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ABOUT G-BUNDLES OVER ELLIPTIC CURVES

by Yves LASZLO(*)

1. Introduction.

In this note, we study principal bundles over a complex elliptic curve X with reductive structure group G. As in the vector bundle case, we first show that a non semistable bundle has a canonical semistable L-structure with L some Levi subgroup of G reducing the study of G-bundles to the study of semistable bundles (Proposition 3.2). We then look at the coarse moduli space M_G of topologically trivial semistable bundles on X (there is not any stable topologically trivial G-bundle) and prove that it is isomorphic to the quotient $[\Gamma(T) \otimes_{\mathbf{Z}} X]/W$ where $\Gamma(T)$ is the group of one parameter subgroups of a maximal torus T and $W = N(G,T)/T$ is the Weyl group (Theorem 4.16). Suppose that G is simple and simply connected and let θ be the longest root (relative to some basis $(\alpha_1, \ldots, \alpha_l)$ of the root system $\Phi(G,T)$). The coroot $\theta^\vee$ of θ has a decomposition $\theta^\vee = \sum_i g_i \alpha_i^\vee$ with g_i a positive integer. Using Theorem 4.16 and Looijenga's isomorphism

$$[\Gamma(T) \otimes_{\mathbf{Z}} X]/W \xrightarrow{\sim} \mathbf{P}(1, g_1, \ldots, g_l),$$

one gets that M_G is isomorphic to the weighted projective space $\mathbf{P}(1, g_1, \ldots, g_l)$ (see 4.17), generalizing the well-known isomorphism $M_{\mathbf{SL}_{l+1}} \xrightarrow{\sim} \mathbf{P}^l$ (see [T] for instance). One recovers for instance the Verlinde formula in this case.

We know that these results are certainly well-known from experts, but we were unable to find any reference in the literature, except of course when G is either **SL** or **GL**. For another point of view, see [BG].

During the referee process of this paper, an independent paper of Friedman, Morgan and Witten has appeared in Duke's eprints (see [FMW]), where the link between Looijenga's result and bundles on elliptic curves is studied from another point of view.

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Notations. — By scheme, we implicitly mean a complex scheme. Let G be a reductive group with a Borel subgroup (resp. a maximal torus) B (resp. $T \subset B$). The corresponding Lie algebras will be denoted by $\mathfrak{t}, \mathfrak{b}$ and $\mathfrak{g}$.

We denote by $W = N(G,T)/T$ the Weyl group and by $\Gamma(T)$ the W-module $\mathrm{Hom}(G_m, T)$.

2. Review on the Harder-Narasimhan reduction.

Let X be an algebraic curve which is smooth, projective and connected and E be a G-bundle on X. Recall that E is semistable if and only if the adjoint bundle $\mathcal{E} = \mathrm{Ad}(E)$ is a semistable vector bundle. Following [AB], let me recall how to define the Harder-Narasimhan reduction of E. Pick a non degenerate invariant quadratic form q on the Lie algebra of G. Then q defines a non degenerate quadratic form on $\mathcal{E}$.

LEMMA 2.1. — *The length $r-1$ of the Harder-Narasimhan filtration*

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_i \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

of $\mathcal{E}$ is even.

Proof. — Let

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_i \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

be the Harder-Narasimhan filtration of $\mathcal{E}$. The Harder-Narasimhan filtration of $\mathcal{E}^*$ is

$$0 \subset \ldots \subset (\mathcal{E}/\mathcal{E}_{r-i})^* \subset \ldots \subset \mathcal{E}^*.$$

