tubular neighborhood (viewing the tubular neighborhood as a normal bundle). Let $\rho_j : \mathcal{N}_j \to \gamma_j$ be the projection along the fibers; in particular, we have $z(\rho_j(z_1, z_2)) = z_1$. We can choose the $\mathcal{N}_j$ to be disjoint, $j = 1, 2, \cdots, N$. For all sufficiently small neighborhoods $\mathcal{U}$ of bW in $\mathbb{T}^2$, $X \subseteq \mathcal{U}$ implies that $\hat{X} \cap (b\Delta(\beta, r) \times \mathbb{C}) \subseteq \cup_{j=1}^N \mathcal{N}_j$; this is because $\widehat{bW} \cap (\ b\Delta(\beta, r) \times \mathbb{C}) = W \cap (\ b\Delta(\beta, r) \times \mathbb{C}) \subseteq \cup_{j=1}^N \mathcal{N}_j$. Fix such a $\mathcal{U}$. Suppose that $X \subseteq \mathcal{U}$. Arguing by contradiction, suppose that there is a continuous complex valued function f defined on $b\Delta(\beta, r)$ such that $\mathrm{Gr}(f) \equiv \{(\lambda, f(\lambda)) : \lambda \in b\Delta(\beta, r)\} \subseteq \hat{X}$. Then $\mathrm{Gr}(f) \subseteq \cup_{j=1}^N \mathcal{N}_j$. By the connectedness of $\mathrm{Gr}(f)$, $\mathrm{Gr}(f) \subseteq \mathcal{N}_k$, for some k, $1 \le k \le N$. Set $g(\lambda) = \rho_k((\lambda, f(\lambda)))$. Then g is a continuous section of the covering projection (of order n_k) $z : \gamma_k \to b\Delta(\beta, r)$. Since γ_k is connected and $n_k > 1$, this is a contradiction. □

Choose a dense sequence $\{\alpha'_n\}$ in Δ and let $\{\alpha_n\}$ be a sequence in Δ in which each α'_k is repeated infinitely often. Choose $\delta_n > 0$ such that $\delta_n < 1 - |\alpha_n|$ and $\delta_n \to 0$. Set $\Delta_n = \Delta(\alpha_n, \delta_n)$. We construct three sequences for $n \geq 0$: a sequence of subvarieties $\{V_n\}$ of Δ^2 with $bV_n \subseteq \mathbb{T}^2$, a sequence of compact subsets $\{X_n\}$ of $\mathbb{T}^2$ and a sequence of disks $\Delta(\beta_n, r_n)$ such that

(a) $bV_n \subseteq$ the interior in $\mathbb{T}^2$ of X_n for $n \geq 1$.

(b) $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq \cdots$

(c) The diameters of the components of the fibers $(\hat{X}_n)_z$ of $\hat{X}_n$ are less that $1/n$ for each $n \geq 1$ and each $z \in \bar{\Delta}$.

(d) For $n \geq 1$, $\overline{\Delta(\beta_n, r_n)} \subseteq \Delta_n \subseteq \Delta$ is such that there is no continuous section of the map $z : \widehat{X_n} \cap z^{-1}(b\Delta(\beta_n, r_n)) \to b\Delta(\beta_n, r_n)$; *i.e.*, there does not exist a continuous complex valued function defined on $b\Delta(\beta_n, r_n)$ with graph contained in $\widehat{X_n}$.

Construction. — For the sake of a uniform notation we set $V_0 = V$, $\Delta(\beta_0, r_0) = \Delta(0, 1)$. We also choose X_0 to be a sufficiently small compact neighborhood of bV in $\mathbb{T}^2$ so that $\hat{X}_0 \subseteq \Omega$—this is possible since $V \cup bV = \widehat{bV} \subseteq \Omega$. We proceed by induction. We assume that we have already defined $V_0, V_1, V_2, \cdots, V_{n-1}$ and $\Delta(\beta_0, r_0), \Delta(\beta_1, r_1), \Delta(\beta_2, r_2), \cdots, \Delta(\beta_{n-1}, r_{n-1})$ and $X_0, X_1, X_2, \cdots, X_{n-1}$ and that this data satisfies (a)-(d) up to index $n-1$. Then, for $n \geq 1$, we define (i) $\Delta(\beta_n, r_n)$, (ii) V_n and (iii) X_n. Choose a point $\lambda_n \in \Delta_n$ such that V_{n-1} is unramified over λ_n. By moving λ_n slightly, we can arrange so that (*) of Lemma 3 also holds. For sufficiently small ϵ, Lemma 3, applied to V_{n-1}, yields a variety W, β_n and r_n such $\Delta(\beta_n, r_n) \subseteq \Delta_n$, $bW \subseteq \operatorname{int}(X_{n-1})$ (since $bV_{n-1} \subseteq \operatorname{int}(X_{n-1})$ by the induction hypothesis) and such that the map $z : W \cap z^{-1}(b\Delta(\beta_n, r_n)) \to b\Delta(\beta_n, r_n)$ is a union of irreducible double covers. We take $V_n = W$. By Lemma 4 there exists a neighborhood $\mathcal{U}$ of bV_n in $\mathbb{T}^2$ such that: $\mathcal{U} \subseteq \operatorname{int}(X_{n-1})$ and if K is a compact subset of $\mathcal{U}$ then $\hat{K}$ has no continuous section over $b\Delta(\beta_n, r_n)$. Now take X_n to be a compact neighborhood of bV_n with $X_n \subseteq \mathcal{U}$ and we get (a), (b) and (d). Moreover since the fibers of $\widehat{bV_n} = bV_n \cup V_n$ are finite, by taking X_n to be a sufficiently small neighborhood of bV_n, it follows that the connected components of the fibers of $\widehat{X_n}$ each have diameter less that $1/n$. This gives (c) and completes the construction.