Because $\mathcal{E}$ is self-dual, the Harder-Narasimhan filtration is self-dual and one has an isomorphism

$$\mathcal{E}_i \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}_{r-i})^* \tag{2.1}$$

inducing isomorphisms

$$gr_i \xrightarrow{\sim} (gr_{r+1-i})^*, i = 1, \ldots r \tag{2.2}$$

where $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. Assume that the length $r-1$ is odd. Let us consider the morphisms

$$m_k : \mathcal{E}_{r/2} \otimes \mathcal{E}_k \to \mathcal{E}/\mathcal{E}_{k-1},\ 0 \leq k \leq r-1$$

deduced from the Lie bracket of $\mathcal{E}$. The equality (2.2) gives the inequalities

$$\mu_1 > \ldots > \mu_{r/2} > -\mu_{r/2} > \ldots > -\mu_1 \tag{2.3}$$

where μ_i is the slope of the semistable vector bundle $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. In particular, the slopes $\mu_i + \mu_j$ of the subquotients $gr_i \otimes gr_j, i \leq r/2$ and $j \leq k$ which appear in $\mathcal{E}_{r/2} \otimes \mathcal{E}_k$ are not less than $\mu_{r/2} + \mu_k > \mu_k$ though the slopes of the subquotients $gr_i, i \geq k$ which appear $\mathcal{E}/\mathcal{E}_{k-1}$ are $\leq \mu_k$. This shows that m_k is zero for all k and that all elements of$\mathcal{E}_{r/2}$ are *nilpotent*. By (2.1), this algebra is also lagrangian. Suppose that the center of G is of positive dimension. Then $\mathcal{E}$ contains a trivial sub-bundle (of positive rank) as a direct summand which implies that some of the μ_i's is zero, contradicting (2.3). The Lie algebra bundle $\mathcal{E}$ is therefore semisimple and therefore has no no non trivial lagrangian sub-Lie algebra (with respect of the Killing form form) consisting of nilpotent elements, just by a dimension argument. □

It follows that one can index the Harder-Narasimhan filtration of $\mathcal{E}$ such that

$$0 = \mathcal{E}_{-r} \subset \mathcal{E}_{-r+1} \subset \ldots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_{r-1} = \mathcal{E} \tag{2.4}$$

where $\mathcal{E}_{-j}$ is the orthogonal of $\mathcal{E}_{j-1}$. One checks that $\mathcal{E}_0$ is a subalgebra of $\mathcal{E}$. Notice that $\mathcal{E}_0/\mathcal{E}_{-1}$ is self-dual and therefore has slope zero. In particular, the slope of $gr_{-j}, j > 0$ is > 0. As in the proof of the preceding lemma, this immediately implies the sequence of inclusions

$$[\mathcal{E}_{-j}, \mathcal{E}_{-1}] \subset \mathcal{E}_{-j-1} \text{ for all } j. \tag{2.5}$$

In particular, all elements of $\mathcal{E}_{-1}$ are nilpotent. For sake of completeness, let me prove this easy lemma (cf. Th VIII.10.1 of [Bo]).

LEMMA 2.2. — *Let $\mathfrak{g}$ be a reductive algebra endowed with an invariant non degenerate bilinear form. Let $\mathfrak{n}'$ be a subalgebra of $\mathfrak{g}$ whose elements are nilpotent. Then, if the orthogonal of $\mathfrak{n}'$ is a sub-Lie algebra of $\mathfrak{g}$, it is parabolic.*

Proof. — The Lie algebra $\mathfrak{n}'$ is nilpotent. Let $\mathfrak{b}$ be a maximal solvable Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{n}'$ and $\mathfrak{n}$ its nilpotent ideal. By [Bo], Definition VIII.3.3.1, $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. Because all elements of $\mathfrak{n}'$ are nilpotent, $\mathfrak{n}'$ is contained in $\mathfrak{n}$. By [Bo], proposition VII.1.3.10 (iii), the orthogonal of $\mathfrak{n}$ is $\mathfrak{b}$ and the lemma follows. □

Because the orthogonal of $\mathcal{E}_{-1}$ is $\mathcal{E}_0$, the Lie subalgebra $\mathcal{E}_0$ is therefore parabolic with radical $\mathcal{E}_{-1}$ (see [AB], p. 589). Let P be the unique standard parabolic subgroup defined by $\mathcal{E}_0$. If F is the bundle of local trivialization $G_S \xrightarrow{\sim} E_S$ ($S \to X$ étale) whose differential sends $\mathrm{Lie}(P)_S$ to $\mathcal{E}_0$, then F is a P-structure of E. Let us denote by U the unipotent radical of P and by $\bar{P}$ (resp. $\bar{F}$) the quotient P/U (resp. F/U). By construction, $\bar{F}$ is semistable (because $\mathrm{Ad}(\bar{F}) = \mathcal{E}_0/\mathcal{E}_{-1}$).

DEFINITION 2.3. — *With the notation above, the P-bundle F is the Harder-Narasimhan reduction of E.*

Remark 2.4. — It is easy to check that the filtration and therefore the corresponding reduction does not depend on the particular choice of the invariant non degenerate quadratic form on $\mathrm{Lie}(G)$.

Example 2.5. — Suppose that E is the $\mathbf{GL}_n$-bundle of local frames of a vector bundle $\mathcal{E}$ on X with Harder-Narasimhan filtration $0 = \mathcal{E}_0 \ldots \subset \mathcal{E}_i \ldots \subset \mathcal{E}_k = \mathcal{E}$. Let P be the quasi-triangular subgroup of $\mathbf{GL}_n$-defined by the partition $[r_i = \mathrm{rk}(\mathcal{E}_{i+1}) - \mathrm{rk}(\mathcal{E}_i)]_{0 \le i < k}$ of $\mathrm{rk}(\mathcal{E})$. Then F is the P-bundle of local frames compatible with the filtration and $\bar{F}$ is the $\times_i \mathbf{GL}_{r_i}$-bundle of local frames of $\bigoplus_i \mathcal{E}_{i+1}/\mathcal{E}_i$.

We suppose once for all that X is an elliptic curve.

3. Non semistable G-bundles.

Let E be a G-bundle on X and let F be the Harder-Narasimhan reduction of E. Let us consider a Levi factor of P thought as a section $\sigma : \bar{P} \to P$ of the canonical projection $P \twoheadrightarrow \bar{P} = P/U$.

Remark 3.1. — Following Humphreys (see [Hu], 30.2) a Levi factor is a factor of the unipotent radical and not of the radical itself (as in Bourbaki for instance).

PROPOSITION 3.2. — *With the notations above, the P-bundle $\sigma_*(\bar{F})$ is isomorphic to F.*

Remark 3.3. — This is the generalization of the well-known (and easy) fact that any vector bundle on X is a direct sum of semistable vector bundles.

Proof. — Let us denote by

$$1 \to \mathcal{U} \to \mathcal{P} \to \bar{\mathcal{P}} \to 1$$

be the twist of

$$1 \to U \to P \to \bar{P} \to 1$$

by F (see [S], chap. I & 5). Geometrically, $\mathcal{P}$ (resp. $\bar{\mathcal{P}}$) is the group scheme $\mathcal{A}ut_P(F)$ (resp. $\mathcal{A}ut_{\bar{P}}(\bar{F})$). The twisted group $\mathcal{U}$ is the unipotent radical of $\mathcal{P}$ and is isomorphic $F \times_P U$ (P acts on the normal subgroup U by conjugation). As usual, the map

$$\begin{cases} H^1(X,P) & \to & H^1(X,\mathcal{P}) \\ F'' & \longmapsto & \mathcal{I}som_P(F,F'') \end{cases} \text{resp.} \begin{cases} H^1(X,\bar{P}) & \to & H^1(X,\bar{\mathcal{P}}) \\ \bar{F}'' & \longmapsto & \mathcal{I}som_{\bar{P}}(\bar{F},\bar{F}'') \end{cases}$$

are bijective. The image of $\mathcal{I}som_P(F,\sigma_*\bar{F})$ in $H^1(X,\bar{\mathcal{P}})$ is the trivial torsor $\mathcal{I}som_{\bar{P}}(\bar{F},\bar{F})$ and it is enough to show the equality $H^1(X,\mathcal{U}) = \{[\mathcal{U}]\}$ to prove the isomorphism $F \xrightarrow{\sim} \sigma_*(\bar{F})$. With the notations of (2.4), the Lie algebra of $\mathcal{U}$ is $\mathcal{E}_{-1}$. By (2.5), the Lie algebra $\mathcal{E}_{-j}/\mathcal{E}_{-j-1}$ is abelian for any $j \geq 1$. This induces a filtration