Continuing the proof of the theorem, we let $X = \bigcap_{n=1}^{\infty} X_n$. Then

$\hat{X} = \bigcap_{n=1}^{\infty} \widehat{X_n}$. Hence $\hat{X} \subseteq \Omega$, since $\widehat{X_0} \subseteq \Omega$. Also $\hat{X} \setminus X$ is non-empty. To see this note that $\hat{X} \cap \{z = 0\}$ is the intersection of the sets $\widehat{X_n} \cap \{z = 0\}$. And these sets are non-empty since $\widehat{X_n} \cap \{z = 0\} \supseteq V_n \cap \{z = 0\} \neq \emptyset$.

Finally we need to show that $\hat{X} \setminus X$ does not contain analytic structure. We argue by contradiction and suppose that $\hat{X} \setminus X$ contains a 1-dimensional analytic set A. We can assume that A is connected. Then $z(A)$ is open in $\mathbb{C}$. For if not, then $z|A$ is constant $\equiv z_0$. Hence A is contained in the set $\hat{X}_{z_0}$, which is totally disconnected by (c)—a contradiction.

Thus we can choose a regular point $p \in A$ so that z maps a neighborhood of p in A biholomorphically to an open set ω in Δ. Hence there is an analytic function f on ω whose graph is in A. There exists n such that $\overline{\Delta_n} \subseteq \omega$. This is because $\delta_n \to 0$ and each α'_n is repeated infinitely often in $\{\alpha_n\}$. Hence $\overline{\Delta(\beta_n, r_n)} \subseteq \omega$. Then f gives a section of $\hat{X}$ over $b\Delta(\beta_n, r_n)$. Hence f gives a section of $\widehat{X_n} \supseteq \hat{X}$ over $b\Delta(\beta_n, r_n)$. This is a contradiction of (d) and gives the theorem. □

5. Concluding comments.

If the variety V in the theorem is assumed to be irreducible, then the set $X \subseteq \mathbb{T}^2$ constructed in the proof is a minimal set having a non-empty hull without analytic structure. Minimal here means that every proper closed subset of X is polynomially convex. We omit the straightforward proof.

By a well-known result of B. Shiffman [Sh], a (pure one-dimensional) subvariety of Δ^2 with boundary in $\mathbb{T}^2$ can be reflected across $\mathbb{T}^2$ to yield a subvariety of $\mathbb{C}^2$ (or of $\mathbb{P}^2$). The local version was given in [A]. In fact, this reflection procedure works more generally for pseudoconcave subsets Z of Δ^2 with boundary in $\mathbb{T}^2$. Namely, the set $\bar{Z} \cup \tau(Z)$ is pseudoconcave across $\mathbb{T}^2$, where τ is the reflection map $\tau((z,w)) = (1/\bar{z}, 1/\bar{w})$. The local version also holds. This pseudoconcavity across $\mathbb{T}^2$ can be shown by adapting the proof of the Lemma in [A], in part, by replacing the use of the maximum principle by the use of the local maximum modulus principle. In particular, we can apply reflection to sets $Z = \Delta^2 \cap \hat{X}$ for X a compact subset of $\mathbb{T}^2$. More specifically the varieties V_n and the sets X_n and X constructed in the proof of the Theorem can be reflected across $\mathbb{T}^2$. The convergence of X_n to X on Δ^2 clearly extends to convergence, on compact subsets of $\mathbb{C}^n$, of the sets extended by reflection.

Duval and Sibony [DSib] employed Wermer's example [W] to produce extreme points in the cone of positive closed $(1,1)$ currents on $\mathbb{P}^2$ such that these extreme points have no analytic structure in their supports. (First Demailly [D] found extreme points that were not supported by algebraic varieties.) Their construction requires a Wermer set given in all of $\mathbb{C}^2$, not just in the polydisk. As noted in the previous paragraph, by reflecting in $\mathbb{T}^2$, the constructions of the present paper yield sets where the convergence in all of $\mathbb{C}^2$ is evident.

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