$$1 = \mathcal{U}_{-r} \subset \mathcal{U}_{-r+1} \subset \ldots \subset \mathcal{U}_{-2} \subset \mathcal{U}_{-1} = \mathcal{U}$$

by unipotent group schemes where the exponential defines isomorphisms

$$\mathcal{U}_{-j}/\mathcal{U}_{-j-1} \xrightarrow{\sim} \mathcal{E}_{-j}/\mathcal{E}_{-j-1} \; j \geq 1$$

of abelian group schemes. By construction, $\mathcal{E}_{-j}/\mathcal{E}_{-j-1}, j \geq 1$ is semistable of positive slope and therefore

$$H^1(X,\mathcal{E}_{-j}/\mathcal{E}_{-j-1}) = 0, j \geq 1$$

because $g(X) = 1$. This implies the equality

$$H^1(X,\mathcal{U}) = \{[\mathcal{U}]\}.$$

□

4. The coarse moduli space M_G.

Let M_G be the coarse moduli space of semistable G-bundle of trivial topological type (what is the same, the component containing the trivial torsor G_X). Recall that the (closed) points of M_G are S-equivalence classes of semistable G-bundles. The only thing which will be needed about this equivalence relation is the following (cf. [Ra1], Corollary 3.12.1):

4.1. Every class ξ defines a Levi subgroup L such that there exists a stable L-bundle F with $F(G) \in \xi$. Moreover, $F(G)$ is well defined up to isomorphism.

Remark 4.2. — Ramanathan's construction of M_G is written for a curve of genus ≥ 2, but the construction can be made in general (see for instance [LeP] in the case of $G = GL_n$ from which the general case follows).

4.3. We denote by $a \otimes b$ the product of two T-bundles a and b (for the natural structure of abelian group of $H^1(X, T)$). Let $\underline{\psi} = (\psi)_{i \in I}$ be a finite family of one parameter subgroups and $\underline{L} = (L_i \in I)$ a family of line bundles of degree 0 on X (thought as G_m-torsors). Then, $\underset{i \in I}{\otimes} L_i(\psi_i)$ is a T-structure of a G-bundle $\underline{L}_{\underline{\psi}}$ on X which is semistable. This defines a morphism of abelian groups

$$p : \Gamma(T) \otimes_{\mathbf{Z}} X \to H^1(X, T).$$

Chose a (closed) point x of X which defines an isomorphism $\mathrm{Pic}^0(X) \xrightarrow{\sim} X$ and a Poincaré line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^0(X)$. This allows to construct a universal semistable T-bundle $\mathbf{L}$ on $X \times \Gamma(T) \underset{\mathbf{Z}}{\otimes} X$.

Remark 4.4. — The theta line bundle Θ on $X = \mathrm{Pic}^1(X)$ becomes through the isomorphism $X \xrightarrow{\sim} \mathrm{Pic}^0(X)$ the determinant bundle $\det(R\Gamma\mathcal{P})^*$.

The family of semistable bundles $\mathbf{L}(G)$ defines a morphism of (reduced) schemes

$$\Gamma(T) \otimes_{\mathbf{Z}} X \to M_G.$$

The action of the Weyl group W on $\Gamma(T)$ defines an action $\Gamma(T) \underset{\mathbf{Z}}{\otimes} X$ such that $w.L_\psi \xrightarrow{\sim} L_\psi$ for all $w \in W$. Let

$$\pi : \ [\Gamma(T) \otimes_{\mathbf{Z}} X]/W \to M_G$$

be the induced morphism. We want to prove that π is an isomorphism.

4.5. Let us prove that π is finite. Let $G \to \mathbf{SL}_N$ be a faithful representation of G inducing a morphism $M_G \to M_{\mathbf{SL}_N}$. Let L be the inverse of the determinant bundle on $M_{\mathbf{SL}_N}$.

Remark 4.6. — Notice that in this case, $M_{\mathbf{SL}_N} = \mathbf{P}^{N-1}$ and that the determinant bundle is just $\mathcal{O}(1)$ (see [Tu], Theorem 7 for instance).

LEMMA 4.7. — *The line bundle $\pi^*(L)$ is ample.*

Proof. — One can assume that G is semisimple. Let q be the natural morphism

$$q : \ \Gamma(T) \otimes_{\mathbf{Z}} X \to M_G.$$

It is enough to prove that $q^*(L)$ is ample. Let us choose a basis of $\Gamma(L)$ identifying $\Gamma(T) \underset{\mathbf{Z}}{\otimes} X$ with X^l (l is the rank of G). Let $\gamma : G_m \to T$ be a non trivial element in $\Gamma(T)$. Let $q_\gamma : X \to M_G$ be the morphism defined by γ. One can assume that $\gamma(z) = \mathrm{diag}(z^{\gamma_1}, \dots, z^{\gamma_l})$ for $z \in \mathbf{C}^*$ (with $\sum \gamma_i = 0$). Then (see Remark 4.4),

$$q_\gamma^*(L) = \Theta^{\sum_i \gamma_i^2}$$

which is ample because $\sum_i \gamma_i^2 > 0$ (recall that γ is non trivial). The rank-N vector bundle bundle parameterized by $(x_1, \dots, x_l)$ is

$$\bigoplus_i \mathcal{O}\Big(\sum_{\gamma \in \gamma} \gamma_i (x_\gamma - x) \Big).$$

By additivity of the determinant bundle, $q^*(L)$ is of the form

$$\boxtimes_{1 \le i \le l} \Theta^{b_i} \text{ with } b_i > 0$$

and therefore is ample. ☐

The fibers of π are therefore finite, and the proper morphism π is finite.

4.8. Let $\pi^{-1}(0)$ be the fiber of π at the trivial bundle G_X. Let us first prove that $\pi^{-1}(0)$ is set-theoretically reduced to [0], the class $W.0$. Let us first prove the following general result.

4.9. Let us consider the following situation: let $p : \mathcal{X} \to S$ be a proper morphism such that $\mathcal{O}_S \to p_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism. Assume that p has a section $\sigma : S \to \mathcal{X}$. Let $A \subset B$ be a reductive subgroup of a linear group B.

LEMMA 4.10. — *Let α be an A-bundle trivial along σ. Then, if the associated B-bundle $\beta = \alpha(B)$ is trivial, the A-bundle α is so.*

Proof. — Let s be the section of β/A defined by α. Because β/A is affine over S, the section s factors through p in a section $\tilde{s}$. Because α is trivial along σ, the section $\tilde{s}$ comes from a section of the restriction to σ of the trivial bundle β and can be lifted to a section s' of β. The section s' mod A ofβ/A is equal to s and defines a trivialization of $\alpha = s^*\beta$. ☐

4.11. Choose an embedding G in a product $G' = \prod_i \mathbf{GL}_{n_i}$ of linear groups such that $Z_0(G) \subset Z_0(G')$ (Z_0 denotes the neutral component). Let T' be a maximal torus of G' containing T. Let $f : M_G \to M_{G'}$ be a natural morphism (see [Ra2], Corollary of Theorem 7.1). Let E be a T-bundle such that $E \in \pi^{-1}(0)$ and let E' be the corresponding T'-bundle.

Because $f(E(G)) = [E'(G')]$, the semistable bundle is equivalent to the trivial bundle and is therefore trivial (a direct sum of line bundles of degree 0 is equivalent to the trivial bundle if and only if all summands are trivial). Applying the preceding lemma with $\alpha = E, A = T, B = G'$ and $\mathcal{X} = X$ for instance, one gets that E is trivial.

4.12. It remains to show that π is étale at the origin: this will follow from the fact that the completion of π at the origin can be identified to the completion at the origin of the Chevalley isomorphism $\mathfrak{t}/W \xrightarrow{\sim} \mathfrak{g}/G$.

LEMMA 4.13. — *The morphism* $\pi : (\Gamma(T) \otimes \mathrm{Pic}^0(X))/W \to M_G$ *is étale at the origin.*

Proof. — Let's briefly recall how to construct the moduli space M_G (see [Ra1], [BLS]), or better of an affine neighborhood M of the trivial bundle $X \times G$ as a GIT quotient Y/H of a smooth affine scheme Y by some reductive group H (with Lie algebra $\mathfrak{h}$). One choose first a faithful representation $G \hookrightarrow \mathbf{GL}_n$ inducing an embedding $\Gamma(T) \otimes \mathrm{Pic}^0(X) \hookrightarrow (\mathrm{Pic}^0(X))^n$. For m big enough, one knows that the canonical morphism

$$\iota_P : \; H^0(P(\mathbf{C}^n) \otimes \mathcal{O}(mx)) \otimes \mathcal{O} \to P(\mathbf{C}^n) \otimes \mathcal{O}(mx)$$

is surjective for all semistable bundles P and that $H^0(\iota_P)$ is bijective. Let χ be the Euler characteristic of some $P(\mathbf{C}^n) \otimes \mathcal{O}(mx)$. By the theory of Hilbert schemes, the pairs (P, ι) where P is a semistable G-bundle and ι an isomorphism

$$H^0(P(\mathbf{C}^n) \otimes \mathcal{O}(mx)) \xrightarrow{\sim} \mathbf{C}^\chi$$

are parameterized by a smooth scheme Y and M_G is a GIT quotient of this scheme by $H = \mathbf{GL}_\chi$ (see [BLS]). Notice that the stabilizer of the "trivial pair" is G itself.

Let $\mathcal{U}$ be the universal T-bundle on $X \times (\Gamma(T) \otimes \mathrm{Pic}^0(X))$. Let us chose a trivialization of the vector bundle $R\Gamma(\mathcal{L} \boxtimes \mathcal{O}(mx))$ on some symmetric affine neighborhood S^0 of 0 in $\mathrm{Pic}^0(X)$. Therefore, the direct image of $\mathcal{U}(\mathbf{C}^n) \boxtimes \mathcal{O}(m)$ is trivial on $S = (S^0)^n \cap \Gamma(T) \otimes \mathrm{Pic}^0(X)$ and the trivialization is W-equivariant. The induced morphism $\pi \colon S \to M_G = Y/H$ is therefore induced by a W-equivariant morphism $S \to Y$ mapping 0 to y. Notice that the orbit $H.y$ is closed. By considering some H-invariant affine open neighborhood of y, one can assume that Y is affine (the quotient Y/H is now a neighborhood of $H.y$ in M_G).

Let's consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathbf{C}[Y]_+ & \longrightarrow & \mathbf{C}[S]_+ \\
\downarrow & & \downarrow \\
V = (T_y^* Y)/\mathfrak{h} & \xrightarrow{k} & T_0^* S
\end{array}
\tag{4.1}
$$

where $\mathbf{C}[Y]_+$ (resp. $\mathbf{C}[S]_+$) denotes the maximal ideal of y (resp. 0). The transpose of k is the tangent map of $S \to \mathcal{M}_G$ from S to the stack of G-bundles on X, namely the Kodaira-Spencer map

$$k: \ \mathfrak{t} = \mathfrak{t} \otimes H^1(X, \mathcal{O}_X) = T_0 S \to (T_y Y)/\mathfrak{h} = \mathfrak{g} \otimes H^1(X, \mathcal{O}) = \mathfrak{g}.$$

LEMMA 4.14. — *The Kodaira-Spencer map k is the canonical inclusion $\mathfrak{t} \hookrightarrow \mathfrak{g}$.*

Proof. — By functoriality, one is reduced to the case where $G = \mathbf{GL}_n$ and T is the torus of invertible diagonal matrices. Consider the one parameter subgroup of differential $aE_{i,i}$ for some integer a ($E_{i,i}$ is the standard diagonal rank 1 matrix). If $(\lambda_{\alpha,\beta})$ is a Cech-cocycle representing $\lambda \in H^1(\mathcal{O})$, the derivative

$$\frac{\partial \pi}{\partial(\gamma \otimes \lambda)}(0)$$

is defined by the vector bundle on $X[\epsilon]/(\epsilon^2 = 0)$ with cocycle $1 + a\epsilon\lambda_{\alpha,\beta}E_{i,i}$. In other words,

$$\frac{\partial \pi}{\partial(\gamma \otimes \lambda)}(0) = \lambda \mathrm{d}\gamma,$$

which proves the lemma. □

Notice that k is $N(G,T)$-equivariant. By Luna's results ([Lu]), one obtains an étale slice of $Y \to Y/H$ as follows.

One choose an $H_y = \mathrm{Aut}_G(X \times G) = G$-invariant section σ of $\mathbf{C}[Y]_+ \to (T_y^* Y)/\mathfrak{h}$ and the induced morphism $Y \to V$ identifies étale locally Y/H and V/H_y. The group $N(G,T)$ being reductive and ${}^t k$ being surjective, one pick an invariant section τ of ${}^t k : \mathfrak{g}^* \twoheadrightarrow \mathfrak{t}^*$ which defines a morphism (still denoted by τ)

$$\mathfrak{t}^* \to \mathbf{C}[Y]_+ \to \mathbf{C}[S]_+$$

which is W-equivariant. This is a W-equivariant section of $\mathbf{C}[S]_+ \to T_0^*S$ and therefore defines an étale slice of $S \to S/W$. Shrinking S and Y if necessary, one obtains from the diagram (4.1) the commutative diagram

$$\begin{array}{ccc} \mathbf{C}[Y]_+^H & \xrightarrow{\pi} & \mathbf{C}[S]_+^W \\ {\scriptstyle \mathbf{S}(\sigma)}\uparrow & & \uparrow{\scriptstyle \mathbf{S}(\tau)} \\ (\mathbf{S}\,\mathfrak{g}^*)^G & \xrightarrow{\mathbf{S}({}^t k)} & (\mathbf{S}\,\mathfrak{t}^*)^W \end{array}$$

where $\mathbf{S}(\sigma)$ and $\mathbf{S}(\tau)$ are étale. By Chevalley's theorem, π is therefore étale at the origin. □

4.15. The morphism π is therefore a finite morphism between normal varieties and is of degree 1. We have proved the

THEOREM 4.16. — *The morphism*

$$\pi : \ [\Gamma(T) \otimes_{\mathbf{Z}} X]/W \to M_G$$

is an isomorphism.

4.17. Assume that G is simple and simply connected. Let θ be the longest root and $\alpha_i, i = 1, \dots l$ the basis of the root system $\Phi(B, G)$. The coroot $\theta^\vee$ is a sum

$$\theta^\vee = \sum_i g_i \alpha_i^\vee$$

where $\alpha_i^\vee$ is the coroot of α. Then Looijenga [Lo] has proved that $[\Gamma(T) \otimes_{\mathbf{Z}} X]/W$ is the weighted projective space $\mathbf{P}(1, g_1, \dots g_l)$.

Remark 4.18. — The proof in [Lo] is not correct. See [BS] for a more general result and hints for a complete proof.

4.19. It is interesting to remark ([D], remarques 1.8) that $\mathcal{O}(l)$ is locally free if and only if l is a multiple of $\mathrm{lcm}(g_i)$ although it is reflexive (Lemme 4.1 of *loc. cit.*). In particular, M_G is locally factorial if and only if $\mathrm{lcm}(g_i) = 1$, condition which is equivalent to G special in the sense of Serre (look at the table of [Bo]). If one notice that $\mathrm{lcm}(g_i)$ is also the minimal Dynkin index of the representations of G (see [LS]), this funny characterization of special groups in terms of M_G is the version in the genus one case of Proposition 13.2 of [BLS] (which deals with the genus > 1). In all the cases, one has the formula

$$\dim H^0(M_G, \mathcal{O}(l)) = \mathrm{card}(P_l) \tag{4.2}$$

where P_l is the number of dominant weights w such that $< \theta^\vee, w > \leq l$, as predicted by the Verlinde formula (see [Be]).

4.20. Let us explain briefly the link between the theorem of Narasimhan and Seshadri and our description of M_G. Suppose that G is semisimple with maximal compact subgroup K. The theorem of Narasimhan and Seshadri says that the complex points of M_G are parameterized by equivalence classes of pairs of elements of K which commutes (K acting on these pairs diagonally through the adjoint action). Suppose further that G is simply connected. Then such a class has a representative in $T_{\mathbf{R}} \times T_{\mathbf{R}}$ (where $T_{\mathbf{R}}$ is the maximal torus of K). Suppose that $X(\mathbf{C})$ is a complex torus $\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$ of period τ in the Poincaré upper half plane. The complex structure $(a, b) \to a - \tau b$ on $\mathbf{R} \times \mathbf{R}$ induces a complex structure on $T_{\mathbf{R}} \times T_{\mathbf{R}}$ which is naturally the maximal torus T of G. We get a diagram

$$\begin{array}{ccc} \Gamma(T) \otimes \mathbf{C} & \xrightarrow{\sim} & T_{\mathbf{R}} \times T_{\mathbf{R}} \\ \downarrow & & \downarrow \\ \Gamma(T) \otimes X(\mathbf{C}) & \xrightarrow{\pi} & M_G(\mathbf{C}) \end{array}$$

One checks easily that this diagram commutes.

BIBLIOGRAPHY

[AB] M.F. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. Lond., A 308 (1982), 523-615.

[Be] A. Beauville, Conformal blocks, fusion rules and the Verlinde formula, Israel Math. Conf. Proc., 9 (1996), 75-96.

[BS] I. N. Bernshtein, O. V. Shvartsman, Chevalley's theorem for complex crystallographic Coxeter groups, Funkt. Anal. i Ego Prilozheniya, 12 (1978), 79-80.

[BLS] A. Beauville, Y. Laszlo, C. Sorger, The Picard group of the moduli stack of G-bundles on a curve, preprint alg-geom/9608002, to appear in Compos. Math.

[Bo] N. Bourbaki, Groupes et algèbres de Lie, chap. 7, 8 (1990), Masson.

[BG] V. Baranovsky, V. Ginzburg Conjugacy classes in loop groups and G-bundles on elliptic curves, Int. Math. Res. Not., 15 (1966), 733-752.

[D] C. Delorme Espaces projectifs anisotropes, Bull. Soc. Math. France, 103 (1975), 203-223.

[FMW] R. Friedman, J. Morgan, E. Witten Vector bundles and F Theory, eprint hep-th 9701162.

[Hu] J. E. HUMPHREYS, Linear algebraic groups, GTM 21, Berlin, Heidelberg, New-York, Springer (1975).

[LS] Y. LASZLO, C. SORGER Picard group of the moduli stack of G-bundles, Ann. Scient. Éc. Norm. Sup., 4e série, 30 (1997), 499-525.

[LeP] J. LE POTIER, Fibrés vectoriels sur les courbes algébriques, Publ. Math. Univ. Paris 7, 35 (1995).

[Lo] E. LOOIJENGA, Root systems and elliptic curves, Invent. Math., 38 (1976), 17-32.

[Ra1] A. RAMANATHAN, Moduli for principal bundles over algebraic curves, I and II, Proc. Indian Acad. Sci. Math. Sci., 106 (1996), 301-328 and 421-449.

[Ra2] A. RAMANATHAN, Stable principal bundles on a compact Rieman surface, Math. Ann., 213 (1975), 129-152.

[S] J.-P. SERRE, Cohomologie galoisienne, LNM 5(1964).

[T] L. TU, Semistable bundles over an elliptic curve, Adv. Math., 98 (1993), 1–26.

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